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Research article

Pullback dynamics and robustness for the 3D Navier-Stokes-Voigt equations with memory

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Abstract: The tempered pullback dynamics and robustness of the 3D Navier-Stokes-Voigt equations with memory and perturbed external force are considered in this paper. Based on the global well-posedness results and energy estimates involving memory, a suitable tempered universe is constructed, the robustness is finally established via the upper semi-continuity of tempered pullback attractors when the perturbation parameter epsilon tends to zero.

Keywords: Navier-Stokes-Voigt equations; memory; robustness

1. Introduction

The Navier-Stokes equations are a typical nonlinear system, which model the mechanics law for fluid flow and have been applied in many fields. There are many research findings on the Navier-Stokes system, involving well-posedness, long-time behavior, etc., [1–10]. Furthermore, to simulate the fluid movement modeled by the Navier-Stokes equations, some regularized systems are proposed, such as the Navier-Stokes-Voigt equations. The Navier-Stokes-Voigt equations were introduced by Oskolkov in 1973, which describe the motion of Kelvin-Voigt viscoelastic incompressible fluid. Based on the global well-posedness of 3D Navier-Stokes-Voigt equations in [11], many interesting results on long-time behavior of solutions have been obtained, such as the existence of global attractor and pullback attractors, determining modes and estimate on fractal dimension of attractor [12–14] and references therein for details.

The influence of past history term on dynamical system is well known, we refer to [15–18] for interesting conclusions, such as the global well-posedness, the existence of attractors and so on. In 2013, Gal and Tachim-Medjo [17] studied the Navier-Stokes-Voigt system with instantaneous viscous term and memory-type viscous term, and obtained the well-posedness of solution and exponential attractors of finite dimension. In 2018, Plinio et al. [18] considered the Navier-Stokes-Voigt system in [18], in which the instantaneous viscous term was completely replaced by the memory-type viscous term and the Ekman damping βu was presented. The authors showed the existence of regular global and exponential attractors with finite dimension. The presence of Ekman damping was to eliminate the difficulties brought by the memory term in deriving the dissipation of system.

Some convergence results of solutions or attractors as perturbation vanishes for the non-autonomous dynamical systems without memory can be seen in [19–22]. However, there are few convergence results on the system with memory. Therefore, our purpose is to study the tempered pullback dynamics and robustness of the following 3D incompressible Navier-Stokes-Voigt equations on the bounded domain Ω with memory and the Ekman damping:

$$\begin{cases} \frac{\partial}{\partial t}(u-\alpha\Delta u)-\int_0^\infty g(s)\Delta u(t-s)ds+(u\cdot\nabla)u+\beta u+\nabla p=f_\varepsilon(t,x),\ (t,x)\in\Omega_\tau,\\ \mathrm{div}u=0,\ (t,x)\in\Omega_\tau,\\ u(t,x)=0,\ (t,x)\in\partial\Omega_\tau,\\ u(\tau,x)=u(\tau),\ x\in\Omega,\\ u(\tau-s,x)=\varphi(s,x),\ (s,x)\in\Omega_0, \end{cases} \tag{1.1}$$

where $\Omega_\tau=(\tau,+\infty)\times\Omega$, $\partial\Omega_\tau=(\tau,+\infty)\times\partial\Omega$, $\Omega_0=(0,\infty)\times\Omega$, $\tau\in\mathbb{R}^+$ is the initial time, $\alpha>0$ is a length scale parameter characterizing the elasticity of fluid, $\beta>0$ is the Ekman dissipation constant, $u=(u_1(t,x),u_2(t,x),u_3(t,x))$ is the unknown velocity field of fluid, and p is the unknown pressure. The non-autonomous external force is $f_\varepsilon(t,x)=f_1(x)+\varepsilon f_2(t,x)$ $(0\le\varepsilon<\varepsilon_0)$, where ε_0 is a fixed constant small enough. In addition, $u(\tau)$ is the initial velocity, and $\varphi(s,x)$ denotes the past history of velocity. The memory kernel $g:\ [0,\infty)\to[0,\infty)$ is supposed to be convex, smooth on $(0,\infty)$ and satisfies that

$$g(\infty)=0,\ \int_0^\infty g(s)ds=1.$$

In general, we give the past history variable

$$\eta=\eta^t(s)=\int_0^s u(t-\sigma)d\sigma,\ s\ge 0,$$

which satisfies

$$\frac{\partial}{\partial t}\eta=-\frac{\partial}{\partial s}\eta+u(t).$$

Also, η has the explicit representation

$$\begin{cases} \eta^t(s) &= \int_0^s u(t-\sigma)d\sigma,\ 0<s\le t,\\ \eta^t(s) &= \eta^0(s-t)+\int_0^t u(t-\sigma)d\sigma,\ s>t, \end{cases} \tag{1.2}$$

and

$$\eta^\tau(s)=\int_0^s \varphi(\sigma)d\sigma.$$

Next, we give the main features of this paper as follows.

1) Inspired by [18, 23], we provide a detailed representation and Gronwall type estimates for the energy of (1.1) dependent on ε in Lemma 2.1, with a focus on the parameters $\omega,\ \Lambda$ and the increasing

function $J(*)$. Using these parameters, we construct the universe $\mathcal{D}$ and derive the existence of $\mathcal{D}-$pullback absorbing sets, see Lemma 4.9.

2) Via the decomposition method, we show that the process of the system has the property of $\mathcal{D}-\kappa-$pullback contraction in the space N_V, and the $\mathcal{D}-$pullback asymptotic compactness is obtained naturally. Based on the theory of attractor in [1,24], the $\mathcal{D}$-pullback attractors for the process $\{S^{\varepsilon}(t,\tau)\}$ in N_V are derived, see Theorem 3.3.

3) When the perturbation parameter $\varepsilon \to 0$ with the non-autonomous external force, the robustness is obtained via the upper semi-continuity of pullback attractors of (1.1) by using the technique in [18,21,22], see Theorems 3.2 in Section 3.

This paper is organized as follows. Some preliminaries are given in Section 2, and the main results are stated in Section 3, which contains the global well-posedness of solution, the existence of pullback attractors and robustness. Finally, the detailed proofs are provided in Sections 4 and 5.

2. Preliminaries

2.1. Some functional spaces

• The Sobolev spaces

Let $E = \{u|u \in (C_0^\infty(\Omega))^3, \mathrm{div} u = 0\}$, H is the closure of E in $(L^2(\Omega))^3$ topology with the norm and inner product as

$$|u| = \|u\|_H = (u,v)^{1/2},\ (u,v) = \sum_{j=1}^{3}\int_\Omega u_j(x)v_j(x)dx,\ \forall\, u,v \in H.$$

V is the closure of E in $(H^1(\Omega))^3$ topology with the norm and inner product as

$$\|u\| = \|u\|_V = ((u,u))_V^{1/2},\ ((u,v))_V = \sum_{i,j=1}^{3}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx,\ \forall\, u,v \in V.$$

Also, we denote

$$((u,v))_{V_\alpha} = (u,v) + \alpha((u,v))_V,\ \|u\|_{V_\alpha}^2 = |u|^2 + \alpha\|u\|^2.$$

H and V are Hilbert spaces with their dual spaces H and V' respectively, $\|\cdot\|_*$ and $\langle\cdot,\cdot\rangle$ denote the norm in V' and the dual product between V and V' respectively, and also H to itself.

• The fractional power functional spaces

Let P_L be the Helmholz-Leray orthogonal projection in $(L^2(\Omega))^3$ onto H [3,7], and

$$P_L:\ H \oplus H^\perp \to H,$$

where

$$H^\perp = \{u \in (L^2(\Omega))^3;\ \exists\, \chi \in (L^2_{loc}(\Omega))^3:\ u = \nabla\chi\}.$$

$A = -P_L\Delta$ is the Stokes operator with eigenvalues $\{\lambda_j\}_{j=1}^\infty$ and orthonormal eigenfunctions $\{\omega_j\}_{j=1}^\infty$.

Define the fractional operator A^s by

$$A^s u = \sum_j \lambda_j^s(u,\omega_j)\omega_j,\quad s \in \mathbb{R},\quad j \in \mathbb{Z}^+$$

for $u = \sum_j (u,\omega_j)\omega_j$ with the domain $D(A^s) = \{u | A^s u \in H\}$, and we use the norm of $D(A^s)$ as

$$\|u\|_{2s}^2 = |A^s u|^2 = \sum_j \lambda_j^{2s} |(u,\omega_j)|^2.$$

Especially, denote $W = D(A)$, and $V = D(A^{1/2})$ with norm $\|u\|_1 = |A^{1/2}u| = \|u\|$ for any $u \in V$.

• The memory spaces

For any $s \in (0,\infty)$, we define $\mu(s) = -g'(s)$, which is nonnegative, absolutely continuous, decreasing ($\mu' \leq 0$ almost everywhere) and

$$\kappa = \int_0^\infty \mu(s)ds > 0. \tag{2.1}$$

Also, there exists $\delta > 0$ such that

$$\mu'(s) + \delta\mu(s) \leq 0, \ a.e. \ s \in (0,\infty). \tag{2.2}$$

Let

$$M_X = L_\mu^2(\mathbb{R}^+; X), \ X = V \text{ or } W,$$

which is a Hilbert space on $\mathbb{R}^+$ with inner product and norm

$$((\eta,\zeta))_{M_X} = \int_0^\infty \mu(s)((\eta(s),\zeta(s))_X ds, \ \|\eta\|_{M_X} = \Big(\int_0^\infty \mu(s)\|\eta(s)\|_X^2 ds\Big)^{1/2}.$$

Moreover, the extended memory space can be defined as

$$N_X = X \times M_X$$

equipped with the norm

$$\|(u,\eta)\|_{N_X}^2 = \|u\|_X^2 + \|\eta\|_{M_X}^2.$$

2.2. *Some inequalities and conclusions*

• The bilinear and trilinear operators

The bilinear and trilinear operators are defined as follows [8]

$$B(u,v) := P_L((u\cdot\nabla)v), \ \ \forall\, u,v \in V, \tag{2.3}$$

$$b(u,v,w) =< B(u,v), w >= \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx. \tag{2.4}$$

Denote $B(u) = B(u,u)$, $B(u,v)$ is a continuous operator from $V \times V$ to V', and there hold

$$b(u,v,v) = 0, \ b(u,v,w) = -b(u,w,v), \ \forall\, u,v,w \in V. \tag{2.5}$$

• Some useful lemmas

Lemma 2.1. *([23]) Assume that*

1) A nonnegative function h is locally summable on $\mathbb{R}^+$*, and for any* $\varepsilon \in (0, \varepsilon_0]$ *and any* $t \geq \tau \geq 0$ *there holds*

$$\varepsilon \int_\tau^t e^{-\varepsilon(t-s)} h(s)ds \leq \frac{8}{5} \sup_{t\geq 0} \int_t^{t+1} h(s)ds < \infty.$$

2) The nonnegative function $y_\varepsilon(t)$ *is absolutely continuous on* $[\tau, \infty)$*, and satisfies for some constants* $R, C_0 \geq 0$ *that*

$$y_\varepsilon(t) \leq Re^{-\varepsilon(t-\tau)} + \varepsilon^p \int_\tau^t e^{-\varepsilon(t-s)} h(s) y_\varepsilon(s)^q ds + \frac{C_0}{\varepsilon^{1+r}},$$

where $p, q, r \geq 0$*, and* $p - 1 > (q - 1)(1 + r) \geq 0$.

3) Let $z(t) \geq 0$ *be a continuous function on* $(0, \infty)$ *equivalent to* $y_\varepsilon(t)$*, which means there exist some constants* $M \geq 1,\ L \geq 0$ *such that*

$$z(t) \leq My_\varepsilon(t) \leq M(z(t) + L).$$

Then, there exist $\omega,\ \Lambda > 0$ *and an increasing function* $J(*) :\ \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$z(t) \leq J(MR)e^{-\omega(t-\tau)} + \Lambda(MC_0 + L).$$

Remark 2.1. *Under the assumptions in Lemma 2.1, there exists a constant* $\theta \in (0, 1)$ *satisfying*

$$p_1 = p\theta - \theta + 1 - q > 0,\ p_2 = 1 - \theta - r\theta > 0.$$

Denote

$$p_3 = max\{\varepsilon_0^{-1/\theta}, (2 \sup_{t\geq 0} \int_t^{t+1} h(s)ds)^{1/p_1}, C_0^{1/p_2}\},\ p_4 = 2max\{6Rp_3^{-1}, 1\},$$

then

$$\omega = \omega_{\theta,p,q,r,C_0} = \frac{1}{2} p_3^{-\theta},\ \Lambda = \Lambda_{\theta,p,q,r,C_0} = 5p_3^{1-p_2},$$

$$J(R) = J_{\theta,p,q,r,C_0}(R) = 2p_4^q p_3 exp(\frac{p_4^\theta}{1 - 2^{-\theta}} ln(6p_4^q)).$$

Lemma 2.2. *([15]) Let* η *be the past history variable and* (1.2) *holds. Then*

$$\|\eta^t(s)\|_{M_X}^2 - \|\eta^\tau(s)\|_{M_X}^2 \leq 2 \int_\tau^t ((\eta^\sigma, u(\sigma)))_{M_X} d\sigma.$$

3. Main results

3.1. Some assumptions

We assume that $f_1(x)$ and $f_2(t, x)$ satisfy the following hypotheses:
(C1) The function $f_1 \in H$.

(C2) $f_2(t,x)$ is translation bounded in $L^2_{loc}(\mathbb{R}, H)$, which means there exists a constant $K > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} |f_2(s)|^2 ds < K,$$

and for any $t \in \mathbb{R}$, there also holds

$$\int_{-\infty}^{t} e^{\iota s}\|f_2(s)\|^2 ds < \infty,\ 0 < \iota \le \nu\varepsilon_0,\ \nu = min\{\frac{\alpha\kappa\delta}{72}, 1\}, \tag{3.1}$$

where κ, δ are the same as parameters in (2.1) and (2.2) respectively.

3.2. Equivalent problem

Construct the infinitesimal generator of right-translation semigroup on M_X

$$T\eta = -\frac{\partial}{\partial s}\eta,$$

whose domain is

$$D(T) = \{\eta \in M_X :\ \frac{\partial}{\partial s}\eta \in M_X,\ \ \eta(0) = 0\}.$$

Given initial datum $U(\tau) = (u(\tau), \eta^\tau) \in N_V$, then (1.1) can be transformed into the following abstract form

$$\begin{cases} \frac{\partial}{\partial t}(u + \alpha Au) + \int_0^\infty \mu(s)A\eta(s)ds + B(u,u) + \beta u = P_L f_\varepsilon(t,x),\ (t,x) \in \Omega_\tau, \\ \frac{\partial}{\partial t}\eta = T\eta + u, \\ \mathrm{div}u = 0,\ (t,x) \in \Omega_\tau, \\ u(t,x) = 0,\ (t,x) \in \partial\Omega_\tau, \\ u(\tau,x) = u(\tau),\ x \in \Omega, \\ \eta^\tau(s) = \int_0^s \varphi(\sigma)d\sigma. \end{cases} \tag{3.2}$$

3.3. Main results for the system (3.2)

• Global well-posedness of solution

Definition 3.1. *A function $U(t) = (u(t), \eta^t) :\ [\tau, +\infty) \to N_V$ is called the weak solution to* (3.2)*, if for any fixed $T > \tau$ there hold*

(i) $U(t) \in C([\tau,T]; N_V)$, $\frac{\partial u}{\partial t} \in L^2(\tau,T;V)$.

(ii) $U(\tau) = (u(\tau), \eta^\tau)$.

(iii) for any $w \in C^1([\tau,T];V)$ with $w(T,x) = 0$, there holds

$$-\int_\tau^T \langle u + \alpha Au, w_t\rangle dt + \int_\tau^T\int_0^\infty \mu(s)((\eta(s), w))_V ds dt + \int_\tau^T b(u,u,w)dt + \int_\tau^T (\beta u, w)dt$$
$$= ((u(\tau), w(\tau)))_{V_\alpha} + \int_\tau^T (P_L f_\varepsilon, w)dt. \tag{3.3}$$

Theorem 3.2. *Let $U(\tau) \in N_V$, and the hypotheses (C1)–(C2) hold. Then the global weak solution $U(t,x)$ to system* (3.2) *uniquely exists on (τ, T), which generates a strongly continuous process*

$$S^\varepsilon(t,\tau) :\ N_V \to N_V,\ \forall\, t \ge \tau, \quad 0 \le \varepsilon < \varepsilon_0$$

and $S^\varepsilon(t,\tau)U(\tau) = U(t)$.

Proof. The global well-posedness of solution can be obtained by the Galerkin approximation method, energy estimates and compact scheme. The detailed proof can be found in [15, 18] and is omitted here.

• Existence of $\mathcal{D}$-pullback attractors

Theorem 3.3. *Assume $U(\tau) \in N_V$ and the hypotheses (C1)–(C2) hold. Then the process $S^\varepsilon(t,\tau) : N_V \to N_V$ generated by the system* (3.2) *possesses a minimal family of $\mathcal{D}$-pullback attractors $\mathcal{A}^\varepsilon = \{A^\varepsilon(t)\}_{t\in\mathbb{R}}$ in N_V.*

Proof. See Section 4.2.

When $\varepsilon = 0$, the system (3.2) can be reduced to the following autonomous system

$$\begin{cases} \frac{\partial}{\partial t}(u + \alpha Au) + \int_0^\infty \mu(s)A\eta(s)ds + B(u,u) + \beta u = P_L f_1(x),\ (t,x) \in \Omega_\tau, \\ \frac{\partial}{\partial t}\eta = T\eta + u, \\ \text{div}u = 0,\ (t,x) \in \Omega_\tau, \\ u(t,x) = 0,\ (t,x) \in \partial\Omega_\tau, \\ u(\tau,x) = u(\tau),\ x \in \Omega, \\ \eta^\tau(s) = \int_0^s \varphi(\sigma)d\sigma. \end{cases} \tag{3.4}$$

Remark 3.1. *The existence of global attractor $\mathcal{A}^0$ in N_V can be achieved for the semigroup $S^0(t-\tau)$ generated by* (3.4).

• Robustness: upper semi-continuity of $\mathcal{D}$-pullback attractors

Let $\bigvee$ be a metric space, and $\{\mathcal{A}^\lambda\}_{\lambda\in\bigvee}$ is a family of subsets in X. Then it is said that $\{\mathcal{A}^\lambda\}$ has the property of upper semi-continuity as $\lambda \to \lambda_0$ in X if

$$\lim_{\lambda\to\lambda_0} \text{dist}_X(\mathcal{A}^\lambda, \mathcal{A}^{\lambda_0}) = 0.$$

The upper semi-continuity of attractors and related conclusions can be referred to [1, 19, 20, 22] for more details.

In the following way, we intend to establish some results on the convergence between $\mathcal{D}$-pullback attractors $\mathcal{A}^\varepsilon$ to system (3.2) and global attractor $\mathcal{A}^0$ to system (3.4) as $\varepsilon \to 0$.

Theorem 3.4. *Let $U(\tau) \in N_V$, $\mathcal{A}^\varepsilon$ is the family of $\mathcal{D}$-pullback attractors of $S^\varepsilon(t,\tau)$ in N_V to system* (3.2), *and $\mathcal{A}^0$ is the global attractor of $S^0(t-\tau)$ in N_V to system* (3.4). *Then the robustness of system is obtained by the following upper semi-continuity*

$$\lim_{\varepsilon\to 0} dist_{N_V}(\mathcal{A}^\varepsilon, \mathcal{A}^0) = 0.$$

Proof. See Section 5.

4. Pullback dynamics

4.1. Theory of dynamics

In this section, we first give the fundamental theory of attractors for dissipative systems, and the related conclusions can be seen in [1–3, 7].

• Some relevant definitions

Definition 4.1. *Assume that $\mathcal{P}(X)$ is the family of all nonempty subsets in a metric space X. If $\mathcal{D}$ is some nonempty class of families in the form $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $D(t) \subset X$ is nonempty and bounded, then $\mathcal{D}$ is said to be a universe in $\mathcal{P}(X)$.*

Definition 4.2. *The family $\widehat{D_0} = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is $\mathcal{D}$-pullback absorbing for the process $S(\cdot,\cdot)$ on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that*

$$S(t,\tau)D(\tau) \subset D_0(t),\ \forall\, \tau \leq \tau_0(t, \widehat{D}).$$

Definition 4.3. *A process $S(\cdot,\cdot)$ on X is said to be $\mathcal{D}$-pullback asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$, the sequence $\{S(t,\tau_n)x_n\}$ is relatively compact in X.*

The $\mathcal{D}$-pullback asymptotic compactness can be characterized by the Kuratowski measure of noncompactness $\kappa(B)$ $(B \subset X)$, relating definition and properties can be referred to [25, 26], and the definition of $\mathcal{D} - \kappa$-pullback contraction will be given as follows.

Definition 4.4. *For any $t \in \mathbb{R}$ and $\varepsilon > 0$, a process $S(t,\tau)$ on X is said to be $\mathcal{D}-\kappa$-pullback contracting if there exists a constant $T_{\mathcal{D}}(t,\varepsilon) > 0$ such that*

$$\kappa(S(t,t-\tau)D(t-\tau)) \leq \varepsilon,\ \forall\, \tau \geq T_{\mathcal{D}}(t,\varepsilon).$$

Definition 4.5. *A family $\mathcal{A}(t) = \{A(t)\}_{t\in\mathbb{R}}$ is called the $\mathcal{D}$-pullback attractors of process $S(t,\tau)$, if for any $t \in \mathbb{R}$ and any $\{D(t)\} \in \mathcal{D}$, the following properties hold.*

(i) $A(t)$ is compact in X.

(ii) $S(t,\tau)A(\tau) = A(t)$, $t \geq \tau$.

(iii) $\lim\limits_{\tau\to-\infty} \mathrm{dist}_X(S(t,\tau)D(\tau), A(t)) = 0.$

In addition, $\mathcal{D}$-pullback attractor $\mathcal{A}$ is said to be minimal if whenever $\hat{C}$ is another $\mathcal{D}$-attracting family of closed sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

• Some conclusions

Theorem 4.6. *([1, 27]) Let $S(\cdot,\cdot) : \mathbb{R}^2_d \times X \to X$ be a continuous process, where $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 |\, t \geq \tau\}$, $\mathcal{D}$ is a universe in $\mathcal{P}(X)$, and a family $\widehat{D_0} = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is $\mathcal{D}$-pullback absorbing for $S(\cdot,\cdot)$, which is $\mathcal{D}$-pullback asymptotically compact. Then, the family of $\mathcal{D}$-pullback attractors $\mathcal{A}_{\mathcal{D}} = \{A_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ exists and*

$$A_{\mathcal{D}}(t) = \bigcap_{s\geq 0} \overline{\bigcup_{\tau\geq s} S(t,t-\tau)D(t-\tau)}^{X},\ t \in \mathbb{R}.$$

Remark 4.1. *If $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}$ is minimal family of closed subsets attracting pullback to $\mathcal{D}$. It is said to be unique provided that $\widehat{D}_0 \in \mathcal{D}$, $D_0(t)$ is closed for any $t \in \mathbb{R}$, and $\mathcal{D}$ is inclusion closed.*

Theorem 4.7. *([21]) Assume that $\tilde{\mathcal{D}} = \{\widehat{\tilde{D}}(t)\}$ is a family of sets in X, $S(\cdot,\cdot)$ is continuous, and, for any $t \in \mathbb{R}$, there exists a constant $T(t, \mathcal{D}, \tilde{\mathcal{D}})$ such that*

$$S(t, t-\tau)D(t-\tau) \subset \tilde{D}(t),\ \forall\, \tau \geq T(t, \mathcal{D}, \tilde{\mathcal{D}}).$$

If $S(\cdot,\cdot)$ is $\mathcal{D}$-pullback absorbing and $\hat{\mathcal{D}} - \kappa$-pullback contracting, then the $\mathcal{D}$-pullback attractors $\mathcal{A}_{\mathcal{D}} = \{A_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ exist for $S(\cdot,\cdot)$.

Lemma 4.8. *([28]) Assume that $S(\cdot,\cdot) = S_1(\cdot,\cdot) + S_2(\cdot,\cdot)$, $\tilde{\mathcal{D}} = \{\widehat{\tilde{D}}(t)\}$ is a family of subsets in X, and for any $t \in \mathbb{R}$ and any $\tau \in \mathbb{R}^+$ there hold*

(i) For any $u(t-\tau) \in \tilde{D}(t-\tau)$,

$$\|S_1(t, t-\tau)u(t-\tau)\|_X \leq \Phi(t, \tau) \to 0\ (\tau \to +\infty).$$

(ii) For any $T \geq \tau$, $\cup_{0\leq\tau\leq T} S_2(t, t-\tau)\tilde{D}(t-\tau)$ is bounded, and $S_2(t, t-\tau)\tilde{D}(t-\tau)$ is relatively compact in X.
Then $S(\cdot,\cdot)$ is $\hat{\mathcal{D}} - \kappa$-pullback contracting in X.

4.2. Proof of Theorem 3.3

From Theorems 3.2, we know that the system (3.2) generates a continuous process $S^{\varepsilon}(t,\tau)$ in N_V. To obtain the $\mathcal{D}$-pullback attractors, we need to establish the existence of $\mathcal{D}$-pullback absorbing set and the $\mathcal{D}$-pullback asymptotic compactness of $S^{\varepsilon}(t,\tau)$.

- Existence of $\mathcal{D}$-pullback absorbing set in N_V

Let $\mathcal{D}$ denote a family of all $\{D(t)\}_{t\in\mathbb{R}} \subset \mathcal{P}(N_V)$ satisfying

$$\lim_{\tau\to-\infty} e^{\omega\tau} \sup_{U(\tau)\in D(\tau)} J(2|U(\tau)|^2) = 0,$$

where $\omega = \omega_{3/4,1,4,3,f_\varepsilon} > 0$ and $J(\cdot) = J_{3/4,1,4,3,f_\varepsilon}(\cdot)$. Next, we establish the existence of $\mathcal{D}$-pullback absorbing set.

Lemma 4.9. *Let $(u(\tau), \eta^{\tau}) \in N_V$, then the process $\{S^{\varepsilon}(t,\tau)\}$ to system* (3.2) *possesses a $\mathcal{D}$-pullback absorbing set $\widehat{D_0^{\varepsilon}}(t) = \{D_0^{\varepsilon}(t)\}_{t\in\mathbb{R}}$ in N_V, where*

$$D_0^{\varepsilon}(t) = \bar{B}_{N_V}(0, \rho^{\varepsilon}_{N_V}(t)),$$

with radius

$$\rho^{\varepsilon}_{N_V}(t) \quad = \quad \sqrt{2}\sqrt{\Lambda_{3/4,1,4,3,f_\varepsilon}(2C(|f_1|^2 + \varepsilon K) + 1)}. \tag{4.1}$$

Proof. Multiplying (3.2) by u, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{V_\alpha}^2 + \int_0^\infty \mu(s)A\eta(s)u(t)ds + \beta|u|^2 = (P_L f_\varepsilon, u), \tag{4.2}$$

that is

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|_{V_\alpha}^2 + \int_0^\infty \mu(s)A\eta(s)(\partial_t\eta(s) + \partial_s\eta(s))ds + \beta|u|^2 \\ =& \frac{1}{2}\frac{d}{dt}(\|u\|_{V_\alpha}^2 + \|\eta\|_{M_V}^2) + \frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\|\eta\|^2 ds + \beta|u|^2 \le |(f_\varepsilon, u)|. \end{aligned} \tag{4.3}$$

Multiplying (3.2) by u_t, we have

$$\|u_t\|_{V_\alpha}^2 + ((\eta, u_t))_{M_V} + \frac{1}{2}\beta\frac{d}{dt}|u|^2 + b(u, u, u_t) = (P_L f_\varepsilon, u_t). \tag{4.4}$$

Then, the interpolation inequality and Young inequality lead to

$$\begin{aligned} &\beta\frac{d}{dt}|u|^2 + 2\|u_t\|_{V_\alpha}^2 \\ &\le 2|((\eta, u_t))_{M_V}| + 2|(f_\varepsilon, u_t)| + 2|b(u, u, u_t)| \\ &\le 2|((\eta, u_t))_{M_V}| + 2|(f_\varepsilon, u_t)| + C\|u\|_{L^3}\|u\|\|u\|_{L^6} \\ &\le 2|((\eta, u_t))_{M_V}| + 2|(f_\varepsilon, u_t)| + C|u|^{1/2}\|u\|^{1/2}\|u\|\|u\|_{L^6} \\ &\le \alpha\|u_t\|^2 + C\|\eta\|_{M_V}^2 + C|u|\|u\|^3 + C|f_\varepsilon|^2. \end{aligned} \tag{4.5}$$

To estimate the term $\int_0^\infty \mu(s)\frac{d}{ds}\|\eta\|^2 ds$ in (4.3) and avoid the possible singularity of μ at zero, we refer to [18] and construct the following new function

$$\tilde{\mu}(s) = \begin{cases} \mu(\tilde{s}), 0 < s \le \tilde{s}, \\ \mu(s), s > \tilde{s}, \end{cases}$$

where $\tilde{s}$ is fixed such that $\int_0^{\tilde{s}} \mu(s)ds \le \kappa/2$. Also, if we set

$$\Phi(t) = \frac{-4}{\kappa}\int_0^\infty \tilde{\mu}(s)((\eta(s), u(t)))ds,$$

then differentiating in t leads to

$$\frac{d}{dt}\Phi(t) + \|u\|^2 \le \frac{4\mu(\tilde{s})}{\kappa^2}\int_0^\infty \mu(s)\frac{d}{ds}\|\eta\|^2 ds + \frac{4}{\alpha\kappa\varepsilon}\|\eta\|_{M_V}^2 + \alpha\varepsilon\|u_t\|^2. \tag{4.6}$$

We use the technique in [18] and set

$$y_\varepsilon(t) = E(t) + \nu\varepsilon\Phi(t) + \varepsilon^2\Psi(t),$$

where

$$E(t) = \frac{1}{2}(\|u\|_{V_\alpha}^2 + \|\eta\|_{M_V}^2),\ \Psi(t) = 2\beta|u|^2.$$

For sufficient small ε, it leads to

$$E(t) \le 2y_\varepsilon(t) \le 2(E(t)+1),$$

where we choose ε_0 satisfying

$$\nu\varepsilon_0 \sup_{t\in[\tau,T]} \Phi(t) + \varepsilon_0^2 \sup_{t\in[\tau,T]} \Psi(t) = 1,$$

and $0 \le \varepsilon < \varepsilon_0$. Then, there holds

$$\frac{d}{dt}y_\varepsilon(t) + C\varepsilon y_\varepsilon(t) \le C\varepsilon^4 y_\varepsilon(t)^3 + C|f_\varepsilon|^2, \tag{4.7}$$

and

$$\begin{aligned} y_\varepsilon(t) &\le y_\varepsilon(\tau)e^{-\varepsilon(t-\tau)} + C\varepsilon^4\int_\tau^t e^{-\varepsilon(t-s)}\cdot 1\cdot y_\varepsilon(s)^3 ds + C\sup_{t\in\mathbb{R}}\int_t^{t+1}|f_\varepsilon|^2\varepsilon^{-1}ds \\ &\le y_\varepsilon(\tau)e^{-\varepsilon(t-\tau)} + C\varepsilon^4\int_\tau^t e^{-\varepsilon(t-s)}\cdot 1\cdot y_\varepsilon(s)^3 ds + C(|f_1|^2+\varepsilon K)\varepsilon^{-1}. \end{aligned} \tag{4.8}$$

Then by Lemma 2.1, there exist

$$\omega = \omega_{3/4,1,4,3,f_\varepsilon} > 0,\ \Lambda = \Lambda_{3/4,1,4,3,f_\varepsilon} > 0,$$

and an increasing function

$$J(\cdot) = J_{3/4,1,4,3,f_\varepsilon}(\cdot):\ \mathbb{R}^+ \to \mathbb{R}^+$$

such that

$$E(t) \le J(2E(\tau))e^{-\omega(t-\tau)} + \Lambda(2C(|f_1|^2+\varepsilon K)+1),$$

which implies the conclusion holds.

Remark 4.2. *For the semigroup $S^0(t-\tau)$, it has the global absorbing set D_0^0 in N_V, where*

$$D_0^0 = \{U \in N_V;\ \|U\|_{N_V} \le \rho_{N_V}^0 = \sqrt{2}\sqrt{\Lambda_{3/4,1,4,3,f_1}(2C|f_1|^2+1)}\} \tag{4.9}$$

and

$$\limsup_{\varepsilon\to 0}\rho_{N_V}^\varepsilon(t) = \rho_{N_V}^0. \tag{4.10}$$

- $\mathcal{D}-\kappa$-pullback contraction of $S^\varepsilon(t,\tau)$ in N_V

To verify the pullback contraction of $S^\varepsilon(t,\tau)$, we decompose $S^\varepsilon(t,\tau)$ as follows

$$S^\varepsilon(t,\tau)U(\tau) = S_1^\varepsilon(t-\tau)U_1(\tau) + S_2^\varepsilon(t,\tau)U_2(\tau) =: U_1(t) + U_2(t),$$

which solve the following two problems respectively

$$\begin{cases} \frac{\partial}{\partial t}(u_1+\alpha Au_1) + \int_0^\infty \mu(s)A\eta_1(s)ds + B(u,u_1) = 0,\ (t,x)\in\Omega_\tau, \\ \frac{\partial}{\partial t}\eta_1 = T\eta_1 + u_1, \\ \mathrm{div} u_1 = 0,\ (t,x)\in\Omega_\tau, \\ u_1(t,x) = 0,\ (t,x)\in\partial\Omega_\tau, \\ u_1(\tau,x) = u(\tau),\ x\in\Omega, \\ \eta_1^\tau(s) = \int_0^s \varphi(\sigma)d\sigma, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \frac{\partial}{\partial t}(u_2+\alpha Au_2)+\int_0^\infty \mu(s)A\eta_2(s)ds+B(u,u_2)+\beta u_2=P_Lf_\varepsilon-\beta u_1,\ (t,x)\in\Omega_\tau, \\ \frac{\partial}{\partial t}\eta_2=T\eta_2+u_2, \\ \mathrm{div}u_2=0,\ (t,x)\in\Omega_\tau, \\ u_2(t,x)=0,\ (t,x)\in\partial\Omega_\tau, \\ u_2(\tau,x)=0,\ x\in\Omega, \\ \eta_2^\tau(s)=0. \end{cases} \tag{4.12}$$

Lemma 4.10. *Let $U(\tau)\in D_0^\varepsilon(\tau)$, then the solution $S_1^\varepsilon(t-\tau)U(\tau)$ to the system* (4.11) *satisfies*

$$\|S_1^\varepsilon(t-\tau)U(\tau)\|_{N_V}\le J(2E(\tau))e^{-\omega(t-\tau)}\to 0\ (\tau\to-\infty).$$

Proof. Multiplying (4.11) by u_1 and $\frac{\partial}{\partial t}u_1$ respectively, and repeating the reasonings as shown as in Lemma 4.9, in which $\beta=0$ and $f_\varepsilon=0$, we can derive the conclusion finally. The parameter ω is dependent on ε and the increasing function $J(*)$ is different from the one in Lemma 4.9. Despite all this, these parameters can be unified in same representation, and the concrete details are omitted here.

Lemma 4.11. *Let $U(\tau)\in D_0^\varepsilon(\tau)$, then for any $t\in\mathbb{R}$, there exists $C^\varepsilon(t)>0$ such that the solution $S_2^\varepsilon(t,\tau)U(\tau)$ to the system* (4.12) *satisfies*

$$\|S_2^\varepsilon(t,\tau)U(\tau)\|_{N_W}\le C^\varepsilon(t).$$

Proof. Multiplying (4.12) by Au_2, we have

$$\frac{1}{2}\frac{d}{dt}(\|u_2\|^2+\alpha|Au_2|^2)+\int_0^\infty \mu(s)A\eta_2(s)Au_2(t)ds+\beta\|u_2\|^2+b(u,u_2,Au_2)=(P_Lf_\varepsilon-\beta u_1,Au_2), \tag{4.13}$$

from the existence of pullback absorbing set, Lemma 4.10, the interpolation inequality and Young inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u_2\|^2+\alpha|Au_2|^2+\|\eta_2\|_{M_W}^2)+\frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}|A\eta_2|^2ds+\beta\|u_2\|^2 \\ \le\ &|(f_\varepsilon,Au_2)|+|(\beta u_1,Au_2)|+\|u\|_{L^6}\|\nabla u_2\|_{L^3}|Au_2| \\ \le\ &\frac{\nu\varepsilon}{4}|Au_2|^2+C\|u\|\|u_2\|^{1/2}|Au_2|^{1/2}|Au_2|+C|f_\varepsilon|^2 \\ \le\ &\frac{\nu\varepsilon}{2}|Au_2|^2+C|f_\varepsilon|^2+C. \end{aligned} \tag{4.14}$$

Multiplying (4.12) by $A\partial_t u_2$, we have

$$\|\partial_t u_2\|^2+\alpha|A\partial_t u_2|^2+((\eta_2,\partial_t u_2))_{M_W}+\frac{1}{2}\beta\frac{d}{dt}\|u_2\|^2+b(u,u_2,\partial_t u_2)=(P_Lf_\varepsilon-\beta u_1,A\partial_t u_2). \tag{4.15}$$

By the existence of pullback absorbing set, Lemma 4.10 and Young inequality, one has

$$
\begin{aligned}
&\beta\frac{d}{dt}\|u_2\|^2 + 2\|\partial_t u_2\|^2 + 2\alpha|A\partial_t u_2|^2\\
&\le 2|((\eta_2, \partial_t u_2))_{M_W}| + 2|(P_L f_\varepsilon - \beta u_1, A\partial_t u_2)| + 2|b(u, u_2, \partial_t u_2)|\\
&\le 2|((\eta_2, \partial_t u_2))_{M_W}| + 2|(P_L f_\varepsilon - \beta u_1, A\partial_t u_2)| + C|Au_2||A\partial_t u_2|\\
&\le \alpha|A\partial_t u_2|^2 + C\|\eta_2\|^2_{M_W} + C|Au_2|^2 + C|f_\varepsilon|^2. \qquad (4.16)
\end{aligned}
$$

To estimate the term $\int_0^\infty \mu(s)\frac{d}{ds}|A\eta_2|^2 ds$ in (4.14), we set

$$
\Phi_2(t) = \frac{-6}{\kappa}\int_0^\infty \tilde{\mu}(s)(A\eta_2(s), Au_2(t))ds,
$$

and differentiating in t leads to

$$
\begin{aligned}
&\frac{d}{dt}\Phi_2(t) + \frac{6}{\kappa}\int_0^\infty \tilde{\mu}(s)(Au_2(t), Au_2(t))ds\\
\le& -\frac{6}{\kappa}\int_0^\infty \tilde{\mu}(s)(-A\partial_s\eta_2, Au_2(t))ds - \frac{6}{\kappa}\int_0^\infty \tilde{\mu}(s)(A\eta_2(s), A\partial_t u_2(t))ds, \qquad (4.17)
\end{aligned}
$$

where

$$
\frac{6}{\kappa}\int_0^\infty \tilde{\mu}(s)(Au_2(t), Au_2(t))ds \ge \frac{6}{\kappa}\int_{\tilde{s}}^\infty \tilde{\mu}(s)ds\cdot|Au_2(t)|^2, \qquad (4.18)
$$

and

$$
\begin{aligned}
&-\frac{6}{\kappa}\int_0^\infty \tilde{\mu}(s)(-A\partial_s\eta_2, Au_2(t))ds = -\frac{6}{\kappa}\int_{\tilde{s}}^\infty \mu'(s)(A\eta_2, Au_2(t))ds\\
&\le \frac{6}{\kappa}\int_{\tilde{s}}^\infty -\mu'(s)|A\eta_2||Au_2(t)|ds \le \frac{6}{\kappa}\Big(\int_{\tilde{s}}^\infty -\mu'(s)|A\eta_2|^2 ds\Big)^{1/2}\Big(\int_{\tilde{s}}^\infty -\mu'(s)|Au_2(t)|^2 ds\Big)^{1/2}\\
&\le \frac{6}{\kappa}\Big(\int_{\tilde{s}}^\infty \mu(s)\frac{d}{ds}|A\eta_2|^2 ds\Big)^{1/2}(2\mu(\tilde{s}))^{1/2}|Au_2(t)|\\
&\le |Au_2(t)|^2 + \frac{18\mu(\tilde{s})}{\kappa^2}\int_{\tilde{s}}^\infty \mu(s)\frac{d}{ds}|A\eta_2|^2 ds, \qquad (4.19)
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{6}{\kappa}\int_0^\infty \tilde{\mu}(s)(A\eta_2(s), A\partial_t u_2(t))ds &\le \frac{6}{\kappa}\int_0^\infty \mu(s)|A\eta_2(s)||A\partial_t u_2(t)|ds\\
&\le \alpha\varepsilon|A\partial_t u_2(t)|^2 + \frac{9}{\alpha\kappa^2\varepsilon}\|\eta\|^2_{M_W}. \qquad (4.20)
\end{aligned}
$$

Also, from the fact that $\mu'(s) + \delta\mu(s) \le 0$, we have

$$
\int_0^\infty \mu(s)\frac{d}{ds}|A\eta_2|^2 ds \ge \int_0^\infty \delta\mu(s)|A\eta_2|^2 ds = \delta\|\eta_2\|^2_{M_W}. \qquad (4.21)
$$

Thus

$$\frac{d}{dt}\Phi_2(t) + 2|Au_2|^2 \leq \frac{18\mu(\tilde{s})}{\kappa^2}\int_0^\infty \mu(s)\frac{d}{ds}|A\eta_2|^2 ds + \frac{9}{\alpha\kappa\varepsilon}\|\eta_2\|_{M_W}^2 + \alpha\varepsilon|A\partial_t u_2|^2. \tag{4.22}$$

We use the technique in [18] and set

$$z_\varepsilon(t) = E_2(t) + \nu\varepsilon\Phi_2(t) + \varepsilon^2\Psi_2(t),$$

where

$$E_2(t) = \frac{1}{2}(\|u_2\|^2 + \alpha|Au_2|^2 + \|\eta_2\|_{M_W}^2),\ \Psi_2(t) = \beta\|u\|^2.$$

For sufficient small enough ε, it leads to

$$E_2(t) \leq 2z_\varepsilon(t) \leq 2(E_2(t) + 1),$$

and there holds

$$\frac{d}{dt}z_\varepsilon(t) + \nu\varepsilon z_\varepsilon(t) \leq C + C|f_\varepsilon|^2, \tag{4.23}$$

it follows from the Gronwall lemma that

$$\begin{aligned} E_2(t) &\leq Ce^{-\nu\varepsilon(t-\tau)}E_2(\tau) + C\varepsilon\int_\tau^t e^{\nu\varepsilon(s-t)}|f_2(s,x)|^2 ds + C|f_1|^2 + C \\ &\leq C\varepsilon e^{-\nu\varepsilon t}\int_\tau^t e^{\nu\varepsilon_0 s}|f_2(s,x)|^2 ds + C|f_1|^2 + C, \end{aligned} \tag{4.24}$$

which means the conclusion holds.

Above all, Lemmas 4.10, 4.11 and 4.8 lead to

Lemma 4.12. *Let $U(\tau) \in N_V$, then the process $S^\varepsilon(t,\tau) : N_V \to N_V$ generated by the system* (3.2) *is $\mathcal{D} - \kappa$-pullback contracting in N_V.*

Consequently, from Theorem 4.7, we can finish the proof of Theorem 3.3.

5. Robustness

5.1. Theories on robustness

By the definition of upper semi-continuity, the following lemmas can be used to obtain the robustness of pullback attractors for evolutionary systems.

Lemma 5.1. *([20]) Let $\varepsilon \in (0,\varepsilon_0]$, $\{S^\varepsilon(t,\tau)\}$ is the process of evolutionary system with non-autonomous term (depending on ε), which is obtained by perturbing the semigroup $S^0(\tau)$ of system without ε, and, for any $t \in \mathbb{R}$, there also hold that*

(i) $S^\varepsilon(t,\tau)$ has the pullback attractors $\mathcal{A}^\varepsilon(t)$, and $\mathcal{A}^0$ is the global attractor for $S^0(\tau)$.

(ii) For any $\tau \in \mathbb{R}^+$ and any $u \in X$, there holds uniformly that

$$\lim_{\varepsilon\to 0} d_X(S^\varepsilon(t, t-\tau)u, S^0(\tau)u) = 0.$$

(iii) There exists a compact subset $G \subset X$ such that

$$\lim_{\varepsilon\to 0} dist_X(A^\varepsilon(t), G) = 0.$$

Then, for any $t \in \mathbb{R}$, there holds

$$\lim_{\varepsilon\to 0} dist_X(\mathcal{A}^\varepsilon, \mathcal{A}^0) = 0.$$

Lemma 5.2. *([21]) For any $t \in \mathbb{R}$, $\tau \in \mathbb{R}^+$, and $\varepsilon \in (0, \varepsilon_0]$, $\widehat{D_0^\varepsilon}(t) = \{D_0^\varepsilon(t) : t \in \mathbb{R}\}$ is the pullback absorbing set for $S^\varepsilon(t,\tau)$, and $\widehat{C_0^\varepsilon}(t) = \{C_0^\varepsilon(t) : t \in \mathbb{R}\}$ is a family of compact subsets in X. Assume that $S^\varepsilon(\cdot,\cdot) = S_1^\varepsilon(\cdot,\cdot) + S_2^\varepsilon(\cdot,\cdot)$, and there hold*

(i) For any $u_{t-\tau} \in D_0^\varepsilon(t-\tau)$,

$$\|S_1^\varepsilon(t, t-\tau)u_{t-\tau}\|_X \le \Phi(t,\tau) \to 0 \ (\tau \to \infty).$$

(ii) For any $T \ge \tau$, $\cup_{0\le\tau\le T} S_2^\varepsilon(t, t-\tau)D_0^\varepsilon(t-\tau)$ is bounded, and there exists a constant $T_{\mathcal{D}_0^\varepsilon}(t)$, independent of ε, such that

$$S_2^\varepsilon(t, t-\tau)D_0^\varepsilon(t-\tau) \subset C_0^\varepsilon(t), \ \forall\, \tau > T_{\mathcal{D}_0^\varepsilon}(t).$$

(iii) There is a compact subset $G \subset X$ such that

$$\lim_{\varepsilon\to 0} dist_X(C_0^\varepsilon(t), G) = 0.$$

Then, the process $S^\varepsilon(t,\tau)$ has the pullback attractors $\mathcal{A}^\varepsilon(t)$, and

$$\lim_{\varepsilon\to 0} dist_X(\mathcal{A}^\varepsilon, G) = 0.$$

5.2. Proof of Theorem 3.4

We give the following procedure to verify Theorem 3.4.

Lemma 5.3. *Let $(u^\varepsilon, \eta^\varepsilon) = S^\varepsilon(t,\tau)U(\tau)$ be the solution to system* (3.2)*, and $(u,\eta) = S^0(t-\tau)U(\tau)$ is the solution to system* (3.4)*, then, for any bounded subset $B \subset N_V$, there holds*

$$\lim_{\varepsilon\to 0} \sup_{U(\tau)\in B} d_{N_V}(S^\varepsilon(t,\tau)U(\tau), S^0(t-\tau)U(\tau)) = 0.$$

Proof. We know

$$\begin{cases} \frac{\partial}{\partial t}(u^\varepsilon + \alpha A u^\varepsilon) + \int_0^\infty \mu(s) A\eta^\varepsilon(s) ds + B(u^\varepsilon, u^\varepsilon) + \beta u^\varepsilon = P_L f_\varepsilon(t,x), \ (t,x) \in \Omega_\tau, \\ \frac{\partial}{\partial t}\eta^\varepsilon = T\eta^\varepsilon + u^\varepsilon, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \frac{\partial}{\partial t}(u + \alpha Au) + \int_0^\infty \mu(s)A\eta(s)ds + B(u, u) + \beta u = P_L f_1(x), \ (t, x) \in \Omega_\tau, \\ \frac{\partial}{\partial t}\eta = T\eta + u. \end{cases} \tag{5.2}$$

Let $w^\varepsilon = u^\varepsilon - u$ and $\xi^\varepsilon = \eta^\varepsilon - \eta$, we can derive

$$\frac{\partial}{\partial t}(w^\varepsilon + \alpha Aw^\varepsilon) + \int_0^\infty \mu(s)A\xi^\varepsilon(s)ds + B(u^\varepsilon, w^\varepsilon) + B(w^\varepsilon, u) + \beta w^\varepsilon = \varepsilon P_L f_2(t, x), \tag{5.3}$$

and multiplying it by w^ε leads to

$$\frac{1}{2}\frac{d}{dt}\|w^\varepsilon\|_\alpha^2 + \int_0^\infty \mu(s)(A\xi^\varepsilon(s), w^\varepsilon(t))ds + b(w^\varepsilon, u, w^\varepsilon) + \beta|w^\varepsilon|^2 = \varepsilon(P_L f_2(t, x), w^\varepsilon), \tag{5.4}$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\|w^\varepsilon\|_\alpha^2 + \int_0^\infty \mu(s)((\xi^\varepsilon(s), w^\varepsilon))ds \le |b(w^\varepsilon, u, w^\varepsilon)| + \varepsilon|(f_2(t, x), w^\varepsilon)|. \tag{5.5}$$

Integrating (5.5) over $[\tau, t]$, from Lemma 2.2 we derive that

$$\begin{aligned} &\|w^\varepsilon(t)\|_\alpha^2 + \|\xi^\varepsilon(t-s)\|_{M_V}^2 \\ \le\ &\|w^\varepsilon(\tau)\|_\alpha^2 + \|\xi^\varepsilon(\tau - s)\|_{M_V}^2 + 2\int_\tau^t |b(w^\varepsilon, u, w^\varepsilon)|ds + 2\varepsilon\int_\tau^t |(f_2(t, x), w^\varepsilon)|ds \\ \le\ &\|(w^\varepsilon, \xi^\varepsilon)|_\tau\|_{N_V}^2 + C\int_\tau^t \|u\|\|w^\varepsilon\|^2 ds + 2\varepsilon\int_\tau^t |(f_2(t, x), w^\varepsilon)|ds \\ \le\ &\|(w^\varepsilon, \xi^\varepsilon)|_\tau\|_{N_V}^2 + \varepsilon^2\int_\tau^t |f_2|^2 ds + C\int_\tau^t \|w^\varepsilon\|^2 ds, \end{aligned} \tag{5.6}$$

that is

$$\|(w^\varepsilon, \xi^\varepsilon)|_t\|_{N_V}^2 \le \|(w^\varepsilon, \xi^\varepsilon)|_\tau\|_{N_V}^2 + \varepsilon^2\int_\tau^t |f_2|^2 ds + C\int_\tau^t \|(w^\varepsilon, \xi^\varepsilon)|_s\|_{N_V}^2 ds, \tag{5.7}$$

and the Gronwall inequality leads to

$$\|(w^\varepsilon, \xi^\varepsilon)|_t\|_{N_V}^2 \le C(\|(w^\varepsilon, \xi^\varepsilon)|_\tau\|_{N_V}^2 + \varepsilon^2\int_\tau^t |f_2|^2 ds) \to 0\ (\varepsilon \to 0), \tag{5.8}$$

which means that the conclusion is finished.

Proof of Theorem 3.4. From (4.24) and the fact that $W \hookrightarrow V$ is compact, we know that there exists a compact subset $G \subset N_V$ such that

$$\lim_{\varepsilon\to 0} \mathrm{dist}_X(C_0^\varepsilon(t), G) = 0. \tag{5.9}$$

Combining Lemma 4.10, Lemma 5.2 and (5.9), we have

$$\lim_{\varepsilon\to 0} dist_X(\mathcal{A}^\varepsilon, G) = 0.$$

In addition, the confirmation of condition (*ii*) in Lemma 5.1 is finished from Lemma 5.3, and we have

$$\lim_{\varepsilon\to 0} dist_X(\mathcal{A}^\varepsilon, \mathcal{A}^0) = 0.$$

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Conflict of interest

The authors declare there is no conflict of interest.

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