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Research article

Lie algebras with differential operators of any weights

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Abstract: In this paper, we define a cohomology theory for differential Lie algebras of any weight. As applications of the cohomology, we study abelian extensions and formal deformations of differential Lie algebras of any weight. Finally, we consider homotopy differential operators on L_∞ algebras and 2-differential operators of any weight on Lie 2-algebras, and we prove that the category of 2-term L_∞ algebras with homotopy differential operators of any weight is same as the category of Lie 2-algebras with 2-differential operators of any weight.

Keywords: cohomology; extension; deformation; derivation

1. Introduction

Derivations play a crucial role in studying deformation formulas [1], differential Galois theory [2] and homotopy algebras [3]. They also are useful of control systems theory [4,5] and gauge algebras [6]. The authors studied the operad of associative algebras with derivation in [7]. Recently, the authors introduced Lie algebras with derivations, and studied their cohomology and deformations, extensions in [8]. Later, Das [9] considered the similar results for Leibniz algebras with derivations. The authors studied cohomology of Leibniz triple systems with derivations in [10].

More and more scholars have begun to pay close attention to the structures of any weight thanks to the result of outstanding work [11, 12], all kinds of Rota-Baxter algebras of any weight [13–18] appear successively. In order to study non-abelian extensions of Lie algebras. The notion of crossed homomorphisms of Lie algebras was introduced by Lue [19], which was applied to study the representations of Cartan Lie algebras [20]. For $\lambda \in \mathbf{k}$, the notion of a differential algebra of weight λ was first introduced by Guo and Keigher [21], which generalizes simultaneously the concept of the classical differential algebra and difference algebra [22]. Applying the same method as for differential Lie algebras of weight λ. Later, the authors defined the cohomology of relative difference Lie algebras, and studied some properties in [23]. Our aim in this paper is to consider Lie algebras with differential operators of weight λ (also known as differential Lie algebras). More precisely, we define a cohomology theory for differential Lie algebras and consider some properties.

The paper is organized as follows. In Section 2, we consider the representations of differential Lie algebras of any weight. In Section 3, we define a cohomology theory for differential Lie algebras of any weight. In Section 4, we study central extensions of differential Lie algebras of any weight. In Section 5, we study formal deformations of differential Lie algebras of any weight. In Section 6, we consider homotopy differential operators on L_∞ algebras and 2-differential operators of any weight on Lie 2-algebras. In Section 7, we prove that the category of 2-term L_∞ algebras with homotopy differential operator of any weight and the category of Lie 2-algebras with 2-differential operators of any weight are equivalent.

Throughout this paper, $\mathbf{k}$ denotes a field of characteristic zero. All the vector spaces, algebras, linear maps and tensor products are taken over $\mathbf{k}$ unless otherwise specified.

2. Representations of differential Lie algebras of any weight

For $\lambda \in \mathbf{k}$. A differential operator of weight λ on a Lie algebra $\mathfrak{g}$ is a linear operator $d_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ such that

$$d_\mathfrak{g}([a, b]) = [d_\mathfrak{g}(a), b] + [a, d_\mathfrak{g}(b)] + \lambda[d_\mathfrak{g}(a), d_\mathfrak{g}(b)], \quad \forall a, b \in \mathfrak{g}. \tag{2.1}$$

We denote by $\mathrm{Der}_\lambda(\mathfrak{g})$ the set of differential operators of weight λ of the Lie algebra $\mathfrak{g}$.

Definition 2.1. Denote a Lie algebra $\mathfrak{g}$ with a differential operator $d_\mathfrak{g} \in \mathrm{Der}_\lambda(\mathfrak{g})$ by $(\mathfrak{g}, d_\mathfrak{g})$ and we call it a differential Lie algebra.

Definition 2.2. Given two differential Lie algebras $(\mathfrak{g}, d_\mathfrak{g})$, $(\mathfrak{h}, d_\mathfrak{h})$, a homomorphism of differential Lie algebras is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\varphi \circ d_\mathfrak{g} = d_\mathfrak{h} \circ \varphi$. We denote by $\mathbf{LieD}_\lambda$ the category of differential Lie algebras and their morphisms.

To simply notations, for all the above notions, we will often suppress the mentioning of the weight λ unless it needs to be specified.

Definition 2.3. (i) A representation over the differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$ is a pair (V, d_V), where $d_V \in \mathrm{End}_\mathbf{k}(V)$, and (V, ρ) is a representation over the Lie algebra $\mathfrak{g}$, such that $\forall x \in \mathfrak{g}, v \in V$, the following identity holds:

$$d_V(\rho(x)v) \quad = \quad \rho(d_\mathfrak{g}(x))v + \rho(x)d_V(v) + \lambda\rho(d_\mathfrak{g}(x))d_V(v).$$

(ii) Given two representations (U, ρ_U, d_U), (V, ρ_V, d_V) over $(\mathfrak{g}, d_\mathfrak{g})$, a linear map $f : U \to V$ is called a homomorphism of representations, if $f \circ d_U = d_V \circ f$ and

$$f \circ \rho_U(a) = \rho_V(a) \circ f, \quad \forall a \in \mathfrak{g}.$$

One denotes by $(\mathfrak{g}, d_\mathfrak{g})$-Rep the category of representations over the differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$.

Example 2.4. Any differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$ is a representation over itself with

$$\rho : \mathfrak{g} \to \mathrm{End}_\mathbf{k}(\mathfrak{g}),\ a \mapsto (b \mapsto [a, b]).$$

It is called the adjoint representation over the differential Lie algebras $(\mathfrak{g}, d_\mathfrak{g})$.

Example 2.5. Let (V, ρ) be a representation of a Lie algebra $\mathfrak{g}$. Then the pair (V, Id_V) is a representation of the differential Lie algebra $(\mathfrak{g},\mathrm{Id}_{\mathfrak{g}})$ of weight -1.

Example 2.6. Let $(\mathfrak{g}, d_{\mathfrak{g}})$ be a differential Lie algebra of weight λ and (V, d_V) be a representation of it. Then for $\kappa \neq 0 \in \mathbf{k}$, the pair $(V, \kappa d_V)$ is a representation of the differential Lie algebra $(\mathfrak{g}, \kappa d_{\mathfrak{g}})$ of any weight $\frac{1}{\kappa}\lambda$.

The following result is easily to check and we omit it.

Proposition 2.7. *Let (V, d_V) be a representation of the differential Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}})$ of weight λ. Then $(\mathfrak{g} \oplus V, d_{\mathfrak{g}} \oplus d_V)$ is a differential Lie algebra, where*

$$[a + u, b + v] = [a, b] + \rho(a)v - \rho(b)u, \quad \forall a, b \in \mathfrak{g},\ u, v \in V.$$

3. Cohomology of differential Lie algebras of any weight

Recall that the cochain complex of Lie algebra $\mathfrak{g}$ with coefficients in representation V is the cochain complex

$$(\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, V) = \oplus_{n=0}^{\infty}\mathrm{C}^n_{\mathrm{Lie}}(\mathfrak{g}, V), \partial^*_{\mathrm{Lie}}),$$

and the coboundary operator

$$\partial^n_{\mathrm{Lie}} : \mathrm{C}^n_{\mathrm{Lie}}(\mathfrak{g}, V) \longrightarrow \mathrm{C}^{n+1}_{\mathrm{Lie}}(\mathfrak{g}, V), n \geqslant 0$$

is given by

$$\begin{aligned}\partial^n_{\mathrm{Lie}} f(x_1, \ldots, x_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i+n}\rho(x_i)f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})\\ &+ \sum_{1\leq i<j\leq n+1}(-1)^{i+j+n+1} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}),\end{aligned}$$

for all $f \in \mathrm{C}^n_{\mathrm{Lie}}(\mathfrak{g}, V)$, $x_1, \ldots, x_{n+1} \in \mathfrak{g}$. The corresponding cohomology is denoted by $\mathrm{H}^*_{\mathrm{Lie}}(\mathfrak{g}, V)$. When V is the adjoint representation, we write $\mathrm{H}^n_{\mathrm{Lie}}(\mathfrak{g}) = \mathrm{H}^n_{\mathrm{Lie}}(\mathfrak{g}, V), n \geq 0$.

In the following, we will define the cohomology of the differential Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}})$ of weight λ with coefficients in the representation (V, d_V).

Define

$$C^n_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V) := \begin{cases} C^n_{\mathrm{Lie}}(\mathfrak{g}, V) \oplus C^{n-1}_{\mathrm{Lie}}(\mathfrak{g}, V), & n \geq 2,\\ C^1_{\mathrm{Lie}}(\mathfrak{g}, V) = \mathrm{Hom}(\mathfrak{g}, V), & n = 1,\\ C^0_{\mathrm{Lie}}(\mathfrak{g}, V) = V, & n = 0.\end{cases} \tag{3.1}$$

and define a linear map $\delta : C^n_{\mathrm{Lie}}(\mathfrak{g}, V) \to C^n_{\mathrm{Lie}}(\mathfrak{g}, V)$ $(n \geq 1)$ by

$$\delta f_n(x_1, \ldots, x_n) := \sum_{k=1}^{n} \lambda^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n} f_n(x_1, \ldots, d_{\mathfrak{g}}(x_{i_1}), \ldots, d_{\mathfrak{g}}(x_{i_k}), \ldots, x_n) - d_V f_n(x_1, \ldots, x_n),$$

for any $f_n \in C^n_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V)$ and

$$\delta v = -d_V(v), \quad \forall v \in C^0_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V) = V.$$

Lemma 3.1. *We have* $\partial_{\mathrm{Lie}} \circ \delta = \delta \circ \partial_{\mathrm{Lie}}$.

Theorem 3.2. *The pair* $(C^*_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V), \partial_{\mathrm{LieD}_\lambda})$ *is a cochain complex. So* $\partial^2_{\mathrm{LieD}_\lambda} = 0$.

Proof. For any $v \in C^0_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V)$, we have

$$\partial^2_{\mathrm{LieD}_\lambda} v = \partial_{\mathrm{LieD}_\lambda}(\partial_{\mathrm{LieD}_\lambda} v, \delta v) = (\partial^2_{\mathrm{Lie}} v, \partial_{\mathrm{Lie}}\delta v - \delta\partial_{\mathrm{Lie}} v) = 0.$$

Given any $f \in C^n_{\mathrm{Lie}}(\mathfrak{g}, V)$, $g \in C^{n-1}_{\mathrm{Lie}}(\mathfrak{g}, V)$ with $n \geq 1$, we have

$$\partial^2_{\mathrm{LieD}_\lambda}(f, g) = \partial_{\mathrm{LieD}_\lambda}(\partial_{\mathrm{Lie}} f, \partial_{\mathrm{Lie}} g + (-1)^n \delta f) = (\partial^2_{\mathrm{Lie}} f, \partial_{\mathrm{Lie}}(\partial_{\mathrm{Lie}} g + (-1)^n \delta f) + (-1)^{n+1}\delta\partial_{\mathrm{Lie}} f) = 0.$$

Hence, the proof is finished.

Definition 3.3. The cohomology of the cochain complex $(C^*_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V), \partial_{\mathrm{LieD}_\lambda})$, denoted by $H^*_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V)$, is called the cohomology of the differential Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}})$ of weight λ.

4. Abelian extensions of differential Lie algebras of any weight

In this section, we show that abelian extensions of differential Lie algebras are classified by the second cohomology.

Definition 4.1. An abelian extension of differential Lie algebras is a short exact sequence of homomorphisms of differential Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_V} & & \downarrow{\scriptstyle d_{\hat{\mathfrak{g}}}} & & \downarrow{\scriptstyle d_{\mathfrak{g}}} & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

such that $[u, v]_V = 0$ for all $u, v \in V$.

$(\hat{\mathfrak{g}}, d_{\hat{\mathfrak{g}}})$ is called an abelian extension of $(\mathfrak{g}, d_{\mathfrak{g}})$ by (V, d_V).

Definition 4.2. Let $(\hat{\mathfrak{g}}_1, d_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2, d_{\hat{\mathfrak{g}}_2})$ be two abelian extensions of $(\mathfrak{g}, d_{\mathfrak{g}})$ by (V, d_V). They are said to be isomorphic if there exists $\zeta : (\hat{\mathfrak{g}}_1, d_{\hat{\mathfrak{g}}_1}) \leftrightarrow (\hat{\mathfrak{g}}_2, d_{\hat{\mathfrak{g}}_2})$ is an isomorphism of differential Lie algebras such that:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V, d_V) & \xrightarrow{i} & (\hat{\mathfrak{g}}_1, d_{\hat{\mathfrak{g}}_1}) & \xrightarrow{p_1} & (\mathfrak{g}, d_{\mathfrak{g}}) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \zeta} & & \| & & \\ 0 & \longrightarrow & (V, d_V) & \xrightarrow{i} & (\hat{\mathfrak{g}}_2, d_{\hat{\mathfrak{g}}_2}) & \xrightarrow{p_2} & (\mathfrak{g}, d_{\mathfrak{g}}) & \longrightarrow & 0. \end{array}$$

A section of an abelian extension $(\hat{\mathfrak{g}}, d_{\hat{\mathfrak{g}}})$ of $(\mathfrak{g}, d_{\mathfrak{g}})$ by (V, d_V) is a linear map $s : \mathfrak{g} \to \hat{\mathfrak{g}}$ satisfying $p \circ s = \mathrm{Id}_{\mathfrak{g}}$.

Given a section $s : \mathfrak{g} \to \hat{\mathfrak{g}}$, define $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbf{k}}(V)$ by

$$\rho(x)v := \rho(s(x))v, \quad \forall x \in \mathfrak{g}, v \in V.$$

Proposition 4.3. *With the above notations, (V, d_V) is a representation over the differential Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}})$.*

Proof. Firstly, we prove that ρ is a Lie algebra homomorphism, in fact, for any $x, y \in \mathfrak{g},\ v \in V$, we have

$$\rho([x,y])(v) = \rho(s([x,y]))v = \rho([s(x), s(y)])v = \rho(x)\rho(y)(v) - \rho(y)\rho(x)(v).$$

Moreover, we obtain

$$\begin{aligned} d_V(\rho(x)v) &= d_V(\rho(s(x))v) = d_{\hat{\mathfrak{g}}}(\rho(s(x))v) \\ &= \rho(d_{\hat{\mathfrak{g}}})(s(x))v + \rho(s(x))d_{\hat{\mathfrak{g}}}(v) + \lambda\rho(d_{\hat{\mathfrak{g}}}(s(x)))d_{\hat{\mathfrak{g}}}(v) \\ &= \rho(s(d_{\mathfrak{g}}(x)))v + \rho(s(x))d_V(v) + \lambda\rho(s(d_{\mathfrak{g}}(x)))d_V(v) \\ &= \rho(d_{\mathfrak{g}}(x))v + \rho(x)d_V(v) + \lambda\rho(d_{\mathfrak{g}}(x))d_V(v). \end{aligned}$$

Hence, (V, d_V) is a representation over $(\mathfrak{g}, d_{\mathfrak{g}})$.

We further consider linear maps $\psi : \mathfrak{g} \otimes \mathfrak{g} \to V$ and $\chi : \mathfrak{g} \to V$ by

$$\begin{aligned} \psi(x,y) &= [s(x), s(y)] - s([x,y]), \quad \forall x, y \in \mathfrak{g}, \\ \chi(x) &= d_{\hat{\mathfrak{g}}}(s(x)) - s(d_{\mathfrak{g}}(x)), \quad \forall x \in \mathfrak{g}. \end{aligned}$$

The differential Lie algebra structure on $\mathfrak{g}\oplus V$ with a multiplication $[\cdot,\cdot]_\psi$ and the differential operator d_χ defined by

$$[x+u, y+v]_\psi = [x,y] + \rho(x)v - \rho(y)u + \psi(x,y),\ \forall x, y \in \mathfrak{g},\ u, v \in V, \tag{4.1}$$

$$d_\chi(x+v) = d_{\mathfrak{g}}(x) + \chi(x) + d_V(v),\ \forall x \in \mathfrak{g},\ v \in V. \tag{4.2}$$

Proposition 4.4. *The triple $(\mathfrak{g} \oplus V, [\cdot,\cdot]_\psi, d_\chi)$ is a differential Lie algebra if and only if (ψ, χ) is a 2-cocycle.*

Proof. For any $x, y, z \in \mathfrak{g}$, By (4.1), we have

$$\psi(x,[y,z]) + \psi(x,\psi(y,z)) + \psi(y,[z,x]) + \psi(y,\psi(z,x)) + \psi(z,[x,y]) + \psi(z,\psi(x,y)) = 0. \tag{4.3}$$

Since d_χ satisfies Eq (2.1), we deduce that

$$\begin{aligned} &\chi([x,y]) - \rho(x)\chi(y) - \lambda\rho(d_{\mathfrak{g}}(x))\chi(y) + \rho(y)\chi(x) + \lambda\rho(d_{\mathfrak{g}}(y))\chi(x) \\ &+ d_V(\psi(x,y)) - \psi(d_{\mathfrak{g}}(x), y) - \psi(x, d_{\mathfrak{g}}(y)) - \lambda\psi(d_{\mathfrak{g}}(x), d_{\mathfrak{g}}(y)) = 0. \end{aligned} \tag{4.4}$$

Therefore, (ψ, χ) is a 2-cocycle.

Conversely, if (ψ, χ) satisfies Eqs (4.3) and (4.4), direct verification that $(\mathfrak{g} \oplus V, [\cdot,\cdot]_\psi, d_\chi)$ is a differential Lie algebra.

In the following, we will classify abelian extensions of differential Lie algebras.

Theorem 4.5. *Let V be a vector space and $d_V \in \mathrm{End}_{\boldsymbol{k}}(V)$. Then abelian extensions of a differential Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}})$ by (V, d_V) are classified by $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V)$ of $(\mathfrak{g}, d_{\mathfrak{g}})$.*

Proof. Let $(\hat{\mathfrak{g}}, d_{\hat{\mathfrak{g}}})$ be an abelian extension of $(\mathfrak{g}, d_{\mathfrak{g}})$ by (V, d_V). We choose a section $s : \mathfrak{g} \to \hat{\mathfrak{g}}$ to obtain a 2-cocycle (ψ, χ) and let s_1 and s_2 be two distinct sections providing 2-cocycles (ψ_1, χ_1) and (ψ_2, χ_2) respectively. Define $\phi : \mathfrak{g} \to V$ by $\phi(x) = s_1(x) - s_2(x)$, we have

$$\begin{aligned}
\psi_1(x, y) &= [s_1(x), s_1(y)] - s_1([x, y]) \\
&= [s_2(x) + \phi(x), s_2(y) + \phi(y)] - (s_2([x, y]) + \phi([x, y])) \\
&= ([s_2(x), s_2(y)] - s_2([x, y])) + [s_2(x), \phi(y)] + [\phi(x), s_2(y)] - \phi([x, y]) \\
&= ([s_2(x), s_2(y)] - s_2([x, y])) + [x, \phi(y)] + [\phi(x), y] - \phi([x, y] \\
&= \psi_2(x, y) + \partial\phi(x, y)
\end{aligned}$$

and

$$\begin{aligned}
\chi_1(x) &= d_{\hat{\mathfrak{g}}}(s_1(x)) - s_1(d_{\mathfrak{g}}(x)) \\
&= d_{\hat{\mathfrak{g}}}(s_2(x) + \phi(x)) - (s_2(d_{\mathfrak{g}}(x)) + \phi(d_{\mathfrak{g}}(x))) \\
&= (d_{\hat{\mathfrak{g}}}(s_2(x)) - s_2(d_{\mathfrak{g}}(x))) + d_V(\phi(x)) - \phi(d_{\mathfrak{g}}(x)) \\
&= \chi_2(x) + d_V(\phi(x)) - \phi(d_{\mathfrak{g}}(x)) \\
&= \chi_2(x) - \delta\phi(x).
\end{aligned}$$

That is, $(\psi_1, \chi_1) = (\psi_2, \chi_2) + \partial_{\mathrm{LieD}_\lambda}(\phi)$. Thus (ψ_1, χ_1) and (ψ_2, χ_2) are in the same cohomological class in $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V)$.

Next, we prove that isomorphic abelian extensions give rise to the same element in $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, V)$. Assume that $(\hat{\mathfrak{g}}_1, d_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2, d_{\hat{\mathfrak{g}}_2})$ are two isomorphic abelian extensions of $(\mathfrak{g}, d_{\mathfrak{g}})$ by (V, d_V) with the associated homomorphism $\zeta : (\hat{\mathfrak{g}}_1, d_{\hat{\mathfrak{g}}_1}) \to (\hat{\mathfrak{g}}_2, d_{\hat{\mathfrak{g}}_2})$. Let s_1 be a section of $(\hat{\mathfrak{g}}_1, d_{\hat{\mathfrak{g}}_1})$. As $p_2 \circ \zeta = p_1$, we have

$$p_2 \circ (\zeta \circ s_1) = p_1 \circ s_1 = \mathrm{Id}_{\mathfrak{g}}.$$

Therefore, $\zeta \circ s_1$ is a section of $(\hat{\mathfrak{g}}_2, d_{\hat{\mathfrak{g}}_2})$. Denote $s_2 := \zeta \circ s_1$. Since ζ is a homomorphism of differential Lie algebras such that $\zeta|_V = \mathrm{Id}_V$, we have

$$\begin{aligned}
\psi_2(x, y) &= [s_2(x), s_2(y)] - s_2([x, y]) = [\zeta(s_1(x)), \zeta(s_1(y))] - \zeta(s_1([x, y])) \\
&= \zeta([s_1(x), s_1(y)] - s_1([x, y])) = \zeta(\psi_1(x, y)) \\
&= \psi_1(x, y)
\end{aligned}$$

and

$$\begin{aligned}
\chi_2(x) &= d_{\hat{\mathfrak{g}}_2}(s_2(x)) - s_2(d_{\mathfrak{g}}(x)) = d_{\hat{\mathfrak{g}}_2}(\zeta(s_1(x))) - \zeta(s_1(d_{\mathfrak{g}}(x))) \\
&= \zeta(d_{\hat{\mathfrak{g}}_1}(s_1(x)) - s_1(d_{\mathfrak{g}}(x))) = \zeta(\chi_1(x)) \\
&= \chi_1(x).
\end{aligned}$$

Therefore, the result can be obtained.

Conversely, given two 2-cocycles (ψ_1, χ_1) and (ψ_2, χ_2), we can construct two abelian extensions $(\mathfrak{g} \oplus V, [\cdot, \cdot]_{\psi_1}, d_{\chi_1})$ and $(\mathfrak{g} \oplus V, [\cdot, \cdot]_{\psi_2}, d_{\chi_2})$ via Eqs (4.1) and (4.2), and then there exists a linear map $\phi : \mathfrak{g} \to V$ such that

$$(\psi_1, \chi_1) = (\psi_2, \chi_2) + \partial_{\mathrm{LieD}_\lambda}(\phi).$$

Define $\zeta : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ by

$$\zeta(x, v) := (x, \phi(x) + v).$$

Then ζ is an isomorphism of these two abelian extensions.

5. Deformations of differential Lie algebras of any weight

In this section, we show that if $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, \mathfrak{g}) = 0$, then the differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$ is rigid.

Let $(\mathfrak{g}, d_\mathfrak{g})$ be a differential Lie algebra. Denote by $\mu_\mathfrak{g}$ the multiplication of $\mathfrak{g}$. Consider the 1-parameterized family

$$\mu_t = \sum_{i=0}^{\infty} \mu_i t^i,\ \mu_i \in C^2_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}), \quad d_t = \sum_{i=0}^{\infty} d_i t^i,\ d_i \in C^1_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}).$$

Definition 5.1. A 1-parameter formal deformation of a differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$ is a pair (μ_t, d_t) which endows the $\mathbf{k}[[t]]$-module $(\mathfrak{g}[[t]], \mu_t, d_t)$ with the differential Lie algebra over $\mathbf{k}[[t]]$ such that $(\mu_0, d_0) = (\mu_\mathfrak{g}, d_\mathfrak{g})$.

Given any differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$, interpret $\mu_\mathfrak{g}$ and $d_\mathfrak{g}$ as the formal power series μ_t and d_t with $\mu_i = \delta_{i,0}\mu_\mathfrak{g}$ and $d_i = \delta_{i,0} d_\mathfrak{g}$ respectively for all $i \geq 0$. Then $(\mathfrak{g}[[t]], \mu_\mathfrak{g}, d_\mathfrak{g})$ is a 1-parameter formal deformation of $(\mathfrak{g}, d_\mathfrak{g})$.

The pair (μ_t, d_t) generates a 1-parameter formal deformation of the differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$ if and only if the following identities hold:

$$0 = \mu_t(x, \mu_t(y, z)) + \mu_t(y, \mu_t(z, x)) + \mu_t(z, \mu_t(x, y)), \tag{5.1}$$

$$d_t(\mu_t(x, y)) = \mu_t(d_t(x), y) + \mu_t(x, d_t(y)) + \lambda\mu_t(d_t(x), d_t(y)), \forall x, y, z \in \mathfrak{g}. \tag{5.2}$$

Expanding these identities and collecting coefficients of t^n, we see that Eqs (5.1) and (5.2) are equivalent to the systems of identities:

$$0 = \sum_{i=0}^{n} \mu_i(x, \mu_{n-i}(y, z)) + \mu_i(y, \mu_{n-i}(z, x)) + \mu_i(z, \mu_{n-i}(x, y)), \tag{5.3}$$

$$\sum_{\substack{k,l \geq 0 \\ k+l=n}} d_l \mu_k(x, y) = \sum_{\substack{k,l \geq 0 \\ k+l=n}} (\mu_k(d_l(x), y) + \mu_k(x, d_l(y))) + \lambda \sum_{\substack{k,l,m \geq 0 \\ k+l+m=n}} \mu_k(d_l(x), d_m(y)). \tag{5.4}$$

Remark 5.2. For $n = 0$, Eq (5.3) is equal to the Jabobi identity of $\mu_\mathfrak{g}$, and Eq (5.4) is equal to the fact that $d_\mathfrak{g}$ is a differential operator of weight λ.

Proposition 5.3. *Let $(\mathfrak{g}[[t]], \mu_t, d_t)$ be a 1-parameter formal deformation of a differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$. Then (μ_1, d_1) is a 2-cocycle of the differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$ with the coefficient in the adjoint representation $(\mathfrak{g}, d_\mathfrak{g})$.*

Proof. For $n = 1$, Eq (5.3) is equal to $\partial_{\mathrm{Lie}}\mu_1 = 0$, and Eq (5.4) is equal to

$$\partial_{\mathrm{Lie}} d_1 + \delta\mu_1 = 0.$$

Thus for $n = 1$, Eqs (5.3) and (5.4) imply that (μ_1, d_1) is a 2-cocycle.

If $\mu_t = \mu_\mathfrak{g}$ in the above 1-parameter formal deformation of the differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$, we obtain a 1-parameter formal deformation of the differential operator $d_\mathfrak{g}$. Consequently, we have

Corollary 5.4. *Let d_t be a* 1*-parameter formal deformation of the differential operator $d_\mathfrak{g}$. Then d_1 is a 1-cocycle of the differential operator $d_\mathfrak{g}$ with coefficients in the adjoint representation* $(\mathfrak{g}, d_\mathfrak{g})$.

Proof. In the special case when $n = 1$, Eq (5.4) is equal to $\partial_{\mathrm{Lie}} d_1 = 0$, which implies that d_1 is a 1-cocycle of the differential operator $d_\mathfrak{g}$ with coefficients in the adjoint representation $(\mathfrak{g}, d_\mathfrak{g})$.

Definition 5.5. The 2-cocycle (μ_1, d_1) is called the infinitesimal of the 1-parameter formal deformation $(\mathfrak{g}[[t]], \mu_t, d_t)$ of $(\mathfrak{g}, d_\mathfrak{g})$.

Definition 5.6. Two 1-parameter formal deformations $(\mathfrak{g}[[t]], \mu_t, d_t)$ and $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$ of $(\mathfrak{g}, d_\mathfrak{g})$ are said to be equivalent if there exists a formal isomorphism from $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$ to $(\mathfrak{g}[[t]], \mu_t, d_t)$ is a power series $\Phi_t = \sum_{i\geq 0} \phi_i t^i : \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$, where $\phi_i : \mathfrak{g} \to \mathfrak{g}$ are linear maps with $\phi_0 = \mathrm{Id}_\mathfrak{g}$, such that

$$\Phi_t \circ \bar{\mu}_t = \mu_t \circ (\Phi_t \times \Phi_t), \tag{5.5}$$

$$\Phi_t \circ \bar{d}_t = d_t \circ \Phi_t. \tag{5.6}$$

Theorem 5.7. *The infinitesimals of two equivalent* 1*-parameter formal deformations of* $(\mathfrak{g}, d_\mathfrak{g})$ *are in the same cohomology class* $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, \mathfrak{g})$.

Proof. Let $\Phi_t : (\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t) \to (\mathfrak{g}[[t]], \mu_t, d_t)$ be a formal isomorphism. For all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} \Phi_t \circ \bar{\mu}_t(x, y) &= \mu_t \circ (\Phi_t \times \Phi_t)(x, y), \\ \Phi_t \circ \bar{d}_t(x) &= d_t \circ \Phi_t(x). \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} \bar{\mu}_1(x, y) &= \mu_1(x, y) + [\phi_1(x), y] + [x, \phi_1(y)] - \phi_1([x, y]), \\ \bar{d}_1(x) &= d_1(x) + d_\mathfrak{g}(\phi_1(x)) - \phi_1(d_\mathfrak{g}(x)). \end{aligned}$$

Thus, we have

$$(\bar{\mu}_1, \bar{d}_1) = (\mu_1, d_1) + \partial_{\mathrm{LieD}_\lambda}(\phi_1),$$

which implies that $[(\bar{\mu}_1, \bar{d}_1)] = [(\mu_1, d_1)]$ in $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, \mathfrak{g})$.

Definition 5.8. A 1-parameter formal deformation $(\mathfrak{g}[[t]], \mu_t, d_t)$ of $(\mathfrak{g}, d_\mathfrak{g})$ is said to be trivial if it is equal to the deformation $(\mathfrak{g}[[t]], \mu_\mathfrak{g}, d_\mathfrak{g})$, that is, there exists $\Phi_t = \sum_{i\geq 0} \phi_i t^i : \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$, where $\phi_i : \mathfrak{g} \to \mathfrak{g}$ are linear maps with $\phi_0 = \mathrm{Id}_\mathfrak{g}$, such that

$$\Phi_t \circ \mu_t = \mu_\mathfrak{g} \circ (\Phi_t \times \Phi_t), \tag{5.7}$$

$$\Phi_t \circ d_t = d_\mathfrak{g} \circ \Phi_t. \tag{5.8}$$

Definition 5.9. A differential Lie algebra $(\mathfrak{g}, d_\mathfrak{g})$ is said to be rigid if every 1-parameter formal deformation is trivial.

Theorem 5.10. *Regarding* $(\mathfrak{g}, d_\mathfrak{g})$ *as the adjoint representation over itself, if* $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, \mathfrak{g}) = 0$, *the differential Lie algebra* $(\mathfrak{g}, d_\mathfrak{g})$ *is rigid.*

Proof. Let $(\mathfrak{g}[[t]], \mu_t, d_t)$ be a 1-parameter formal deformation of $(\mathfrak{g}, d_\mathfrak{g})$. By Proposition 5.3, (μ_1, d_1) is a 2-cocycle. By $H^2_{\mathrm{LieD}_\lambda}(\mathfrak{g}, \mathfrak{g}) = 0$, there exists a 1-cochain $\phi_1 \in C^1_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$ such that

$$(\mu_1, d_1) = -\partial_{\mathrm{LieD}_\lambda}(\phi_1). \tag{5.9}$$

Then setting $\Phi_t = \mathrm{Id}_\mathfrak{g} + \phi_1 t$, we have a deformation $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$, where

$$\begin{aligned} \bar{\mu}_t(x, y) &= (\Phi_t^{-1} \circ \mu_t \circ (\Phi_t \times \Phi_t))(x, y), \\ \bar{d}_t(x) &= (\Phi_t^{-1} \circ d_t \circ \Phi_t)(x). \end{aligned}$$

Thus, $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$ is equivalent to $(\mathfrak{g}[[t]], \mu_t, d_t)$. Furthermore, we have

$$\begin{aligned} \bar{\mu}_t(x, y) &= (\mathrm{Id}_\mathfrak{g} - \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i + \cdots)(\mu_t(x + \phi_1(x)t, y + \phi_1(y)t)), \\ \bar{d}_t(x) &= (\mathrm{Id}_\mathfrak{g} - \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i + \cdots)(d_t(x + \phi_1(x)t)). \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{\mu}_t(x, y) &= [x, y] + (\mu_1(x, y) + [x, \phi_1(y)] + [\phi_1(x), y] - \phi_1([x, y]))t + \bar{\mu}_2(x, y)t^2 + \cdots, \\ \bar{d}_t(x) &= d_\mathfrak{g}(x) + (d_\mathfrak{g}(\phi_1(x)) + d_1(x) - \phi_1(d_\mathfrak{g}(x)))t + \bar{d}_2(x)t^2 + \cdots. \end{aligned}$$

By Eq (5.9), we have

$$\begin{aligned} \bar{\mu}_t(x, y) &= [x, y] + \bar{\mu}_2(x, y)t^2 + \cdots, \\ \bar{d}_t(x) &= d_\mathfrak{g}(x) + \bar{d}_2(x)t^2 + \cdots. \end{aligned}$$

Then by repeating the argument, we can show that $(\mathfrak{g}[[t]], \mu_t, d_t)$ is equivalent to $(\mathfrak{g}[[t]], \mu_\mathfrak{g}, d_\mathfrak{g})$. Thus, $(\mathfrak{g}, d_\mathfrak{g})$ is rigid.

6. Homotopy differential operators of any weight on 2-term L_∞-algebras

In this section, we pay our attention to the homotopy differential operator of any weight on 2-term L_∞-algebras introduced by [24].

Definition 6.1. A 2-term L_∞-algebra consists of
- a complex of vector spaces $L_1 \xrightarrow{d} L_0$,
- bilinear maps $l_2 : L_i \otimes L_j \to L_{i+j}$, where $i + j \leq 1$,
- a skew-symmetric trilinear map $l_3 : L_0 \otimes L_0 \otimes L_0 \to L_1$, satisfying:

$$\begin{aligned} &(a)\ l_2(a, b) = -l_2(b, a),\ l_2(a, u) = -l_2(u, a), \\ &(b)\ dl_2(a, u) = l_2(a, du),\ l_2(du, v) = l_2(u, dv), \\ &(c)\ dl_3(a, b, c) = l_2(l_2(a, b), c) - l_2(l_2(a, c), b) - l_2(a, l_2(b, c)), \\ &(d)\ l_3(a, b, du) = l_2(l_2(a, b), u) - l_2(a, l_2(b, u)) - l_2(l_2(a, u), b), \\ &(e)\ l_2(xa, l_3(b, c, w)) + l_2(l_3(a, c, w), b) - l_2(l_3(a, b, w), c) + l_2(l_3(a, b, c), w) = l_3(l_2(a, b), c, w) \\ &-l_3(l_2(a, c), b, w) + l_3(l_2(a, w), b, c) + l_3(a, l_2(b, c), w) + l_3(a, l_2(b, w), c) + l_3(a, b, l_2(c, w)). \end{aligned}$$

for any $a, b, c, w \in L_0$ and $u, v \in L_1$.

One denotes a 2-term L_∞-algebra as above by $(L_1 \xrightarrow{d} L_0, l_2, l_3)$. A 2-term L_∞-algebra is called **skeletal** if $d = 0$.

Definition 6.2. Let $L = (L_1 \xrightarrow{d} L_0, l_2, l_3)$ and $L' = (L'_1 \xrightarrow{d'} L'_0, l'_2, l'_3)$ be two 2-term L_∞-algebras. A morphism $f : L \to L'$ consists of
• a chain map $f : L \to L'$ (which consists of linear maps $f_0 : L_0 \to L'_0$ and $f_1 : L_1 \to L'_1$ with $f_0 \circ d = d' \circ f_1$),
• a bilinear map $f_2 : L_0 \otimes L_0 \to L'_1$ satisfying

$$(a)\ d(f_2(a, b)) = f_0(l_2(a, b)) - l'_2(f_0(a), f_0(b)),$$
$$(b)\ f_2(a, du) = f_1(l_2(a, u)) - l'_2(f_0(a), f_1(u)),$$
$$\begin{aligned}(c)\ & f_1(l_3(a, b, c)) + l'_2(f_0(a, b), f_0(c)) - l'_2(f_2(a, c), f_0(b)) - l'_2(f_0(a), f_2(b, c))\\ &+ f_2(l_2(a, b), c) - f_2(l_2(a, c), b) - f_2(a, l_2(b, c)) - l'_3(f_0(a), f_0(b), f_0(c)) = 0,\end{aligned}$$

for any $a, b, c \in L_0$ and $u \in L_1$.

If $f = (f_0, f_1, f_2) : L \to L'$ and $g = (g_0, g_1, g_2) : L' \to L''$ are two morphism of 2-term L_∞-algebras, their composition $g \circ f : L \to L''$ is defined by $(g \circ f)_0 = g_0 \circ f_0, (g \circ f)_1 = g_1 \circ f_1$ and

$$(g \circ f)_2(a, b) = g_2(f_0(a), f_0(b)) + g_1(f_2(a, b)), \quad \forall a, b \in L_0.$$

For any 2-term L_∞-algebra L, the identity morphism $\mathrm{Id}_L : L \to L$ is given by the identity chain map $L \to L$ together with $(\mathrm{Id}_L)_2 = 0$.

The collection of 2-term L_∞-algebras and morphisms between them form a category. We denote this category by $\mathbf{2Lie}_\infty$.

Definition 6.3. Let $L = (L_1 \xrightarrow{d} L_0, l_2, l_3)$ be a 2-term L_∞-algebra. A homotopy differential operator of weight λ on it consists of a chain map of the underlying chain complex (i.e., linear maps $\theta_0 : L_0 \to L_0$ and $\theta_1 : L_1 \to L_1$ with $\theta_0 \circ d = d \circ \theta_1$) and a bilinear map $\theta_2 : L_0 \otimes L_0 \to L_1$ such that for any $a, b, c \in L_0$ and $u \in L_1$, the following identities are hold

$$(a)\ d(\theta_2(a, b)) = \theta_0(l_2(a, b)) - l_2(\theta_0(a), b) - l_2(a, \theta_0(b)) - \lambda l_2(\theta_0(a), \theta_0(b)),$$
$$(b)\ \theta_2(a, du) = \theta_1(l_2(a, u)) - l_2(\theta_0(a), u) - l_2(a, \theta_1(u)) - \lambda l_2(\theta_0(a), \theta_1(u)),$$
$$\begin{aligned}(c)\ & l_3(\theta_0(a), b, c) + l_3(a, \theta_0(b), c) + l_3(a, b, \theta_0(c)) - \theta_1(l_3(a, b, c))\\ &= l_2(\theta_2(a, b), c) - l_2(\theta_2(a, c), b) - l_2(a, \theta_2(b, c)) + \theta_2(l_2(a, b), c) - \theta_2(l_2(a, c), b) - \theta_2(a, l_2(b, c)).\end{aligned}$$

A 2-term L_∞-algebra with a homotopy differential operator of weight λ as above denoted by the pair $((L_1 \xrightarrow{d} L_0, l_2, l_3), (\theta_0, \theta_1, \theta_2))$. A 2-term L_∞-algebra with a homotopy differential operator of weight λ is said to be skeletal if the underlying 2-term L_∞-algebra is skeletal, i.e., $d = 0$.

Definition 6.4. Let $((L_1 \xrightarrow{d} L_0, l_2, l_3), (\theta_0, \theta_1, \theta_2))$ and $((L'_1 \xrightarrow{d'} L'_0, l'_2, l'_3), (\theta'_0, \theta'_1, \theta'_2))$ be two 2-term L_∞-algebras with homotopy differential operators of weight λ. A morphism between them consists of a morphism (f_0, f_1, f_2) between the underlying 2-term L_∞-algebras and a linear map $\Psi : L_0 \to L'_1$ satisfying

$$(1) \quad \Psi \circ \phi_0 = \phi'_1 \circ \Psi,$$

(2) $f_0(\theta_0(a)) - \theta'_0(f_0(a)) = d'(\Psi(a))$,

(3) $f_1(\theta_1(u)) - \theta'_1(f_1(u)) = \Psi(da)$,

(4) $f_1(\theta_2(a,b)) - \theta'_2(f_0(a), f_0(b)) = \theta'_1(f_2(a,b)) - f_2(\theta_0(a), b) - f_2(a, \theta_0(b))$
$+\Psi(l_2(a,b)) - l'_2(\Psi(a), f_0(b)) - l'_2(f_0(a), \Psi(b))$.

We denote the category of 2-term L_∞-algebras with homotopy differential operators of weight λ and morphisms between them by $\mathbf{2LieD}_{\lambda\infty}$.

Theorem 6.5. *There is a one-to-one correspondence between skeletal 2-term L_∞-algebras with homotopy differential operators with weight λ and tuples $((\mathfrak{g}, d_\mathfrak{g}), (V, d_V), (\theta, \overline{\theta}))$, where $(\mathfrak{g}, d_\mathfrak{g})$ is a differential Lie algebra of weight λ , (V, d_V) is a representation and $(\theta, \overline{\theta})$ is a 3-cocycle of the differential Lie algebra of weight λ with coefficients in the representation.*

Proof. Let $(L_1 \stackrel{0}{\longrightarrow} L_0, l_2, l_3, (\theta_0, \theta_1, \theta_2))$ be a skeletal 2-term L_∞-algebra with a homotopy differential operator of weight λ. Then θ_0 is a differential operator of weight λ for the Lie algebra (L_0, l_2). We have that (L_1, θ_1) is a representation of the differential Lie algebra (L_0, θ_0) of weight λ from Definition 6.3. According to the condition (c) in Definition 6.3, we have $\partial_{\mathrm{LieD}_\lambda}(\theta_2) + \delta(l_3) = 0$. Therefore $(l_3, -\theta_2)$ is a 3-cocycle.

Conversely, define $L_0 = L, L_1 = V$ and $\theta_0 = d_\mathfrak{g}, \theta_1 = d_V, \theta_2 = -\overline{\theta}$. We define multiplications $l_2 : L_i \otimes L_j \to L_{i+j}$ and $l_3 : L_0 \otimes L_0 \otimes L_0 \to L_1$ by

$$l_2(a,b) = [a,b], l_2(a,u) = [a,u], l_2(u,a) = [u,a], l_3 = 0,$$

for $a, b, c \in L_0 = L$ and $u \in L_1 = V$. Then it is easy to verify that $((L_1 \stackrel{0}{\longrightarrow} L_0, l_2, l_3), (\theta_0, \theta_1, \theta_2))$ is a skeletal 2-term L_∞-algebra with a homotopy differential operator of weight λ. Hence, the proof is finished.

A 2-term L_∞-algebra with a homotopy differential operator of weight λ is said to be strict if the underlying 2-term L_∞-algebra is strict, i.e., $\theta_2 = 0$. Next we introduce crossed modules of differential Lie algebras of weight λ and show that strict 2-term L_∞-algebra with a homotopy differential operator of weight λ are in one-to-one correspondence with crossed module of differential Lie algebras of weight λ.

Definition 6.6. A crossed module of differential Lie algebras of weight λ consist of $((\mathfrak{g}, d_\mathfrak{g}), (\mathfrak{h}, d_\mathfrak{h}), dt, \Lambda)$ where $(\mathfrak{g}, d_\mathfrak{g})$ and $(\mathfrak{h}, d_\mathfrak{h})$ are differential Lie algebras of weight λ, $dt : \mathfrak{g} \to \mathfrak{h}$ is a differential Lie algebra morphism and

$$\Lambda : \mathfrak{h} \to gl(\mathfrak{g}), \quad a \mapsto \Lambda_a,$$

such that for $u, v \in \mathfrak{g}, a, b \in \mathfrak{h}$,

(a) $dt(\Lambda_a(u)) = [a, dt(u)]_\mathfrak{h}$,

(b) $\Lambda_{dt(u)}(v) = [u, v]_\mathfrak{g}$,

(c) $\Lambda_{[a,b]_\mathfrak{h}} = \Lambda_a\Lambda_b - \Lambda_b\Lambda_a$,

(d) $d_\mathfrak{g}(\Lambda_a(u)) = \Lambda_{d_\mathfrak{h}(a)}(u) + \Lambda_a(d_\mathfrak{g}(u)) + \lambda\Lambda_{d_\mathfrak{h}(a)}(d_\mathfrak{g}(u))$.

Theorem 6.7. *There is a one-to-one correspondence between strict 2-term L_∞-algebras with homotopy differential operators of weight λ and crossed module of differential Lie algebras of weight λ.*

Proof. Let $(L_1 \xrightarrow{d} L_0, l_2, l_3 = 0, (\theta_0, \theta_1, \theta_2))$ be a strict 2-term L_∞-algebra with a homotopy differential operator of weight λ. Then θ_0 is a differential operator of weight λ for the Lie algebra (L_0, l_2) and θ_1 is a differential operator of weight λ for the Lie algebra (L_1, l_2) from Definition 6.3. Thus (L_0, θ_0) and (L_1, θ_1) are both differential Lie algebras of weight λ. Since $\theta_0 \circ d = d \circ \theta_1$, the map $dt = d : L_1 \to L_0$ is a morphism of differential Lie algebras of weight λ. Finally, the condition (b) of Definition 6.3 is equal to the condition (d) of Definition 6.6. Hence, the results are obtained.

7. Categorification of differential Lie algebras of any weight

In this section, we study categorified differential operators of any weight (also called 2-differential operator) on Lie 2-algebras.

Definition 7.1. A Lie 2-algebra is a 2-vector space L equipped with
- a bilinear functor $[\cdot, \cdot] : L \otimes L \to L$,
- a trilinear natural isomorphism, called the Jacobiator

$$\mathcal{J}_{a,b,c} : [[a, b], c] \to [[a, c], b] + [a, [b, c]],$$

satisfying

$$\begin{array}{ccc}
[[[a,b],c],w] & \xrightarrow{\mathcal{J}_{[a,b],c,w}} & [[[a,b],w],c] + [[a,b],[c,w]] \\
\Big\downarrow {\scriptstyle [\mathcal{J}_{a,b,c},w]} & & \Big\downarrow {\scriptstyle [\mathcal{J}_{a,b,w},c]+1} \\
[[[a,c],b] + [a,[b,c]],w] & & R \\
\Big\downarrow {\scriptstyle \mathcal{J}_{[a,c],b,w}+\mathcal{J}_{a,[b,c],w}} & & \Big\downarrow {\scriptstyle \Theta} \\
P & \xrightarrow[{[\mathcal{J}_{a,c,w},b]+1+1+[a,\mathcal{J}_{b,c,w}]}]{} & Q,
\end{array}$$

where Θ, R, P and Q are given by

$$\begin{aligned}
\Theta &= \mathcal{J}_{[a,w],b,c} + \mathcal{J}_{a,[b,w],c} + \mathcal{J}_{a,b,[c,w]} \\
R &= [[[a, w], b], c] + [[a, [b, w]], c] + [[a, b], [c, w]], \\
P &= [[[a, c], w], b] + [[a, c], [b, w]] + [[a, w], [b, c]] + [a, [[b, c], w]], \\
Q &= [[[a, w], c], b] + [[a, [c, w]], b] + [[a, c], [b, w]] + [[a, w], [b, c]] + [a, [[b, w], c]] + [a, [b, [c, w]]].
\end{aligned}$$

Definition 7.2. Let $(L, [\cdot, \cdot], \mathcal{J})$ and $(L', [\cdot, \cdot]', \mathcal{J}')$ be two Lie 2-algebras. A Lie 2-algebra morphism consists of
- a linear functor (F_0, F_1) from the underlying 2-vector space L to L';
- a bilinear natural transformation

$$F_2(a, b) : [F_0(a), F_0(b)]' \to F_0([a, b])$$

satisfying

$$
\begin{array}{ccc}
[[F_0(a), F_0(b)]', F_0(c)]' & \xrightarrow{\mathcal{J}_{F_0(a),F_0(b),F_0(c)}} & [[F_0(a), F_0(c)]', \Phi' F_0(b)]' + [F_0(a), [F_0(b), F_0(c)]']' \\
\Big\downarrow {\scriptstyle [F_2(a,b),1]'} & & \Big\downarrow {\scriptstyle [F_2(a,c),1]'+[1,F_2(b,c)]'} \\
[F_0[a,b], F_0(c)]' & & [F_0[a,c], F_0(b)]' + [F_0(a), F_0[b,c]]' \\
\Big\downarrow {\scriptstyle F_2([a,b],c)} & & \Big\downarrow {\scriptstyle F_2([a,c],b)+F_2(a,[b,c])} \\
F_0[[a,b],c] & \xrightarrow[F_0(\mathcal{J}_{a,b,c})]{} & F_0([[a,c],b] + [a,[b,c]]).
\end{array}
$$

Let L, L' and L'' be three Lie 2-algebras and $F : L \to L', G : L' \to L''$ be Lie 2-algebra morphisms. Their composition $G \circ F : L \to L''$ is a Lie 2-algebra morphism whose components are given by $(G \circ F)_0 = G_0 \circ F_0, (G \circ F)_1 = G_1 \circ F_1$ and $(G \circ F)_2$ is given by

$$
\begin{array}{ccc}
[G_0 \circ F_0(\xi), G_0 \circ F_0(\eta)]'' & \xrightarrow{(G\circ F)_2(\xi,\eta)} & (G_0 \circ F_0)([\xi, \eta]) \\
& {\scriptstyle G_2(F_0(\xi),F_0(\eta))} \searrow \quad \nearrow {\scriptstyle G_0(F_2(\xi,\eta))} & \\
& G_0([F_0(\xi), F_0(\eta)]'). &
\end{array}
$$

For any Lie 2-algebra L, the identity morphism $\mathrm{Id}_L : L \to L$ is given by the identity functor as its linear functor together with the identity natural transformation as $(\mathrm{Id}_L)_2$.

Lie 2-algebras and Lie 2-algebra morphisms form a category. We denote this category by **Lie2**.

In the next, we define 2-differential operators of weight λ on Lie 2-algebras. They are categorification of differential operators on Lie algebras.

Definition 7.3. Let $(L, [\cdot,\cdot], \mathcal{J})$ be a Lie 2-algebra. A 2-differential operator of weight λ on it consists of a linear map functor $D : L \to L$ and a natural isomorphism

$$\mathcal{D}_{a,b} : D[a,b] \to [Da,b] + [a,Db] + \lambda[Da,Db], \forall a,b \in L$$

satisfying

$$
\begin{array}{ccc}
D[[a,b],c] & \xrightarrow{\mathcal{J}} & D([[a,c],b] + [a,[b,c]]) \\
\Big\downarrow {\scriptstyle \mathcal{D}_{[a,b],c}} & & \Big\downarrow \\
[D[a,b],c] + [[a,b],D(c)] + \lambda[D[a,b],Dc] & & P \\
\Big\downarrow {\scriptstyle [\mathcal{D},1]+1+1} & & \Big\downarrow {\scriptstyle [\mathcal{D},1]+1+1+1+1+[1,\mathcal{D}]} \\
[[Da,b] + [a,Db] + \lambda[Da,Db],c] + [[a,b],D(c)] + \lambda[D[a,b],Dc] & \xrightarrow[\mathcal{J}+\mathcal{J}+1+\mathcal{J}+1]{} & Q,
\end{array}
$$

where

$$
\begin{aligned}
P &= [D[a,c],b] + [[a,c],D(b)] + [D(a),[b,c]] + [a,D[b,c]] + \lambda[D[a,c],Db] + \lambda[Da,D[b,c]] \\
Q &= [[Da,c],b] + [[a,Dc],b] + \lambda[[Da,Dc],b] + [[a,c],D(b)] + [D(a),[b,c]] \\
&\quad +[a,[Db,c]] + [a,[b,Dc]] + \lambda[a,[Db,Dc]] + \lambda[D[a,c],Db] + \lambda[Da,D[b,c]].
\end{aligned}
$$

Definition 7.4. Let $(L, [\cdot,\cdot], \mathcal{J}, D, \mathcal{D})$ and $(L', [\cdot,\cdot]', \mathcal{J}', D', \mathcal{D}')$ be two Lie 2-algebras with 2-differential operators of weight λ. A morphism between them consists of a Lie 2-algebras mophism $(F = (F_0, F_1), F_2)$ and a natural isomorphism

$$\Theta_a : D'(F_0(a)) \to F_0(D(a)), \forall a \in L_0$$

satisfying

$$\begin{array}{ccc}
D'([F_0(a), F_0(b)]') & \xrightarrow{\quad F_2 \quad} & D'(F_0[a,b]) \\
\downarrow{\scriptstyle \mathcal{D}'} & & \downarrow{\scriptstyle \Theta_{[a,b]}} \\
[D'(F_0(a)), F_0(b)]' + [F_0(a), D'(F_0(b))]' & & F_0(D[a,b]) \\
\downarrow{\scriptstyle [\Theta_a,1]'+[1,\Theta_b]'} & & \downarrow{\scriptstyle \mathcal{D}} \\
[F_0(D(a)), F_0(b)]' + [F_0(a), F_0(D(b))]' & \xrightarrow[F_2+F_2]{} & F_0([Da,b] + [a,Db]).
\end{array}$$

We denote the category of Lie 2-algebras with 2-differential operators of weight λ and morphisms between them by $\mathbf{LieD2}_\lambda$.

In the following, we will give our main result of this section.

Theorem 7.5. *The categories* $\mathbf{2LieD}_{\lambda\infty}$ *and* $\mathbf{LieD}2_\lambda$ *are equivalent.*

Proof. First we construct a functor $T : \mathbf{2LieD}_{\lambda\infty} \to \mathbf{LieD2}_\lambda$ as follows. Given a 2-term L_∞-algebra with a homotopy differential operator of weight λ $((L_1 \xrightarrow{d} L_0, l_2, l_3), (\theta_0, \theta_1, \theta_2))$, we obtain the 2-vector space $C = (L_0 \oplus L_1 \rightrightarrows L_0)$. Define a bilinear functor $[\cdot,\cdot] : C \otimes C \to C$ by

$$[(a,u),(b,v)] = (l_2(a,b), l_2(a,v) + l_2(u,b) + l_2(du,v)),$$

for $(a,u),(b,v) \in C_1 = L_0 \oplus L_1$. Define

$$\mathcal{J}_{a,b,c} = ([[a,b],c], l_3(a,b,c)).$$

According to the identities (a)–(e), we can check that $(C, [\cdot,\cdot], \mathcal{J})$ is a Lie 2-algebra. Moreover, we define a 2-differential operator of weight λ $(D, \mathcal{D}')$ by

$$D(a,u) := (\theta_0(a), \theta_1(u)), \mathcal{D}_{a,b} := ([a,b], \theta_2(a,b)).$$

Given any 2-term L_∞-algebra with a homotopy differential operator of weight λ morphism (f_0, f_1, f_2, Ψ) from L to L', for any $F_0 = f_0, F_1 = f_1$ and

$$F_2(a,b) = ([f_0(a), f_0(b)]', f_2(a,b)), \Theta = \Psi.$$

Direct verification that F is a morphism from C to C'. Furthermore, we can check that T preserve the identity morphisms and composition of morphisms. Hence, T is a functor from $\mathbf{2LieD}_{\lambda\infty}$ to $\mathbf{LieD2}_\lambda$.

Conversely. Given a Lie 2-algebra $C = (C_1 \oplus C_0, \mathcal{J}, D, \mathcal{D})$ with a 2-differential operator of weight λ, we have the 2-term chain complex

$$L_1 = kers \stackrel{d=t|_{kers}}{\longrightarrow} C_0 = L_0.$$

Define $l_2 : L_i \otimes L_j \to L_{i+j}$ by

$$l_2(a,b) = [a,b], \quad l_2(a,u) = [a,u], \quad l_2(u,a) = [u,a].$$

The map $l_3 : L_0 \otimes L_0 \otimes L_0 \to L_1$ is defined by

$$l_3(a,b,c) = pr(\mathcal{J}_{a,b,c}), \quad \forall a,b,c \in L_0,$$

where pr denote the projection on $ker(s)$. Moreover, we define a homotopy differential operator by

$$\theta_0(a) := D(i(u)), \theta_1(u) := D|_{ker(s)}(u), \theta_2(a,b) := pr(\mathcal{D}_{a,b}).$$

For any Lie 2-algebra morphism $(F_0, F_1, F_2, \Theta) : C \to C'$, then $f_0 = F_0$, $f_1 = F_1|_{L_1} = kers$ with a 2-differential operator of weight λ and define f_2 by

$$f_2(a,b) = prF_2(a,b), \Psi = \Theta.$$

Moreover, S preserve the identity morphisms and composition of morphisms. Therefore, S is a functor from $\mathbf{LieD2}_\lambda$ to $\mathbf{2LieD}_{\lambda\infty}$.

Finally, it is easy to prove that $T \circ S \cong 1_{\mathbf{LieD2}_\lambda}$, and the composite $S \circ T \cong 1_{\mathbf{2LieD}_{\lambda\infty}}$ and we omit them.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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