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Research article

Solving the conformable Huxley equation using the complex method

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Abstract: We apply the complex method to build new exact solutions for the conformable Huxley equation. The results show that abundant new exact solutions were constructed, which extends the results of Cevikel, Bekir and Zahran. Further, we extend the undetermined form for the exponential function solutions in the complex method.

Keywords: analytical solutions; exact solutions; exponential function solutions; conformable derivative

1. Introduction

Differential equations are used to describe dynamic evolutionary processes in natural sciences, engineering and technology. There are many mathematicians working to construct methods for computing exact solutions to differential equations: for example, the Bäcklund transformation method [1], the inverse scattering method [2], the Darboux transformation method [3], the Hirota bilinear method [4], the tanh-function method [5], the Homotopy analysis method [6], etc. In recent years, there have been many results in constructing exact solutions of partial differential equations with a finite number of integer-order derivatives, such as the conformable fractional derivative, the M-fractional derivative, the alternative fractional derivative, the local fractional derivative and the Caputo-Fabrizio fractional derivatives with exponential kernels.

In Tarasov's points, the above mentioned differential operators are not fractional, with exponential kernels that cannot be considered as fractional derivatives of non-integer orders [7]. Therefore, the method designed for differential operators with integer orders can be embedded in the above mentioned type of differential equations with derivative variants.

For the question of how to distinguish between differential equations with integer-order and with fractional order, Tarasov introduced the nonlocality principle to prove that the conformable fractional derivative with exponential kernels cannot be considered as fractional derivatives of non-integer orders [7]. The derivatives of integer orders are determined by properties of differentiable functions only in the infinitely small neighborhoods of the considered point but not nonlocal. In the sense of conformable derivatives, the fractional differential problems become differential problems with integer-order derivatives that may no longer properly describe the original fractional physical phenomena [8]. Many mathematicians have worked on improving the conformable derivative to make it have more complete properties or apply more; for examples, see [9–12].

There are many ways of find exact solutions of differential equations, but no single method can handle all types of solutions. The complex method [13–15] and its variants, such as the extended complex method [16], can be used to construct meromorphic solutions for certain partial differential equations. Several researchers have tried to apply the complex method to find exact solutions of some higher-order or higher-dimension partial differential equations; for examples, see [17, 18]. The complex method will be used here because this kind of method can construct more types of solutions in the complex domain, such as elliptic function solutions, simply periodic function solutions and exponential function solutions. Also, this is a good attempt to solve differential equations with conformal derivatives by the complex method.

Shallow waters exhibit nonlinear phenomena in the propagation and transformation of waves. Nonlinear differential equations, such as the Huxley equation, describe spectral energy transfer for waves of finite amplitude in shallow waters above a flat seafloor [19, 20]. Chaotic oscillations usually occur in nonlinear dynamical systems. These systems can be represented by the Huxley equation with nonlinear oscillations and external periodic excitation [21].

Although the sub-equation method, the Kudryashov method and the exp-function method have been applied to build exact solutions for the conformable Huxley equation [22] (see Eq (3.1)), we remain committed to finding abundant new exact solutions using the complex method. The new exact solutions may contribute to a much better understanding of the features of the solutions of the conformable Huxley equation.

2. Preliminaries

The basic definition and theorems of conformable derivatives [23] are as follows.

Definition 2.1. *Let $g : (0, \infty) \to R$ be a function. For all $k > 0, \alpha \in (0, 1)$, the CFD of g for order α is defined by*

$$T_\alpha(g(k)) = \lim_{\varepsilon\to 0} \frac{g\left(k + \varepsilon k^{1-\alpha}\right) - g(k)}{\varepsilon}. \tag{2.1}$$

Lemma 2.1. *For $\alpha \in (0, 1]$, if a function $g : (0, +\infty) \to R$ is α-differentiable at $t_0 > 0$, then g is continuous at t_0.*

Lemma 2.2. *Let f and g be α-differential at a point $t > 0$, $\alpha \in (0, 1]$.*

$$\begin{aligned} T_\alpha(\alpha f + bg) &= aT_\alpha(f) + bT_\alpha(g), \textit{ for all } a, b \in R,\\ T_\alpha\left(t^p\right) &= pt^{p-\alpha}, \textit{ for all } p \in R,\\ T_\alpha(\lambda) &= 0, \textit{ for all constant functions } f(t) = \lambda,\\ T_\alpha(fg) &= fT_\alpha(g) + gT_\alpha(f) \textit{ and}\\ T_\alpha\left(\frac{f}{g}\right) &= \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}. \end{aligned} \tag{2.2}$$

If g is differentiable,

$$T_\alpha(g)(t) = t^{1-\alpha}\frac{dg}{dt}(t). \tag{2.3}$$

Giving a nonlinear conformable partial differential equation with two independent variables,

$$P\left(\frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) = 0, 0 < \alpha \leq 1. \tag{2.4}$$

Taking a traveling wave transformation $u(x,t) = w(z), z = x - c\frac{t^\alpha}{\alpha}$ (c is the speed of wave), on Eq (2.4) with

$$\frac{\partial^\alpha}{\partial t^\alpha} = -c\frac{\partial}{\partial \xi}, \frac{\partial^{2\alpha}}{\partial t^{2\alpha}} = c^2\frac{\partial^2}{\partial \xi^2}, \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}, \cdots, \tag{2.5}$$

Eq (2.4) will be reduced to a nonlinear ordinary differential equation (ODE),

$$Q(w, w', w'', w''', \cdots) = 0. \tag{2.6}$$

Weierstrass elliptic function [24] $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with two periods $2\omega_1,\ 2\omega_2$ and satisfies

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{2.7}$$

with the invariants $g_2 = 60s_4,\ g_3 = 140s_6$ and discriminant $\Delta(g_2, g_3) \neq 0$.

Furthermore, $\wp'(-z) = -\wp'(z)$, $2\wp''(z) = 12\wp^2(z) - g_2$, $\wp'''(z) = 12\wp(z)\wp'(z), \cdots$, any kth derivatives of $\wp$ can be deduced by these identities, and $\wp$ has the Laurent series expansion $\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + O(|z|^6)$, with the addition formula

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}]^2. \tag{2.8}$$

The basic related definitions and lemmas of the complex method [13] are as follows.

Given a nonlinear ODE

$$P(w, w', \cdots, w^{(m)}) = 0, \tag{2.9}$$

P is a polynomial in $w(z)$ and its derivatives with constant coefficients.

We assume that the Laurent series expansion of meromorphic solutions of Eq (2.9) are in the form of

$$w(z) = \sum_{k=-q}^{\infty} c_k(z - z_0)^k \ (q > 0). \tag{2.10}$$

Definition 2.2. *If there are exactly p distinct formal meromorphic Laurent series*

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k \tag{2.11}$$

satisfying Eq (2.9), we say Eq (2.9) satisfies the $\langle p, q\rangle$ condition. If only determine p distinct principal parts $\sum_{k=-q}^{-1} c_k z^k$, we say Eq (2.9) satisfies the weak $\langle p, q\rangle$ condition.

Eremenko [25] defined that a meromorphic function $f(z)$ belongs to the class W if $f(z)$ is an elliptic function, a rational function of $e^{\alpha z}(\alpha \in \mathbb{C})$ or a rational function of z.

Lemma 2.3. *([13, 15]) Suppose that an equation*

$$P(w, w', \cdots, w^{(m)}) = bw^n \tag{2.12}$$

satisfies the $\langle p, q\rangle$ *condition, where* $p,\ l,\ m,\ n \in \mathbb{N},\ \deg P(w, w', \cdots, w^{(m)}) < n$. *Then, all meromorphic solutions* w *belong to the class* W. *Furthermore, each elliptic solution with a pole at* $z = 0$ *can be written as*

$$w(z) = \sum_{i=1}^{l-1}\sum_{j=2}^{q} \frac{(-1)^j c_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+B_i}{\wp(z)-A_i}\right]^2 - \wp(z)\right) + \sum_{i=1}^{l-1}\frac{c_{-i1}}{2}\frac{\wp'(z)+B_i}{\wp(z)-A_i} + \sum_{j=2}^{q}\frac{(-1)^j c_{-lj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + c_0, \tag{2.13}$$

where c_{-ij} *are given by (2.11),* $B_i^2 = 4A_i^3 - g_2 A_i - g_3$, $\sum_{i=1}^{l} c_{-i1} = 0$, *and* $c_0 \in \mathbb{C}$.

Each rational function solution $w := R(z)$ *is of the form*

$$R(z) = \sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(z-z_i)^j} + c_0, \tag{2.14}$$

with $l(\leq p)$ *distinct poles of multiplicity* q.

Each simply periodic solution is a rational function $R(\xi)$ *of* $\xi = e^{\alpha z}(\alpha \in \mathbb{C})$. $R(\xi)$ *is of the form*

$$R(\xi) = \sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(\xi-\xi_i)^j} + c_0, \tag{2.15}$$

where $R(\xi)$ *has* $l(\leq p)$ *distinct poles with multiplicity* q.

By the former discussion, the complex method can be described concerning Eq (2.4) as follows:

Step 1 Substituting the transform $u(x,t) = w(z), z = Kx - c\frac{t^\alpha}{\alpha}$, with Eq (2.5) into Eq (2.4) and obtaining the nonlinear ODE Eq (2.6).

Step 2 Substituting (2.11) into Eq (2.6) to determine that the $\langle p, q\rangle$ condition holds.

Step 3 By indeterminant relations (2.13)–(2.15), building the elliptic, rational and simply periodic solutions $w(z)$ of Eq (2.6) with pole at $z = 0$, respectively.

Step 4 By Lemma 2.3 mainly, obtaining all meromorphic solutions $w(z - z_0)$.

Step 5 Substituting the inverse transform T^{-1} into these meromorphic solutions $w(z-z_0)$, we get all exact solutions $u(x,t)$ of the original given Eq (2.4).

3. Proof

The conformable equation is defined as

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = \beta u(x,t)(1-u(x,t))(u(x,t)-\gamma), \tag{3.1}$$

where $\alpha \in (0,1], \beta$ is a non zero constant, and $\gamma \in (0,1)$. Equation (3.1) can be written as

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) + \beta u^3(x,t) - \beta(1+\gamma)u^2(x,t) + \beta\gamma u(x,t) = 0, \quad 0 < \alpha \leq 1. \tag{3.2}$$

Using the transformations $u(x,t) = u(z), z = Kx - \frac{\lambda t^\alpha}{\alpha}$ in Eq (3.2), where K and λ are non-zero constants, it follows that

$$K^2u'' + \lambda u' - \beta\gamma u + \beta(1+\gamma)u^2 - \beta u^3 = 0. \tag{3.3}$$

If $\lambda = 0$ and $\beta = 0$, Eq (3.3) will reduce to the equation $K^2u'' - \beta u^3 = 0$. Then, multiply u', and it will reduce to a first-order Briot-Bouquet differential equation.

If $\gamma = -1$, Eq (3.3) will reduce to the equation $K^2u'' + \lambda u' + \beta u - \beta u^3 = 0$, and its solutions were investigated in [26].

For Eq (3.3), we assume $\gamma \neq -1$. Obviously, Eq (3.3) has no nonconstant polynomial solution and has no nonconstant transcendental entire solution by using the Wiman-Valiron theory. Therefore, we need to consider the nonconstant meromorphic solutions of Eq (3.3) with at least one pole on $\mathbb{C}$.

Assume that a meromorphic solution $u(z)$ satisfies Eq (3.3), and if $u(z)$ has a movable pole at $z = 0$, then in a neighborhood of $z = z_0$, the Laurent series of u is of the form $\sum_{k=-q}^{\infty} c_k(z-z_0)^k$ $(q > 0, c_{-q} \neq 0)$. Substituting the Laurent series into Eq (3.3), balancing the terms u'' and w^3, we have $p = 2, q = 1$, and then

$$\begin{aligned} c_{-1} = \sqrt{2}\sqrt{\frac{1}{\beta}}K, c_0 = \frac{2K\gamma\sqrt{\beta} + 2K\sqrt{\beta} - \lambda\sqrt{2}}{6K\sqrt{\beta}}, c_1 = \frac{\sqrt{2}\left(\beta\left(\gamma^2-\gamma+1\right)K^2 - \frac{\lambda^2}{2}\right)}{18\sqrt{\beta}K^3}, \\ c_2 = \frac{\left(\gamma-\frac{1}{2}\right)K^3(\gamma-2)(\gamma+1)\beta^{\frac{3}{2}} + \frac{3}{2}\left(\beta\left(\gamma^2-\gamma+1\right)K^2 - \frac{2\lambda^2}{3}\right)\lambda\sqrt{2}}{54\sqrt{\beta}K^5}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} c_{-1} = -\sqrt{2}\sqrt{\frac{1}{\beta}}K, c_0 = \frac{2K\gamma\sqrt{\beta} + 2K\sqrt{\beta} + \lambda\sqrt{2}}{6K\sqrt{\beta}}, c_1 = -\frac{\sqrt{2}\left(\beta\left(\gamma^2-\gamma+1\right)K^2 - \frac{\lambda^2}{2}\right)}{18\sqrt{\beta}K^3}, \\ c_2 = \frac{\left(\gamma-\frac{1}{2}\right)K^3(\gamma-2)(\gamma+1)\beta^{\frac{3}{2}} - \frac{3}{2}\left(\beta\left(\gamma^2-\gamma+1\right)K^2 - \frac{2\lambda^2}{3}\right)\lambda\sqrt{2}}{54\sqrt{\beta}K^5}. \end{aligned} \tag{3.5}$$

By comparing the coefficients of z in the expansion of $K^2u'' + \lambda u'$ and $-\beta\gamma u + \beta(1+\gamma)u^2 - \beta u^3$, we have

$$\begin{aligned} &\left(0\cdot c_3 - \frac{2\lambda\beta}{27K^2} - \frac{2\lambda\beta\gamma^3}{27K^2} + \frac{\lambda\beta\gamma^2}{9K^2} + \frac{\lambda\beta\gamma}{9K^2} - \frac{\sqrt{\beta}\lambda^2\sqrt{2}}{9K^3} + \frac{2\sqrt{2}\lambda^4}{27K^5\sqrt{\beta}} - \frac{\sqrt{\beta}\lambda^2\sqrt{2}\gamma^2}{9K^3} + \frac{\sqrt{\beta}\lambda^2\sqrt{2}\gamma}{9K^3}\right) \\ &\cdot\left(0\cdot c_3 - \frac{2\lambda\beta}{27K^2} + \frac{\sqrt{\beta}\lambda^2\sqrt{2}\gamma^2}{9K^3} - \frac{\sqrt{\beta}\lambda^2\sqrt{2}\gamma}{9K^3} - \frac{2\sqrt{2}\lambda^4}{27K^5\sqrt{\beta}} + \frac{\sqrt{\beta}\lambda^2\sqrt{2}}{9K^3} + \frac{\lambda\beta\gamma^2}{9K^2} + \frac{\lambda\beta\gamma}{9K^2} - \frac{2\lambda\beta\gamma^3}{27K^2}\right) \\ &= 0. \end{aligned} \tag{3.6}$$

Then, Eq (3.6) can be reduced to

$$\begin{aligned} &\left(\left(-\frac{2\lambda}{27K^2} - \frac{2\lambda\gamma^3}{27K^2} + \frac{\lambda\gamma^2}{9K^2} + \frac{\lambda\gamma}{9K^2}\right)\beta + \left(-\frac{\lambda^2\sqrt{2}}{9K^3} - \frac{\lambda^2\sqrt{2}\gamma^2}{9K^3} + \frac{\lambda^2\sqrt{2}\gamma}{9K^3}\right)\sqrt{\beta} + \frac{2\sqrt{2}\lambda^4}{27K^5\sqrt{\beta}}\right) \\ &\cdot\left(\left(-\frac{2\lambda\gamma^3}{27K^2} + \frac{\lambda\gamma^2}{9K^2} + \frac{\lambda\gamma}{9K^2} - \frac{2\lambda}{27K^2}\right)\beta + \left(\frac{\lambda^2\sqrt{2}\gamma^2}{9K^3} - \frac{\lambda^2\sqrt{2}\gamma}{9K^3} + \frac{\lambda^2\sqrt{2}}{9K^3}\right)\sqrt{\beta} - \frac{2\sqrt{2}\lambda^4}{27K^5\sqrt{\beta}}\right) = 0, \end{aligned} \tag{3.7}$$

or

$$4\left(K^2(\gamma-2)^2\beta-2\lambda^2\right)\left(\left(\gamma-\frac{1}{2}\right)^2K^2\beta-\frac{\lambda^2}{2}\right)\left(K^2(\gamma+1)^2\beta-2\lambda^2\right)\lambda^2=0. \tag{3.8}$$

Equation (3.3) has two integer Fuchs indexes, $-1, 4$. From Eq (3.6), we know the coefficient c_3 is an arbitrary constant, and the other coefficients $c_4, c_5, \cdots$ can be represented using c_3. Then, Eq (3.3) satisfies the $< p, q >$ condition, and Eq (3.3) is integrable. Therefore, by Lemma 2.3, all meromorphic solutions of Eq (3.3) belong to the class W.

According to the complex method, we will build the meromorphic solutions of Eq (3.3).

Case 1. Rational solutions.

According to (2.14), we assume that the undetermined form of rational solutions of Eq (3.3) with a pole at $z_0 \in \mathbb{C}$ is given by

$$u(z)=c_{-1}(z-z_0)^{-1}+c_0=\pm\frac{\sqrt{2}\sqrt{\frac{1}{\beta}}K}{z-z_0}+\frac{2K\gamma\sqrt{\beta}+2K\sqrt{\beta}\mp\lambda\sqrt{2}}{6K\sqrt{\beta}}. \tag{3.9}$$

From Eq (3.8), we have $\gamma = 0$, $\lambda^2 = 2\beta K^2$ or $\lambda^2 = \frac{1}{2}\beta K^2$; $\gamma = 1$, $\lambda^2 = 2\beta K^2$ or $\lambda^2 = \frac{1}{2}\beta K^2$; $\gamma = 2, \lambda^2 = \frac{9}{2}\beta K^2$; $\gamma = \frac{1}{2}, \lambda^2 = \frac{9}{8}\beta K^2$. From Eq (3.9), only the cases $\gamma = 0, 1$ make Eq (3.3) have the following rational solutions.

When $\gamma = 0$ and $\lambda = \pm\sqrt{2}\sqrt{\beta}K$, we have the following rational solution:

$$u(z)=\pm\frac{\sqrt{2}\sqrt{\frac{1}{\beta}}K}{z-z_0}, \tag{3.10}$$

where z_0 is an arbitrary constant.

When $\gamma = 1$ and $\lambda = \pm\sqrt{2}\sqrt{\beta}K$, we have the following rational solution:

$$u(z)=\mp\frac{\sqrt{2}\sqrt{\frac{1}{\beta}}K}{z-z_0}+1. \tag{3.11}$$

Then, substituting $z = x - c\frac{t^\alpha}{\alpha}$ into the rational solutions (3.10) and (3.11), we get the following exact solutions for Eq (3.1):

$$u(z)=\pm\frac{\sqrt{2}\sqrt{\frac{1}{\beta}}K}{x-c\frac{t^\alpha}{\alpha}-z_0}, \tag{3.12}$$

$$u(z)=\mp\frac{\sqrt{2}\sqrt{\frac{1}{\beta}}K}{x-c\frac{t^\alpha}{\alpha}-z_0}+1. \tag{3.13}$$

Case 2. Elliptic function solutions.

Case 2.1

By (2.13) and $\sum_{i=1}^{2} c_{-1} = 0$, we can assume the following undetermined form of elliptic solutions will satisfy Eq (3.3):

$$u(z)=\sum_{i=1}^{l-1}\frac{c_{-i1}}{2}\frac{\wp'(z)+B_i}{\wp(z)-A_i}+c_0=\frac{\sqrt{2}\,\sqrt{\frac{1}{\beta}}\,K\,(\mathcal{P}'(z;g2,g3)+B_1)}{2\mathcal{P}(z;g2,g3)-A_1}-\frac{\sqrt{2}\,\sqrt{\frac{1}{\beta}}\,K\,(\mathcal{P}'(z;g2,g3)+B_2)}{2\,(\mathcal{P}(z;g2,g3)-A_2)}+c_0, \tag{3.14}$$

where c_0 is a constant. We noted that the Laurent series of (3.14) are as follows:

$$u(z)=-\sqrt{2}\,\sqrt{\frac{1}{\beta}}\,K\,(-A_2+A_1)\,z+\frac{\sqrt{2}\,\sqrt{\frac{1}{\beta}}\,K\,(B_1-B_2)\,z^2}{2}+O\big(z^3\big). \tag{3.15}$$

However, the term of z_{-1} vanished in (3.14), compared to the Laurent series (3.4). It follows that Eq (3.3) has no elliptic solution in the form of (3.14).

Case 2.2

To reduce the complexity of the calculation, rewrite Eq (3.3) into the following form:

$$u''+Au'+Bu+Cu^2+Du^3=0, \tag{3.16}$$

where $A=\frac{\lambda}{K^2}, B=-\frac{\beta\gamma}{K^2}, C=\frac{\beta(1+\gamma)}{k^2}, D=-\frac{\beta}{k^2}$. By (2.13), we assume that the following forms of elliptic solutions satisfy Eq (3.3):

$$w(z)=\frac{1}{\sqrt{-2D}}\frac{\wp'(z,g_2,g_3)+B_1}{\wp(z,g_2,g_3)-A_1}+c_0, \tag{3.17}$$

$$w(z)=-\frac{1}{\sqrt{-2D}}\frac{\wp'(z,g_2,g_3)+B_1}{\wp(z,g_2,g_3)-A_1}+c_0. \tag{3.18}$$

Comparing the coefficients in the Laurent series of solutions $w(z)$ in (3.17) and (3.18) with (3.4) and (3.5), we have the following solutions:

$$w(z)=\frac{1}{\sqrt{-2D}}\frac{\wp'(z,g_2,g_3)+B_1}{\wp(z,g_2,g_3)-A_1}+\frac{A}{\sqrt{-2D}}=-\frac{2}{\sqrt{-2D}}z^{-1}+\frac{A}{\sqrt{-2D}}, \tag{3.19}$$

where $C=A\sqrt{-2D}, A_1=\frac{A^2}{12}+\frac{B}{6}, B_1=0, g_2=\frac{\left(A^2+2B\right)^2}{12}, g_3=-\frac{13\left(A^2+2B\right)^3}{3240}$ and $B=-\frac{A^2}{2}$. Solution (3.19) must degenerate into a rational function, and

$$w(z)=-\frac{1}{\sqrt{-2D}}\frac{\wp'(z,g_2,g_3)+B_1}{\wp(z,g_2,g_3)-A_1}-\frac{A}{\sqrt{-2D}}, \tag{3.20}$$

where $C=-A\sqrt{-2D}, A_1=\frac{A^2}{12}+\frac{B}{6}, B_1=0, g_2=\frac{\left(A^2+2B\right)^2}{12}, g_3=-\frac{\left(A^2+2B\right)^3}{216}$.

Therefore, we obtain the following solutions of Eq (3.3):

$$w(z)=-\frac{\sqrt{2K^2}}{\sqrt{\beta}}\frac{1}{z-z_0}+\frac{\lambda}{\sqrt{2\beta K^2}}, \tag{3.21}$$

where $\beta(1+\gamma)=\frac{\lambda\sqrt{2\beta}}{K^2}, \beta\gamma=\frac{\lambda^2}{2K^2}$, and

$$w(z) = -\frac{1}{\sqrt{-2D}}\frac{\wp'(z-z_0,g_2,g_3)+B_1}{\wp(z-z_0,g_2,g_3)-A_1} - \frac{\lambda}{\sqrt{2\beta K^2}}, \tag{3.22}$$

where $\beta(1+\gamma) = -\frac{\lambda\sqrt{2\beta}}{K^2}$, $A_1 = \frac{\lambda^2}{12K^4} - \frac{\beta\gamma}{6K^2}$, $B_1 = 0$, $g_2 = \frac{(2K^2\beta\gamma-\lambda^2)^2}{12K^8}$, $g_3 = \frac{(2K^2\beta\gamma-\lambda^2)^3}{216K^{12}}$, and z_0 is arbitrary.

Case 2.3

Rewrite Eq (3.3) into the following form:

$$u'' + \frac{\lambda}{K^2}u' - \frac{\beta}{K^2}u(u-1)(u-\gamma) = 0. \tag{3.23}$$

By Lemma 4.5 in the [27], we have $q_1 = 0, q_2 = 1, q_3 = \gamma$, and Eq (3.23) has nonconstant meromorphic solutions if and only if

$$\frac{\lambda}{K^2}\prod\left(\frac{\lambda\mu}{K^2} + q_i + q_j - 2q_k\right)\left(-\frac{\lambda\mu}{K^2} + q_i + q_j - 2q_k\right) = 0, \tag{3.24}$$

where $\mu = \pm\sqrt{\frac{2K^2}{\beta}}$, (ijk) is any permutation of (123). Further, for $\lambda \neq 0$ and $\frac{\lambda}{K^2} = \frac{2q_i-q_j-q_k}{\mu} = \frac{-q_i+2q_j-q_k}{-\mu}$, Eq (3.23) has the following elliptic solutions:

$$w(z) = q_k - \frac{q_i-q_k}{2}e^{-\frac{q_i-q_k}{\mu}z}\frac{\wp'\left(e^{-\frac{q_i-q_k}{\mu}z} - z_0; g_2, 0\right)}{\wp\left(e^{-\frac{q_i-q_k}{\mu}z} - z_0; g_2, 0\right)}, z_0, g_2 \text{ arbitrary}. \tag{3.25}$$

Then substituting $z = x - c\frac{t^\alpha}{\alpha}$ into the solution (3.25), yielding the corresponding exact solutions for Eq (3.1) instantly.

Case 3. Exponential function solutions.

We only consider the case of $c_{-1} = \sqrt{2}\sqrt{\frac{1}{\beta}}K$, for in the case of $c_{-1} = -\sqrt{2}\sqrt{\frac{1}{\beta}}K$, which we omit here, the operation is the same.

Case 3.1

By (2.15), we assume that the undetermined form of the simply periodic solutions of Eq (3.3) is given by

$$w(z) = \frac{c_{-1}}{e^{\theta z}-\xi} + c_0 = \frac{\sqrt{2}\sqrt{\frac{1}{\beta}}K}{e^{\theta z}-\xi} + \frac{2K\gamma - \lambda\sqrt{2}\sqrt{\frac{1}{\beta}} + 2K}{6K}, \tag{3.26}$$

where $\xi \in \mathbb{C}$ is a constant.

Substituting (3.26) into Eq (3.3), combining the similar terms in the expansion of Eq (3.3) and balancing the coefficients, we have

$$\frac{2\sqrt{2}\sqrt{\frac{1}{\beta}}K^3\left(e^{2\alpha z}\alpha^2 - 1\right)}{(e^{\alpha z}-\xi)^3} = 0. \tag{3.27}$$

Equation (3.27) obviously has no algebraic solution of α. This means that Eq (3.3) does not have a simply periodic solution in the shape of (3.26).

Case 3.2

We assume that Eq (3.3) has the following undetermined form of exponential solutions:

$$u(z) = \frac{A(z)}{e^{\alpha z} - \xi} + c_0, \tag{3.28}$$

where $\xi, c_0 \in \mathbb{C}$ are constants, and $A(z)$ is an undetermined function.

Then, substituting (3.28) into Eq (3.3), we have

$$\frac{A\left(-2K^2 e^{2\alpha z}\alpha^2 + \beta A^2\right)}{(-e^{\alpha z} + \xi)^3} = 0. \tag{3.29}$$

Therefore, $A(z) = \pm\frac{\sqrt{2}\,\alpha\, e^{\alpha z} K}{\sqrt{\beta}}$ and $A = 0$ is omitted here.

Case 3.2.1

Substituting (3.28) with $A(z) = \frac{\sqrt{2}\,\alpha\, e^{\alpha z} K}{\sqrt{\beta}}$ into Eq (3.3), we have

$$\frac{3\left(-\frac{2K(\gamma-3c_0+1)\sqrt{\beta}}{3} + \sqrt{2}\left(K^2\alpha + \frac{\lambda}{3}\right)\right)\alpha^2 K\, e^{2\alpha z}}{\sqrt{\beta}\,(e^{\alpha z} - \xi)^2} = 0. \tag{3.30}$$

By solving Eq (3.30), we have

$$c_0 = -\frac{3K^2\sqrt{2}\,\alpha - 2K\sqrt{\beta}\,\gamma - 2K\sqrt{\beta} + \sqrt{2}\,\lambda}{6K\sqrt{\beta}}. \tag{3.31}$$

Then, substituting (3.28) with (3.31) into Eq (3.3), we have

$$\alpha = \pm\frac{\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}}{3K^2}. \tag{3.32}$$

Then, substituting (3.28) with (3.31), (3.32) into Eq (3.3), we have

$$\left(\frac{2}{27}\gamma^3 - \frac{1}{9}\gamma^2 - \frac{1}{9}\gamma + \frac{2}{27}\right)\beta + \left(\frac{\gamma^2\sqrt{2}\,\lambda}{9K} - \frac{\gamma\sqrt{2}\,\lambda}{9K} + \frac{\sqrt{2}\,\lambda}{9K}\right)\sqrt{\beta} - \frac{2\sqrt{2}\,\lambda^3}{27K^3\sqrt{\beta}} = 0. \tag{3.33}$$

Therefore, we have the following solution:

$$u(z) = \frac{\frac{\sqrt{2}\,\alpha\, e^{\alpha z} K}{\sqrt{\beta}}}{e^{\alpha z} - \xi} - \frac{3K^2\sqrt{2}\,\alpha - 2K\sqrt{\beta}\,\gamma - 2K\sqrt{\beta} + \sqrt{2}\,\lambda}{6K\sqrt{\beta}}, \tag{3.34}$$

provided that $\alpha = \pm\frac{\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}}{3K^2}$, and $\left(\frac{2}{27}\gamma^3 - \frac{1}{9}\gamma^2 - \frac{1}{9}\gamma + \frac{2}{27}\right)\beta + \left(\frac{\gamma^2\sqrt{2}\,\lambda}{9K} - \frac{\gamma\sqrt{2}\,\lambda}{9K} + \frac{\sqrt{2}\,\lambda}{9K}\right)\sqrt{\beta} - \frac{2\sqrt{2}\,\lambda^3}{27K^3\sqrt{\beta}} = 0.$

The exponential function solution (3.34) can be reduced to

$$\begin{aligned} u(z) = {} & \pm\frac{\sqrt{2}\,\sqrt{6K^2\beta\gamma^2 - 6K^2\beta\gamma + 6K^2\beta - 3\lambda^2}\, e^{\pm\frac{\sqrt{6K^2\beta\gamma^2 - 6K^2\beta\gamma + 6K^2\beta - 3\lambda^2}\, z}{3K^2}}}{3\sqrt{\beta}\,K\left(e^{\pm\frac{\sqrt{6K^2\beta\gamma^2 - 6K^2\beta\gamma + 6K^2\beta - 3\lambda^2}\, z}{3K^2}} - \xi\right)} \\ & - \frac{\pm\sqrt{2}\,\sqrt{6K^2\beta\gamma^2 - 6K^2\beta\gamma + 6K^2\beta - 3\lambda^2} - 2\sqrt{\beta}\,K\gamma - 2\sqrt{\beta}\,K + \lambda\sqrt{2}}{6\sqrt{\beta}\,K}, \end{aligned} \tag{3.35}$$

provided that $\left(\frac{2}{27}\gamma^3 - \frac{1}{9}\gamma^2 - \frac{1}{9}\gamma + \frac{2}{27}\right)\beta + \left(\frac{\gamma^2\sqrt{2}\lambda}{9K} - \frac{\gamma\sqrt{2}\lambda}{9K} + \frac{\sqrt{2}\lambda}{9K}\right)\sqrt{\beta} - \frac{2\sqrt{2}\lambda^3}{27K^3\sqrt{\beta}} = 0$.

Then, substituting $z = x - c\frac{t^\alpha}{\alpha}$ into the exponential function solution (3.34), we get the following exact solution for Eq (3.1):

$$u(x,t) = \frac{\frac{\sqrt{2}\alpha\, \mathrm{e}^{\alpha K x - \lambda t^\alpha} K}{\sqrt{\beta}}}{e^{\alpha K x - \lambda t^\alpha} - \xi} - \frac{3K^2\sqrt{2}\alpha - 2K\sqrt{\beta}\gamma - 2K\sqrt{\beta} + \sqrt{2}\lambda}{6K\sqrt{\beta}}, \tag{3.36}$$

provided that $\alpha = \pm\frac{\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}}{3K^2}$, and $\left(\frac{2}{27}\gamma^3 - \frac{1}{9}\gamma^2 - \frac{1}{9}\gamma + \frac{2}{27}\right)\beta + \left(\frac{\gamma^2\sqrt{2}\lambda}{9K} - \frac{\gamma\sqrt{2}\lambda}{9K} + \frac{\sqrt{2}\lambda}{9K}\right)\sqrt{\beta} - \frac{2\sqrt{2}\lambda^3}{27K^3\sqrt{\beta}} = 0$.

Case 3.2.2

Substituting (3.28) with $A(z) = -\frac{\sqrt{2}\alpha\, \mathrm{e}^{\alpha z} K}{\sqrt{\beta}}$ into Eq (3.3), we have

$$\frac{3\left(-\frac{2K(\gamma - 3c_0 + 1)\sqrt{\beta}}{3} + \sqrt{2}\left(K^2\alpha + \frac{\lambda}{3}\right)\right)\alpha^2 K\, \mathrm{e}^{2\alpha z}}{\sqrt{\beta}\,(\mathrm{e}^{\alpha z} - \xi)^2} = 0. \tag{3.37}$$

By solving Eq (3.37), we have

$$c_0 = \frac{3K^2\sqrt{2}\alpha + 2K\sqrt{\beta}\gamma + 2K\sqrt{\beta} + \sqrt{2}\lambda}{6K\sqrt{\beta}}. \tag{3.38}$$

Then, substituting (3.28) with (3.38) in Eq (3.3), we have

$$\alpha = \pm\frac{\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}}{3K^2}. \tag{3.39}$$

Then, substituting (3.28) with (3.38) and (3.39) into Eq (3.3), we have

$$\left(-\frac{1}{9}\gamma^2 + \frac{2}{27}\gamma^3 - \frac{1}{9}\gamma + \frac{2}{27}\right)\beta + \left(-\frac{\gamma^2\sqrt{2}\lambda}{9K} + \frac{\gamma\sqrt{2}\lambda}{9K} - \frac{\sqrt{2}\lambda}{9K}\right)\sqrt{\beta} + \frac{2\sqrt{2}\lambda^3}{27K^3\sqrt{\beta}} = 0. \tag{3.40}$$

Therefore, we have the following solution:

$$u(z) = \frac{-\sqrt{2}\alpha\, e^{\alpha z} K}{\sqrt{\beta}(e^{\alpha z} - \xi)} + \frac{3K^2\sqrt{2}\alpha + 2K\sqrt{\beta}\gamma + 2K\sqrt{\beta} + \sqrt{2}\lambda}{6K\sqrt{\beta}}, \tag{3.41}$$

provided that $\alpha = \pm\frac{\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}}{3K^2}$, and $\left(-\frac{1}{9}\gamma^2 + \frac{2}{27}\gamma^3 - \frac{1}{9}\gamma + \frac{2}{27}\right)\beta + \left(-\frac{\gamma^2\sqrt{2}\lambda}{9K} + \frac{\gamma\sqrt{2}\lambda}{9K} - \frac{\sqrt{2}\lambda}{9K}\right)\sqrt{\beta} + \frac{2\sqrt{2}\lambda^3}{27K^3\sqrt{\beta}} = 0$.

Exponential function solution (3.41) can be reduced to

$$\begin{aligned} u(z) = {} & \mp\frac{\sqrt{2}\,\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}\,\mathrm{e}^{\pm\frac{\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}\,z}{3K^2}}}{3K\sqrt{\beta}\left(\mathrm{e}^{\pm\frac{\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2}\,z}{3K^2}} - \xi\right)} \\ & + \frac{\pm\sqrt{2}\,\sqrt{6K^2\beta\gamma^2 - 6\beta\gamma K^2 + 6K^2\beta - 3\lambda^2} + 2K\sqrt{\beta}\gamma + 2K\sqrt{\beta} + \sqrt{2}\lambda}{6K\sqrt{\beta}}. \end{aligned} \tag{3.42}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solution (3.41), we get the following exact solution for Eq (3.1):

$$u(x,t) = \frac{-\sqrt{2}\,\alpha\,\mathrm{e}^{\alpha Kx - ct^\alpha} K}{\sqrt{\beta}(e^{\alpha Kx-ct^\alpha} - \xi)} + \frac{3K^2\sqrt{2}\,\alpha + 2K\sqrt{\beta}\,\gamma + 2K\sqrt{\beta} + \sqrt{2}\,\lambda}{6K\sqrt{\beta}}. \tag{3.43}$$

Case 3.3

Substituting

$$u(z) = \frac{\sqrt{2}\,\alpha\,\mathrm{e}^{\alpha z} K}{\sqrt{\beta}\,(\mathrm{e}^{\alpha z} - \xi)} - \frac{3K^2\sqrt{2}\,\alpha - 2K\sqrt{\beta}\,\gamma - 2K\sqrt{\beta} + \sqrt{2}\,\lambda}{6K\sqrt{\beta}} \tag{3.44}$$

into Eq (3.3), we have

$$-\frac{\left(K^4\alpha^2 - \frac{2\beta(\gamma^2-\gamma+1)K^2}{3} + \frac{\lambda^2}{3}\right)\alpha\sqrt{2}\,\mathrm{e}^{\alpha z}}{2\sqrt{\beta}\,(\mathrm{e}^{\alpha z} - \xi)\,K} = 0. \tag{3.45}$$

It follows that

$$\beta = \frac{3K^4\alpha^2 + \lambda^2}{2K^2\,(\gamma^2 - \gamma + 1)}. \tag{3.46}$$

Substituting (3.44) with (3.46) into Eq (3.3), Eq (3.3) reduces to an algebraic equation:

$$\frac{K\,(\gamma+1)\left(K^4\alpha^2 + \frac{\lambda^2}{3}\right)\left(\gamma - \frac{1}{2}\right)(\gamma - 2)\sqrt{\frac{3K^4\alpha^2+\lambda^2}{K^2}} + 3\left(\gamma^2-\gamma+1\right)^{\frac{3}{2}}\left(K^2\alpha - \frac{\lambda}{3}\right)\left(K^2\alpha + \frac{\lambda}{3}\right)\lambda}{9\,(\gamma^2-\gamma+1)\sqrt{\frac{3K^4\alpha^2+\lambda^2}{K^2}}\,K^3} = 0. \tag{3.47}$$

By solving Eq (3.47), we have

$$\alpha = \pm\frac{\lambda}{K^2\,(2\gamma-1)}, \pm\frac{\gamma\lambda}{K^2\,(\gamma-2)}, \pm\frac{\lambda\,(\gamma-1)}{K^2\,(\gamma+1)}. \tag{3.48}$$

Case 3.3.1

Substituting (3.44) with (3.46) and $\alpha = \frac{\lambda}{K^2(2\gamma-1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z) = \frac{\mathrm{e}^{\frac{\lambda z}{K^2(2\gamma-1)}}}{\mathrm{e}^{\frac{\lambda z}{K^2(2\gamma-1)}} - \xi}. \tag{3.49}$$

Solution (3.49) can be converted to the following form:

$$u(z) = \frac{\cosh\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) + \sinh\left(\frac{\lambda z}{K^2(2\gamma-1)}\right)}{\cosh\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) + \sinh\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) - \xi}. \tag{3.50}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.49) and (3.50), we get the following exact solutions for Eq (3.1):

$$u((x,t) = \frac{\mathrm{e}^{\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}}}{\mathrm{e}^{\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}} - \xi}, \tag{3.51}$$

$$u(x,t)=\frac{\cosh\left(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(2\gamma-1)}\right)+\sinh\left(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(2\gamma-1)}\right)}{\cosh\left(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(2\gamma-1)}\right)+\sinh\left(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(2\gamma-1)}\right)-\xi}. \tag{3.52}$$

Case 3.3.2

Substituting (3.44) with (3.46) and $\alpha=-\frac{\lambda}{K^2(2\gamma-1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z)=\frac{\xi}{-e^{-\frac{\lambda z}{K^2(2\gamma-1)}}+\xi}. \tag{3.53}$$

Solution (3.53) can be converted to the following form:

$$u(z)=\frac{\xi}{-\cosh\left(\frac{\lambda z}{K^2(2\gamma-1)}\right)+\sinh\left(\frac{\lambda z}{K^2(2\gamma-1)}\right)+\xi}. \tag{3.54}$$

Then, substituting $z=Kx-c\frac{t^{\alpha}}{\alpha}$ into the exponential function solutions (3.53) and (3.54), we get the following exact solutions for Eq (3.1):

$$u(x,t)=\frac{\xi}{-e^{-\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(2\gamma-1)}}+\xi}, \tag{3.55}$$

$$u(x,t)=\frac{\xi}{-\cosh\left(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(2\gamma-1)}\right)+\sinh\left(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(2\gamma-1)}\right)+\xi}. \tag{3.56}$$

Case 3.3.3

Substituting (3.44) with (3.46) and $\alpha=\frac{\gamma\lambda}{K^2(2\gamma-1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z)=\frac{\xi\gamma}{-e^{\frac{\gamma\lambda z}{K^2(\gamma-2)}}+\xi}. \tag{3.57}$$

Solution (3.57) can be converted to the following form:

$$u(z)=\frac{\xi\gamma}{-\cosh(\frac{\gamma\lambda z}{K^2(\gamma-2)})-\sinh(\frac{\gamma\lambda z}{K^2(\gamma-2)})+\xi}. \tag{3.58}$$

Then, substituting $z=Kx-c\frac{t^{\alpha}}{\alpha}$ into the exponential function solutions (3.57) and (3.58), we get the following exact solutions for Eq (3.1):

$$u(x,t)=\frac{\xi\gamma}{-e^{\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(\gamma-2)}}+\xi}, \tag{3.59}$$

$$u(x,t)=\frac{\xi\gamma}{-\cosh(\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(\gamma-2)})-\sinh(\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^2(\gamma-2)})+\xi}. \tag{3.60}$$

Case 3.3.4

Substituting (3.44) with (3.46) and $\alpha = -\frac{\gamma\lambda}{K^2(2\gamma-1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z) = \frac{\gamma\, \mathrm{e}^{-\frac{\gamma\lambda z}{K^2(\gamma-2)}}}{\mathrm{e}^{-\frac{\gamma\lambda z}{K^2(\gamma-2)}} - \xi}. \tag{3.61}$$

Solution (3.61) can be converted to the following form:

$$u(z) = \frac{\gamma\left(\cosh\left(\frac{\gamma\lambda z}{K^2(\gamma-2)}\right) - \sinh\left(\frac{\gamma\lambda z}{K^2(\gamma-2)}\right)\right)}{\cosh\left(\frac{\gamma\lambda z}{K^2(\gamma-2)}\right) - \sinh\left(\frac{\gamma\lambda z}{K^2(\gamma-2)}\right) - \xi}. \tag{3.62}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.61) and (3.62), we get the following exact solutions for Eq (3.1):

$$u(x,t) = \frac{\gamma\, \mathrm{e}^{-\frac{\gamma\lambda(Kx - c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}}}{\mathrm{e}^{-\frac{\gamma\lambda z}{K^2(\gamma-2)}} - \xi}, \tag{3.63}$$

$$u(x,t) = \frac{\gamma\left(\cosh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right) - \sinh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right)\right)}{\cosh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right) - \sinh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right) - \xi}. \tag{3.64}$$

Case 3.3.5

Substituting (3.44) with (3.46) and $\alpha = \frac{\lambda(\gamma-1)}{K^2(\gamma+1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to $4\,(\gamma-2)\,(2\gamma-1)\,\lambda^2$. Therefore, we obtain the following solution:

$$u(z) = \frac{(-2\gamma+1)\,\mathrm{e}^{\frac{\lambda(\gamma-1)z}{K^2(\gamma+1)}} - \xi\,(\gamma-2)}{-3\,\mathrm{e}^{\frac{\lambda(\gamma-1)z}{K^2(\gamma+1)}} + 3\xi}, \tag{3.65}$$

provided that $2\left(K\sqrt{\frac{\lambda^2}{K^2}} + \lambda\right)(\gamma-2)\,\lambda^2\,(2\gamma-1) = 0$.

Solution (3.65) can be converted to the following form:

$$u(z) = \frac{(-2\gamma+1)\left(\cosh\left(\frac{\lambda(\gamma-1)z}{K^2(\gamma+1)}\right) + \sinh\left(\frac{\lambda(\gamma-1)z}{K^2(\gamma+1)}\right)\right) - \xi\,(\gamma-2)}{-3\cosh\left(\frac{\lambda(\gamma-1)z}{K^2(\gamma+1)}\right) - 3\sinh\left(\frac{\lambda(\gamma-1)z}{K^2(\gamma+1)}\right) + 3\xi}. \tag{3.66}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.65) and (3.66), we get the following exact solutions for Eq (3.1):

$$u(x,t) = \frac{(-2\gamma+1)\,\mathrm{e}^{\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma+1)}} - \xi\,(\gamma-2)}{-3\,\mathrm{e}^{\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma+1)}} + 3\xi}, \tag{3.67}$$

$$u(x,t)=\frac{(-2\gamma+1)\left(\cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma+1)}\right)+\sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma+1)}\right)\right)-\xi(\gamma-2)}{-3\cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma+1)}\right)-3\sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma+1)}\right)+3\xi}. \tag{3.68}$$

Case 3.3.6

Substituting (3.44) with (3.46) and $\alpha=-\frac{\lambda(\gamma-1)}{K^{2}(\gamma+1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to $4(\gamma-2)(2\gamma-1)\lambda^{2}=0$. Therefore, we obtain the following solution:

$$u(z)=\frac{(\gamma-2)\,\mathrm{e}^{-\frac{\lambda(\gamma-1)z}{K^{2}(\gamma+1)}}+\xi(2\gamma-1)}{-3\,\mathrm{e}^{-\frac{\lambda(\gamma-1)z}{K^{2}(\gamma+1)}}+3\xi}, \tag{3.69}$$

provided that $(\gamma-2)(2\gamma-1)\lambda^{2}=0$.

Solution (3.69) can be converted to the following form:

$$u(z)=\frac{(\gamma-2)\left(\cosh\left(\frac{\lambda(\gamma-1)z}{K^{2}(1+\gamma)}\right)-\sinh\left(\frac{\lambda(\gamma-1)z}{K^{2}(1+\gamma)}\right)\right)+\xi(2\gamma-1)}{-3\cosh\left(\frac{\lambda(\gamma-1)z}{K^{2}(1+\gamma)}\right)+3\sinh\left(\frac{\lambda(\gamma-1)z}{K^{2}(1+\gamma)}\right)+3\xi}. \tag{3.70}$$

Then, substituting $z=Kx-c\frac{t^{\alpha}}{\alpha}$ into the exponential function solutions (3.69) and (3.70), we get the following exact solutions for Eq (3.1):

$$u(x,t)=\frac{(\gamma-2)\,\mathrm{e}^{-\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma+1)}}+\xi(2\gamma-1)}{-3\,\mathrm{e}^{-\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma+1)}}+3\xi}, \tag{3.71}$$

$$u(x,t)=\frac{(\gamma-2)\left(\cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(1+\gamma)}\right)-\sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(1+\gamma)}\right)\right)+\xi(2\gamma-1)}{-3\cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(1+\gamma)}\right)+3\sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(1+\gamma)}\right)+3\xi}. \tag{3.72}$$

Case 3.4

Substituting

$$u(z)=-\frac{\sqrt{2}\,\alpha\,\mathrm{e}^{\alpha z}K}{\sqrt{\beta}\,(\mathrm{e}^{\alpha z}-\xi)}+\frac{3K^{2}\sqrt{2}\,\alpha+2K\sqrt{\beta}\,\gamma+2K\sqrt{\beta}+\sqrt{2}\,\lambda}{6K\sqrt{\beta}} \tag{3.73}$$

into Eq (3.3), we have

$$-\frac{\left(K^{4}\alpha^{2}-\frac{2\beta(\gamma^{2}-\gamma+1)K^{2}}{3}+\frac{\lambda^{2}}{3}\right)\alpha\sqrt{2}\,\mathrm{e}^{\alpha z}}{2\sqrt{\beta}\,(\mathrm{e}^{\alpha z}-\xi)K}=0. \tag{3.74}$$

It follows that

$$\beta=\frac{3K^{4}\alpha^{2}+\lambda^{2}}{2K^{2}(\gamma^{2}-\gamma+1)}. \tag{3.75}$$

Substituting (3.73) with (3.75) into Eq (3.3), Eq (3.3) reduces to an algebraic equation:

$$\frac{\left(\gamma-\frac{1}{2}\right)(1+\gamma)(\gamma-2)\left(K^{4}\alpha^{2}+\frac{\lambda^{2}}{3}\right)K\sqrt{\frac{3K^{4}\alpha^{2}+\lambda^{2}}{K^{2}}}-3\lambda\left(\gamma^{2}-\gamma+1\right)^{\frac{3}{2}}\left(K^{2}\alpha-\frac{\lambda}{3}\right)\left(K^{2}\alpha+\frac{\lambda}{3}\right)}{9\sqrt{\frac{3K^{4}\alpha^{2}+\lambda^{2}}{K^{2}}}\,(\gamma^{2}-\gamma+1)K^{3}}=0. \tag{3.76}$$

By solving Eq (3.76), we have

$$\alpha = \pm\frac{\lambda}{K^2\,(2\gamma-1)}, \pm\frac{\gamma\lambda}{K^2\,(\gamma-2)}, \pm\frac{\lambda\,(\gamma-1)}{K^2\,(\gamma+1)}. \tag{3.77}$$

Case 3.4.1

Substituting (3.73) with (3.75) and $\alpha = \frac{\lambda}{K^2(2\gamma-1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z) = \frac{(-2\gamma+1)\,\mathrm{e}^{\frac{\lambda z}{K^2(2\gamma-1)}} + 2\xi\,(1+\gamma)}{-3\,\mathrm{e}^{\frac{\lambda z}{K^2(2\gamma-1)}} + 3\xi}. \tag{3.78}$$

Solution (3.78) can be converted to the following form:

$$u(z) = \frac{(-2\gamma+1)\left(\cosh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) + \sinh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right)\right) + 2\xi\,(1+\gamma)}{-3\cosh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) - 3\sinh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) + 3\xi}. \tag{3.79}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.78) and (3.79), we get the following exact solutions for Eq (3.1):

$$u(x,t) = \frac{(-2\gamma+1)\,\mathrm{e}^{\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}} + 2\xi\,(1+\gamma)}{-3\,\mathrm{e}^{\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}} + 3\xi}, \tag{3.80}$$

$$u(x,t) = \frac{(-2\gamma+1)\left(\cosh\!\left(\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}\right) + \sinh\!\left(\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}\right)\right) + 2\xi\,(1+\gamma)}{-3\cosh\!\left(\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}\right) - 3\sinh\!\left(\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}\right) + 3\xi}. \tag{3.81}$$

Case 3.4.2

Substituting (3.73) with (3.75) and $\alpha = -\frac{\lambda}{K^2(2\gamma-1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z) = \frac{2\,\mathrm{e}^{-\frac{\lambda z}{K^2(2\gamma-1)}}\gamma - 2\xi\gamma + 2\,\mathrm{e}^{-\frac{\lambda z}{K^2(2\gamma-1)}} + \xi}{3\,\mathrm{e}^{-\frac{\lambda z}{K^2(2\gamma-1)}} - 3\xi}. \tag{3.82}$$

Solution (3.82) can be converted to the following form:

$$u(z) = \frac{(2\gamma+2)\sinh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) + (-2\gamma-2)\cosh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) + 2\xi\gamma - \xi}{3\sinh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) - 3\cosh\!\left(\frac{\lambda z}{K^2(2\gamma-1)}\right) + 3\xi}. \tag{3.83}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.82) and (3.83), we get the following exact solutions for Eq (3.1):

$$u(x,t) = \frac{2\,\mathrm{e}^{-\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}}\gamma - 2\xi\gamma + 2\,\mathrm{e}^{-\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}} + \xi}{3\,\mathrm{e}^{-\frac{\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(2\gamma-1)}} - 3\xi}, \tag{3.84}$$

$$u(x,t)=\frac{(2\gamma+2)\sinh\Big(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(2\gamma-1)}\Big)+(-2\gamma-2)\cosh\Big(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(2\gamma-1)}\Big)+2\xi\gamma-\xi}{3\sinh\Big(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(2\gamma-1)}\Big)-3\cosh\Big(\frac{\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(2\gamma-1)}\Big)+3\xi}. \tag{3.85}$$

Case 3.4.3

Substituting (3.73) with (3.75) and $\alpha=\frac{\gamma\lambda}{K^{2}(2\gamma-1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z)=\frac{(-2\gamma-2)\,e^{\frac{\gamma\lambda z}{K^{2}(\gamma-2)}}-\xi(\gamma-2)}{-3\,e^{\frac{\gamma\lambda z}{K^{2}(\gamma-2)}}+3\xi}, \tag{3.86}$$

provided that $2\left(K\sqrt{\frac{\lambda^{2}}{K^{2}}}+\lambda\right)(2\gamma-1)\,\lambda^{2}(1+\gamma)=0$.

Solution (3.86) can be converted to the following form:

$$u(z)=\frac{(-2\gamma-2)\left(\cosh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)+\sinh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)\right)-\xi(\gamma-2)}{-3\cosh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)-3\sinh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)+3\xi}. \tag{3.87}$$

Then, substituting $z=Kx-c\frac{t^{\alpha}}{\alpha}$ into the exponential function solutions (3.86) and (3.87), we get the following exact solutions for Eq (3.1):

$$u(x,t)=\frac{(-2\gamma-2)\,e^{\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma-2)}}-\xi(\gamma-2)}{-3\,e^{\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma-2)}}+3\xi}, \tag{3.88}$$

$$u(x,t)=\frac{(-2\gamma-2)\left(\cosh\Big(\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma-2)}\Big)+\sinh\Big(\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma-2)}\Big)\right)-\xi(\gamma-2)}{-3\cosh\Big(\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma-2)}\Big)-3\sinh\Big(\frac{\gamma\lambda(Kx-c\frac{t^{\alpha}}{\alpha})}{K^{2}(\gamma-2)}\Big)+3\xi}. \tag{3.89}$$

Case 3.4.4

Substituting (3.73) with (3.75) and $\alpha=-\frac{\lambda(\gamma-1)}{K^{2}(\gamma+1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z)=\frac{(\gamma-2)\,e^{-\frac{\gamma\lambda z}{K^{2}(\gamma-2)}}+2\xi(1+\gamma)}{-3\,e^{-\frac{\gamma\lambda z}{K^{2}(\gamma-2)}}+3\xi}, \tag{3.90}$$

provided that $2\lambda^{2}(2\gamma-1)\left(K\sqrt{\frac{\lambda^{2}}{K^{2}}}+\lambda\right)(1+\gamma)=0$.

Solution (3.90) can be converted to the following form:

$$u(z)=\frac{(\gamma-2)\left(\cosh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)-\sinh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)\right)+2\xi(1+\gamma)}{-3\cosh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)+3\sinh\Big(\frac{\gamma\lambda z}{K^{2}(\gamma-2)}\Big)+3\xi}. \tag{3.91}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.90) and (3.91), we get the following exact solutions for Eq (3.1):

$$u(x,t) = \frac{(\gamma-2)\,\mathrm{e}^{-\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}} + 2\xi\,(1+\gamma)}{-3\,\mathrm{e}^{-\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}} + 3\xi}, \tag{3.92}$$

$$u(x,t) = \frac{(\gamma-2)\left(\cosh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right) - \sinh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right)\right) + 2\xi\,(1+\gamma)}{-3\cosh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right) + 3\sinh\left(\frac{\gamma\lambda(Kx-c\frac{t^\alpha}{\alpha})}{K^2(\gamma-2)}\right) + 3\xi}. \tag{3.93}$$

Case 3.4.5

Substituting (3.73) with (3.75) and $\alpha = \frac{\lambda(\gamma-1)}{K^2(\gamma+1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z) = \frac{\xi\gamma - \mathrm{e}^{\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}}}{-\mathrm{e}^{\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}} + \xi}, \tag{3.94}$$

provided that $2\,(2\gamma-1)\,\lambda^2\left(K\sqrt{\frac{\lambda^2}{K^2}} - \lambda\right)(\gamma-2) = 0$.

Solution (3.94) can be converted to the following form:

$$u(z) = \frac{\xi\gamma - \cosh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right) - \sinh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right)}{-\cosh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right) - \sinh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right) + \xi}. \tag{3.95}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.94) and (3.95), we get the following exact solutions for Eq (3.1):

$$u(x,t) = \frac{\xi\gamma - \mathrm{e}^{\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}}}{-\mathrm{e}^{\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}} + \xi}, \tag{3.96}$$

$$u(x,t) = \frac{\xi\gamma - \cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right) - \sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right)}{-\cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right) - \sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right) + \xi}. \tag{3.97}$$

Case 3.4.6

Substituting (3.73) with (3.75) and $\alpha = -\frac{\lambda(\gamma-1)}{K^2(\gamma+1)}$ into Eq (3.3), the left hand side of Eq (3.3) reduces to zero. Therefore, we obtain the following solution:

$$u(z) = \frac{\mathrm{e}^{-\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}}\gamma - \xi}{\mathrm{e}^{-\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}} - \xi}, \tag{3.98}$$

provided that $2\lambda^2\,(2\gamma-1)\left(K\sqrt{\frac{\lambda^2}{K^2}} - \lambda\right)(\gamma-2) = 0$.

Solution (3.98) can be converted to the following form:

$$u(z)=\frac{\left(\cosh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right)-\sinh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right)\right)\gamma-\xi}{\cosh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right)-\sinh\left(\frac{\lambda(\gamma-1)z}{K^2(1+\gamma)}\right)-\xi}. \tag{3.99}$$

Then, substituting $z = Kx - c\frac{t^\alpha}{\alpha}$ into the exponential function solutions (3.98) and (3.99), we get the following exact solutions for Eq (3.1):

$$u(x,t)=\frac{e^{-\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}}\gamma-\xi}{e^{-\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}}-\xi}, \tag{3.100}$$

$$u(x,t)=\frac{\left(\cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right)-\sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right)\right)\gamma-\xi}{\cosh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right)-\sinh\left(\frac{\lambda(\gamma-1)(Kx-c\frac{t^\alpha}{\alpha})}{K^2(1+\gamma)}\right)-\xi}. \tag{3.101}$$

4. Discussion

4.1. New solutions

The traveling wave solutions (3.12), (3.13), (3.21), (3.22), (3.25), (3.36), (3.43), (3.51), (3.52), (3.55), (3.56), (3.59), (3.60) (3.63), (3.64), (3.67), (3.68), (3.71), (3.72), (3.80), (3.81), (3.84), (3.85), (3.88), (3.89), (3.92), (3.93), (3.96), (3.97), (3.100), (3.101) appear to be new, comparing the results in [22] and other open literature.

4.2. New form of exponential solutions

We find a new shape of exponential function solution for Eq (2.12) in the shape of (3.28), where $A(z)$ is an undetermined function, which is not of the shape of (2.15). This has resulted in an important extension to the complex method to build new exponential function solutions for PDEs, since the new shape of the exponential function solution cannot be degenerated by the elliptic function solution.

5. Conclusions

By traveling wave transformation, the conformable Huxley equation is reduced to an ordinary differential equation. Then, by using the complex method and the extended form of exponential solutions $u(z) = \frac{A(z)}{e^{\alpha z}-\xi}$ + const, where $A(z)$ is an undetermined function, we can build some new exact exponential solutions and hyperbolic solutions. Therefore, by using the complex method and the extended form of exponential solutions, more exact solutions can be built for partial differential equations.

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Conflict of interest

The authors declare there are no conflicts of interest.

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