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Research article

Side-side-angle triangle congruence axiom and the complete quadrilaterals

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Abstract: Among the triangle congruence axioms, the side-side-angle (SsA) axiom states that two triangles are congruent if and only if two pairs of corresponding sides and the angles opposite the longer sides are equal. We construct two triangle sequences in which the items satisfy a modified condition. We require that the opposite angles of the shorter sides be equal. The locus of the intersection points of other sides of triangles is derived to be a hyperbola, and in a generalized form defined by a complete quadrilateral, it is a conic section.

Keywords: triangle congruence; side-side-angle axiom; complete quadrilateral; conic section

1. Introduction and main results

Triangle congruences play a significant role in Euclidean geometry. Some of them are axioms in almost all geometric structures. The triangle congruences are as follows, where S, A, s, R, and H represent, respectively, the side, angle, shorter side, right-angle, and hypotenuse.

SSS. Two triangles are congruent iff their corresponding sides are equal.

ASA. Two triangles are congruent iff two pairs of corresponding angles and the sides between them are equal.

SAS. Two triangles are congruent iff two pairs of corresponding sides and the angles between those sides are equal.

AAS. Two triangles are congruent iff two pairs of corresponding angles and a non-included side are equal.

SsA. Two triangles are congruent iff two pairs of corresponding sides and the angles opposite the longer sides are equal.

RHS. Two right-angled triangles are congruent iff their corresponding hypotenuse and one side are equal.

Currently, the triangle congruence axioms have attracted the interest of a number of researchers, such as Donnelly [1, 2] and Rigby [3], who examined their role in absolute geometry, and Hähl and Peters [4], who derived a variant of Hilbert's axioms from a subset of them.

The SsA triangle congruence is the most complex of the triangle congruences. Some researchers disagree that it is a fundamental triangle congruence axiom. We understand that two triangles are not necessarily congruent if two pairs of corresponding sides and the angles opposite the shorter sides are equal. In this case, two non-congruent triangles are possible under these conditions. It led us to examine the connections between these two triangles satisfying the following condition.

Condition sSA. Two triangle hold the condition sSA if their two pairs of corresponding sides and the angles opposite the shorter sides are equal.

In this article, we fix and superimpose the shorter sides of two triangles under the condition sSA, and we demonstrate that the locus of the intersection points of the side lines is a hyperbola.

Theorem 1. *Let A and B be two fixed points. If the triangles ABC and ABC', with Condition sSA have their corresponding shorter sides AB, and the common lengths of their corresponding longer sides AC and AC', respectively, are fixed, then the locus of the intersection points of the corresponding lines of sides is a hyperbola.*

Although the results and proofs in this article, as well as the tools used, handle mostly with elementary mathematical concepts, in the following we clarify some geometric configurations for readers less familiar with projective geometry. For example, a *complete quadrilateral* is a configuration of four straight lines in general position with their six intersection points; a *cross-ratio* $(ABCD)$ of four points A, B, C, D gives the ratio of distance ratios AC/CB and AD/DB; two points are *conjugate* with respect to a conic, when their cross-ratio with the intersection points of their line and the conic is -1; a line is *polar* of a point (*pole*) with respect to a conic, when all the points of the line is conjugate to the pole with respect to the conic. In textbooks [5, 6], readers find more precise definitions and examples. Furthermore, as another main result, we generalize this construction and prove the following theorem.

Theorem 2. *Let A, B, and D be three different collinear fixed points, and let $\mathcal{K}$ be a conic. Moreover, let C be a point of $\mathcal{K}$, and let C' be the intersection point ($\neq C$) of line CD and $\mathcal{K}$. If A, B, C, and C' are not collinear, then they form a complete quadrilateral with diagonal point D. Consequently, the locus of the other two diagonal points is a conic if C is moving along $\mathcal{K}$.*

In addition, the loci equations are derived in this article.

We used GeoGebra software to visually verify our assumptions and to create the figures for the article, as well as Maple software [7] to precisely validate some highly complex computations. The GeoGebra and Maple files are available [8]. See some examples of Maple source code in Appendix.

2. Base construction

2.1. Definitions

Consider two triangles ABC and $A'B'C'$ with the same orientation. We denote their sides and angles by a, b, c, α, β, γ, and a', b', c', α', β', γ', as appropriate. If $b = b'$, $c = c'$ with $b > c$ and $\gamma = \gamma'$, then these triangles satisfy the condition sSA. We are aware that these triangles are not necessarily congruent. Let $A = A'$, $B = B'$, and the third vertices of both triangles be in the same half-plane bordered by the line $c = c'$. Figure 1 depicts our structure. Let us fix b and c. Let M and $\overline{M}$ be the intersection points of lines BC and $A'C'$, and lines AC and $B'C'$, respectively. In the following, we will look at this construction, specifically the locus of the points M and $\overline{M}$ when γ is the variable. For our analytical calculations, we select an appropriate coordinate system. Let the origin O be A and the y-axis be the line AB, so that B lies at the positive end of the line, as shown in Figure 1. Moreover, without compromising generality, let $b = 1$ and then $0 < c < 1$. The points C and C' lie on the circle $\mathcal{K}$, whose equation is $x^2 + y^2 = 1$. Now consider the expressions $A(0, 0)$, $B(0, c)$, and $C(\sin\alpha, \cos\alpha)$, where we extend $\alpha \in [0, 2\pi]$ and $\alpha \neq \pi$.

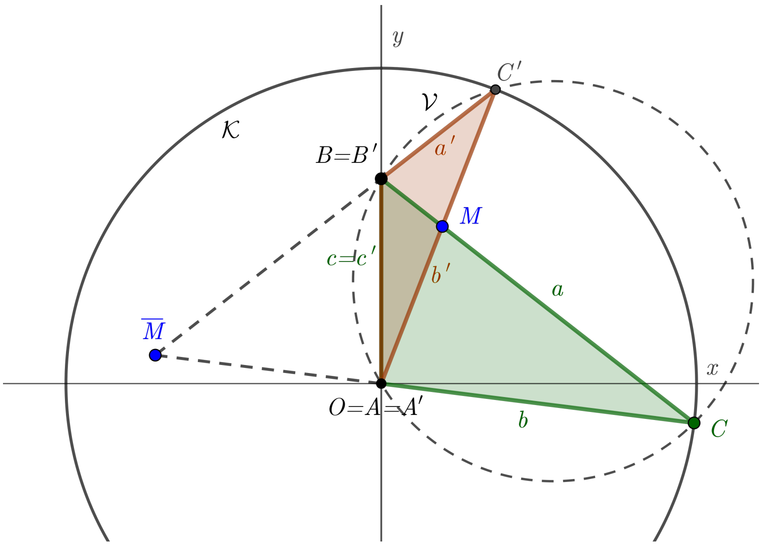

Figure 1. Base construction of sSA.

2.2. *Locus of M*

Let D be the intersection point of the lines AB and CC'. We show that D is a fixed point, independent of C (see Figure 2).

Lemma 2.1. *Regarding the circle $\mathcal{K}$, the point D with coordinates $(0, 1/c)$ is conjugate of point B.*

Proof. Because the angles ACB and $AC'B$ are equal, C and C' are on an arc of a circle whose endpoints are A and B. Let $(k, c/2)$ and r represent the center and radius, respectively, of this circle $\mathcal{V}$. It has the equation $(x - k)^2 + (y - c/2)^2 = r^2$. If $A(0, 0) \in \mathcal{V}$, then

$$k^2 + \left(\frac{c}{2}\right)^2 = r^2. \tag{2.1}$$

Subtracting equations $\mathcal{K}$ and $\mathcal{V}$ yields the equation for line CC', which is

$$2kx + cy = k^2 + \frac{c^2}{4} - r^2 + 1. \tag{2.2}$$

Using Eq (2.1), the intersection of CC' and axis $x = 0$ yields $cy = k^2 + \frac{c^2}{4} - r^2 + 1 = 1$. Consequently, D's coordinates are $(0, 1/c)$, and D and B are conjugate with respect to $\mathcal{K}$.

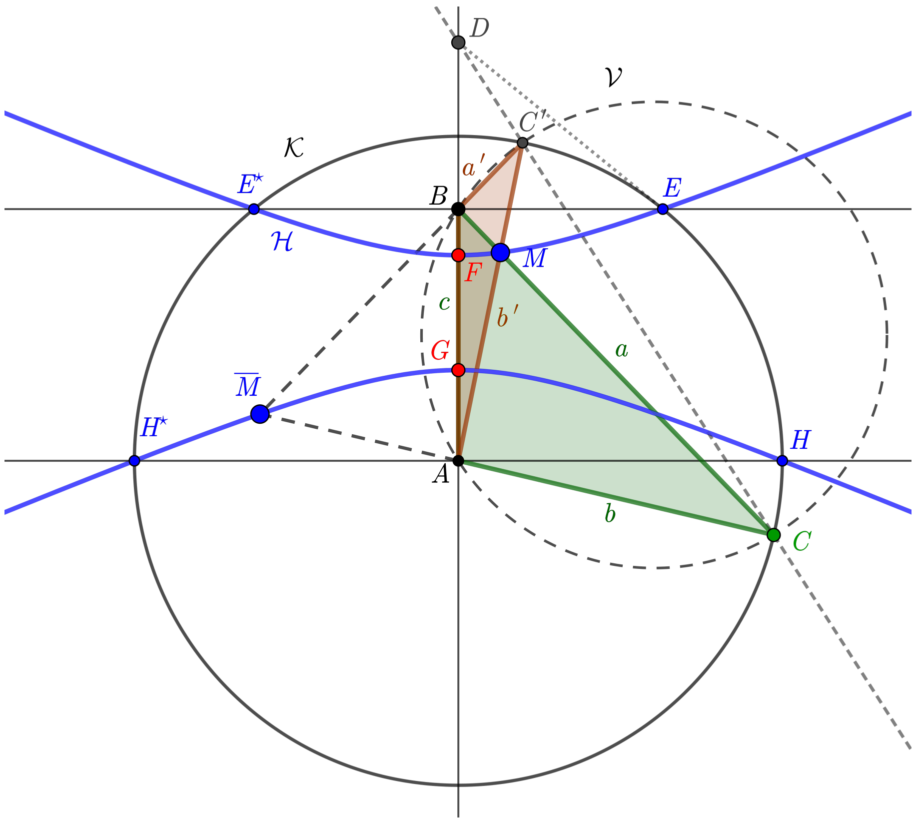

Figure 2. Locus of M.

Let the intersection points of $\mathcal{K}$ and the line $y = c$ be E and $E^\star$. They are at coordinates $E(\sqrt{1-c^2}, c)$ and $E^\star(-\sqrt{1-c^2}, c)$. Due to the fact that D and B are conjugate, the tangent line to $\mathcal{K}$ at E passes through D.

Let the points H and $H^\star$ be the intersections of the x-axis and the circle $\mathcal{K}$. Then $H(1, 0)$ and $H^\star(-1, 0)$. Furthermore, consider $F(0, c/(2-c))$ and $G(0, c/(2+c))$ (see Figure 2).

Based on the previous construction and results, we formulate the following theorem.

Theorem 3. *The locus of the points M and $\overline{M}$ is the hyperbola $\mathcal{H}$ defined by the points E, $E^\star$, H, $H^\star$, F, and G, and its equation is*

$$\frac{(y-2s)^2}{(cs)^2} - \frac{x^2}{cs} = 1, \tag{2.3}$$

where $s = c/(4-c^2)$.

Proof. Recall $0 < c < 1$ and $C(\sin\alpha, \cos\alpha)$, where $\alpha \in (0, 2\pi)$ and $\alpha \neq \pi$. The equation of line CD is

$$\left(\cos\alpha - \frac{1}{c}\right)x = \sin\alpha\left(y - \frac{1}{c}\right). \tag{2.4}$$

Then we obtain

$$C'\left(\frac{(1-c^2)\sin\alpha}{c^2 - 2c\cos\alpha + 1}, \frac{-c^2\cos\alpha + 2c - \cos\alpha}{c^2 - 2c\cos\alpha + 1}\right), \tag{2.5}$$

where $\cos\alpha \neq (c^2+1)/2c$, because $c \neq 1$ and $c^2 + 1 \geq 2c$.

The equations of BC and AC' are

$$(\cos\alpha - c)x = \sin\alpha(y - c) \tag{2.6}$$

and

$$(-c^2\cos\alpha + 2c - \cos\alpha)x = y(1 - c^2)\sin\alpha, \tag{2.7}$$

respectively. Their intersection point is

$$M(x_M, y_M) = M\left(\frac{c(c^2-1)\sin\alpha}{c^3 - 3c + 2\cos\alpha}, \frac{c(c^2\cos\alpha - 2c + \cos\alpha)}{c^3 - 3c + 2\cos\alpha}\right).$$

The point F is the limit point of M when α tends to π. Thus, because

$$\lim_{\alpha\to\pi} x_M = 0 \quad \text{and} \quad \lim_{\alpha\to\pi} y_M = \frac{c}{2-c},$$

$F(0, c/(2-c))$ determines the locus of M. Similarly, if α tends to 0, we obtain $G(0, c/(2+c))$.

The points H, E, $H^\star$ and $E^\star$ are special cases of C. Now, we examine whether these points satisfy the equation of the hyperbola $\mathcal{H}$. Since $c/s = 4 - c^2$ and $4s/c - 1 = cs$, we conclude that

$$\frac{(2s)^2}{(cs)^2} - \frac{1}{cs} = \frac{4s^2 - cs}{(cs)^2} = \frac{\frac{4s}{c} - 1}{cs} = 1.$$

for points H and H^*. In the case of E and $E^\star$ we have

$$\frac{(c-2s)^2}{(cs)^2} - \frac{1-c^2}{cs} = \frac{c^2 + c^3 s - 5cs + 4s^2}{(cs)^2} = \frac{\frac{c}{s} + c^2 - 5 + \frac{4s}{c}}{cs} = \frac{\frac{4s}{c} - 1}{cs} = 1.$$

In the case of F, as $c/(2-c) - 2s = (c(c+2) - 2c)/(4-c^2) = cs$ we have

$$\frac{\left(\frac{c}{2-c} - 2s\right)^2}{(cs)^2} = 1.$$

Similarly, G also satisfies the Eq (2.3).

Finally, the coordinates of M are substituted into (2.3). We find that

$$\frac{\left(\frac{c(c^2\cos\alpha - 2c + \cos\alpha)}{c^3 - 3c + 2\cos\alpha} - 2s\right)^2}{(cs)^2} - \frac{\left(\frac{c(c^2-1)\sin\alpha}{c^3-3c+2\cos\alpha}\right)^2}{cs} = 1. \tag{2.8}$$

If we exchange the positions of the points C and C', we will obtain the point $\overline{M}$ instead of M. Consequently, $\overline{M}$ is one of the points of $\mathcal{H}$.

For the infinite points of $\mathcal{H}$, the parallelism of lines BC and AC'must be examined. From their Eqs (2.6) and (2.7), we can obtain, for a suitable μ, the system of equations

$$\mu(\cos\alpha - c) = -c^2\cos\alpha + 2c - \cos\alpha,$$
$$\mu\sin\alpha = (1 - c^2)\sin\alpha.$$

Then $\sin\alpha = 0$, or $\mu = 1 - c^2$ and $\cos\alpha = (3c - c^3)/2$, where $0 < 3c - c^3 < 2$ since $0 < c < 1$ and $3c - c^3 - 2 < 3c - c^2 - 2 < 0$. The cases $\alpha = 0$ and $\alpha = \pi$ provide the points F and G, and when $\alpha = \pm\arccos((3c - c^3)/2)$ we obtain the two infinity points.

We have continuously assigned all the points of the circle $\mathcal{K}$ to the points of the hyperbola $\mathcal{H}$ in a one-to-one correspondence, so the locus of points M and $\overline{M}$ is the hyperbola $\mathcal{H}$.

Corollary 1. *The equations of the asymptotes of hyperbola $\mathcal{H}$ are*

$$x\cos\varphi = \pm\sin\varphi(y - 2s),$$

where $\varphi = \pm\arccos((3c + c^3)/2)$ *and* $s = c/(4 - c^2)$.

Proof. The proof of Theorem 3 proves that the asymptote directions are defined by $\varphi = \pm\arccos((3c + c^3)/2)$. The direction vectors are therefore $\overline{v}(\sin\varphi, \cos\varphi)$. The hyperbola's center is the midpoint of segment FG, or $(0, 2c/(4 - c^2))$. These observations yield the equations.

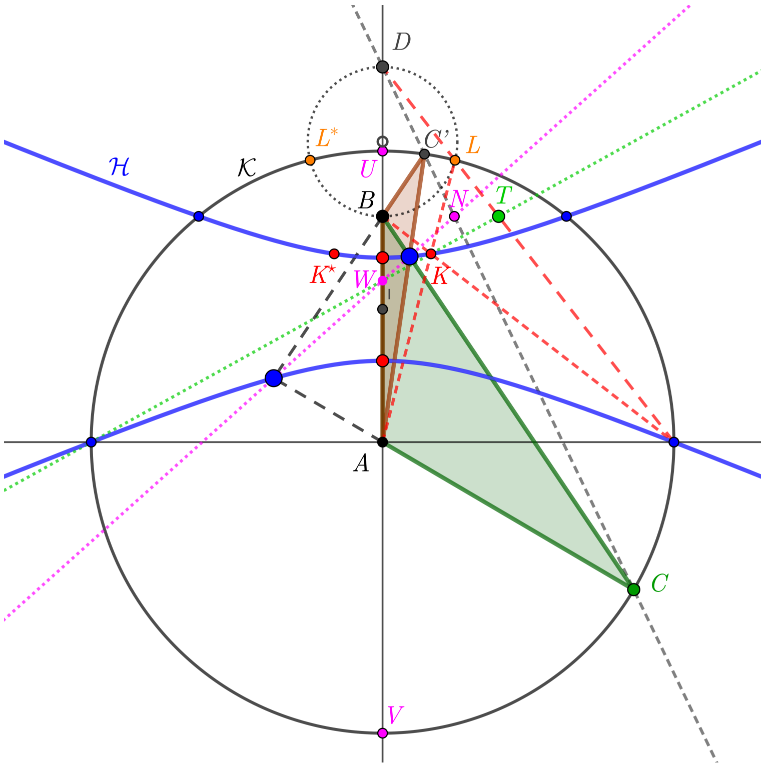

Figure 3. Hyperbola $\mathcal{H}$.

Now, we examine the special case where C and H coincide. The line HD has the equation $1-x = cy$, and its other point of intersection with $\mathcal{K}$ is $L((1 - c^2)/(1 + c^2), 2c/(1 + c^2))$. Then the equations of line BH and AL are $y = -cx + c$ and $y = 2cx/(1 - c^2)$, respectively, and their intersection point is

$$K\left(\frac{1 - c^2}{3 - c^3}, \frac{2c}{3 - c^3}\right)$$

(see Figure 3). Now we obtain the following corollary.

Corollary 2. *The point K and its reflection point $K^\star$ over the y-axis lie on the hyperbola $\mathcal{H}$.*

Proof. The points K and $K^\star$ lie on the hyperbola defined by Eq (2.3) because

$$\frac{\left(\frac{2c}{3-c^3} - 2s\right)^2}{(cs)^2} - \frac{\left(\frac{1-c^2}{3-c^3}\right)^2}{cs} = \frac{4}{c^2\left(3 - c^2\right)^2} - \frac{\left(1 - c^2\right)^2\left(4 - c^2\right)}{c^2\left(3 - c^2\right)^2} = 1.$$

As a result of the aforementioned facts, the following theorem can be derived.

Theorem 4. *The cyclic quadrilateral $ACC'B$ defines a complete quadrilateral with the diagonal points D, M, and $\overline{M}$. In a specific case, the quadrilateral $AHLB$ also forms a complete quadrilateral with diagonal points D, K, and $H^\star$.*

Let T and W be the intersection points of lines $EE^\star$ and DH, and lines $TH^\star$ and AB, respectively. Following a short calculation, we have

$$T(1-c^2, c) \quad \text{and} \quad W\left(0, \frac{c}{2-c^3}\right).$$

Theorem 5. *The points T, K, W, and $H^\star$ are collinear.*

Proof. All four points are located on the line $TH^\star$, which has the equation $c(x+1) = y(2-c^2)$.

Let N represent the intersection of lines $EE^\star$ and CD. The following theorem is now available.

Theorem 6. *The points N, M, W, and $\overline{M}$ are collinear, and W is a fixed point (regardless of the position of point C).*

Proof. From (2.4) and $y = c$ we have

$$N\left(\frac{\sin\alpha\left(c-\frac{1}{c}\right)}{\cos\alpha-\frac{1}{c}}, c\right).$$

And we can verify that M and $\overline{M}$ are located on the line NW.

Corollary 3. *The tangent lines of $\mathcal{H}$ at points E and $E^\star$ intersect at point W.*

Proof. When $C = C' = E$, the tangent line is the line $M\overline{M}$.

Based on the properties of complete quadrilaterals, the following cross ratios can be determined.

Corollary 4. $(NWM\overline{M}) = -1$, $(TWKH^\star) = -1$, $(DTLH) = -1$, $(CC'ND) = -1$, $(ABWD) = -1$.

In summary, the fixed points E, $E^\star$, H, $H^\star$, K, $K^\star$, F, and G are on the hyperbola $\mathcal{H}$, and its tangents at points E and $E^\star$ are the lines WE and $WE^\star$ (see Figure 4). In the subsequent section, we generalize our construction and demonstrate that some of these data are also fixed.

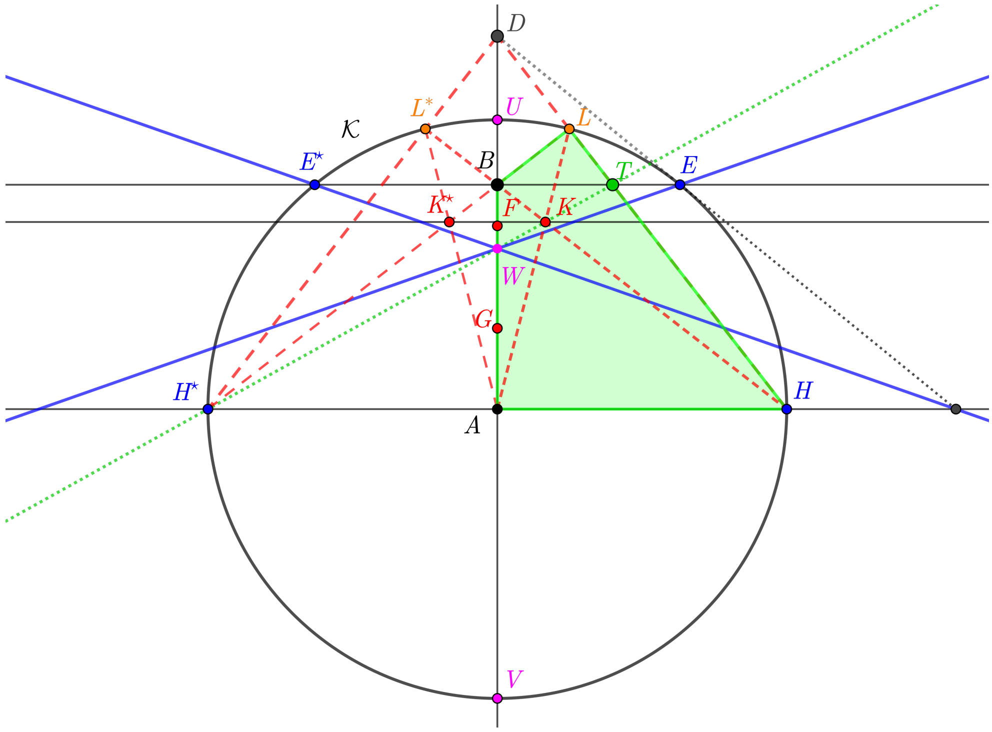

Figure 4. Fixed items.

3. Generalization of condition sSA

In this subsection, we examine our construction in a broader sense. Assume that A, B, and D are fixed collinear points. Consider the complete quadrilateral $ACC'B$ with diagonal points D, M, and $\overline{M}$. When C and C' are moving on a given circle $\mathcal{K}$, the locus of the points M and $\overline{M}$ is a conic $\mathcal{H}$, as demonstrated (see Figure 5).

Let the center of the circle $\mathcal{K}$ with the equation $x^2 + y^2 = 1$ serve as the origin O of our suitable coordinate system. Let D be a fixed y-axis point with coordinates $(0, 1/c)$, where $c \neq 1$ and $c \neq 0$. Let (a_x, a_y) be the coordinates of A, and since B is on the line AD with equation $(a_y c - 1)x - a_x c y + a_x = 0$, then $B(\lambda a_x, \lambda a_y + (1-\lambda)(1/c))$, where λ is fixed and $\lambda \neq 0, \lambda \neq 1$. Consider $C(\sin\alpha, \cos\alpha)$ once more, where $\alpha \in [0, 2\pi]$ and $\alpha \neq \pi$, then (2.5) also holds for the coordinates of C'. Similar to Section 2, α is the only variable in our construction.

Let W represent the intersection of lines AD and $M\overline{M}$. Then, we know that W is the projective harmonic conjugate of D with respect to A and B, based on the properties of complete quadrilaterals. Consequently, their cross ratio is $(ABWD) = -1$. A quick calculation for the coordinates of the fixed W (that is independent from the position of C) yields (if $\lambda \neq -1$)

$$W\left(\frac{2a_x\lambda}{\lambda+1}, \frac{2a_y c\lambda - \lambda + 1}{c(\lambda+1)}\right). \tag{3.1}$$

(When $\lambda = -1$, W is at infinity.) In addition, let point $P(P_x, P_y)$ be the pole of line AD relative to the circle $\mathcal{K}$. Let $E(\sqrt{1-c^2}, c)$ and $E^\star(-\sqrt{1-c^2}, c)$ be the points of intersection of $\mathcal{K}$ and the polar of D with respect to $\mathcal{K}$, respectively.

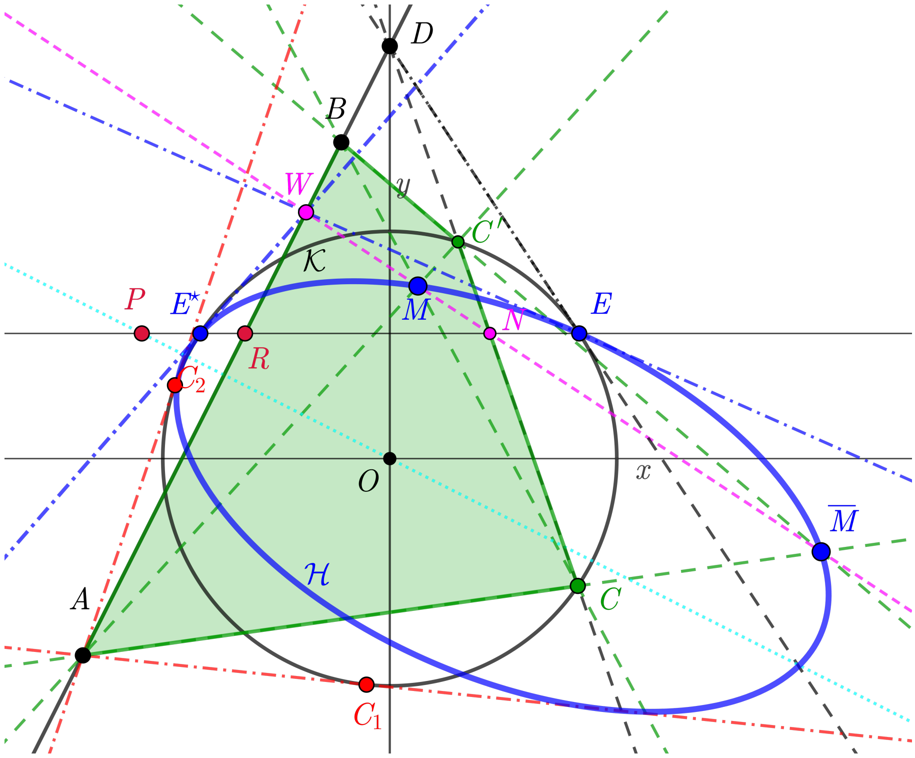

Figure 5. Generalization of condition sSA.

Theorem 7. *The locus of the points M and $\overline{M}$ is the conic section $\mathcal{H}$ defined by the points E, $E^{\star}$, lines EW, $E^{\star}W$ as tangents at points E, $E^{\star}$, respectively, and the pole-polar pair P, AD with respect to $\mathcal{H}$. Moreover, the equation of $\mathcal{H}$ is*

$$h_1x^2 + h_2y^2 + h_3xy + h_4x + h_5y + h_6 = 0. \tag{3.2}$$

where if $\Theta = (2a_yc - c^2 - 1)\lambda - c^2 + 1$, then

$$\begin{aligned}
h_1 &= \Theta^2, \\
h_2 &= (4a_x^2c^2 + c^4 - 2c^2 + 1)\lambda^2 + (2c^4 - 2)\lambda + c^4 - 2c^2 + 1, \\
h_3 &= -4a_xc\lambda\Theta, \\
h_4 &= 4a_xc^2\lambda\Theta, \\
h_5 &= -4c(2a_x^2c^2 + a_yc^3 - a_yc - c^2 + 1)\lambda^2 - 4c(a_yc^3 - a_yc + c^2 - 1)\lambda, \\
h_6 &= (4a_x^2c^4 + 4a_y^2c^4 - 4a_y^2c^2 - 4a_yc^3 - c^4 + 4a_yc + 2c^2 - 1)\lambda^2 \\
&\quad +2(2a_yc^3 + c^4 - 2a_yc - 2c^2 + 1)\lambda - c^4 + 2c^2 - 1.
\end{aligned}$$

Proof. The coordinates (M_x, M_y) of M are now given as the intersection of lines BC and AC'. Following some computations, we have

$$M_x = \frac{(-c^2\lambda+c^2+\lambda-1)\sin(\alpha)+2c\lambda a_x\cos(\alpha)-2a_xc^2\lambda}{-c^2\lambda+2\cos(\alpha)c-c^2+\lambda-1},$$

$$M_y = \frac{(2a_yc\lambda-c^2\lambda+c^2-\lambda+1)\cos(\alpha)-2a_yc^2\lambda+2c\lambda-2c}{-c^2\lambda+2\cos(\alpha)c-c^2+\lambda-1}.$$

Using the substitution $t = \tan(\alpha/2)$ for the trigonometric functions, $\cos\alpha = ((1-t^2)/(1+t^2)$ and $\sin\alpha = (2t/(1+t^2)$, we obtain

$$\begin{aligned} M_x &= \tfrac{a_x c\lambda(c+1)t^2+(c^2-1)(\lambda-1)t+a_x c\lambda(c-1)}{(c^2\lambda+c^2+2c-\lambda+1)t^2+c^2\lambda+c^2-2c-\lambda+1}, \\ M_y &= \tfrac{(c+1)(2a_y c\lambda-(c+1)(\lambda-1))t^2+(c-1)(2a_y c\lambda+(c-1)(\lambda-1))}{(c^2\lambda+c^2+2c-\lambda+1)t^2+c^2\lambda+c^2-2c-\lambda+1}. \end{aligned} \tag{3.3}$$

We can see that the coordinates of M are rational functions of t, and that all the numerators and denominators are second-order functions; therefore, the locus of points M is a conic section that we'll designate as $\mathcal{H}$. If we switch the roles of C and C', we obtain the same coordinates for $\overline{M}$.

Let the equation of $\mathcal{H}$ be given by the form

$$h_1x^2 + h_2y^2 + h_3xy + h_4x + h_5y + h_6 = 0. \tag{3.4}$$

We substitute $x = M_x$ and $y = M_y$ into (3.4), simplify the equation, and collect all coefficients with the same rational power of t. Now that all coefficients must equal to zero, we have five linear equations with the variable h_i. The solution of the system reveals that h_2 is a free variable. We obtain the solutions to the Eq (3.4) by selecting $h_2 = (4a_x^2c^2 + c^4 - 2c^2 + 1)\lambda^2 + (2c^4 - 2)\lambda + c^4 - 2c^2 + 1$. We used the program Maple for the calculation. (See Appendix.)

If point C is moved to point E, then $E = C = C^\star = M = \overline{M}$ is a fixed point of $\mathcal{H}$, and the tangent line passes through W. The argument for point $E^\star$ is comparable. We must verify that these two points satisfy the equation of $\mathcal{H}$. By substituting the coordinates of E and $E^\star$ into the expression (3.4), we analytically determine that they are points of $\mathcal{H}$. Additionally, it is simple to confirm that the tangents of $\mathcal{H}$ at points E and $E^\star$ are the lines EW and $E^\star W$. The equations of these tangents are, respectively,

$$\left((-2a_yc + c^2 + 1)\lambda + c^2 - 1\right)x + c\,(\mp e_x(\lambda+1) + 2a_x\lambda)\,y \pm e_x(2a_yc\lambda - \lambda + 1) - 2a_xc^2\lambda = 0,$$

where $e_x = \sqrt{1-c^2}$. See signs $\mp$ and $\pm$. The top $-$ and $+$ yield the first equation and the bottom $+$ and $-$ do the second one.

Remember that if (x_0, y_0) is a point that serves as a pole with respect to $\mathcal{H}$, then its polar equation is

$$h_1x_0x + h_2y_0y + \frac{1}{2}h_3(x_0y + xy_0) + \frac{1}{2}h_4(x + x_0) + \frac{1}{2}h_5(y + y_0) + h_6 = 0. \tag{3.5}$$

If (x_0, y_0) is on $\mathcal{H}$, then its polar is a tangent to $\mathcal{H}$ at (x_0, y_0).

Since the equation of line AD is $(a_yc - 1)x - a_xcy + a_x = 0$, and as P is the pole of AD with respect to $\mathcal{K}$, then $P(p_x, p_y) = P((1 - a_yc)/a_x, c)$, because $xp_y + yp_y - 1 = 0$ is equivalent to the equation of AD. Let R be the point where lines $EE^\star$ and AD intersect. As the points E, $E^\star$, R, and P are fixed, and P and AD are a pole-polar pair with respect to $\mathcal{K}$, $(EE^\star RP) = -1$ is valid, and the same holds true for P and AD with respect to $\mathcal{H}$, as well. Analytically, we also discover their pole-polar connection in relation to $\mathcal{K}$.

Corollary 5. *If λ tends to* 0*, then B tends to D and the Eq* (3.2) *of $\mathcal{H}$ tends to the equation $x^2 + y^2 = 1$ of $\mathcal{K}$.*

Proof. If $\lambda = 0$, then $h_1 = h_2 = (c^2 - 1)^2$ and $h_6 = -(c^2 - 1)^2$. Which is equivalent to the equation of the circle.

We note that the type of conic $\mathcal{H}$ depends on λ.

Corollary 6. *The type of conic section $\mathcal{H}$ is determined by λ. Let us denote*

$$\lambda_1 = \frac{1-c}{1+c} \quad \textit{and} \quad \lambda_2 = \frac{1+c}{1-c} = \lambda_1^{-1},$$

then the conic section $\mathcal{H}$ is

1). a parabola, iff $\lambda = \lambda_1$ or $\lambda = \lambda_2$,

2). an ellipse, iff

- *λ is between λ_1 and λ_2, and $|c| \geq 1$, or*
- *λ is not between λ_1 and λ_2, and $|c| \leq 1$,*

3). a hyperbola, iff

- *λ is between λ_1 and λ_2, and $|c| \leq 1$, or*
- *λ is not between λ_1 and λ_2, and $|c| \geq 1$.*

Proof. For the type of conic section $\mathcal{H}$, we sought out its ideal points. Point M represents the arbitrarily chosen point of $\mathcal{H}$ (see (3.3)). The denominator of the coordinates of point M is a quadratic polynomial of t. If this denominator tends to zero, both coordinates of point M tend to infinity. The number of ideal points of $\mathcal{H}$ is therefore dependent on the discriminant Δ of this quadratic polynomial of t which can be written as

$$\Delta = -4\left((c+1)^2 + (c^2-1)\lambda\right)\left((c-1)^2 + (c^2-1)\lambda\right).$$

The discriminant Δ as a quadratic polynomial of λ has roots $\lambda_1 = (1-c)/(1+c)$ and $\lambda_2 = (1+c)/(1-c) = \lambda_1^{-1}$.

If and only if $\lambda = \lambda_1$ or $\lambda = \lambda_2$, the discriminant Δ is zero, there is only one t when the denominators of the coordinates of M tend to zero, so the conic section $\mathcal{H}$ has one point at infinity. Thus, $\mathcal{H}$ is a parabola.

If the discriminant Δ has a positive value, the denominators of the corresponding coordinates have two distinct real roots, and the conic section $\mathcal{H}$ has two distinct ideal points (the conic section is a hyperbola).

If the discriminant Δ has a negative value, the denominators of the corresponding coordinates do not have real roots, and the conic section $\mathcal{H}$ does not have any ideal points (the conic section is an ellipse).

For real numbers $\lambda \in (\lambda_1, \lambda_2)$ and λ outside this interval ($\lambda \neq \lambda_1$ and $\lambda \neq \lambda_1$), the signs of the discriminant Δ are different. The value $\lambda = 1$ is always between λ_1 and λ_2 for all real values of c; therefore, for $\lambda = 1$, the substituting value of the discriminant $\Delta = 16c^c(1-c^2)$, which is positive if and only if $|c| \leq 1$. The conditions 2 and 3 of the corollary follow from this statement.

Let C_1 and C_2 be the tangent points of the circle and its tangents through A (see Figure 5). If we move the point C into C_1 and C_2, then the lines AC_1 and AC_2 will also be tangents to $\mathcal{H}$. Similarly, we obtain two tangent lines to $\mathcal{H}$ through B.

Corollary 7. *The lines that are tangent to $\mathcal{K}$ at points A and B are also tangent to $\mathcal{H}$.*

Now we can prove our primary theorem.

Proof of Theorem 2. Using an appropriate projective transformation, we can transform the circle from Theorem 7 into the given conic; consequently, the transformation maps $\mathcal{H}$ to a conic. We note that we did not use metric steps when proving Theorem 7.

4. Conclusions and future work

We examined the so-called side-side-angle triangle congruence axiom in a new sense. Our objective was to determine the locus of the intersection points of non-fixed triangle sides. We determined that it is a hyperbola, and upon generalization, a conic section.

In addition, as a result of our work, we are able to define a new type of transformation among conics. Conic $\mathcal{H}$ is the sSA transformation of conic $\mathcal{K}$ with respect to the points A, B and D. The base of the transformation is the complete quadrilateral $ACC'B$ with fixed points A and B and the diagonal point D. If the points C and C' lie on $\mathcal{K}$, then the other diagonal points M and $\overline{M}$ determine the conic $\mathcal{H}$,

$$(C, C') \in \mathcal{K} \mapsto (M, \overline{M}) \in \mathcal{H}.$$

Moreover, when the point B is moved along the line AD, so that the parameter λ is considered a variable, then the expression (3.2) describes a pencil of conics with two common points E, $E^\star$, two common tangents AC_1, AC_2, and a common pole-polar pair P, AD (see Figure 6).

We intend to investigate the projective properties of both the transformation and this conic pencil.

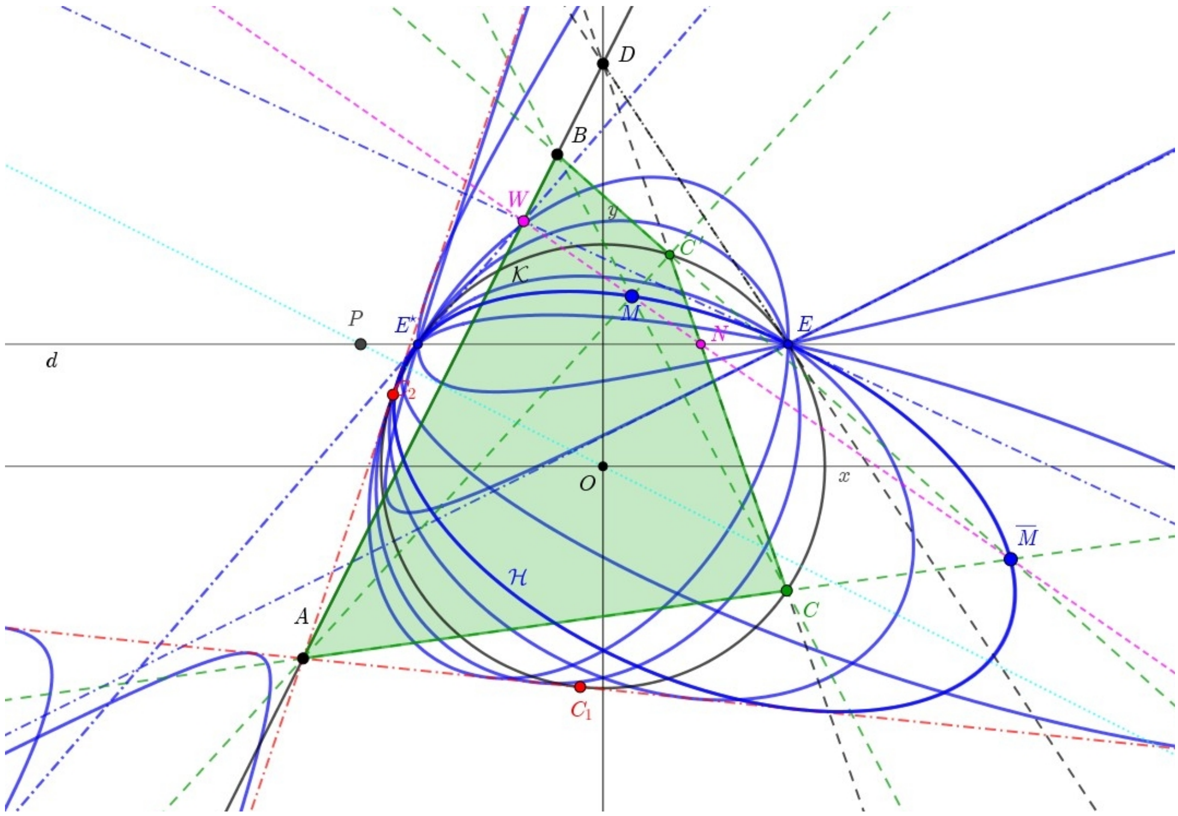

Figure 6. Pencil of conics.

Notably, the phenomena and relationships described in this article can be effectively and easily illustrated using interactive geometric software such as GeoGebra, and can thus be presented to students, prospective or practicing mathematics teachers. Even more so due to the fact that the mathematical apparatus employed in the proofs does not exceed the standard curriculum of geometry, mathematical analysis, and algebra courses used in mathematics teacher education. In particular, we would like to emphasize that knowledge from other mathematical subfields is utilized in a geometric topic that provides another geometrically correct definition of conic sections.

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Conflict of interest

The authors declare there is no conflicts of interest.

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Appendix

In this appendix, we give the Maple source codes containing the calculation of the coordinates of points, the equations of lines, and the coefficients of the conic section of $\mathcal{H}$ in Theorem 7. The Maple source file is available from webpage [8].

Listing 1. Calculation for Theorem 3.

```
restart; with(linalg);
cir := x^2+y^2 = 1;
s := c/(-c^2+4); hyp := (y-2*s)^2/(c*s)^2-x^2/(c*s)-1;
e_x := sqrt(-c^2+1); e_y := c; d_x := 0; d_y := 1/c; b_x := 0; b_y := c;
c_x := sin(alpha); c_y := cos(alpha);
h_x := 1; h_y := 0; f_x := 0; f_y := c/(2-c); g_x := 0; g_y := c/(2+c);
lineCD := collect(simplify(det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) = 1, (2, 1) = c_x
    ↪ , (2, 2) = c_y, (2, 3) = 1, (3, 1) = d_x, (3, 2) = d_y, (3, 3) = 1})) = 0), [x, y],
    ↪ recursive);
solve({cir, lineCD}, {x, y});
CC_x := sin(alpha)*(c^2-1)/(2*cos(alpha)*c-c^2-1);
CC_y := (cos(alpha)*c^2+cos(alpha)-2*c)/(2*cos(alpha)*c-c^2-1);
lineBC := collect(simplify(det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) = 1, (2, 1) = b_x
    ↪ , (2, 2) = b_y, (2, 3) = 1, (3, 1) = c_x, (3, 2) = c_y, (3, 3) = 1})) = 0), [x, y],
    ↪ recursive);
lineACC := collect(simplify((2*cos(alpha)*c-c^2-1)*det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y,
    ↪ (1, 3) = 1, (2, 1) = 0, (2, 2) = 0, (2, 3) = 1, (3, 1) = CC_x, (3, 2) = CC_y, (3, 3) =
    ↪  1})) = 0), [x, y, sin(alpha), cos(alpha)], recursive);
M := solve({lineACC, lineBC}, {x, y});
M_x := rhs(M[1]); M_y := rhs(M[2]);
# proof
simplify(subs(x = M_x, y = M_y, hyp));
simplify(subs(x = e_x, y = e_y, hyp));
simplify(subs(x = h_x, y = h_y, hyp));
simplify(subs(x = g_x, y = g_y, hyp));
```

Listing 2. Coordinates of points and equations of lines for Section 3.

```
restart; with(linalg);
cir := x^2+y^2 = 1;
rac := (w_x-a_x)*(b_x-d_x)/((b_x-w_x)*(d_x-a_x)) = -1;
# coordinates of W
w_x := solve(rac, w_x); w_y := (2*a_y*b_y-a_y*d_y-b_y*d_y)/(a_y+b_y-2*d_y);
w_x := simplify(subs(b_x = lambda*a_x, d_x = 0, w_x));
w_y := simplify(subs(b_y = 1/c+lambda*(a_y-1/c), d_y = 1/c, w_y));
# coordinates of points E, B, C
e_x := sqrt(-c^2+1); e_y := c;
b_x := lambda*a_x; b_y := 1/c+lambda*(a_y-1/c);
c_x := sin(alpha); c_y := cos(alpha);
# equations of lines
lineEW1 := collect(simplify(c*(lambda+1)*det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) =
    ↪ 1, (2, 1) = e_x, (2, 2) = e_y, (2, 3) = 1, (3, 1) = w_x, (3, 2) = w_y, (3, 3) = 1})) =
    ↪  0), [x, y, lambda], recursive);
```

```
lineEW2 := collect(simplify(c*(lambda+1)*det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) =
    ↪ 1, (2, 1) = -e_x, (2, 2) = e_y, (2, 3) = 1, (3, 1) = w_x, (3, 2) = w_y, (3, 3) = 1}))
    ↪ = 0), [x, y, lambda], recursive);
lineAD := collect(simplify(c*det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) = 1, (2, 1) =
    ↪ 0, (2, 2) = 1/c, (2, 3) = 1, (3, 1) = a_x, (3, 2) = a_y, (3, 3) = 1})) = 0), x);
# coordinates of P
ADortho := (-a_y*c+1)*y-a_x*c*x = 0;
solve({ADortho, y = c}, {x, y}); p_x := (-a_y*c+1)/a_x; p_y := c;
# coordinates of C'
lineCD := collect(simplify(c*det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) = 1, (2, 1) =
    ↪ 0, (2, 2) = 1/c, (2, 3) = 1, (3, 1) = c_x, (3, 2) = c_y, (3, 3) = 1})) = 0), x);
S := solve({cir, lineCD}, {x, y});
cc1_x := sin(alpha)*(c^2-1)/(2*cos(alpha)*c-c^2-1);
cc1_y := (cos(alpha)*c^2+cos(alpha)-2*c)/(2*cos(alpha)*c-c^2-1);
cc_x := collect(simplify(subs(cos(alpha) = (-t^2+1)/(t^2+1), sin(alpha) = 2*t/(t^2+1), cc1_x)
    ↪ ), t);
cc_y := collect(simplify(subs(cos(alpha) = (-t^2+1)/(t^2+1), sin(alpha) = 2*t/(t^2+1), cc1_y)
    ↪ ), t);
# coordinates of M
lineACC := collect(simplify(c*det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) = 1, (2, 1) =
    ↪ cc1_x, (2, 2) = cc1_y, (2, 3) = 1, (3, 1) = a_x, (3, 2) = a_y, (3, 3) = 1})) = 0), y);
lineBC := collect(simplify(c*det(Matrix(3, 3, {(1, 1) = x, (1, 2) = y, (1, 3) = 1, (2, 1) =
    ↪ b_x, (2, 2) = b_y, (2, 3) = 1, (3, 1) = c_x, (3, 2) = c_y, (3, 3) = 1})) = 0), x);
sm := solve({lineACC, lineBC}, {x, y});
denomM:=(-c^2*lambda+2*cos(alpha)*c-c^2+lambda-1);
M1_x := collect(2*lambda*a_x*c*cos(alpha)-sin(alpha)*c^2*lambda-2*a_x*c^2*lambda+sin(alpha)*c
    ↪ ^2+sin(alpha)*lambda-sin(alpha), sin(alpha))/denomM;
M1_y := collect(2*cos(alpha)*a_y*c*lambda-cos(alpha)*c^2*lambda-2*a_y*c^2*lambda+cos(alpha)*c
    ↪ ^2-cos(alpha)*lambda+2*c*lambda+cos(alpha)-2*c, cos(alpha))/ denomM;
M_x := collect(simplify(subs(cos(alpha) = (-t^2+1)/(t^2+1), sin(alpha) = 2*t/(t^2+1), M1_x)),
    ↪  t);
M_y := collect(simplify(subs(cos(alpha) = (-t^2+1)/(t^2+1), sin(alpha) = 2*t/(t^2+1), M1_y)),
    ↪  t);
simplify(subs(lambda = 1, M_x)-a_x); simplify(subs(lambda = 1, M_y)-a_y); # check of point A
```

Listing 3. Coefficients of the equation of $\mathcal{H}$ in Section 3 using the results of Listing 2.

```
con := h_1*x^2+h_2*y^2+h_3*x*y+h_4*x+h_5*y+h_6 = 0; # general equation of conic section
con_tan := h_1*x*x_0+h_2*y*y_0+(1/2)*h_3*(x*y_0+x_0*y)+(1/2)*h_4*(x+x_0)+(1/2)*h_5*(y+y_0)+
    ↪ h_6 = 0;
T := collect(simplify(subs(x = M_x, y = M_y, con)*((c^2*lambda+c^2+2*c-lambda+1)*t^2+c^2*
    ↪ lambda+c^2-2*c-lambda+1)^2), t);
eq0 := subs(t = 0, T);
eq1 := coeff(lhs(T), t, 1); eq2 := coeff(lhs(T), t, 2);
```

```
eq3 := coeff(lhs(T), t, 3); eq4 := coeff(lhs(T), t, 4);
S := solve({eq0, eq1, eq2, eq3, eq4, h_2 = 4*a_x^2*c^2*lambda^2+c^4*lambda^2+2*c^4*lambda+c
    ↪ ^4-2*c^2*lambda^2-2*c^2+lambda^2-2*lambda+1}, {h_1, h_2, h_3, h_4, h_5, h_6});
for i to 6 do S[i] end do;
# checking
assign(S);
simplify(subs(x = M_x, y = M_y, con));
E1 := lhs(collect(simplify(subs(x_0 = e_x, y_0 = e_y, con_tan)), [x, y], recursive));
E2 := lhs(simplify(lineEW1));
Coef10 := E1-coeff(E1, x)*x-coeff(E1, y)*y;
Coef20 := E2-coeff(E2, x)*x-coeff(E2, y)*y;
collect(simplify(coeff(E1, x)/Coef10-coeff(E2, x)/Coef20), [x, y], recursive); # tangent line
    ↪  at E is ok
collect(simplify(coeff(E1, y)/Coef10-coeff(E2, y)/Coef20), [x, y], recursive); # tangent line
    ↪  at E is ok
E1 := lhs(collect(simplify(subs(x_0 = p_x, y_0 = p_y, con_tan)), [x, y], recursive));
E2 := lhs(simplify(lineAD));
Coef10 := E1-coeff(E1, x)*x-coeff(E1, y)*y;
Coef20 := E2-coeff(E2, x)*x-coeff(E2, y)*y;
collect(simplify(coeff(E1, x)/Coef10-coeff(E2, x)/Coef20), [x, y], recursive); # pole-polar
    ↪ is ok
collect(simplify(coeff(E1, y)/Coef10-coeff(E2, y)/Coef20), [x, y], recursive); # pole-polar
    ↪ is ok
```

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