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Research article

New sharp estimates of the interval length of the uniqueness results for several two-point fractional boundary value problems

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Abstract: This paper investigates the existence and uniqueness of solutions for several two-point fractional BVPs, including hybrid fractional BVP, sequential fractional BVP and so on. Using the Banach contraction mapping theorem, some sharp conditions that depend on the length of the given interval are presented, which ensure the uniqueness of solutions for the considered BVPs. Illustrative examples are also constructed. The results obtained in this study are noteworthy extensions of earlier works.

Keywords: fractional differential equation; two-point boundary condition; uniqueness of solution

1. Introduction

Bailey, Shampine and Waltman analyzed the following classical two-point boundary value problems (BVPs) [1]:

$$\begin{cases} y''(t) + f(t, y(t)) = 0, \quad t \in (a, b), \\ y(a) = A, \quad y(b) = B, \quad \text{and} \end{cases} \tag{1.1}$$

$$\begin{cases} y''(t) + f(t, y(t), y'(t)) = 0, \quad t \in (a, b), \\ y(a) = A, \quad y(b) = B, \end{cases} \tag{1.2}$$

where $A, B \in \mathbb{R}$. The authors presented the following results.

Theorem 1.1. [1] Let $f(t, y)$ be continuous on $[a, b] \times \mathbb{R}$ and satisfy Lipschitz condition with Lipschitz constant K,

$$|f(t, y) - f(t, x)| \leq K,$$

for all $(t, y), (t, x) \in [a, b] \times \mathbb{R}$. Then BVP (1.1) has a unique solution whenever

$$b - a < \frac{2\sqrt{2}}{\sqrt{K}}. \tag{1.3}$$

Theorem 1.2. [1] Let $f(t, y, y')$ be continuous on $[a, b] \times \mathbb{R}^2$ and satisfy Lipschitz condition with Lipschitz constants K and L,

$$|f(t, y, y') - f(t, x, x')| \leq K|y - x| + L|y' - x'|,$$

for all $(t, y, y'), (t, x, x') \in [a, b] \times \mathbb{R}^2$. Then BVP (1.2) has a unique solution if

$$\frac{K(b-a)^2}{8} + \frac{L(b-a)}{2} < 1. \tag{1.4}$$

Fractional calculus has risen in many fields of science and engineering over the past few decades. Numerous problems in physics, chemistry, biology, economics, signal and image processing, fluid dynamics, economics and control theory can be modeled in the form of fractional models, especially to describe processes with memory effects [2–6]. Consequently, solving fractional BVPs has always received considerable attention, and various interesting results dealing with the existence and uniqueness results for fractional differential equations involving a variety of boundary conditions have been established [7–11].

Recently, several scholars have proposed to extend the result of Theorem 1.1 by considering a fractional derivative. Additionally, based on different definitions of the fractional derivative, inequality (1.3) has been generalized to various forms. Examples include the Riemann-Liouville derivative [12], the Caputo fractional derivative [13], the Conformable fractional derivative [14], and the Hadamard fractional derivative [15]. For example, Ferreira [12] extended the result of Theorem 1.1 by using the Riemann-Liouville fractional derivative, that is, the following two-point fractional BVP was studied:

$$\begin{cases} D^{\alpha}_{a+}y(t) + f(t, y(t)) = 0, \quad a < t < b, \\ y(a) = 0, \quad y(b) = B, \end{cases} \tag{1.5}$$

where $1 < \alpha \leq 2$, D^{α}_{a+} is the Riemann-Liouville fractional derivative of order α. The following result was obtained.

Theorem 1.3. [12] Let $f : [a, b]\times\mathbb{R} \to \mathbb{R}$ be continuous and satisfy the Lipschitz condition on $[a, b]\times\mathbb{R}$ with Lipschitz constant K,

$$|f(t, x) - f(t, y)| \leq K|x - y|,$$

for all $(t, x), (t, y) \in [a, b] \times \mathbb{R}$. Then BVP (1.5) has a unique solution if

$$b - a < (\Gamma(\alpha))^{1/\alpha} \frac{\alpha^{(\alpha+1)/\alpha}}{K^{1/\alpha}(\alpha - 1)^{(\alpha-1)/\alpha}}.$$

Laadjal, Abdeljawad and Jarad [14] extended the result of Theorem 1.1 involving a conformable fractional derivative. The author investigated the following two-point fractional BVP:

$$\begin{cases} T^{a}_{\beta}u(t) + f(t, u(t)) = 0, \quad t \in (a, b), \\ u(a) = A, \quad u(b) = B, \quad A, B \in \mathbb{R}, \end{cases} \tag{1.6}$$

where $1 < \beta \leq 2$ and T_β^a is the conformable fractional derivative of order β. The following result was obtained.

Theorem 1.4. [14] Let $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ be continuous and satisfy the Lipschitz condition on $[a,b]\times\mathbb{R}$ with Lipschitz constant K,

$$|f(t,x) - f(t,y)| \leq K|x-y|,$$

for all $(t,x),(t,y) \in [a,b]\times\mathbb{R}$. Then BVP (1.6) has a unique solution if

$$b - a < \frac{\beta^{(2\beta-1)/\beta(\beta-1)}}{K^{1/\beta}}.$$

However, no result exists in the literature that extends Theorem 1.2 to fractional differential equations. The main objective of this study is to bridge this gap. To this end, inspired by the above literature, the following fractional BVPs are considered.

Motivated by the above-mentioned works, the sharp estimate for the unique solution of the following two-point hybrid fractional BVP is investigated:

$$\begin{cases} D_{a+}^{\alpha}\left[\dfrac{x(t)}{f(t,x(t))}\right] + g(t,x(t)) = 0, \quad t \in (a,b), \\ x(a) = 0, \quad x(b) = B, \end{cases} \tag{1.7}$$

where $1 < \alpha \leq 2$, D_{a+}^{α} is the Riemann-Liouville fractional derivative of order α, $B \in \mathbb{R}$ is a constant, $f : [a,b]\times\mathbb{R} \to \mathbb{R}\backslash\{0\}$ and $g : [a,b]\times\mathbb{R} \to \mathbb{R}$ are two continuous functions.

As a second problem, inspired by the above ideas and by [1], this paper aims to extend Theorem 1.2 by considering a fractional derivative. The sharp estimate for the unique solution of the following two-point fractional BVPs were studied with a non-linear term depending on the fractional derivative given by

$$\begin{cases} D_{a+}^{\alpha}y(t) + \mathfrak{f}(t,(t-a)^{2-\alpha}y(t), D_{a+}^{\alpha-1}y(t)) = 0, \quad t \in (a,b), \\ \lim\limits_{t\to a+}(t-a)^{2-\alpha}y(t) = A, \quad y(b) = B, \end{cases} \tag{1.8}$$

and the sequential fractional BVP

$$\begin{cases} D_{a+}^{\beta}\,{}^{C}D_{a+}^{\gamma}z(t) + \mathfrak{g}(t,z(t),(t-a)^{1-\beta}\,{}^{C}D_{a+}^{\gamma}z(t)) = 0, \quad t \in (a,b), \\ z(a) = A, \quad z(b) = B, \end{cases} \tag{1.9}$$

where $1 < \alpha \leq 2$, $0 < \gamma,\beta \leq 1$, $1 < \gamma+\beta \leq 2$, D_{a+}^{κ} is the Riemann-Liouville fractional derivative of order $\kappa = \alpha,\beta$, ${}^{C}D_{a+}^{\gamma}$ is the Caputo fractional derivative of order γ; $A, B \in \mathbb{R}$ are two constants, and $\mathfrak{f},\mathfrak{g} : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ are two continuous functions.

The following assumptions were considered throughout the present analysis:

(A_1) There exist Lipschitz constants L_1, L_2, such that, for all $(t,x_i) \in [a,b]\times\mathbb{R}$, $(i = 1,2)$,

$$|f(t,x_1) - f(t,x_2)| \leq L_1|x_1 - x_2|, \quad |g(t,x_1) - g(t,x_2)| \leq L_2|x_1 - x_2|.$$

(A_2) There exist Lipschitz constants K, L, such that, for any $(t,u_i,v_i) \in [a,b]\times\mathbb{R}^2$, $(i = 1,2)$,

$$|\mathfrak{f}(t,u_1,v_1) - \mathfrak{f}(t,u_2,v_2)| \leq K|u_1 - u_2| + L|v_1 - v_2|.$$

(A_3) There exist Lipschitz constants P, Q, such that, for any $(t, w_i, z_i) \in [a, b] \times \mathbb{R}^2$, $(i = 1, 2)$,

$$|g(t, w_1, z_1) - g(t, w_2, z_2)| \leq P|w_1 - w_2| + Q|z_1 - z_2|.$$

Based on the above interpretation, the contribution of this work is summarized as follows:

- A new condition in terms of the end points of the given interval is presented, which ensures the uniqueness of the solution for a two-point hybrid fractional BVP and generalizes the result of Theorem 1.3.
- The sharp estimate for the unique solution of the two-point fractional BVPs with a non-linear term depending on a lower fractional order derivative is established, which extends the classical integer order results of Theorem 1.2.
- The sequential two-point fractional BVP is considered, providing a means to solve the open problem (30) proposed in [14].

The rest of the paper is organized as follows. In Section 2, some basic results related to the fractional calculus are given. In Section 3, by using the Banach contraction mapping theorem the estimate for the uniqueness results of the two-point fractional BVPs (1.7)–(1.9) are investigated. In Section 4, we present some examples which illustrate the efficiency of the main results. Finally, Section 5 addresses the conclusion of the work.

2. Preliminaries

In this section, we recall some basic definitions and lemmas on fractional calculus, we refer the reader to [2].

Definition 2.1. [2] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $u : [a, b] \to \mathbb{R}$ is defined as

$$I^{\alpha}_{a+}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} u(s)ds,\ t \in [a, b].$$

Definition 2.2. [2] The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $u : [a, b] \to \mathbb{R}$ is defined as

$$D^{\alpha}_{a+}u(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t - s)^{n-\alpha-1} u(s)ds,\ \ t \in [a, b],\ \ n - 1 < \alpha < n,\ \ n = [\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. [2] The Caputo fractional derivative of order $\alpha > 0$ for a $(n - 1)$-times absolutely continuous function $u : [a, b] \to \mathbb{R}$ is defined as

$${}^{C}D^{\alpha}_{a+}u(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} u^{(n)}(s)ds,\ t \in [a, b],\ \ n - 1 < \alpha < n,\ \ n = [\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1. [2,7] Let $\alpha > 0$. If $u \in C(a,b) \cap L^1(a,b)$, then

$$I^{\alpha}_{a+}D^{\alpha}_{a+}u(t) = u(t) - c_1(t-a)^{\alpha-1} - c_2(t-a)^{\alpha-2} - \cdots - c_n(t-a)^{\alpha-n},$$

for some constants $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, and $n = [\alpha] + 1$.

Lemma 2.2. [2] Let $\alpha > 0$. If $x, {}^C D^{\alpha}_{a+}x \in L([a,b], \mathbb{R})$, then

$$I^{\alpha}_{a^+}\,{}^C D^{\alpha}_{a+}x(t) = x(t) - c_0 - c_1(t-a) - \cdots - c_{n-1}(t-a)^{n-1},$$

for some constants $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$, and $n = [\alpha]+1$.

Lemma 2.3. [2,7] Let $\alpha, \beta > 0$, $n = [\alpha] + 1$. Then,

$$I^{\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\alpha+\beta-1}, \quad t > a; \tag{2.1}$$

$$D^{\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \quad t > a; \tag{2.2}$$

$$D^{\alpha}_{a+}(t-a)^{\alpha-j} = 0, \quad t > a, \quad j = 1, 2, \cdots, n; \tag{2.3}$$

$${}^C D^{\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \quad t > a; \tag{2.4}$$

$${}^C D^{\alpha}_{a+}(t-a)^{k} = 0,\ t > a,\ k = 0, 1, 2, \cdots, n-1, \text{ and } {}^C D^{\alpha}_{a+}1 = 0. \tag{2.5}$$

Lemma 2.4. [2] Let $\alpha > \beta > 0$, $u(t) \in C(a,b)$. Then,

$$I^{\alpha}_{a+}I^{\beta}_{a+}u(t) = I^{\alpha+\beta}_{a+}u(t), \quad D^{\alpha}_{a+}I^{\alpha}_{a+}x(t) = u(t) = {}^C D^{\alpha}_{a+}I^{\alpha}_{a+}u(t), \tag{2.6}$$

$$D^{\beta}_{a+}I^{\alpha}_{a+}u(t) = I^{\alpha-\beta}_{a+}u(t). \tag{2.7}$$

3. Main results

Define Banach space X of continuous functions on $[a,b]$ with the norm $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$. Let $\alpha, \beta, \gamma \in \mathbb{R}$, $1 < \alpha \le 2$, $0 < \gamma, \beta \le 1$, $1 < \gamma + \beta \le 2$ be fixed and $I = [a,b]$. For any $y : (a,b] \to \mathbb{R}$ and $z : [a,b] \to \mathbb{R}$, we define functions $y_\alpha : I \to \mathbb{R}$ and $\dot{z}_\beta : I \to \mathbb{R}$ by

$$y_\alpha(t) = \begin{cases} (t-a)^{2-\alpha}y(t), & \text{if } t \in (a,b], \\ \lim\limits_{t\to a+}(t-a)^{2-\alpha}y(t), & \text{if } t = a, \end{cases}$$

$$\dot{z}_\beta(t) = \begin{cases} (t-a)^{1-\beta}\,{}^C D^{\gamma}_{a+}z(t), & \text{if } t \in (a,b], \\ \lim\limits_{t\to a+}(t-a)^{1-\beta}\,{}^C D^{\gamma}_{a+}z(t), & \text{if } t = a, \end{cases}$$

given that the right-hand limits are exist. Define spaces

$$Y := \{y : (a,b] \to \mathbb{R} | y_\alpha, D^{\alpha-1}_{a+}y \in C[a,b]\},$$
$$Z := \{z : [a,b] \to \mathbb{R} | z, \dot{z}_\beta(t) \in C[a,b]\}.$$

It is not difficult to show that Y and Z are two Banach spaces equipped with the norms

$$\|y\|_\alpha = \max_{t\in[a,b]}(K|y_\alpha| + L|D_{a+}^{\alpha-1}y|), \quad \|z\|_\beta = \max_{t\in[a,b]}(P|z| + Q|\dot{z}_\beta|),$$

respectively.

Lemma 3.1. Assume that f and g are continuous functions. Then, a function $x \in C[a,b]$ is a solution of Eq (1.7) if and only if $x(t)$ satisfies

$$x(t) = f(t,x(t))\int_a^b G(t,s)g(s,x(s))ds + \frac{B(t-a)^{\alpha-1}f(t,x(t))}{(b-a)^{\alpha-1}f(b,x(b))}, \tag{3.1}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \dfrac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, \ a \le s \le t \le b, \\ \dfrac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}, \ a \le t \le s \le b. \end{cases} \tag{3.2}$$

Proof. According to Lemma 2.1, applying operator I_{a+}^α on both sides of Eq (1.7) yields

$$\frac{x(t)}{f(t,x(t))} = -I_{a+}^\alpha g(t,x(t)) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2},$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Therefore,

$$x(t) = f(t,x(t))\Big[-I_{a+}^\alpha g(t,x(t)) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2}\Big].$$

Using boundary condition $x(a) = 0$, we get $c_2 = 0$. Then,

$$x(t) = f(t,x(t))\left[-I_{a+}^\alpha g(t,x(t)) + c_1(t-a)^{\alpha-1}\right]. \tag{3.3}$$

From boundary condition $x(b) = B$, it follows that

$$\frac{B}{f(b,x(b))} = -I_{a+}^\alpha g(t,x(t))|_{t=b} + c_1(b-a)^{\alpha-1},$$

that is,

$$c_1 = \frac{B}{(b-a)^{\alpha-1}f(b,x(b))} + \frac{I_{a+}^\alpha g(t,x(t))|_{t=b}}{(b-a)^{\alpha-1}}.$$

Substituting the value of c_1 in Eq (3.3) yields

$$\begin{aligned} x(t) =& f(t,x(t))\Big(-\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s,x(s))ds \\ &+ \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}}\int_a^b (b-s)^{\alpha-1}g(s,x(s))ds\Big) + \frac{B(t-a)^{\alpha-1}f(t,x(t))}{(b-a)^{\alpha-1}f(b,x(b))} \\ =& f(t,x(t))\int_a^b G(t,s)g(s,x(s))ds + \frac{B(t-a)^{\alpha-1}f(t,x(t))}{(b-a)^{\alpha-1}f(b,x(b))}. \end{aligned}$$

Conversely, by direct computation, it can be established that (3.1) satisfies problem (1.7). This completes the proof.

Lemma 3.2. Assume that $\mathfrak{f}$ is a continuous function. Then, a function $y \in Y$ is a solution of Eq (1.8) if and only if $y(t)$ satisfies the integral equations

$$y(t)=\int_a^b G(t,s)\mathfrak{f}(s,(s-a)^{2-\alpha}y(s),D_{a+}^{\alpha-1}y(s))ds+\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}[B-A(b-a)^{\alpha-2}]+A(t-a)^{\alpha-2}, \tag{3.4}$$

and

$$D_{a+}^{\alpha-1}y(t)=\frac{\Gamma(\alpha)[B-A(b-a)^{\alpha-2}]}{(b-a)^{\alpha-1}}+\int_a^b H(t,s)\mathfrak{f}(s,(s-a)^{2-\alpha}y(s),D_{a+}^{\alpha-1}y(s))ds, \tag{3.5}$$

where $G(t,s)$ defined the same as in (3.2),

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\dfrac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}-(t-s)^{\alpha-1}, & a\le s\le t\le b,\\ \dfrac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}, & a\le t\le s\le b,\end{cases}$$

and

$$H(t,s)=\frac{1}{(b-a)^{\alpha-1}}\begin{cases}(b-s)^{\alpha-1}-(b-a)^{\alpha-1}, & a\le s\le t\le b,\\ (b-s)^{\alpha-1}, & a\le t\le s\le b.\end{cases} \tag{3.6}$$

Proof. In view of Lemma 2.1, a general solution of the fractional equation in (1.8) is given by

$$y(t)=-I_{a+}^{\alpha}\mathfrak{f}(t,(t-a)^{2-\alpha}y(t),D_{a+}^{\alpha-1}y(t))+c_0(t-a)^{\alpha-1}+c_1(t-a)^{\alpha-2}, \tag{3.7}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. From the first boundary condition $\lim_{t\to a+}(t-a)^{2-\alpha}y(t)=A$, we obtain $c_1 = A$, which yields

$$y(t)=-I_{a+}^{\alpha}\mathfrak{f}(t,(t-a)^{2-\alpha}y(t),D_{a+}^{\alpha-1}y(t))+c_0(t-a)^{\alpha-1}+A(t-a)^{\alpha-2}. \tag{3.8}$$

From $y(b) = B$ and by using (3.8), we derive

$$c_0=\frac{1}{(b-a)^{\alpha-1}}\left[I_{a+}^{\alpha}f(t,(t-a)^{2-\alpha}y(t),D_{a+}^{\alpha-1}y(t))|_{t=b}+B-A(b-a)^{\alpha-2}\right].$$

Substituting the value of c_0 in (3.8) yields solution (3.4),

$$\begin{aligned}y(t)=&-I_{a+}^{\alpha}\mathfrak{f}(t,(t-a)^{2-\alpha}y(t),D_{a+}^{\alpha-1}y(t))+A(t-a)^{\alpha-2}\\&+\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\left[I_{a+}^{\alpha}f(t,(t-a)^{2-\alpha}y(t),D_{a+}^{\alpha-1}y(t))|_{t=b}+B-A(b-a)^{\alpha-2}\right]\\=&\int_a^b G(t,s)\mathfrak{f}(s,(s-a)^{2-\alpha}y(s),D_{a+}^{\alpha-1}y(s))ds+\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}[B-A(b-a)^{\alpha-2}]+A(t-a)^{\alpha-2}.\end{aligned}$$

The converse follows by direct computation. On the other hand, according to (2.2), (2.3) and (2.7), taking the $(\alpha-1)$-th Riemann-Liouville fractional derivative on the both sides of (3.4) yields (3.5),

$$\begin{aligned} D_{a+}^{\alpha-1}y(t) =& -D_{a+}^{\alpha-1}I_{a+}^{\alpha}\mathfrak{f}(t,(t-a)^{2-\alpha}y(t),D_{a+}^{\alpha-1}y(t))+A[D_{a+}^{\alpha-1}(t-a)^{\alpha-2}] \\ &+\frac{D_{a+}^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\left[I_{a+}^{\alpha}f(t,(t-a)^{2-\alpha}y(t),D_{a+}^{\alpha-1}y(t))|_{t=b}+B-A(b-a)^{\alpha-2}\right] \\ =& -\int_a^t \mathfrak{f}(s,(s-a)^{2-\alpha}y(s),D_{a+}^{\alpha-1}y(s))ds+\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}[B-A(b-a)^{\alpha-2}] \\ &+\frac{1}{(b-a)^{\alpha-1}}\Big[\int_a^b (b-s)^{\alpha-1}f(s,(s-a)^{2-\alpha}y(s),D_{a+}^{\alpha-1}y(s))ds\Big] \\ =&\frac{\Gamma(\alpha)[B-A(b-a)^{\alpha-2}]}{(b-a)^{\alpha-1}}+\int_a^b H(t,s)\mathfrak{f}(s,(s-a)^{2-\alpha}y(s),D_{a+}^{\alpha-1}y(s))ds. \end{aligned}$$

This proves the lemma.

Lemma 3.3. Assume that $\mathfrak{g}$ is a continuous function. Then, a function $z\in Y$ is a solution of equation (1.9) if and only if $z(t)$ satisfies the integral equations

$$z(t)=\int_a^b \mathfrak{G}(t,s)\mathfrak{g}(s,z(s),(s-a)^{1-\beta C}D_{a+}^{\gamma}z(s))ds+\frac{(t-a)^{\gamma+\beta-1}}{(b-a)^{\gamma+\beta-1}}(B-A)+A, \tag{3.9}$$

and

$${}^{C}D_{a+}^{\gamma}z(t)=\frac{(B-A)\Gamma(\gamma+\beta)}{(b-a)^{\gamma+\beta-1}\Gamma(\beta)}(t-a)^{\beta-1}+\int_a^b \mathfrak{H}(t,s)\mathfrak{g}(s,z(s),(s-a)^{1-\beta C}D_{a+}^{\gamma}z(s))ds, \tag{3.10}$$

where kernel functions $\mathfrak{G}(t,s)$ and $\mathfrak{H}(t,s)$ are defined as

$$\mathfrak{G}(t,s)=\frac{1}{\Gamma(\gamma+\beta)}\begin{cases}\dfrac{(t-a)^{\gamma+\beta-1}}{(b-a)^{\gamma+\beta-1}}(b-s)^{\gamma+\beta-1}-(t-s)^{\gamma+\beta-1}, & a\le s\le t\le b,\\ \dfrac{(t-a)^{\gamma+\beta-1}}{(b-a)^{\gamma+\beta-1}}(b-s)^{\gamma+\beta-1}, & a\le t\le s\le b,\end{cases} \tag{3.11}$$

and

$$\mathfrak{H}(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}\dfrac{(t-a)^{\beta-1}}{(b-a)^{\gamma+\beta-1}}(b-s)^{\gamma+\beta-1}-(t-s)^{\beta-1}, & a\le s\le t\le b,\\ \dfrac{(t-a)^{\beta-1}}{(b-a)^{\gamma+\beta-1}}(b-s)^{\gamma+\beta-1}, & a\le t\le s\le b.\end{cases} \tag{3.12}$$

Proof. Applying operators I_{a+}^{β} and I_{a+}^{γ} on the fractional equation in (1.9) and then using Lemmas 2.1–2.4 yields

$$z(t)=-I_{a+}^{\gamma+\beta}\mathfrak{g}(t,z(t),(t-a)^{1-\beta C}D_{a+}^{\gamma}z(t))+\frac{\Gamma(\beta)}{\Gamma(\gamma+\beta)}c_0(t-a)^{\gamma+\beta-1}+c_1, \tag{3.13}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Applying the boundary conditions $z(a) = A,\ z(b) = B$ in (3.13) yields

$$c_1 = A,\ c_0 = \frac{\Gamma(\gamma+\beta)}{(b-a)^{\gamma+\beta-1}\Gamma(\beta)}\left[B - A + I_{a+}^{\gamma+\beta}\mathfrak{g}(t, z(t), (t-a)^{1-\beta C}D_{a+}^{\gamma}z(t))|_{t=b}\right].$$

Substituting the above values into (3.13), the solution given by (3.9) is obtained. The converse of the lemma can be obtained by direct computation. On the other hand, by Lemma 2.3 and Lemma 2.4, taking the γ-th Caputo fractional derivative on both sides of (3.9) yields (3.10). The proof is completed.

Lemma 3.4. The Green's functions $G(t,s)$, $H(t,s)$, $\mathfrak{G}(t,s)$ and $\mathfrak{H}(t,s)$ given by Lemmas 3.1–3.3, respectively, satisfy the following properties:

(i) $G(t,s)$, $H(t,s)$, $\mathfrak{G}(t,s)$ and $\mathfrak{H}(t,s)$ are continuous functions in $[a,b]\times[a,b]$;

(ii) $G(t,s)$ and $\mathfrak{G}(t,s)$ are two nonnegative functions in $[a,b]\times[a,b]$;

(iii) $\displaystyle\int_a^b G(t,s)ds \le \frac{(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(b-a)^{\alpha}$, for any $t \in [a,b]$;

(iv) $\displaystyle\int_a^b (t-a)^{2-\alpha}G(t,s)ds \le \frac{(b-a)^2}{4\Gamma(\alpha+1)}$, for any $t \in [a,b]$;

(v) $\displaystyle\int_a^b |H(t,s)|ds \le \frac{1}{\alpha}(b-a)$, for any $t \in [a,b]$;

(vi) $\displaystyle\int_a^b \mathfrak{G}(t,s)ds \le \frac{(\gamma+\beta-1)^{\gamma+\beta-1}}{\Gamma(\gamma+\beta)(\gamma+\beta)^{\gamma+\beta+1}}(b-a)^{\gamma+\beta}$, for any $t \in [a,b]$;

(vii) $\displaystyle\int_a^b (t-a)^{1-\beta}|\mathfrak{H}(t,s)|ds \le \max\{\beta,\gamma\}\frac{(b-a)}{(\gamma+\beta)\Gamma(\beta+1)}$, for any $t \in [a,b]$.

Proof. It is obvious that (i) is true. For (ii), in view of the definition of $G(t,s)$, let

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1},\ a \le s \le t \le b,$$
$$G_2(t,s) = \frac{1}{\Gamma(\alpha)}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1},\ a \le t \le s \le b.$$

Then, we can easily obtain that

$$G_2(t,s) \ge 0,\ (t,s) \in [a,b]\times[a,b].$$

Differentiating $G_1(t,s)$ with respect to s for every fixed $t \in [a,b]$,

$$\begin{aligned}\frac{\partial G_1(t,s)}{\partial s} &= \frac{\alpha-1}{\Gamma(\alpha)}\Big[-\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-2} + (t-s)^{\alpha-2}\Big]\\ &= \frac{\alpha-1}{\Gamma(\alpha)}(t-s)^{\alpha-2}\Big[1 - \Big(\frac{t-a}{b-a}\Big)^{\alpha-1}\Big(\frac{t-s}{b-s}\Big)^{2-\alpha}\Big] \ge 0,\end{aligned}$$

that is, $G_1(t,s)$ is increasing with respect to $s \in [a,t]$ for any fixed $t \in [a,b]$. Therefore,

$$G_1(t,s) \ge G_1(t,a) = 0,\ t \in [a,b].$$

Thus, we have derived that $G(t,s)$ ia nonnegative in $[a,b]\times[a,b]$. Let $\alpha=\beta+\gamma$, then we can also get $\mathfrak{G}(t,s)$ is nonnegative in $[a,b]\times[a,b]$. For (iii) and (iv), by the expression for the function $G(t,s)$, we obtain

$$\begin{aligned}\int_a^b G(t,s)ds =&\frac{1}{\Gamma(\alpha)}\Big\{\int_a^t\Big[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big]ds\\&+\int_t^b\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}ds\Big\}\\=&\frac{1}{\Gamma(\alpha)}\Big\{\frac{1}{\alpha}\Big[-\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha}+(t-s)^{\alpha}\Big]\Big|_{s=a}^{s=t}\\&-\frac{1}{\alpha}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha}\Big|_{s=t}^{s=b}\Big\}\\=&\frac{1}{\alpha\Gamma(\alpha)}(t-a)^{\alpha-1}(b-t).\end{aligned}$$

It follows that

$$\int_a^b(t-a)^{2-\alpha}G(t,s)ds=(t-a)^{2-\alpha}\int_a^b G(t,s)ds=\frac{1}{\alpha\Gamma(\alpha)}(t-a)(b-t).$$

Define

$$g(t)=(t-a)(b-t),\quad t\in[a,b],$$

and

$$\tilde{g}(t)=(t-a)^{\alpha-1}(b-t),\quad t\in[a,b].$$

Differentiating the functions $g(t)$ and $\tilde{g}(t)$ on (a,b), we immediately find that $g(t)$ and $\tilde{g}(t)$ are achieved their maximum at the following points, respectively,

$$t^*=\frac{a+b}{2},\quad \tilde{t}=\frac{1}{\alpha}[a+(\alpha-1)b].$$

This yields,

$$\max_{t\in[a,b]}g(t)=\frac{1}{4}(b-a)^2,\quad \max_{t\in[a,b]}\tilde{g}(t)=\frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha}}{\alpha^{\alpha}},$$

which completes the proof of (iii) and (iv). Let $\alpha=\beta+\gamma$, then property (vi) can be obtained directly from (iii). We will now show that properties (v) and (vii) are true. First, for (v), in view of the definition of $H(t,s)$, we have

$$\begin{aligned}&\int_a^b|H(t,s)|ds\\&=\frac{1}{(b-a)^{\alpha-1}}\Big[\int_a^t((b-a)^{\alpha-1}-(b-s)^{\alpha-1})ds+\int_t^b(b-s)^{\alpha-1}ds\Big]\\&=\frac{1}{(b-a)^{\alpha-1}}\Big[(b-a)^{\alpha-1}(t-a)+\frac{2}{\alpha}(b-t)^{\alpha}-\frac{1}{\alpha}(b-a)^{\alpha}\Big].\end{aligned}$$

Define

$$h(t)=(b-a)^{\alpha-1}(t-a)+\frac{2}{\alpha}(b-t)^{\alpha}-\frac{1}{\alpha}(b-a)^{\alpha},\quad t\in[a,b].$$

Then,

$$h(a) = \frac{1}{\alpha}(b-a)^{\alpha} \geq h(b) = (b-a)^{\alpha} - \frac{1}{\alpha}(b-a)^{\alpha} > 0.$$

Taking the second-order derivative of function $h(t)$ on (a, b), we obtain

$$h''(t) = 2(\alpha-1)(b-t)^{\alpha-2} \geq 0, \quad t \in [a, b].$$

Therefore, $h(t)$ is convex on (a, b). Hence,

$$\max_{t\in[a,b]} h(t) = \max_{t\in[a,b]}\{h(a), h(b)\} = h(a) = \frac{1}{\alpha}(b-a)^{\alpha}.$$

This completes the proof of (v). Finally, for (vii), for $a \leq s \leq t \leq b$, we can derive that

$$\begin{aligned}&(b-s)^{\gamma+\beta-1}(t-a)^{\beta-1} - (b-a)^{\gamma+\beta-1}(t-s)^{\beta-1}\\ &= (b-a)^{\gamma+\beta-1}(t-s)^{\beta-1}\Big[\Big(\frac{b-s}{b-a}\Big)^{\gamma+\beta-1}\Big(\frac{t-s}{t-a}\Big)^{1-\beta} - 1\Big] \leq 0.\end{aligned}$$

Hence, it follows from the definition of $\mathfrak{H}(t, s)$ that

$$\begin{aligned}&\int_a^b |\mathfrak{H}(t, s)|ds\\ &= \frac{1}{\Gamma(\beta)}\int_a^t \Big[(t-s)^{\beta-1} - \frac{(t-a)^{\beta-1}}{(b-a)^{\gamma+\beta-1}}(b-s)^{\gamma+\beta-1}\Big]ds + \frac{1}{\Gamma(\beta)}\int_t^b \frac{(t-a)^{\beta-1}}{(b-a)^{\gamma+\beta-1}}(b-s)^{\gamma+\beta-1}ds\\ &= \frac{1}{(\gamma+\beta)\Gamma(\beta)}\frac{(t-a)^{\beta-1}}{(b-a)^{\gamma+\beta-1}}\Big[2(b-t)^{\gamma+\beta} - (b-a)^{\gamma+\beta}\Big] + \frac{1}{\Gamma(\beta+1)}(t-a)^{\beta}.\end{aligned}$$

Consequently,

$$\int_a^b (t-a)^{1-\beta}|\mathfrak{H}(t, s)|ds = \frac{2(b-t)^{\gamma+\beta} - (b-a)^{\gamma+\beta}}{(b-a)^{\gamma+\beta-1}(\gamma+\beta)\Gamma(\beta)} + \frac{(t-a)}{\Gamma(\beta+1)}.$$

Define

$$\mathfrak{h}(t) = \frac{2(b-t)^{\gamma+\beta} - (b-a)^{\gamma+\beta}}{(b-a)^{\gamma+\beta-1}(\gamma+\beta)\Gamma(\beta)} + \frac{(t-a)}{\Gamma(\beta+1)}, \quad t \in [a, b].$$

Then,

$$\mathfrak{h}(a) = \frac{\beta(b-a)}{(\gamma+\beta)\Gamma(\beta+1)} > 0, \quad \mathfrak{h}(b) = \frac{\gamma(b-a)}{(\gamma+\beta)\Gamma(\beta+1)} > 0.$$

Taking the second derivative of function $\mathfrak{h}(t)$ on (a, b) leads to

$$\mathfrak{h}''(t) = (\gamma+\beta-1)\frac{2(b-t)^{\gamma+\beta-2}}{(b-a)^{\gamma+\beta-1}\Gamma(\beta)} \geq 0, \quad t \in (a, b).$$

This implies that $\mathfrak{h}(t)$ is convex on (a, b). Therefore,

$$\max_{t\in[a,b]} \mathfrak{h}(t) = \max\{\mathfrak{h}(a), \mathfrak{h}(b)\} = \max\{\beta, \gamma\}\frac{(b-a)}{(\gamma+\beta)\Gamma(\beta+1)}.$$

The proof is completed.

Based on Lemmas 3.1 to 3.3, operators $T_1 : X\to X$, $T_2 : Y\to Y$ and $T_3 : Z\to Z$ are defined as

$$T_1x(t)=f(t,x(t))\int_a^b G(t,s)g(s,x(s))ds+\frac{B(t-a)^{\alpha-1}f(t,x(t))}{(b-a)^{\alpha-1}f(b,x(b))},\quad x(t)\in X,$$

$$T_2y(t)=\int_a^b G(t,s)\mathfrak{f}(s,(s-a)^{2-\alpha}y(s),D_{a+}^{\alpha-1}y(s))ds+\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}[B-A(b-a)^{\alpha-2}]+A(t-a)^{\alpha-2},\quad y(t)\in Y,$$

$$T_3z(t)=\int_a^b \mathfrak{G}(t,s)\mathfrak{g}(s,z(s),(s-a)^{1-\beta C}D_{a+}^{\gamma}z(s))ds+\frac{(t-a)^{\gamma+\beta-1}}{(b-a)^{\gamma+\beta-1}}(B-A)+A,\quad z(t)\in Z.$$

Theorem 3.1. Assume that $f, g : [a, b]\times\mathbb{R}\to\mathbb{R}$ are continuous functions and satisfy condition (A_1). Let

$$M_1=\sup_{t\in[a,b]}|f(t,x(t))|,\quad m_1=\inf_{t\in[a,b]}|f(t,x(t))|,\quad M_2=\sup_{t\in[a,b]}|g(t,x(t))|.$$

If

$$b-a<\left\{\frac{\alpha^{\alpha+1}[m_1^2-BL_1(m_1+M_1)]\Gamma(\alpha)}{m_1^2(M_2L_1+M_1L_2)(\alpha-1)^{\alpha-1}}\right\}^{1/\alpha},\tag{3.14}$$

then problem (1.7) has a unique solution.

Proof. For $x_1(t), x_2(t)\in X$, using condition (A_1), Lemma 3.4 (iii) and together with (3.1), yields

$$\begin{aligned}|Tx_1(t)-Tx_2(t)|\le&\Big|\frac{B(t-a)^{\alpha-1}f(t,x_1(t))}{(b-a)^{\alpha-1}f(b,x_1(b))}-\frac{B(t-a)^{\alpha-1}f(t,x_2(t))}{(b-a)^{\alpha-1}f(b,x_2(b))}\Big|\\
&+\Big|f(t,x_1(t))\int_a^b G(t,s)g(s,x_1(s))ds-f(t,x_2(t))\int_a^b G(t,s)g(s,x_2(s))ds\Big|\\
\le&\frac{B(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\Big(\Big|\frac{f(t,x_1(t))}{f(b,x_1(b))}-\frac{f(t,x_2(t))}{f(b,x_1(b))}\Big|+\Big|\frac{f(t,x_2(t))}{f(b,x_1(b))}-\frac{f(t,x_2(t))}{f(b,x_2(b))}\Big|\Big)\\
&+\Big|f(t,x_1(t))\int_a^b G(t,s)g(s,x_1(s))ds-f(t,x_2(t))\int_a^b G(t,s)g(s,x_1(s))ds\Big|\\
&+\Big|f(t,x_2(t))\int_a^b G(t,s)g(s,x_1(s))ds-f(t,x_2(t))\int_a^b G(t,s)g(s,x_2(s))ds\Big|\\
\le&\frac{B(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}|f(b,x_1(b))|}|f(t,x_1(t))-f(t,x_2(t))|\\
&+\frac{B(t-a)^{\alpha-1}|f(t,x_2(t))|}{(b-a)^{\alpha-1}|f(b,x_1(b))||f(b,x_2(b))|}|f(b,x_2(b))-f(b,x_1(b))|\\
&+|f(t,x_1(t))-f(t,x_2(t))|\int_a^b G(t,s)|g(s,x_1(s))|ds\\
&+\int_a^b G(t,s)|g(s,x_1(s))-g(s,x_2(s))|ds|f(t,x_2(t))|\\
\le&\Big(\frac{B}{m_1}+\frac{BM_1}{m_1^2}\Big)L_1|x_1-x_2|+(M_2L_1+M_1L_2)\frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha}}{\Gamma(\alpha)\alpha^{\alpha+1}}|x_1-x_2|\end{aligned}$$

$$\leq \Big[\frac{BL_1(m_1+M_1)}{m_1^2} + (M_2L_1 + M_1L_2)\frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha}}{\Gamma(\alpha)\alpha^{\alpha+1}}\Big]\|x_1 - x_2\|_{\infty}.$$

Therefore, we conclude from (3.14) that operator T_1 is a contraction mapping. Hence problem (1.7) has a unique solution.

As special cases of Theorem 3.1, we have the following corollary:

Corollary 3.1. Let $g(t, x)$ be continuous on $[a, b] \times \mathbb{R}$ and satisfy Lipschitz condition

$$|g(t, x_1) - g(t, x_2)| \leq K|x_1 - x_2|, \quad \text{for any } x_1, x_2 \in \mathbb{R},\ K > 0.$$

Then the BVP

$$\begin{cases} D^{\alpha}_{a+}x(t) + g(t, x(t)) = 0,\ t \in (a, b),\ 1 < \alpha \leq 2, \\ x(a) = 0, \quad x(b) = B,\ B \in \mathbb{R}, \end{cases}$$

has a unique solution whenever

$$b - a < \left[\frac{\Gamma(\alpha)\alpha^{\alpha+1}}{K(\alpha-1)^{\alpha-1}}\right]^{1/\alpha}. \tag{3.15}$$

Proof. By Theorem 3.1, let $f(t, x) \equiv 1$, $(t, x) \in [a, b] \times \mathbb{R}$, and $L_2 = K$. Then,

$$M_1 = \sup_{t\in[a,b]} |f(t, x(t))| = \inf_{t\in[a,b]} |f(t, x(t))| = m_1 = 1, \quad L_1 = 0.$$

Substituting the above values into (3.14), the desired result (3.15) is obtained. As such, our results match the results of Theorem 2.3 in [12].

Theorem 3.2. Let $\mathfrak{f}(t, (t-a)^{2-\alpha}y(t), D^{\alpha-1}_{a+}y(t))$ be continuous on $[a, b] \times \mathbb{R}^2$ and satisfy condition (A_2). If

$$\frac{K(b-a)^2}{4\Gamma(\alpha+1)} + \frac{L}{\alpha}(b-a) < 1, \tag{3.16}$$

then problem (1.8) has a unique solution.

Proof. To see when T_2 is contracting, we again from

$$\begin{aligned} &(t-a)^{2-\alpha}|T_2y_1(t) - T_2y_2(t)| \\ &\leq \int_a^b (t-a)^{2-\alpha}G(t, s)|\mathfrak{f}(s, (s-a)^{2-\alpha}y_1(s), D^{\alpha-1}_{a+}y_1(s)) - \mathfrak{f}(s, (s-a)^{2-\alpha}y_2(s), D^{\alpha-1}_{a+}y_2(s))|ds, \end{aligned}$$

and from (3.5)

$$\begin{aligned} &|D^{\alpha-1}_{a+}T_2y_1(t) - D^{\alpha-1}_{a+}T_2y_2(t)| \\ &\leq \int_a^b |H(t, s)||\mathfrak{f}(s, (s-a)^{2-\alpha}y_1(s), D^{\alpha-1}_{a+}y_1(s)) - \mathfrak{f}(s, (s-a)^{2-\alpha}y_2(s), D^{\alpha-1}_{a+}y_2(s))|ds. \end{aligned}$$

Using Lipschitz condition (A_2) and Lemma 3.4 (iv), (v),

$$\begin{aligned}
&(t-a)^{2-\alpha}|T_2y_1(t)-T_2y_2(t)| \\
&\le \int_a^b (t-a)^{2-\alpha}G(t,s)(K(s-a)^{2-\alpha}|y_1(s)-y_2(s)|+L|D_{a+}^{\alpha-1}y_1(s)-D_{a+}^{\alpha-1}y_2(s)|)ds \\
&\le \|y_1-y_2\|_\alpha \int_a^b (t-a)^{2-\alpha}G(t,s)ds \le \frac{(b-a)^2}{4\Gamma(\alpha+1)}\|y_1-y_2\|_\alpha,
\end{aligned}$$

and

$$\begin{aligned}
&|D_{a+}^{\alpha-1}T_2y_1(t)-D_{a+}^{\alpha-1}T_2y_2(t)| \\
&\le \int_a^b |H(t,s)|(K(s-a)^{2-\alpha}|y_1(s)-y_2(s)|+L|D_{a+}^{\alpha-1}y_1(s)-D_{a+}^{\alpha-1}y_2(s)|)ds \\
&\le \|y_1-y_2\|_\alpha \int_a^b |H(t,s)|ds \le \frac{1}{\alpha}(b-a)\|y_1-y_2\|_\alpha.
\end{aligned}$$

Together these imply

$$\|T_2y_1-T_2y_2\|_\alpha \le \Big[\frac{K(b-a)^2}{4\Gamma(\alpha+1)}+\frac{L}{\alpha}(b-a)\Big]\|y_1-y_2\|_\alpha.$$

It follows from (3.16) that operator T_2 is a contraction mapping. Hence, problem (1.8) has a unique solution.

Remark 3.1. Let $\alpha \to 2$. Then, Theorem 3.2 can be reduced to Theorem 1.2.

Theorem 3.3. Let $\mathfrak{g}(t, z(t), (t-a)^{1-\beta C}D_{a+}^{\gamma}z(t))$ be continuous on $[a,b]\times\mathbb{R}^2$ and satisfy condition (A_3). If

$$\frac{P(\gamma+\beta-1)^{\gamma+\beta-1}(b-a)^{\gamma+\beta}}{\Gamma(\gamma+\beta)(\gamma+\beta)^{\gamma+\beta+1}}+\frac{Q\max\{\beta,\gamma\}(b-a)}{(\gamma+\beta)\Gamma(\beta+1)}<1, \tag{3.17}$$

then problem (1.9) has a unique solution.

Proof. This result will follow from the Banach contraction mapping theorem if we can show that operator T_3 is a contraction mapping. In fact, for any $z_1(t), z_2(t) \in Z$, we have

$$\begin{aligned}
&|T_3z_1(t)-T_3z_2(t)| \\
&\le \int_a^b \mathfrak{G}(t,s)|\mathfrak{g}(s,z_1(s),(s-a)^{1-\beta C}D_{a+}^{\gamma}z_1(s))-\mathfrak{g}(s,z_2(s),(s-a)^{1-\beta C}D_{a+}^{\gamma}z_2(s))|ds,
\end{aligned}$$

and in view of (3.10),

$$\begin{aligned}
&|(t-a)^{1-\beta C}D_{a+}^{\gamma}T_3z_1(t)-(t-a)^{1-\beta C}D_{a+}^{\gamma}T_3z_2(t)| \\
&\le \int_a^b (t-a)^{1-\beta}|\mathfrak{H}(t,s)||\mathfrak{g}(s,z_1(s),(s-a)^{1-\beta C}D_{a+}^{\gamma}z_1(s))-\mathfrak{g}(s,z_2(s),(s-a)^{1-\beta C}D_{a+}^{\gamma}z_2(s))|ds.
\end{aligned}$$

Using Lipschitz condition (A_3) and Lemma 3.4 (vi), (vii),

$$\begin{aligned}&|T_3z_1(t)-T_3z_2(t)|\\ &\le \int_a^b \mathfrak{G}(t,s)P|z_1(s)-z_2(s)|+Q(s-a)^{1-\beta}|{}^CD^{\gamma}_{a+}z_1(s)-{}^CD^{\gamma}_{a+}z_2(s)|ds\\ &\le \|z_1-z_2\|_\beta\int_a^b \mathfrak{G}(t,s)ds\le \frac{(\gamma+\beta-1)^{\gamma+\beta-1}(b-a)^{\gamma+\beta}}{\Gamma(\gamma+\beta)(\gamma+\beta)^{\gamma+\beta+1}}\|z_1-z_2\|_\beta,\end{aligned}$$

and

$$\begin{aligned}&(t-a)^{1-\beta}|{}^CD^{\gamma}_{a+}T_3z_1(t)-{}^CD^{\gamma}_{a+}T_3z_2(t)|\\ &\le \int_a^b (t-a)^{1-\beta}|\mathfrak{H}(t,s)|P|z_1(s)-z_2(s)|+Q(s-a)^{1-\beta}|{}^CD^{\gamma}_{a+}z_1(s)-{}^CD^{\gamma}_{a+}z_2(s)|ds\\ &\le \|z_1-z_2\|_\beta\int_a^b (t-a)^{1-\beta}|\mathfrak{H}(t,s)|ds\le \frac{\max\{\beta,\gamma\}(b-a)}{(\gamma+\beta)\Gamma(\beta+1)}\|z_1-z_2\|_\beta,\end{aligned}$$

which, by taking the norm for $t\in[a,b]$, implies that

$$\|T_3z_1-T_3z_2\|_\beta\le\Big[\frac{P(\gamma+\beta-1)^{\gamma+\beta-1}(b-a)^{\gamma+\beta}}{\Gamma(\gamma+\beta)(\gamma+\beta)^{\gamma+\beta+1}}+\frac{Q\max\{\beta,\gamma\}(b-a)}{(\gamma+\beta)\Gamma(\beta+1)}\Big]\|z_1-z_2\|_\beta.$$

From (3.17) we conclude that operator T_3 is a contraction mapping. Thus, problem (1.9) has a unique solution.

Remark 3.2. Let $\gamma,\beta\to 1$. Then Theorem 3.3 can be reduced to Theorem 1.2.

4. Example

Example 4.1. Consider the following two-point fractional BVP

$$\begin{cases} D^{4/3}_{1+}\left[\dfrac{x(t)}{(6t/5)+(\cos x(t))/10}\right]+\dfrac{\ln t}{5}\sin x(t)=0, \quad t\in(1,2),\\ x(1)=0,\ x(2)=1.\end{cases}\tag{4.1}$$

Corresponding to BVP (1.7), here

$$\alpha=4/3,\ a=B=1,\ b=2,$$

$$f(t,x(t))=6t/5+\cos x(t)/10,\ g(t,x(t))=((\ln t)/5)\sin x(t).$$

Obviously, we have

$$\begin{aligned}&|f(t,x)-f(t,y)|\le(1/10)|\cos x-\cos y|\le(1/10)|x-y|,\ \text{for } x,y\in\mathbb{R},\\ &|g(t,x)-g(t,y)|\le((\ln t)/5)|\sin x-\sin y|\le((\ln 2)/5)|x-y|,\ \text{for } x,y\in\mathbb{R}.\end{aligned}$$

It is easy to find that $L_1=1/10, L_2=M_2=(\ln 2)/5, M_1=2.5, m_1=1.1$. Thus,

$$\frac{BL_1(m_1+M_1)}{m_1^2}+(M_2L_1+M_1L_2)\frac{(\alpha-1)^{\alpha-1}(b-a)^\alpha}{\Gamma(\alpha)\alpha^{\alpha+1}}\approx 0.4405<1.$$

Clearly, all assumptions of Theorem 3.1 are satisfied. Therefore, BVP (4.1) has a unique solution on $[1,2]$.

Example 4.2. Consider the following two-point fractional BVP

$$\begin{cases} D^{3/2}_{0+}y(t) + \frac{3}{4}\sqrt{\pi}\sin t^{1/2}y(t) + \frac{3}{4}\sin D^{1/2}_{0+}y(t) + \frac{5}{2} = 0, \quad t \in (0,1), \\ \lim\limits_{t\to 0^+} t^{1/2}y(t) = y(1) = 0, \end{cases} \tag{4.2}$$

Corresponding to BVP (1.8), here

$$\alpha = \frac{3}{2},\ a = A = B = 0,\ b = 1,$$

$$\mathfrak{f}(t,(t-a)^{2-\alpha}y(t), D^{\alpha-1}_{a+}y(t)) = \frac{3}{4}\sqrt{\pi}\sin t^{1/2}y(t) + \frac{3}{4}\sin D^{1/2}_{0+}y(t) + \frac{5}{2},$$

Clearly,

$$\begin{aligned} &|\mathfrak{f}(t,(t-a)^{2-\alpha}y_1(t), D^{\alpha-1}_{a+}y_1(t)) - \mathfrak{f}(t,(t-a)^{2-\alpha}y_2(t), D^{\alpha-1}_{a+}y_2(t))| \\ &\le \frac{3}{4}\sqrt{\pi}t^{1/2}|y_1(t) - y_2(t)| + \frac{3}{4}|D^{\alpha-1}_{a+}y_1(t) - D^{\alpha-1}_{a+}y_2(t)|. \end{aligned}$$

Choosing $K = \frac{3}{4}\sqrt{\pi},\ L = \frac{3}{4}$ and consequently, we obtain

$$\frac{K(b-a)^2}{4\Gamma(\alpha+1)} + \frac{L}{\alpha}(b-a) = \frac{3\sqrt{\pi}}{16\Gamma(5/2)} + \frac{1}{2} = \frac{3}{4} < 1.$$

Thus, all the conditions of Theorem 3.2 are satisfied. Hence, BVP (4.2) has a unique solution on $[0,1]$.

Example 4.3. Consider the following two-point fractional BVP

$$\begin{cases} D^{1/2}_{0+}{}^{C}D^{3/4}_{0+}z(t) + \frac{e^t}{1+t} + \frac{3}{10}\sin z(t) + \frac{2}{5}\cdot\frac{|t^{1/2}{}^{C}D^{3/4}_{0+}z(t)|}{1+|t^{1/2}{}^{C}D^{3/4}_{0+}z(t)|} = 0, \quad t \in (0,2), \\ z(0) = z(2) = 0, \end{cases} \tag{4.3}$$

Corresponding to BVP (1.9), here

$$\beta = \frac{1}{2},\ \gamma = \frac{3}{4},\ a = A = B = 0,\ b = 2,$$

$$\mathfrak{g}(t, z(t), (t-a)^{1-\beta}{}^{C}D^{\gamma}_{a+}z(t)) = \frac{e^t}{1+t} + \frac{3}{10}\sin z(t) + \frac{2}{5}\cdot\frac{|t^{1/2}{}^{C}D^{3/4}_{0+}z(t)|}{1+|t^{1/2}{}^{C}D^{3/4}_{0+}z(t)|}.$$

Clearly,

$$\begin{aligned} &|\mathfrak{g}(t, z_1(t), (t-a)^{1-\beta}{}^{C}D^{\gamma}_{a+}z_1(t)) - \mathfrak{g}(t, z_2(t), (t-a)^{1-\beta}{}^{C}D^{\gamma}_{a+}z_2(t))| \\ &\le \frac{3}{10}|z_1(t) - z_2(t)| + \frac{2}{5}t^{1/2}|{}^{C}D^{3/4}_{0+}z_1(t) - {}^{C}D^{3/4}_{0+}z_2(t)|. \end{aligned}$$

Taking $P = \frac{3}{10},\ Q = \frac{2}{5}$, we can find that

$$\begin{aligned}&\frac{P(\gamma+\beta-1)^{\gamma+\beta-1}(b-a)^{\gamma+\beta}}{\Gamma(\gamma+\beta)(\gamma+\beta)^{\gamma+\beta+1}}+\frac{Q\max\{\beta,\gamma\}(b-a)}{(\gamma+\beta)\Gamma(\beta+1)}\\&=\frac{3}{10}\cdot\frac{(1/4)^{1/4}2^{5/4}}{(5/4)^{9/4}\Gamma(5/4)}+\frac{2}{5}\cdot\frac{3/2}{(5/4)\Gamma(3/2)}\approx 0.9628<1.\end{aligned}$$

Thus, the hypotheses of Theorem 3.3 are satisfied. Therefore, the BVP (4.3) has a unique solution on $[0,2]$.

5. Conclusions

In this article, we discussed the uniqueness results for several two-point fractional BVPs. By using the Banach contraction mapping theorem, we obtained the sharp conditions in terms of the end-points of the given interval which ensures the uniqueness of solutions for these fractional BVPs. This seems to have something in common with studying of the Lyapunov inequality for BVPs. In terms of methods, both of them are converted the BVPs into the equivalent integral equations with corresponding Green's functions. By estimating the upper bound of Green's function, the existence of solutions to BVPs is finally characterized. The difference is that Lyapunov inequality directly estimates the upper bound of the Green's function $G(t,s)$ on the interval $[a,b]\times[a,b]$, while this paper estimates the upper bound of $\int_a^b G(t,s)ds$ for any $t\in[a,b]$. Our work is an extension of the classical results of Theorem 1.1 and Theorem 1.2. It is also an extension and supplement to some recent work [10–14]. Compared with the paper [10–14], we discuss the BVP where the nonlinear term of the differential equation has the fractional derivative of unknown function, and obtain new interesting results. Based on the this study, in the forthcoming paper, we will investigate the sharp estimate for the unique solution of the two-point Ψ-Hilfer fractional hybird-Sturm-Liouville equations.

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Conflict of Interest

The authors declare that there is no conflict of interest.

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