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Research article

The minimality of biharmonic hypersurfaces in pseudo-Euclidean spaces

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Abstract: In this paper, we investigate the minimality of biharmonic hypersurfaces with some recurrent operators in a pseudo-Euclidean space.

Keywords: pseudo-Euclidean spaces; biharmonic hypersurfaces; recurrent operators; minimal hypersurfaces

1. Introduction

Let $\phi : M^n_r \to \mathbb{E}^{n+p}_s$ be an isometric immersion of a submanifold M^n_r into the pseudo-Euclidean space $\mathbb{E}^{n+p}_s$. Denoted by Δ and $\vec{H}$ the Laplace-Beltrami operator and the mean curvature vector field of M^n_r, respectively. It is well known that the position vector of M^n_r satisfies

$$\Delta\phi = -n\vec{H}.$$

A submanifold M^n_r is called biharmonic if and only if

$$\Delta\vec{H} = 0.$$

It is easy to see that minimal submanifolds (i.e., $\vec{H} = 0$) are automatically biharmonic. Naturally, we consider the problem to determine if there exist biharmonic submanifolds of $\mathbb{E}^{n+p}_s$, other than the minimal ones. Concerning this problem, B. Y. Chen in 1991 proposed the following

Chen's conjecture: any biharmonic submanifold in a Euclidean space $\mathbb{E}^{n+p}$ is minimal.

The conjecture was proved to be true in the low dimension. We refer to [1, 2] for $n = 3$, [3, 4] for $n = 4$ and [5] for $n = 5$. For higher dimensional cases, the conjecture is still true under additional geometric conditions, and we refer the reader to [6–8] for a review, [9] with references therein for recent progress. We need to point out that Chen's conjecture is still open widely.

When the ambient space is pseudo-Euclidean, Chen's conjecture is not necessarily true. B. Y. Chen and S. Ishikawa gave some nonminimal biharmonic space-like surfaces in $\mathbb{E}^4_s (s = 1, 2)$ (cf. [10]), and nonminimal biharmonic pseudo-Riemannian surfaces in $\mathbb{E}^4_s (s = 1, 2, 3)$ (cf. [1]).

However, the *minimality* of biharmonic hypersurfaces in pseudo-Euclidean spaces is still one of the central topics in this area, and many important signs of progress have been made during the last four decades. For example, B. Y. Chen and S. IshiKawa (cf. [1, 10]) proved that any biharmonic surface in $\mathbb{E}^3_s$ $(s = 1, 2)$ is minimal. Later, F. Defever, G. Kaimakamis, V. Papantoniou in [11] proved that the hypersurface M^3_r with diagonalizable shape operator in $\mathbb{E}^4_s$ is minimal. Y. Fu showed in [12] that such hypersurfaces M^4_r in $\mathbb{E}^5_s$ are minimal. More generally, it is shown in [13] that hypersurfaces M^n_r with diagonalizable shape operator and at most three distinct principal curvatures in $\mathbb{E}^{n+1}_s$ are minimal. The same conclusion holds for the hypersurfaces in $\mathbb{E}^{n+1}_s$ provided that the number of distinct principal curvatures ≤ 6. We refer to [14] for details.

In this paper, we will continue to focus on the minimality of biharmonic hypersurfaces in $\mathbb{E}^{n+1}_s$ with no restriction for the number of distinct principal curvatures and prove the following.

Main theorem *Let M^n_r be a biharmonic hypersurface with a diagonalizable shape operator in a pseudo-Euclidean space. If one among the Ricci operator, the curvature operator, the Jacobi operator or shape operator of M^n_r is recurrent, then M^n_r must be minimal.*

2. Preliminaries

Let $\mathbb{E}^{n+1}_s$, $0 < s < n + 1$, be a pseudo-Euclidean space with metric given by

$$\overline{g} = -\sum_{i=1}^{s} \mathrm{d}y_i^2 + \sum_{j=s+1}^{n+1} \mathrm{d}y_j^2,$$

where $(y_1, y_2, \ldots, y_{n+1})$ is the natural coordinate system of $\mathbb{E}^{n+1}_s$.

Let M^n_r be an n-dimensional hypersurface in $\mathbb{E}^{n+1}_s$. Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M^n_r and $\mathbb{E}^{n+1}_s$, respectively. Let ξ be a local unit normal vector field to M^n_r in $\mathbb{E}^{n+1}_s$, and we denote $\varepsilon = \langle \xi, \xi \rangle = \pm 1$, then the Gauss and Weingarten formulas are given, respectively, by (cf. [15])

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$\overline{\nabla}_X \xi = -A_\xi(X),$$

where h denotes the second fundamental form, and A_ξ denotes the shape operator with respect to ξ. As it is well known, h and A_ξ are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi(X), Y \rangle. \tag{2.1}$$

The mean curvature vector is given by $\vec{H} = H\xi$, with $H = \frac{\langle \xi, \xi \rangle}{n} \mathrm{trace} A_\xi$ the mean curvature of M^n_r in $\mathbb{E}^{n+1}_s$.

Then the Guass and Codazzi equations are given by

$$\begin{aligned} R(X, Y)Z &= \langle A(X), Z \rangle A(Y) - \langle A(Y), Z \rangle A(X), \\ (\nabla_X A)Y &= (\nabla_Y A)X, \end{aligned}$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$(\nabla_X A)Y = (\nabla_X)A(Y) - A(\nabla_X Y). \tag{2.2}$$

Let T be a tensor on the pseudo-Riemannian manifold M_r^n. Then T is said to be *recurrent* if there exists a certain 1-form η on M_r^n satisfying $\nabla_X T = \eta(X)T$ for any $X \in TM_r^n$. Thus, the recurrent (1,1)-tensor can be considered as an extension of the parallel one.

A hypersurface is said to be biharmonic if

$$\Delta\vec{H} = 0.$$

The condition is equivalent to

$$\Delta\vec{H} = 2A\text{grad}H + n\varepsilon H\text{grad}H + (\Delta^{\perp}H + \varepsilon H\text{trace}A^2)\xi = 0.$$

By comparing the vertical and horizontal parts, the above equation is equivalent to the following two equations:

$$\begin{cases} \Delta^{\perp}H + \varepsilon H\text{trace}A^2 = 0, \\ 2A\text{grad}H + n\varepsilon H\text{grad}H = 0, \end{cases} \tag{2.3}$$

where A and $\Delta^{\perp}$ denote by the Weingarten operator and the Laplace operator in the normal bundle of M_r^n in $\mathbb{E}_s^{n+1}$, respectively.

Finally, we give a useful lemma to complete our main theorem.

Lemma 2.1. *Let M_r^n be a biharmonic hypersurface with a diagonalizable shape operator in $\mathbb{E}_s^{n+1}$. Assume that the mean curvature H is not constant, then*

$$\begin{aligned} \nabla_{e_1} e_i &= \sum_{k=1}^{n} \varepsilon_k \omega_{1i}^k e_k = 0, \quad i = 1, 2, \ldots, n, \\ \nabla_{e_i} e_1 &= \varepsilon_i \omega_{ii}^1 e_i, \quad i \neq 1, \end{aligned} \tag{2.4}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame with $\varepsilon_i = \langle e_i, e_i\rangle = \pm 1$, and ω_{ij}^k, $i, j, k = 1, 2, \ldots, n$, are called connection forms.

Proof. Since H is not a constant, there exists a point $p \in U$, where U is an open subset of M_r^n such that grad$H \neq 0$ on U. We know from (2.3) that grad$H$ is a principal direction corresponding to the principal curvature $\frac{-n\varepsilon}{2}H$. Without loss of generality, we denote by

$$\lambda_1 = \frac{-n\varepsilon}{2}H. \tag{2.5}$$

We choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ such that e_1 is parallel to grad$H$, where grad$H = \sum_{i=1}^n \varepsilon_i e_i(H)e_i$, and A takes the following form

$$A(e_i) = \lambda_i e_i, \tag{2.6}$$

which means that e_i is a principal direction of A with the principal curvature λ_i, $i = 1, 2, \ldots, n$. Then

$$e_1(H) \neq 0, \ \ e_i(H) = 0, \ i = 2, 3, \ldots, n. \tag{2.7}$$

Meanwhile, it follows from (2.1) that, for $i, j = 1, 2, \ldots, n$,

$$\begin{aligned} h(e_i, e_i) &= \varepsilon\varepsilon_i\lambda_i\xi, \\ h(e_i, e_j) &= 0, \ \forall i \neq j. \end{aligned} \tag{2.8}$$

We write

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \varepsilon_k \omega_{ij}^k e_k. \tag{2.9}$$

Then, using the Codazzi equation $(\nabla_{e_i} A)e_k = (\nabla_{e_k} A)e_i$, we obtain

$$e_i(\lambda_k)e_k + (\lambda_k - \lambda_j)\varepsilon_j \omega_{ik}^j e_j = e_k(\lambda_i)e_i + (\lambda_i - \lambda_j)\varepsilon_j \omega_{ki}^j e_j,$$

which implies, for distinct $i, j, k = 1, 2, \ldots, n$,

$$e_i(\lambda_j) = \varepsilon_j(\lambda_i - \lambda_j)\omega_{ji}^j. \tag{2.10}$$

$$\omega_{ik}^j(\lambda_k - \lambda_j) = \omega_{ki}^j(\lambda_i - \lambda_j). \tag{2.11}$$

Note that (2.5) and (2.7) lead to

$$e_1(\lambda_1) \neq 0, \;\; e_i(\lambda_1) = 0, \; i = 2, 3, \ldots, n.$$

Then it is easy to compute that

$$0 = [e_i, e_j](\lambda_1) = \varepsilon_1(\omega_{ij}^1 - \omega_{ji}^1)e_1(\lambda_1), \; 2 \leq i \neq j \leq n,$$

which means

$$\omega_{ij}^1 = \omega_{ji}^1, \; 2 \leq i \neq j \leq n. \tag{2.12}$$

Taking $j = 1$ and $2 \leq i, k \leq n$, (2.11) becomes

$$\omega_{ik}^1(\lambda_k - \lambda_1) = \omega_{ki}^1(\lambda_i - \lambda_1),$$

together with (2.12), we see

$$\omega_{ij}^1 = \omega_{ji}^1 = 0, \;\; 2 \leq i, j \leq n, \;\; i \neq j. \tag{2.13}$$

Applying compatibility condition to calculate $\nabla_{e_k}\langle e_i, e_j\rangle = 0$, we conclude

$$\omega_{ki}^i = 0, \;\; \omega_{kj}^i + \omega_{ki}^j = 0, \; i \neq j, \;\; i, j, k = 1, 2, \ldots, n. \tag{2.14}$$

It follows from the first equation in (2.14) that $\omega_{k1}^1 = 0$. Also, it follows from (2.10) and (2.14) that $\omega_{1i}^1 = 0$ and $\omega_{11}^i = 0$, which together with (2.13), we get

$$\omega_{ij}^1 = \omega_{ji}^1 = 0, \;\; 1 \leq i \neq j \leq n. \tag{2.15}$$

Putting this all together, we obtain the claim.

3. Proof of main theorem

Proof of main theorem The idea of the proof is the following. First, we prove that the mean curvature H of M_r^n is a constant by using a proof by contradiction. Then, combining with (2.3), we show that $H = 0$, i.e., M_r^n is minimal.

Now we start to prove that H is a constant.

Suppose on the contrary that grad$H \neq 0$. Recalled, from (2.3) that grad$H$ is a principal direction corresponding to the principal curvature $\frac{-n\varepsilon}{2}H$. We choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$, such that e_1 is parallel to grad$H$, and (2.6) holds.

Case 1: When the Ricci operator is recurrent.

According to [15], we know

$$Ric(Y, Z) = n\langle \vec{H}, h(Y, Z)\rangle - \sum_{i=1}^{n} \varepsilon_i \langle h(Y, e_i), h(Z, e_i)\rangle,$$

and so

$$Ric(e_j, e_j) = n\langle H\xi, h(e_j, e_j)\rangle - \sum_{i=1}^{n} \varepsilon_i \langle h(e_j, e_i), h(e_j, e_i)\rangle. \tag{3.1}$$

Thus, a short calculation together with (2.6) and (2.8) shows

$$Ric(e_j) = Ric(e_j, e_j) = \alpha_j e_j,$$

where

$$\alpha_j = nH\varepsilon_j\lambda_j - \varepsilon\varepsilon_j\lambda_j^2, \quad j = 1, 2, \ldots, n. \tag{3.2}$$

Because the Ricci operator is recurrent, i.e.,

$$(\nabla_X Ric)Y = \eta(X)Ric(Y),$$

for any $X, Y \in TM_r^n$, we have from (2.9) that

$$\nabla e_i Ric(e_j) = \eta(e_i)\alpha_j e_j + \sum_{k=1}^{n} \varepsilon_k \omega_{ij}^k \alpha_k e_k, \quad i, j = 1, 2, \ldots, n. \tag{3.3}$$

Using (2.9) again, it follows from (3.1) that

$$\nabla e_i Ric(e_j) = e_i(\alpha_j)e_j + \alpha_j \sum_{k=1}^{n} \varepsilon_k \omega_{ij}^k e_k, \quad i, j = 1, 2, \ldots, n. \tag{3.4}$$

When $\alpha_j = 0$, for some $j = 1, 2, \ldots, n$, then, (3.3) and (3.4) show

$$\sum_{k=1}^{n} \varepsilon_k \omega_{ij}^k \alpha_k e_k = 0,$$

which tells us that $\alpha_k = 0$, for $k = 2, 3, \ldots, n$. Then it follows from (3.2) that $\lambda_k = 0$ or $\lambda_k = \varepsilon nH$. Thus, M_r^n has three distinct principal curvatures 0, εnH and $\frac{-n\varepsilon}{2}H$.

When $\alpha_j \neq 0$, for all $j = 1, 2, \ldots, n$, then (2.15), (3.3) and (3.4) give, for fix indexes i, j,

$$\sum_{k=2,k\neq j}^{n} \varepsilon_k \omega_{ij}^k (\alpha_k - \alpha_j) e_k + (\eta(e_i)\alpha_j - e_i(\alpha_j)) e_j = 0, \tag{3.5}$$

which implies

$$\begin{cases} \eta(e_i)\alpha_j = e_i(\alpha_j), \quad i \neq j, \\ \alpha_k = \alpha_j, \ k \neq j. \end{cases} \tag{3.6}$$

It follows the first equation of (3.6) that

$$\eta(e_i) = e_i \ln|\alpha_j|, \quad j = 1, 2, \ldots, n, \ j \neq i.$$

Taking $j = 1$ in the above equation, we have

$$\eta(e_i) = e_i \ln|\alpha_1|.$$

The above facts show that $e_i \ln|\alpha_j| = e_i \ln|\alpha_1|$, which implies that $\ln|\frac{\alpha_j}{\alpha_1}| =$ constant. Further, we get that $\alpha_j = c\alpha_1$ (c is a constant), for any $j \neq 1$. Thus, we get that M_r^n has at most three principal curvatures.

In conclusion, M_r^n has at most three distinct principal curvatures. We know from [13] or [14] that H is a constant, then $\text{grad}H = 0$. It is a contradiction.

Case 2: When the curvature operator is recurrent.

Since the curvature operator R is recurrent, there exists a 1-form η such that

$$(\nabla_X R(Y, Z))W = \eta(X) R(Y, Z)W,$$

for any $X, Y, Z, W \in TM_r^n$. Note that $R(e_i, e_j)e_k = 0$, for distinct i, j, k, and so

$$(\nabla_{e_i} R(e_j, e_k))e_l = \eta(e_i) R(e_j, e_k)e_l = 0.$$

Meanwhile, according to the Guass equation, we have from (2.6) and (2.9) that

$$\begin{aligned} 0 = (\nabla_{e_i} R(e_j, e_k))e_l &= e_i(R(e_j, e_k)e_l) - R(e_j, e_k)\nabla_{e_i} e_l \\ &= \langle A(e_k), \nabla_{e_i} e_l \rangle A(e_j) - \langle A(e_j), \nabla_{e_i} e_l \rangle A(e_k) \\ &= \lambda_k \lambda_j \omega_{il}^k e_j - \lambda_j \lambda_k \omega_{il}^j e_k. \end{aligned}$$

Since the linear independent of $\{e_i\}_{i=1}^n$, it follows from (2.15) that, for distinct $i, j, k, l = 2, 3, \ldots, n$,

$$\lambda_k \lambda_j \omega_{il}^k = 0,$$

which means that $\omega_{il}^k = 0$ (otherwise, M_r^n has two distinct principal curvatures), then it follows from (2.11) that

$$(\lambda_l - \lambda_k)\omega_{il}^k = (\lambda_i - \lambda_k)\omega_{li}^k = 0.$$

Then $\lambda_i = \lambda_k$ or $\omega_{li}^k = 0$, for $i \neq k$. In particular, if $\omega_{li}^k = 0$, then we have from (2.11) that $\lambda_l = \lambda_k$. Thus, M_r^n has at most two distinct principal curvatures. The same situation happens as above, and it should be modified.

Case 3: When the Jacobi operator is recurrent.

Because the Jacobi operator is recurrent, i.e.,

$$(\nabla_Y R_X)Z = \eta(Y)R_X(Z),$$

where $R_X(Z) = R(Z, X)X$, for any $X, Y, Z \in TM^n_r$. Then using the Guass equation, and combining with (2.6) and (2.9), we get

$$\begin{aligned}\nabla_{e_i} R_{e_j}(e_k) =&\eta(e_i)R_{e_j}(e_k) + R_{e_j}(\nabla_{e_i} e_k)\\ =&\eta(e_i)R(e_k, e_j)e_j + R(\nabla_{e_i} e_k, e_j)e_j\\ =&\eta(e_i)(\langle A(e_k), e_j\rangle A(e_j) - \langle A(e_j), e_j\rangle A(e_k))\\ &+ (\langle A(\nabla_{e_i} e_k), e_j\rangle A(e_j) - \langle A(e_j), e_j\rangle A(\nabla_{e_i} e_k))\\ =&-\eta(e_i)\varepsilon_j\lambda_j\lambda_k e_k - \varepsilon_j\lambda_j \sum_{l=1,\ l\neq j}^{n} \varepsilon_l\omega^l_{ik}\lambda_l e_l.\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}\nabla_{e_i} R_{e_j}(e_k) =&\nabla_{e_i} R(e_k, e_j)e_j\\ =&e_i(R(e_k, e_j)e_j) - R(e_k, e_j)\nabla_{e_i} e_j\\ =&e_i(\langle A(e_k), e_j\rangle A(e_j) - \langle A(e_j), e_j\rangle A(e_k))\\ &- (\langle A(e_k), \nabla_{e_i} e_j\rangle A(e_j) - \langle A(e_j), \nabla_{e_i} e_j\rangle A(e_k))\\ =&-e_i(\varepsilon_j\lambda_j\lambda_k)e_k - \varepsilon_j\lambda_j\lambda_k \sum_{l=1}^{n} \varepsilon_l\omega^l_{ik} e_l.\end{aligned}$$

By comparing the above two equations, we obtain

$$\varepsilon_j\lambda_j \sum_{l=1,\ l\neq j}^{n} \varepsilon_l\omega^l_{ik}\lambda_l e_l = \varepsilon_j\lambda_j\lambda_k \sum_{l=1}^{n} \varepsilon_l\omega^l_{ik} e_l.$$

When $\lambda_j = 0$, then it is obvious to see that M^n_r has two distinct principal curvatures.

When $\lambda_j \neq 0$, then

$$\sum_{l=2}^{n}(\lambda_l - \lambda_k)\varepsilon_l\omega^l_{ik}e_l - \varepsilon_j\lambda_j\omega^j_{ik}e_j = 0. \tag{3.7}$$

• If $\lambda_l = \lambda_k$, then it follows from (2.11) that

$$(\lambda_i - \lambda_j)\omega^j_{ki} = 0,$$

which means $\lambda_i = \lambda_j$. So M^n_r has two distinct principal curvatures.

• If $\lambda_l \neq \lambda_k$, we will make the inner product of the two sides of (3.7) with e_j and

$$(\lambda_j - \lambda_k)\omega^j_{ik} - \lambda_j\omega^j_{ik} = 0.$$

Then, we have

$$\lambda_k\omega^j_{ik} = 0.$$

Since $\lambda_k \neq 0$, then it follows that $\omega_{ik}^j = 0$. This together with (2.11), we have

$$(\lambda_i - \lambda_j)\omega_{ki}^j = 0.$$

Hence $\lambda_i = \lambda_j$, $i \neq j$. To sum up, we know that M_r^n has at most two distinct principal curvatures. Applying the same arguments as in *Case 2*, we obtain that H is a constant.

Case 4: When the shape operator is recurrent.

Since the shape operator is recurrent, i.e., $(\nabla_X A)Y = \eta(X)A(Y)$, we have from (2.6) that

$$\langle(\nabla_{e_i}A)e_j, e_k\rangle = \eta(e_i)\langle A(e_j), e_k\rangle = 0.$$

Then, we have from (2.2) that

$$\begin{aligned} 0 = \langle(\nabla_{e_i}A)e_j, e_k\rangle &= \langle(\nabla_{e_i})A(e_j), e_k\rangle - \langle A(\nabla_{e_i}e_j), e_k\rangle \\ &= (\lambda_j - \lambda_k)\omega_{ij}^k, \ \ 2 \leq i, j, k \leq n, \end{aligned}$$

which shows that $\lambda_j - \lambda_k = 0$ or $\omega_{ij}^k = 0$.

We claim that $\lambda_j = \lambda_k$.

Suppose on contrary that $\lambda_j \neq \lambda_k$, then $\omega_{ij}^k = 0$. Together with (2.11), it reduces to

$$\omega_{ij}^k(\lambda_j - \lambda_k) = \omega_{ji}^k(\lambda_i - \lambda_k) = 0,$$

which means that $\lambda_i = \lambda_k$, it is impossible. So, M_r^n has two distinct principal curvatures. By making use of the similar methods in *Case 3*, we know that H is a constant.

Summarizing the above four cases, we obtain that H is a constant.

Next, we prove that M_r^n is minimal.

Since H is a constant, it follows from (2.3) and (2.6) that

$$H\sum_{i=1}^{n}\lambda_i^2 = 0,$$

which implies that $H = 0$ or $\lambda_i = 0$, $i = 1, 2, \ldots, n$. Note that $H = \frac{\varepsilon}{n}\sum_{i=1}^{n}\lambda_i$, then, M_r^n must be minimal.

We complete the proof of main theorem.

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Conflict of interest

The authors declare there is no conflicts of interest.

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