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Research article

Limits of sub-bifractional Brownian noises

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Abstract: Let $S^{H,K} = \{S_t^{H,K}, t \geq 0\}$ be the sub-bifractional Brownian motion (sbfBm) of dimension 1, with indices $H \in (0,1)$ and $K \in (0,1]$. We primarily prove that the increment process generated by the sbfBm $\left\{S_{h+t}^{H,K} - S_h^{H,K}, t \geq 0\right\}$ converges to $\left\{B_t^{HK}, t \geq 0\right\}$ as $h \to \infty$, where $\left\{B_t^{HK}, t \geq 0\right\}$ is the fractional Brownian motion with Hurst index HK. Moreover, we study the behavior of the noise associated to the sbfBm and limit theorems to $S^{H,K}$ and the behavior of the tangent process of sbfBm.

Keywords: Sub-bifractional Brownian motion; noise; limit theorem

1. Introduction

Recently, [1] introduced the process $S^{H,K} = \{S_t^{H,K}, t \geq 0\}$ on the probability space (Ω, F, P) with indices $H \in (0,1)$ and $K \in (0,1]$, named the sub-bifractional Brownian motion (sbfBm) and defined as follows:

$$S_t^{H,K} = \frac{1}{2^{(2-K)/2}}(B_t^{H,K} + B_{-t}^{H,K}),$$

where $\{B_t^{H,K}, t \in \mathbf{R}\}$ is a bifractional Brownian motion (bfBm) with indices $H \in (0,1)$ and $K \in (0,1]$, namely, $\{B_t^{H,K}, t \in \mathbf{R}\}$ is a centered Gaussian process, starting from zero, with covariance

$$\mathbf{E}[B_t^{H,K} B_s^{H,K}] = \frac{1}{2^K}\left[(|t|^{2H} + |s|^{2H})^K - |t-s|^{2HK}\right],$$

with $H \in (0,1)$ and $K \in (0,1]$.

Clearly, the sbfBm is a centered Gaussian process such that $S_0^{H,K} = 0$, with probability 1, and $\mathrm{Var}(S_t^{H,K}) = (2^K - 2^{2HK-1})t^{2HK}$. Since $(2H-1)K - 1 < K - 1 \leq 0$, it follows that $2HK - 1 < K$. We can easily verify that $S^{H,K}$ is self-similar with index HK. When $K = 1$, $S^{H,1}$ is the sub-fractional Brownian motion (sfBm). For more on sub-fractional Brownian motion, we can see [2–5] and so on.

The following computations show that for all $s, t \geq 0$,

$$R^{H,K}(t,s) = \mathbf{E}(S_t^{H,K} S_s^{H,K}) = (t^{2H} + s^{2H})^K - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK} \tag{1.1}$$

and

$$C_1|t-s|^{2HK} \leq \mathbf{E}[(S_t^{H,K} - S_s^{H,K})^2] \leq C_2|t-s|^{2HK}, \tag{1.2}$$

where

$$C_1 = \min\{2^K - 1, 2^K - 2^{2HK-1}\}, \quad C_2 = \max\{1, 2 - 2^{2HK-1}\}. \tag{1.3}$$

(See [1]). [6] investigated the collision local time of two independent sub-bifractional Brownian motions. [7] obtained Berry-Esséen bounds and proved the almost sure central limit theorem for the quadratic variation of the sub-bifractional Brownian motion. For more on sbfBm, we can see [8–10].

Reference [11] studied the limits of bifractional Brownian noises. [12] obtained limit results of sub-fractional Brownian and weighted fractional Brownian noises. Motivated by all these studies, in this paper, we will study the increment process $\left\{S_{h+t}^{H,K} - S_h^{H,K}, t \geq 0\right\}$ of $S^{H,K}$ and the noise generated by $S^{H,K}$ and see how close this process is to a process with stationary increments. In principle, since the sub-bifractional Brownian motion is not a process with stationary increments, its increment process depends on h.

We have organized our paper as follows: In Section 2 we prove our main result that the increment process of $S^{H,K}$ converges to the fractional Brownian motion B^{HK}. Section 3 is devoted to a different view of this main result and we analyze the noise generated by the sub-bifractional Brownian motion and study its asymptotic behavior. In Section 4 we prove limit theorems to the sub-bifractional Brownian motion from a correlated non-stationary Gaussian sequence. Finally, Section 5 describes the behavior of the tangent process of sbfBm.

2. The limiting process of increment process of $S^{H,K}$

In this section, we prove the following main result which says that the increment process of the sub-bifractional Brownian motion $S^{H,K}$ converges to the fractional Brownian motion with Hurst index HK.

Theorem 2.1. Let $K \in (0,1)$. Then, as $h \to \infty$,

$$\left\{S_{h+t}^{H,K} - S_h^{H,K}, t \geq 0\right\} \overset{d}{\Rightarrow} \left\{B_t^{HK}, t \geq 0\right\},$$

where $\overset{d}{\Rightarrow}$ means convergence of all finite dimensional distributions and B^{HK} is the fractional Brownian motion with Hurst index HK.

In order to prove Theorem 2.1, we first show a decomposition of the sub-bifractional Brownian motion with parameters H and K into the sum of a sub-fractional Brownian motion with Hurst parameter HK plus a stochastic process with absolutely continuous trajectories. Some similar results were obtained in [13] for the bifractional Brownian motion and in [14] for the sub-fractional Brownian motion. Such a decomposition is useful in order to derive easier proofs for different properties of sbfBm (like variation, strong variation and Chung's LIL).

We consider the following decomposition of the covariance function of the sub-bifractional Brownian motion:

$$R^{H,K}(t,s) = \mathbf{E}(S_t^{H,K} S_s^{H,K}) = (t^{2H} + s^{2H})^K - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK}$$
$$= \left[(t^{2H} + s^{2H})^K - t^{2HK} - s^{2HK}\right]$$
$$+ \left[t^{2HK} + s^{2HK} - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK}\right]. \tag{2.1}$$

The second summand in (2.1) is the covariance of a sub-fractional Brownian motion with Hurst parameter HK. The first summand turns out to be a non-positive definite and with a change of sign it will be the covariance of a Gaussian process. Let $\{W_t, t \geq 0\}$ a standard Brownian motion, for any $0 < K < 1$, define the process $X^K = \{X_t^K, t \geq 0\}$ by

$$X_t^K = \int_0^\infty (1 - e^{-\theta t})\theta^{-\frac{1+K}{2}} dW_\theta. \tag{2.2}$$

Then, X^K is a centered Gaussian process with covariance:

$$\mathbf{E}(X_t^K X_s^K) = \int_0^\infty (1 - e^{-\theta t})(1 - e^{-\theta s})\theta^{-1-K} d\theta$$
$$= \int_0^\infty (1 - e^{-\theta t})\theta^{-1-K} d\theta - \int_0^\infty (1 - e^{-\theta t})e^{-\theta s}\theta^{-1-K} d\theta$$
$$= \int_0^\infty (\int_0^t \theta e^{-\theta u} du)\theta^{-1-K} d\theta - \int_0^\infty (\int_0^t \theta e^{-\theta u} du)e^{-\theta s}\theta^{-1-K} d\theta$$
$$= \int_0^t (\int_0^\infty \theta^{-K} e^{-\theta u} d\theta) du - \int_0^t (\int_0^\infty \theta^{-K} e^{-\theta(u+s)} d\theta) du$$
$$= \frac{\Gamma(1-K)}{K}\left[t^K + s^K - (t+s)^K\right], \tag{2.3}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

Therefore we obtain the following result:

Lemma 2.1. Let $S^{H,K}$ be a sub-bifractional Brownian motion, $K \in (0,1)$ and assume that $\{W_t, t \geq 0\}$ is a standard Brownian motion independent of $S^{H,K}$. Let X^K be the process defined by (2.2). Then the processes $\left\{\sqrt{\frac{K}{\Gamma(1-K)}} X_{t^{2H}}^K + S_t^{H,K}, t \geq 0\right\}$ and $\left\{S_t^{HK}, t \geq 0\right\}$ have the same distribution, where $\left\{S_t^{HK}, t \geq 0\right\}$ is a sub-fractional Brownian motion with Hurst parameter HK.

Proof. Let $Y_t = \sqrt{\frac{K}{\Gamma(1-K)}} X_{t^{2H}}^K + S_t^{H,K}$. Then, from (2.1) and (2.3), we have, for $s, t \geq 0$,

$$\mathbf{E}(Y_s Y_t) = \frac{K}{\Gamma(1-K)}\mathbf{E}\left(X_{s^{2H}}^K X_{t^{2H}}^K\right) + \mathbf{E}\left(S_s^{H,K} S_t^{H,K}\right)$$
$$= t^{2HK} + s^{2HK} - (t^{2H} + s^{2H})^K$$
$$+ (t^{2H} + s^{2H})^K - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK}$$

$$= t^{2HK} + s^{2HK} - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK},$$

which completes the proof.

Lemma 2.1 implies that

$$\left\{S_t^{H,K}, t \geq 0\right\} \stackrel{d}{=} \left\{S_t^{HK} - \sqrt{\frac{K}{\Gamma(1-K)}} X_{t^{2H}}^K, t \geq 0\right\} \tag{2.4}$$

where $\stackrel{d}{=}$ means equality of all finite-dimensional distributions.

By Theorem 2 in [13], the process X^K has a version with trajectories that are infinitely differentiable trajectories on $(0, \infty)$ and absolutely continuous on $[0, \infty)$.

Reference [15] presented a decomposition of the sub-fractional Brownian motion into the sum of a fractional Brownian motion plus a stochastic process with absolutely continuous trajectories. Namely, we have the following lemma.

Lemma 2.2. Let B^H be a fractional Brownian motion with Hurst parameter H, S^H be a sub-fractional Brownian motion with Hurst parameter H and $B = \{B_t, t \geq 0\}$ is a standard Brownian motion. Let

$$Y_t^H = \int_0^\infty (1 - e^{-\theta t}) \theta^{-\frac{1+2H}{2}} dB_\theta. \tag{2.5}$$

(1) If $0 < H < \frac{1}{2}$ and suppose that B^H and B are independent, then the processes $\left\{\sqrt{\frac{H}{\Gamma(1-2H)}} Y_t^H + B_t^H, t \geq 0\right\}$ and $\left\{S_t^H, t \geq 0\right\}$ have the same distribution.

(2) If $\frac{1}{2} < H < 1$ and suppose that S^H and B are independent, then the processes $\left\{\sqrt{\frac{H(2H-1)}{\Gamma(2-2H)}} Y_t^H + S_t^H, t \geq 0\right\}$ and $\left\{B_t^H, t \geq 0\right\}$ have the same distribution.

Proof. See the proof of Theorem 2.2 in [15] or the proof of Theorem 3.5 in [14].

By (2.4) and Lemma 2.2, we get, as $0 < HK < \frac{1}{2}$,

$$\left\{S_t^{H,K}, t \geq 0\right\} \stackrel{d}{=} \left\{B_t^{HK} + \sqrt{\frac{HK}{\Gamma(1-2HK)}} Y_t^{HK} - \sqrt{\frac{K}{\Gamma(1-K)}} X_{t^{2H}}^K, t \geq 0\right\} \tag{2.6}$$

and as $\frac{1}{2} < HK < 1$,

$$\left\{S_t^{H,K}, t \geq 0\right\} \stackrel{d}{=} \left\{B_t^{HK} - \sqrt{\frac{HK(2HK-1)}{\Gamma(2-2HK)}} Y_t^{HK} - \sqrt{\frac{K}{\Gamma(1-K)}} X_{t^{2H}}^K, t \geq 0\right\}. \tag{2.7}$$

The following Lemma 2.3 comes from Proposition 2.2 in [11].

Lemma 2.3. Let X_t^K be defined by (2.2). Then, as $h \to \infty$,

$$\mathbf{E}\left[\left(X_{(h+t)^{2H}}^K - X_{h^{2H}}^K\right)^2\right] = \frac{\Gamma(1-K)}{K} 2^K H^2 K(1-K) t^2 h^{2(HK-1)}(1 + o(1)).$$

Therefore, as $h \to \infty$,

$$\left\{X_{(h+t)^{2H}}^K - X_{h^{2H}}^K, t \geq 0\right\} \stackrel{d}{\Rightarrow} \{X_t \equiv 0, t \geq 0\}.$$

Lemma 2.4. Let Y_t^H be defined by (2.5). Then, as $h \to \infty$,

$$\mathbf{E}\left[\left(Y_{h+t}^{HK} - Y_h^{HK}\right)^2\right] = 2^{2HK-2}\Gamma(2-2HK)t^2 h^{2(HK-1)}(1+o(1)).$$

Therefore, as $h \to \infty$,

$$\left\{Y_{h+t}^{HK} - Y_h^{HK}, t \geq 0\right\} \overset{d}{\Rightarrow} \{Y_t \equiv 0, t \geq 0\}.$$

Proof. By Proposition 2.1 in [15], we have

$$\mathbf{E}\left(Y_t^H Y_s^H\right) = \begin{cases} \frac{\Gamma(1-2H)}{2H}\left[t^{2H} + s^{2H} - (t+s)^{2H}\right], & \text{if } 0 < H < \frac{1}{2}; \\ \frac{\Gamma(2-2H)}{2H(2H-1)}\left[(t+s)^{2H} - t^{2H} - s^{2H}\right], & \text{if } \frac{1}{2} < H < 1. \end{cases}$$

When $0 < HK < \frac{1}{2}$, we get

$$\mathbf{E}\left(Y_t^{HK} Y_s^{HK}\right) = \frac{\Gamma(1-2HK)}{2HK}\left[t^{2HK} + s^{2HK} - (t+s)^{2HK}\right].$$

In particular, for every $t \geq 0$,

$$\mathbf{E}\left[\left(Y_t^{HK}\right)^2\right] = \frac{\Gamma(1-2HK)}{2HK}(2-2^{2HK})t^{2HK}.$$

Hence, we obtain

$$\mathbf{E}\left[\left(Y_{h+t}^{HK} - Y_h^{HK}\right)^2\right] = -\frac{\Gamma(1-2HK)}{2HK}2^{2HK}\left[(h+t)^{2HK} + h^{2HK}\right] + \frac{\Gamma(1-2HK)}{2HK}2(2h+t)^{2HK}.$$

Then, for every large $h > 0$, by using Taylor's expansion, we have

$$\begin{aligned} I &:= \frac{2HK}{\Gamma(1-2HK)}\mathbf{E}\left[\left(Y_{h+t}^{HK} - Y_h^{HK}\right)^2\right] \\ &= -2^{2HK}\left[(h+t)^{2HK} + h^{2HK}\right] + 2(2h+t)^{2HK} \\ &= -2^{2HK}h^{2HK}\left[(1+th^{-1})^{2HK} + 1\right] + 2h^{2HK}(2+th^{-1})^{2HK} \\ &= -2^{2HK}h^{2HK}\left[2 + 2HKth^{-1} + HK(2HK-1)t^2h^{-2}(1+o(1))\right] \\ &\quad +2h^{2HK}\left[2^{2HK} + 2^{2HK-1}2HKth^{-1} + 2^{2HK-2}HK(2HK-1)t^2h^{-2}(1+o(1))\right] \\ &= 2^{2HK-1}HK(1-2HK)t^2h^{2(HK-1)}(1+o(1)). \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{E}\left[\left(Y_{h+t}^{HK} - Y_h^{HK}\right)^2\right] &= 2^{2HK-2}(1-2HK)\Gamma(1-2HK)t^2h^{2(HK-1)}(1+o(1)) \\ &= 2^{2HK-2}\Gamma(2-2HK)t^2h^{2(HK-1)}(1+o(1)). \end{aligned}$$

Similarly, we can prove the case $\frac{1}{2} < HK < 1$. Therefore we finished the proof of Lemma 2.4.

Proof of Theorem 2.1. It is obvious that Theorem 2.1 is the consequence of (2.6), (2.7), Lemma 2.3 and Lemma 2.4.

3. Sub-bifractional Brownian noise

In this section, we can understand Theorem 2.1 by considering the sub-bifractional Brownian noise, which is increments of sub-bifractional Brownian motion. For every integer $n \geq 0$, the sub-bifractional Brownian noise is defined by

$$Y_n := S^{H,K}_{n+1} - S^{H,K}_{n}.$$

Denote

$$R(a, a+n) := \mathbf{E}(Y_a Y_{a+n}) = \mathbf{E}\left[\left(S^{H,K}_{a+1} - S^{H,K}_{a}\right)\left(S^{H,K}_{a+n+1} - S^{H,K}_{a+n}\right)\right]. \tag{3.1}$$

We obtain

$$R(a, a+n) = f_a(n) + g(n) - g(2a+n+1), \tag{3.2}$$

where

$$f_a(n) = \left[(a+1)^{2H} + (a+n+1)^{2H}\right]^K - \left[(a+1)^{2H} + (a+n)^{2H}\right]^K$$
$$- \left[a^{2H} + (a+n+1)^{2H}\right]^K + \left[a^{2H} + (a+n)^{2H}\right]^K$$

and

$$g(n) = \frac{1}{2}\left[(n+1)^{2HK} + (n-1)^{2HK} - 2n^{2HK}\right].$$

We know that the function g is the covariance function of the fractional Brownian noise with Hurst index HK. Thus we need to analyze the function f_a to understand "how far" the sub-bifractional Brownian noise is from the fractional Brownian noise. In other words, how far is the sub-bifractional Brownian motion from a process with stationary increments?

The sub-bifractional Brownian noise is not stationary. However, the meaning of the following theorem is that it converges to a stationary sequence.

Theorem 3.1. For each n, as $a \to \infty$, we have

$$f_a(n) = 2H^2K(K-1)a^{2(HK-1)}(1 + o(1)) \tag{3.3}$$

and

$$g(2a+n+1) = 2^{2HK-2}HK(2HK-1)a^{2(HK-1)}(1 + o(1)). \tag{3.4}$$

Therefore $\lim_{a\to\infty} f_a(n) = 0$ and $\lim_{a\to\infty} g(2a+n+1) = 0$ for each n.

Proof. (3.3) is obtained by Theorem 3.3 in Maejima and Tudor. For (3.4), we have

$$g(2a+n+1) = \frac{1}{2}\left[(2a+n+2)^{2HK} + (2a+n)^{2HK} - 2(2a+n+1)^{2HK}\right]$$
$$= 2^{2HK-1}a^{2HK}\left[\left(1 + \frac{n+2}{2}a^{-1}\right)^{2HK} + \left(1 + \frac{n}{2}a^{-1}\right)^{2HK} - 2\left(1 + \frac{n+1}{2}a^{-1}\right)^{2HK}\right]$$
$$= 2^{2HK-1}a^{2HK}\Bigg[1 + 2HK\frac{n+2}{2}a^{-1} + HK(2HK-1)\left(\frac{n+2}{2}\right)^2 a^{-2}(1 + o(1))$$
$$+1 + 2HK\frac{n}{2}a^{-1} + HK(2HK-1)\left(\frac{n}{2}\right)^2 a^{-2}(1 + o(1))$$

$$-2\left(1+2HK\frac{n+1}{2}a^{-1}+HK(2HK-1)\left(\frac{n+1}{2}\right)^2 a^{-2}(1+o(1))\right)\Bigg]$$
$$=2^{2HK-2}HK(2HK-1)a^{2(HK-1)}(1+o(1)).$$

Hence the proof of Theorem 3.1 is completed.

We are now interested in the behavior of the sub-bifractional Brownian noise (3.1) with respect to n (as $n\to\infty$). We have the following result.

Theorem 3.2. For integers $a,n\ge 0$, let $R(a,a+n)$ be given by (3.1). Then for large n,

$$R(a,a+n)=HK(K-1)\left[(a+1)^{2H}-a^{2H}\right]n^{2(HK-1)+(1-2H)}+o(n^{2(HK-1)+(1-2H)}).$$

Proof. By (3.2), we have

$$R(a,a+n)=f_a(n)+g(n)-g(2a+n+1).$$

By the proof of Theorem 4.1 in [11], we get, for large n, the term $f_a(n)$ behaves as

$$HK(K-1)\left[(a+1)^{2H}-a^{2H}\right]n^{2(HK-1)+(1-2H)}+o(n^{2(HK-1)+(1-2H)}).$$

We know that the term $g(n)$ behaves as $HK(2HK-1)n^{2(HK-1)}$ for large n. For $g(2a+n+1)$, it is similar to the computation for Theorem 3.1, we can obtain $g(2a+n+1)$ also behaves as $HK(2HK-1)n^{2(HK-1)}$ for large n. Hence we have finished the proof of Theorem 3.2.

It is easy to obtain the following corollary.

Corollary 3.1. For integers $a\ge 1$ and $n\ge 0$, let $R(a,a+n)$ be given by (3.1). Then, for every $a\in N$, we have

$$\sum_{n\ge 0}R(a,a+n)<\infty.$$

Proof. By Theorem 3.2, we get that the main term of $R(a,a+n)$ is $n^{2HK-2H-1}$, and since $2HK-2H-1<-1$, the series is convergent.

4. Limit theorems to the sub-bifractional Brownian motion

In this section, we prove two limit theorems to the sub-bifractional Brownian motion. Define a function $g(t,s), t\ge 0, s\ge 0$ by

$$\begin{aligned}g(t,s)=\frac{\partial^2 R^{H,K}(t,s)}{\partial t\partial s}&=4H^2K(K-1)(t^{2H}+s^{2H})^{K-2}(ts)^{2H-1}+HK(2HK-1)|t-s|^{2HK-2}\\&\quad-HK(2HK-1)(t+s)^{2HK-2}\\&=:g_1(t,s)+g_2(t,s)-g_3(t,s),\end{aligned}\tag{4.1}$$

for (t,s) with $t\ne s$, $t\ne 0$, $s\ne 0$ and $t+s\ne 0$.

Theorem 4.1. Assume that $2HK>1$ and let $\left\{\xi_j, j=1,2,\cdots\right\}$ be a sequence of standard normal random variables. $g(t,s)$ is defined by (4.1). Suppose that $\mathbf{E}(\xi_i\xi_j)=g(i,j)$. Then, as $n\to\infty$,

$$\left\{n^{-HK}\sum_{j=1}^{[nt]}\xi_j, t\ge 0\right\}\stackrel{d}{\Rightarrow}\left\{S_t^{H,K}, t\ge 0\right\}.$$

Remark 1. Theorem 4.1 and 4.2 (below) are similar to the central limit theorem and can be used as a basis for many subsequent studies.

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.1. When $2HK > 1$, we have

$$\int_0^t \int_0^s g(u, v)dudv = (t^{2H} + s^{2H})^K - \frac{1}{2}(t + s)^{2HK} - \frac{1}{2}|t - s|^{2HK}.$$

Proof. It follows from the fact that $g(t, s) = \frac{\partial^2 R^{H,K}(t,s)}{\partial t \partial s}$ for every $t \geq 0, s \geq 0$ and by using that $2HK > 1$.

Proof of Theorem 4.1. It is enough to show that, as $n \to \infty$,

$$I_n := \mathbf{E}\left[\left(n^{-HK}\sum_{i=1}^{[nt]}\xi_i\right)\left(n^{-HK}\sum_{j=1}^{[ns]}\xi_j\right)\right] \to \mathbf{E}(S_t^{H,K} S_s^{H,K}).$$

In fact, we have

$$I_n = n^{-2HK}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]}\mathbf{E}(\xi_i\xi_j) = n^{-2HK}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]} g(i, j).$$

Note that

$$\begin{aligned} g\left(\frac{i}{n}, \frac{j}{n}\right) &= 4H^2K(K-1)\left[\left(\frac{i}{n}\right)^{2H} + \left(\frac{j}{n}\right)^{2H}\right]^{K-2}\left(\frac{ij}{n^2}\right)^{2H-1} \\ &+HK(2HK-1)\left|\frac{i}{n} - \frac{j}{n}\right|^{2HK-2} - HK(2HK-1)\left(\frac{i}{n} + \frac{j}{n}\right)^{2HK-2} \\ &= n^{2(1-HK)} g(i, j). \end{aligned} \tag{4.2}$$

Thus, as $n \to \infty$,

$$\begin{aligned} I_n &= n^{-2HK}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]} n^{2HK-2} g\left(\frac{i}{n}, \frac{j}{n}\right) \\ &= n^{-2}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]} g\left(\frac{i}{n}, \frac{j}{n}\right) \\ &\to \int_0^t \int_0^s g(u, v)dudv \\ &= (t^{2H} + s^{2H})^K - \frac{1}{2}(t + s)^{2HK} - \frac{1}{2}|t - s|^{2HK} \\ &= \mathbf{E}(S_t^{H,K} S_s^{H,K}). \end{aligned}$$

Hence, we finished the proof of Theorem 4.1.

We now consider more general sequence of nonlinear functional of standard normal random variables. Let f be a real valued function such that $f(x)$ does not vanish on a set of positive measure,

$\mathbf{E}[f(\xi_1)] = 0$ and $\mathbf{E}[(f(\xi_1))^2] < \infty$. Let H_k denote the k-th Hermite polynomial with highest coefficient 1. We have

$$f(x) = \sum_{k=1}^{\infty} c_k H_k(x),$$

where $\sum_{k=1}^{\infty} c_k^2 k! < \infty$ and $c_k = \mathbf{E}[f(\xi_j) H_k(\xi_j)]$ (see e.g. [16]). Assume that $c_1 \neq 0$. Let $\eta_j = f(\xi_j)$, $j = 1, 2, \cdots$, where $\left\{\xi_j, j = 1, 2, \cdots\right\}$ is the same sequence of standard normal random variables as before.

Theorem 4.2. Assume that $2HK > \frac{3}{2}$ and let $\left\{\xi_j, j = 1, 2, \cdots\right\}$ be a sequence of standard normal random variables. $g(t, s)$ is defined by (4.1). Suppose that $\mathbf{E}(\xi_i \xi_j) = g(i, j)$. Then, as $n \to \infty$,

$$\left\{ n^{-HK} \sum_{j=1}^{[nt]} \eta_j, t \geq 0 \right\} \stackrel{d}{\Rightarrow} \left\{ c_1 S_t^{H,K}, t \geq 0 \right\}.$$

Proof. Note that $\eta_j = f(\xi_j) = c_1 \xi_j + \sum_{k=2}^{\infty} c_k H_k(\xi_j)$. We obtain

$$n^{-HK} \sum_{j=1}^{[nt]} \eta_j = c_1 n^{-HK} \sum_{j=1}^{[nt]} \xi_j + n^{-HK} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} c_k H_k(\xi_j).$$

Using Theorem 4.1, it is enough to show that, as $n \to \infty$,

$$\mathbf{E}\left[\left(n^{-HK} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} c_k H_k(\xi_j)\right)^2\right] \to 0. \tag{4.3}$$

In fact, we get

$$\begin{aligned} J_n &:= \mathbf{E}\left[\left(n^{-HK} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} c_k H_k(\xi_j)\right)^2\right] \\ &= n^{-2HK} \sum_{i=1}^{[nt]} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} c_k c_l \mathbf{E}\left[H_k(\xi_i) H_l(\xi_j)\right]. \end{aligned}$$

We know that, if ξ and η are two random variables with joint Gaussian distribution such that $\mathbf{E}(\xi) = \mathbf{E}(\eta) = 0$, $\mathbf{E}(\xi^2) = \mathbf{E}(\eta^2) = 1$ and $\mathbf{E}(\xi\eta) = r$, then

$$\mathbf{E}\left[H_k(\xi) H_l(\eta)\right] = \delta_{k,l} r^k k!,$$

where

$$\delta_{k,l} = \begin{cases} 1, & \text{if } k = l; \\ 0, & \text{if } k \neq l. \end{cases}$$

Thus,

$$\begin{aligned} J_n &= n^{-2HK} \sum_{i=1}^{[nt]} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} c_k^2 \left(\mathbf{E}(\xi_i \xi_j)\right)^k k! \\ &= n^{-2HK} [nt] \sum_{k=2}^{\infty} c_k^2 k! + n^{-2HK} \sum_{i,j=1; i \neq j}^{[nt]} \sum_{k=2}^{\infty} c_k^2 k! [g(i, j)]^k. \end{aligned}$$

Since $|g(i,j)| \le (\mathbf{E}(\xi_i^2))^{\frac{1}{2}}(\mathbf{E}(\xi_j^2))^{\frac{1}{2}} = 1$, we get, by (4.2),

$$
\begin{aligned}
J_n &\le n^{-2HK}[nt]\sum_{k=2}^{\infty} c_k^2 k! + n^{-2HK}\sum_{i,j=1;i\neq j}^{[nt]}\sum_{k=2}^{\infty} c_k^2 k![g(i,j)]^2 \\
&= n^{-2HK}[nt]\sum_{k=2}^{\infty} c_k^2 k! + n^{-2HK}\sum_{k=2}^{\infty} c_k^2 k! \sum_{i,j=1;i\neq j}^{[nt]} [g(i,j)]^2 \\
&\le tn^{1-2HK}\sum_{k=2}^{\infty} c_k^2 k! + n^{2(HK-1)}\left(\sum_{k=2}^{\infty} c_k^2 k!\right) n^{-2}\sum_{i,j=1;i\neq j}^{[nt]}\left[g\left(\frac{i}{n},\frac{j}{n}\right)\right]^2. \qquad (4.4)
\end{aligned}
$$

On one hand, by $\sum_{k=2}^{\infty} c_k^2 k! < \infty$ and $2HK > \frac{3}{2} > 1$, we get, as $n \to \infty$,

$$
tn^{1-2HK}\sum_{k=2}^{\infty} c_k^2 k! \to 0. \qquad (4.5)
$$

On the other hand, we have

$$
\begin{aligned}
n^{-2}\sum_{i,j=1;i\neq j}^{[nt]}\left[g\left(\frac{i}{n},\frac{j}{n}\right)\right]^2 &= n^{-2}\sum_{i,j=1;i\neq j}^{[nt]}\left[g_1\left(\frac{i}{n},\frac{j}{n}\right) + g_2\left(\frac{i}{n},\frac{j}{n}\right) - g_3\left(\frac{i}{n},\frac{j}{n}\right)\right]^2 \\
&\le 3n^{-2}\sum_{i,j=1;i\neq j}^{[nt]}\left\{\left[g_1\left(\frac{i}{n},\frac{j}{n}\right)\right]^2 + \left[g_2\left(\frac{i}{n},\frac{j}{n}\right)\right]^2 + \left[g_3\left(\frac{i}{n},\frac{j}{n}\right)\right]^2\right\}.
\end{aligned}
$$

Since $|g_1(u,v)| \le C(uv)^{HK-1}$ and $2HK > \frac{3}{2} > 1$, we obtain

$$
n^{-2}\sum_{i,j=1;i\neq j}^{[nt]}\left[g_1\left(\frac{i}{n},\frac{j}{n}\right)\right]^2 \to \int_0^t\int_0^t g_1^2(u,v)dudv \le C\int_0^t\int_0^t (uv)^{2HK-2}dudv < \infty. \qquad (4.6)
$$

We know that

$$
\begin{aligned}
n^{-2}\sum_{i,j=1;i\neq j}^{[nt]}\left[g_3\left(\frac{i}{n},\frac{j}{n}\right)\right]^2 &\to \int_0^t\int_0^t g_3^2(u,v)dudv \\
&= H^2K^2(2HK-1)^2\int_0^t\int_0^t (u+v)^{4HK-4}dudv \\
&< \infty, \qquad (4.7)
\end{aligned}
$$

since $2HK > \frac{3}{2} > 1$.

We have also

$$
n^{-2}\sum_{i,j=1;i\neq j}^{[nt]}\left[g_2\left(\frac{i}{n},\frac{j}{n}\right)\right]^2 \to \int_0^t\int_0^t g_2^2(u,v)dudv
$$

$$= H^2K^2(2HK-1)^2 \int_0^t \int_0^t (u-v)^{4HK-4} dudv$$
$$< \infty, \tag{4.8}$$

since $2HK > \frac{3}{2}$. Thus (4.3) holds from (4.4)–(4.8) and $2HK > \frac{3}{2}$. The proof is completed.

Remark 2. [11] pointed out, when $2HK > 1$, the convergence of

$$n^{2(HK-1)} n^{-2} \sum_{i,j=1; i \neq j}^{[nt]} \left[g_2\left(\frac{i}{n}, \frac{j}{n}\right)\right]^2$$

had been already proved in [16]. But we can not find the details in [16]. Here we only give the proof when $2HK > \frac{3}{2}$, because the holding condition for (4.8) is $2HK > \frac{3}{2}$.

5. The behavior of the tangent process of sbfBm

In this section, we study an approximation in law of the fractional Brownian motion via the tangent process generated by the sbfBm $S^{H,K}$.

Theorem 5.1. Let $H \in (0,1)$ and $K \in (0,1)$. For every $t_0 > 0$, as $\epsilon \to 0$, we have, the tangent process

$$\left\{ \frac{S^{H,K}_{t_0+\epsilon u} - S^{H,K}_{t_0}}{\epsilon^{HK}}, u \geq 0 \right\} \overset{d}{\Rightarrow} \left\{ B^{HK}_u, u \geq 0 \right\}, \tag{5.1}$$

where B^{HK}_u is the fractional Brownian motion with Hurst index HK.

Proof. As $0 < HK < \frac{1}{2}$, by (2.6), we get

$$\left\{ S^{H,K}_t, t \geq 0 \right\} \overset{d}{=} \left\{ B^{HK}_t + \sqrt{\frac{HK}{\Gamma(1-2HK)}} Y^{HK}_t - \sqrt{\frac{K}{\Gamma(1-K)}} X^K_{t^{2H}}, t \geq 0 \right\}.$$

By (2.5) in [12], there exists a constant $C(H,K) > 0$ such that

$$\mathbf{E}\left[\left(\frac{X^K_{(t_0+\epsilon u)^{2H}} - X^K_{(t_0)^{2H}}}{\epsilon^{HK}} \right)^2 \right] = C(H,K) t_0^{2(HK-1)} u^2 \epsilon^{2(1-HK)} (1 + o(1)),$$

which tends to zero, as $\epsilon \to 0$, since $1 - HK > 0$.

On the other hand, similar to the proof of Lemma 2.4, we obtain

$$\mathbf{E}\left[\left(\frac{Y^{HK}_{t_0+\epsilon u} - Y^{HK}_{t_0}}{\epsilon^{HK}} \right)^2 \right] = 2^{2HK-2} \Gamma(2-2HK) t_0^{2(HK-1)} u^2 \epsilon^{2(1-HK)} (1 + o(1)),$$

which also tends to zero, as $\epsilon \to 0$. Therefore (5.1) holds. Similarly, (5.1) also holds for the case $\frac{1}{2} < HK < 1$. We finished the proof.

6. Conclusions

In this paper, we prove that the increment process generated by the sub-bifractional Brownian motion converges to the fractional Brownian motion. Moreover, we study the behavior of the noise associated to the sbfBm and the behavior of the tangent process of the sbfBm. In the future, we will investigate limits of Gaussian noises.

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Conflict of interest

The author declares there is no conflict of interest.

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