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http://www.aimspress.com/journal/era

ERA, 31(2): 904–927.
DOI: 10.3934/era.2023045
Received: 06 October 2022
Revised: 23 November 2022
Accepted: 24 November 2022
Published: 05 December 2022

Research article

Dynamics of stochastic 3D Brinkman-Forchheimer equations on unbounded domains

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Abstract: This paper is concerned with the asymptotic behavior of the stochastic three dimensional Brinkman-Forchheimer equations in some unbounded domains. We first define a continuous random dynamical system for the equations. Then by J. Ball's idea of energy equations, we obtain pullback asymptotic compactness of solutions and prove that the existence of a unique random attractor for the equations.

Keywords: stochastic Brinkman-Forchheimer equation; unbounded domains; random attractors

1. Introduction

Fluid flowing in porous media is widely found in nature. It is a branch of various engineering and disciplines, involving exploration and exploitation of various underground fluid resources such as oil, natural gas and coalbed methane. The Brinkman-Forchheimer equation is a mathematical model that describes the motion of fluids in saturated porous media, so it has been an active research frontier in recent decades. The asymptotic behavior of the deterministic Brinkman-Forchheimer equations has been widely studied. For example, in the autonomous case, B. Wang and S. Lin [1] and D. Ugurlu [2] proved the existence of global attractors for the 3D Brinkman-Forchheimer equation. Moreover, X. G. Yang [3] studied the structure and stability of pullback attractors for three dimensional Brinkman-Forchheimer equation with delay. The uniform attractors for the non-autonomous Brinkman-Forchheimer equation with delay were obtained by Kang in [4]. In addition, the trajectory attractor and the approximation for the convective Brinkman-Forchheimer equations were obtained by C. Zhao et al. in [5, 6].

During the past two decades, the mathematical theories of random dynamical systems [7] have made substantial progress in describing the asymptotic behavior of solutions for some dissipative dynamical systems. For example, in [8–12], the authors have considered the asymptotic behavior of solutions for some dissipative random dynamical systems. In particular, the existence of attractors on unbounded domains has been studied extensively by many authors, see, e.g., [9, 10, 13–18]. Since Sobolev embeddings are no longer compact on unbounded domains, this is the main difficulty in proving the existence of attractors of equations defined on unbounded domains.

In [14], we studied the asymptotic behavior of the stochastic non-autonomous Brinkman-Forchheimer equations driven by linear multiplicative noise in unbounded domains. The existence of random attractors was obtaibed by transforming the stochastic equation into a pathwise random one. Comparing with [14], if we study the asymptotic behavior of stochastic Brinkman-Forchheimer equations driven by additive noise, different transformations will usually be used. This transformation will lead to more difficult calculations when we prove that the uniform estimates and the pullback asymptotic compactness of the solutions.

In this paper, we consider the following stochastic Brinkman-Forchheimer equations with additive noise:

$$\begin{cases} u_t - \nu\Delta u + \alpha u + \beta|u|u + \gamma|u|^2u + \nabla p = g(x) + h\dfrac{dw}{dt}, & (t,x) \in (0,T) \times O, \\ \nabla \cdot u = 0, & (t,x) \in (0,T) \times O, \\ u(x,t) = 0, & (t,x) \in (0,T) \times \partial O, \\ u(x,0) = u_0(x), & x \in O, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the unknown velocity vector. $p = p(x,t)$ is the unknown pressure. $\nu > 0$ and $\alpha > 0$ denote the Brinkman kinematic viscosity and the Darcy coefficient respectively. $\beta > 0$ and $\gamma > 0$ are the Forchheimer coefficients. $g(x)$ is a force field. $h \in (H^2(O))^3 \bigcap (H_0^1(O))^3$ and $w(t), t \in \mathbb{R}$ is a two-sided real-valued Wiener process on a probability space. The domian $O \subset \mathbb{R}^3$ can be an arbitrary open set (bounded or unbounded) with smooth boundary ∂O, and it satisfies the Poincaré inequality: there exists a constant $\lambda_1 > 0$ such that

$$\int_O |\nabla\varphi|^2 dx \geq \lambda_1 \int_O |\varphi|^2 dx, \quad \forall\, \varphi \in H_0^1(O). \tag{1.2}$$

The purpose of this article is to study the asymptotic behavior of the 3D stochastic Brinkman-Forchheimer equation (1.1) on unbounded domains. We first establish a continuous random dynamical system for (1.1), see (3.46). To this end, we need to convert (1.1) into a deterministic equation (with a random parameter) (3.6) and (3.7) and obtain the existence, regularity and stability of weak solution to (3.6) and (3.7), see Theorems 3.1 and 3.2. The difficulty is the convergence of the nonlinear term, and we will use a truncation argument analogously to [13, 19]. Next, we establish the existence of a unique $\mathcal{D}$-random attractor for (1.1), see Theorem 4.1. Since the Sobolev compact embeddings on unbounded domains are not compact, we will use the idea of energy equations, which was introduced by J. Ball [20]. Comparing with [10, 18], we replace the advection term $(u \cdot \nabla)u$ by the damping term $\alpha u + \beta|u|u + \gamma|u|^2u$, and deal with three dimensional case , which will be much harder to deal with.

This paper is organized as follows. Some basic concepts, a number of spaces and some inequalities are given in Section 2. Then a continuous random dynamical system for (1.1) is established in Section 3. The existence of a pullback random attractor is proved for (1.1) in Section 4. Finally, we summarize the main results and give some perspective on the next research.

2. Preliminary

We recall some basic concepts (see [7–9, 11, 12]), and introduce some spaces and inequalities.

2.1. Some basic concepts

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 2.1. *$(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{R}$ and $\theta_t(P) = P$ for all $t \in \mathbb{R}$.*

Definition 2.2. *A mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$ is called a continuous random dynamical system on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$, if ϕ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for P-a.e. $\omega \in \Omega$,*
(i) *$\phi(0, \omega, \cdot)$ is the identity on X;*
(ii) *$\phi(t + s, \omega, \cdot) = \phi(t, \theta_s\omega, \phi(s, \omega, \cdot))$ for all $t, s \in \mathbb{R}^+$;*
(iii) *$\phi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.*

Definition 2.3. *A random bounded set $\{D(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,*

$$\lim_{t\to\infty} e^{-\sigma t} d(D(\theta_{-t}\omega)) = 0 \ \text{ for all } \sigma > 0,$$

where $d(D) = sup_{x\in D}\|x\|_X$.

Definition 2.4. *Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is said to be a random absorbing set for ϕ in $\mathcal{D}$ if for every $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exsits $t_D(\omega) > 0$ such that*

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset K(\omega) \ \text{ for all } t \geq t_D(\omega).$$

Definition 2.5. *Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then ϕ is called $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X for any sequence $t_n \to +\infty$, and $x_n \in D(\theta_{-t_n}\omega)$ with any $\{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.*

Definition 2.6. *A random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for ϕ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,*
(i) *$\mathcal{A}(\omega)$ is compact and the mapping $\omega \to d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;*
(ii) *$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,*

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \ \text{ for all } t \geq 0;$$

(iii) *$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$, i.e., for every $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,*

$$\lim_{t\to+\infty} d(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

Theorem 2.1. *(see [9], Proposition 2.7) Assume that ϕ is a continuous RDS which has a random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$. If ϕ is $\mathcal{D}$-pullback asymptotically compact, then ϕ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

2.2. Some generic functional spaces and inequalities

Denote $\mathbf{L}^p(\mathcal{O}) = (L^p(\mathcal{O}))^3$ and use $\|\cdot\|_p$ to denote the norm in $\mathbf{L}^p(\mathcal{O})$. Denote $\mathcal{V} := \{u|u \in (C_0^\infty(\mathcal{O}))^3,\ \text{div}\ u = 0\}$. H is the closure of $\mathcal{V}$ in $\mathbf{L}^2(\mathcal{O})$ topology, $\|\cdot\|_H$ and $(\cdot,\cdot)$ denote the norm and inner product in H respectively, where

$$(u, v) = \sum_{i=1}^{3} \int_{\mathcal{O}} u_i(x)v_i(x)dx \quad \text{for } u, v \in \mathbf{L}^2(\mathcal{O}).$$

V is the closure of $\mathcal{V}$ in $(H_0^1(\mathcal{O}))^3$ topology, $\|\cdot\|_V$ and $((\cdot,\cdot))$ denote the norm and inner product in V respectively, where

$$((u, v)) = \sum_{i,j=1}^{3} \int_{\mathcal{O}} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx \quad \text{for } u, v \in (H_0^1(\mathcal{O}))^3.$$

By (1.2), $V \hookrightarrow H \equiv H' \hookrightarrow V'$, H' and V' are dual spaces of H and V respectively, where the injection is dense and continuous. $\|\cdot\|_*$ and $\langle\cdot,\cdot\rangle$ denote the norm in V' and the dual product between V and V' respectively.

Denote by P the Helmholtz-Leray orthogonal projection in $\mathbf{L}^2(\mathcal{O})$ onto the space H. Set $A : D(A) \subset \mathbf{L}^2(\mathcal{O}) \to \mathbf{L}^2(\mathcal{O})$, where $D(A) = (H^2(\mathcal{O}))^3 \cap V$ and $Au = -P\Delta u$.

In addition, the Ladyzhenskaya's inequality is as follows:

$$\|u\|_3 \le c\|u\|_H^{\frac{1}{2}}\|u\|_V^{\frac{1}{2}}, \quad \forall\ u \in V,$$

$$\|u\|_4 \le c\|u\|_H^{\frac{1}{4}}\|u\|_V^{\frac{3}{4}}, \quad \forall\ u \in V.$$

3. Random dynamical systems for (1.1)

In this section, we establish a continuous random dynamical system for the 3D stochastic BF equations.

Applying the Helmholtz-Leray projection P onto the first equation in (1.1), we obtain the following abstract formulation of the 3D stochastic BF equations:

$$du + (\nu Au + \alpha u + \beta|u|u + \gamma|u|^2u)dt = g(x)dt + hdw, \tag{3.1}$$

with initial datum $u(0) = u_0$.

In the following, we consider the probability space $(\Omega, \mathcal{F}, P)$ where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we will identify ω with

$$\omega(t) \equiv w(t), \quad \text{for } t \in \mathbb{R}.$$

Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Consider the one dimensional Ornstein-Uhlenbeck equation

$$dy + \mu y dt = dw, \tag{3.2}$$

where $\mu > 0$. One may easily check that a solution to (3.2) is given by

$$y(t) = y(\theta_t\omega) = -\mu \int_{-\infty}^{0} e^{\mu\tau}\theta_t\omega(\tau)d\tau, \quad t \in \mathbb{R}.$$

Note that $y(\theta_t\omega)$ is P-$a.e.$ continuous and the random variable $|y(\omega)|$ is tempered (see [7, 8, 12]). Therefore, it follows from Proposition 4.3.3 in [7] that there exists a tempered function $R(\omega) > 0$ such that

$$|y(\omega)| + |y(\omega)|^p \le R(\omega), \tag{3.3}$$

where $p \ge 2$ and $R(\omega)$ satisfies, for P-$a.e.$ $\omega \in \Omega$,

$$R(\theta_t\omega) \le e^{\frac{\lambda_1\nu}{8}|t|}R(\omega), \ t \in \mathbb{R}. \tag{3.4}$$

Then it follows from (3.3) and (3.4) that, for P-$a.e.$ $\omega \in \Omega$,

$$|y(\theta_t\omega)| + |y(\theta_t\omega)|^p \le e^{\frac{\lambda_1\nu}{8}|t|}R(\omega), \ t \in \mathbb{R}. \tag{3.5}$$

Putting $z(\theta_t\omega) = hy(\theta_t\omega)$, by (3.2) we have

$$dz + \mu z dt = h dw.$$

In addition, by $h \in (H^2(O))^3 \bigcap (H_0^1(O))^3$ and Ladyzhenskaya's inequalities, we get $z(\theta_t\omega) \in L^2(0,T;V) \cap L^3(0,T;\mathbf{L}^3(O)) \cap L^4(0,T;\mathbf{L}^4(O))$ and $Az(\theta_t\omega) \in L^\infty(0,T;H)$.

Now, let us study (3.1) by means of the classical change of variable $v(t,\omega) = u(t,\omega) - z(\theta_t\omega)$, then $v(t,\omega)$ satifies

$$\begin{cases} v_t + \nu Av + \nu Az + \alpha(v+z) + \beta|v+z|(v+z) + \gamma|v+z|^2(v+z) = g(x) + \mu z, & (3.6)\\ v_0(\omega) = u_0(\omega) - z(\omega). & (3.7) \end{cases}$$

In what follows, we give the definition of weak solutions of problems (3.6) and (3.7).

Definition 3.1. *Let $T > 0$, assume that $v_0 \in H$ and $g \in V'$. We shall say that $v(x,t) \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^3(0,T;\mathbf{L}^3(O)) \cap L^4(0,T;\mathbf{L}^4(O))$ is a weak solution to* (3.6) *and* (3.7)*, if it satisfies, for P-$a.e.$ $\omega \in \Omega$,*

$$\begin{cases} (\frac{\partial v}{\partial t}, \xi) + \nu((v,\xi)) + \nu((z,\xi)) + \alpha(v+z,\xi) + \beta(|v+z|(v+z),\xi) + \gamma(|v+z|^2(v+z),\xi) \\ = \langle g(x), \xi\rangle + (\mu z, \xi), \\ v(x,0) = v_0, \end{cases} \tag{3.8}$$

where (3.8) *holds for all $\xi \in V$ in the sense of $D'(0,T)$.*

Since (3.6) and (3.7) is a deterministic equation with a random parameter, we will use the standard Faedo-Galerkin methods in [21] to show the existence of weak solutions to (3.6) and (3.7) in following.

Theorem 3.1. *For any $T > 0$ and $v_0 \in H$, $g \in V'$, for P-a.e. $\omega \in \Omega$, then problems* (3.6) *and* (3.7) *possesses a weak solution $v(x,t) \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^3(0,T;\mathbf{L}^3(O)) \cap L^4(0,T;\mathbf{L}^4(O))$. Moreover, $v \in C([0,T];H)$.*

Proof. Step 1: Constructing the approximated solution of (3.6) and (3.7).

Since V is a subspace of $(H_0^1(O))^3$, then it is separable. Recalling that $\mathcal{V}$ is dense in V and H, so there exists a sequence of linearly independent elements $\{v_i\}_{i\geq 1} \subset \mathcal{V}$ are dense in V and H. Applying the Gram-Schmidt orthonormalization process, one can obtain an orthonormal basis $\{w_j\}_{j=1}^{\infty} \subset \mathcal{V}$ of H such that the linear combinations of these elements are dense in V.

Let V_m= span$\{w_1, \cdots, w_m\}$ and the projector $P_m : H \to V_m$ be given by

$$P_m v = \sum_{j=1}^{m} (v, w_j) w_j \quad \text{for} \quad v \in \mathbf{L}^2(O). \tag{3.9}$$

We construct the approximated solution $v_m(t) = \sum\limits_{j=1}^{m} h_{j,m}(t) w_j$ satisfying the following Cauchy problem

$$\begin{cases} (\dfrac{\partial v_m}{\partial t}, w_j) + \nu((v_m, w_j)) + \nu((z(\theta_t\omega), w_j)) + \alpha(v_m + z(\theta_t\omega), w_j) \\ + \beta(|v_m + z(\theta_t\omega)|(v_m + z(\theta_t\omega)), w_j) + \gamma(|v_m + z(\theta_t\omega)|^2(v_m + z(\theta_t\omega)), w_j) \\ = \langle g(x), w_j \rangle + (\mu z(\theta_t\omega), w_j), \ \forall\, t \geq 0, \ 1 \leq j \leq m, \\ v_m(x,0) = P_m v_0. \end{cases} \tag{3.10}$$

The problem (3.10) is a well-known ordinary functional differential equations with respect to the unknown variables $\{h_{j,m}(t)\}_{j=1}^{m}$, which has a unique local solution (in an interval $[0, t^*]$ with $0 < t^* \leq T$). In fact, the global solution ($t^* = T$) can be deduced by the a *priori* estimates below.

Step 2: Establishing *a priori* estimates for $\{v_m\}$.

Multiplying the first equation in (3.10) by $h_{j,m}(t)$ and summing in j, we obtain that for a.e. $t \in [0,T]$,

$$\begin{aligned} &\frac{d}{dt}\|v_m\|_H^2 + 2\nu\|v_m\|_V^2 + 2\alpha(v_m + z(\theta_t\omega), v_m + z(\theta_t\omega)) \\ &+ 2\beta(|v_m + z(\theta_t\omega)|(v_m + z(\theta_t\omega)), v_m + z(\theta_t\omega)) \\ &+ 2\gamma(|v_m + z(\theta_t\omega)|^2(v_m + z(\theta_t\omega)), v_m + z(\theta_t\omega)) \\ =\ & 2(g + \mu z(\theta_t\omega), v_m) + 2\alpha(v_m + z(\theta_t\omega), z(\theta_t\omega)) + 2\beta(|v_m + z(\theta_t\omega)|(v_m + z(\theta_t\omega)), z(\theta_t\omega)) \\ &+ 2\gamma(|v_m + z(\theta_t\omega)|^2(v_m + z(\theta_t\omega)), z(\theta_t\omega)) - 2\nu((z(\theta_t\omega), v_m)) \\ \leq\ & 2(\frac{1}{\nu}\|g\|_*^2 + \frac{\nu}{4}\|v_m\|_V^2) + 2\mu(\frac{\mu}{\lambda_1\nu}\|z(\theta_t\omega)\|_2^2 + \frac{\lambda_1\nu}{4\mu}\|v_m\|_2^2) + 2\alpha(\frac{1}{2}\|v_m + z(\theta_t\omega)\|_2^2 + \frac{1}{2}\|z(\theta_t\omega)\|_2^2) \\ &+ 2\beta(\frac{1}{2}\|v_m + z(\theta_t\omega)\|_3^3 + \frac{16}{27}\|z(\theta_t\omega)\|_3^3) + 2\gamma(\frac{1}{2}\|v_m + z(\theta_t\omega)\|_4^4 + \frac{27}{32}\|z(\theta_t\omega)\|_4^4) \\ &+ 2\nu(\|\nabla z(\theta_t\omega)\|_2^2 + \frac{1}{4}\|\nabla v_m\|_2^2) \\ \leq\ & \frac{2}{\nu}\|g\|_*^2 + \frac{\nu}{2}\|v_m\|_V^2 + (\frac{2\mu^2}{\lambda_1\nu} + \alpha)\|z(\theta_t\omega)\|_2^2 + \frac{\nu}{2}\|v_m\|_V^2 + \alpha\|v_m + z(\theta_t\omega)\|_2^2 + \beta\|v_m + z(\theta_t\omega)\|_3^3 \\ &+ \frac{32\beta}{27}\|z(\theta_t\omega)\|_3^3 + \gamma\|v_m + z(\theta_t\omega)\|_4^4 + \frac{27\gamma}{16}\|z(\theta_t\omega)\|_4^4 + 2\nu\|\nabla z(\theta_t\omega)\|_2^2 + \frac{\nu}{2}\|v_m\|_V^2. \end{aligned} \tag{3.11}$$

the above inequality is obtained by using the Young's inequality and Poincaré inequality.

Integrating (3.11) over $[0, t]$ with the time variable, we find

$$\begin{aligned} &\|v_m\|_H^2 + \frac{\nu}{2}\int_0^t \|v_m\|_V^2 ds + \alpha\int_0^t \|v_m + z(\theta_s\omega)\|_2^2 ds + \beta\int_0^t \|v_m + z(\theta_s\omega)\|_3^3 ds \\ &+ \gamma\int_0^t \|v_m + z(\theta_s\omega)\|_4^4 ds \\ \le\ & \|v_m(0)\|_H^2 + \frac{2}{\nu}\int_0^t \|g\|_*^2 ds + (\frac{2\mu^2}{\lambda_1\nu} + \alpha)\int_0^t \|z(\theta_s\omega)\|_2^2 ds + \frac{32\beta}{27}\int_0^t \|z(\theta_s\omega)\|_3^3 ds \\ &+ \frac{27\gamma}{16}\int_0^t \|z(\theta_s\omega)\|_4^4 ds + 2\nu\int_0^t \|\nabla z(\theta_s\omega)\|_2^2 ds. \end{aligned} \tag{3.12}$$

Since $z(\theta_t\omega) \in L^2(0, T; V) \cap L^3(0, T; \mathbf{L}^3(O)) \cap L^4(0, T; \mathbf{L}^4(O))$, we have

$$\begin{aligned} &\{v_m\} \text{ is bounded in } L^\infty(0, T; H) \cap L^2(0, T; V), \\ &\{v_m + z(\theta_t\omega)\} \text{ is bounded in } L^2(0, T; H) \cap L^3(0, T; \mathbf{L}^3(O)) \cap L^4(0, T; \mathbf{L}^4(O)). \end{aligned} \tag{3.13}$$

Moreover, $\int_O |v_m|^3 dx = \int_O |v_m + z(\theta_t\omega) - z(\theta_t\omega)|^3 dx \le \int_O (|v_m + z(\theta_t\omega)| + |z(\theta_t\omega)|)^3 dx \le 4\|v_m + z(\theta_t\omega)\|_3^3 + 4\|z(\theta_t\omega)\|_3^3$, then we get

$$\int_0^t \|v_m\|_3^3 ds \le 4\int_0^t \|v_m + z(\theta_s\omega)\|_3^3 ds + 4\int_0^t \|z(\theta_s\omega)\|_3^3 ds < +\infty. \tag{3.14}$$

Thus, $v_m \in L^3(0, T; \mathbf{L}^3(O))$. Similarly, $v_m \in L^4(0, T; \mathbf{L}^4(O))$.

In conclusion,

$$\{v_m\} \text{ is bounded in } L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^3(0, T; \mathbf{L}^3(O)) \cap L^4(0, T; \mathbf{L}^4(O)). \tag{3.15}$$

Since the domain O maybe unbounded and the boundary ∂O has no any regularity assumption, the compact injection $V \hookrightarrow H$ may not hold. So the way of proving a compactness property on bounded domain is no longer valid here. Next, we will use Corollary 2.34 in [19] to obtain the local compactness result. Based on the estimates (3.15), we just need to prove that the following condition holds, i.e.,

$$\int_0^{T-a} \|v_m(t+a) - v_m(t)\|_H^2 dt \to 0 \text{ as } a \to 0, \text{uniformly for } \{v_m\}. \tag{3.16}$$

From (3.10), for any $0 \le t \le t + a \le T$, one has

$$\begin{aligned} &(v_m(t+a) - v_m(t), w_j) + \nu\int_t^{t+a} (\nabla v_m(s), \nabla w_j) ds + \nu\int_t^{t+a} (\nabla z(\theta_s\omega), \nabla w_j) ds \\ &+ \alpha\int_t^{t+a} (v_m(s) + z(\theta_s\omega), w_j) ds + \beta\int_t^{t+a} (|v_m(s) + z(\theta_s\omega)|(v_m(s) + z(\theta_s\omega)), w_j) ds \\ &+ \gamma\int_t^{t+a} (|v_m(s) + z(\theta_s\omega)|^2(v_m(s) + z(\theta_s\omega)), w_j) ds \\ =\ & \int_t^{t+a} \left\langle g(x), w_j \right\rangle ds + \int_t^{t+a} (\mu z(\theta_s\omega), w_j) ds. \end{aligned} \tag{3.17}$$

Multiplying $h_{j,m}(t+a) - h_{j,m}(t)$ and summing in j, one has

$$\begin{aligned}
&\|v_m(t+a) - v_m(t)\|_H^2\\
&= -\nu \int_t^{t+a} (\nabla v_m(s), \nabla v_m(t+a) - \nabla v_m(t))ds\\
&\quad - \nu \int_t^{t+a} (\nabla z(\theta_s\omega), \nabla v_m(t+a) - \nabla v_m(t))ds\\
&\quad - \alpha \int_t^{t+a} (v_m(s) + z(\theta_s\omega), v_m(t+a) - v_m(t))ds\\
&\quad - \beta \int_t^{t+a} (|v_m(s) + z(\theta_s\omega)|(v_m(s) + z(\theta_s\omega)), v_m(t+a) - v_m(t))ds\\
&\quad - \gamma \int_t^{t+a} (|v_m(s) + z(\theta_s\omega)|^2(v_m(s) + z(\theta_s\omega)), v_m(t+a) - v_m(t))ds\\
&\quad + \int_t^{t+a} \langle g(x), v_m(t+a) - v_m(t)\rangle\, ds + \int_t^{t+a} (\mu z(\theta_s\omega), v_m(t+a) - v_m(t))ds\\
&\le \nu\|v_m(t+a) - v_m(t)\|_V \int_t^{t+a} \|v_m(s)\|_V ds + \nu\|v_m(t+a) - v_m(t)\|_V \int_t^{t+a} \|z(\theta_s\omega)\|_V ds\\
&\quad + \alpha\|v_m(t+a) - v_m(t)\|_H \int_t^{t+a} \|v_m(s) + z(\theta_s\omega)\|_H ds\\
&\quad + \beta\|v_m(t+a) - v_m(t)\|_3 \int_t^{t+a} \|v_m(s) + z(\theta_s\omega)\|_3^2 ds\\
&\quad + \gamma\|v_m(t+a) - v_m(t)\|_4 \int_t^{t+a} \|v_m(s) + z(\theta_s\omega)\|_4^3 ds + \|v_m(t+a) - v_m(t)\|_V \int_t^{t+a} \|g\|_* ds\\
&\quad + \mu\|v_m(t+a) - v_m(t)\|_H \int_t^{t+a} \|z(\theta_s\omega)\|_H ds\\
&\le \|v_m(t+a) - v_m(t)\|_V \int_t^{t+a} G_m(s)ds + \beta\|v_m(t+a) - v_m(t)\|_3 \int_t^{t+a} \|v_m(s) + z(\theta_s\omega)\|_3^2 ds\\
&\quad + \gamma\|v_m(t+a) - v_m(t)\|_4 \int_t^{t+a} \|v_m(s) + z(\theta_s\omega)\|_4^3 ds,
\end{aligned} \tag{3.18}$$

where $G_m(s) = \nu\|v_m(s)\|_V + \nu\|z(\theta_s\omega)\|_V + \frac{\alpha}{\sqrt{\lambda_1}}\|v_m(s) + z(\theta_s\omega)\|_H + \frac{\mu}{\sqrt{\lambda_1}}\|z(\theta_s\omega)\|_H + \|g\|_*$.

Hence,

$$\begin{aligned}
\int_0^{T-a} \|v_m(t+a) - v_m(t)\|_H^2 dt \le& \int_0^{T-a} \|v_m(t+a) - v_m(t)\|_V dt \int_t^{t+a} G_m(s)ds\\
&+ \beta \int_0^{T-a} \|v_m(t+a) - v_m(t)\|_3 dt \int_t^{t+a} \|v_m(s) + z(\theta_s\omega)\|_3^2 ds\\
&+ \gamma \int_0^{T-a} \|v_m(t+a) - v_m(t)\|_4 dt \int_t^{t+a} \|v_m(s) + z(\theta_s\omega)\|_4^3 ds.
\end{aligned} \tag{3.19}$$

Thanks to the Fubini theorem, one has

$$\begin{aligned}\int_0^{T-a}\|v_m(t+a)-v_m(t)\|_H^2dt \le &\int_0^T G_m(s)ds\int_{\overline{s-a}}^{\overline{s}}\|v_m(t+a)-v_m(t)\|_Vdt\\&+\beta\int_0^T\|v_m(s)+z(\theta_s\omega)\|_3^2ds\int_{\overline{s-a}}^{\overline{s}}\|v_m(t+a)-v_m(t)\|_3dt\\&+\gamma\int_0^T\|v_m(s)+z(\theta_s\omega)\|_4^3ds\int_{\overline{s-a}}^{\overline{s}}\|v_m(t+a)-v_m(t)\|_4dt,\end{aligned}\tag{3.20}$$

where

$$\overline{s}=\begin{cases}0, & \text{if } s\le 0,\\ s, & \text{if } 0<s\le T-a,\\ T-a, & \text{if } s>T-a.\end{cases}\tag{3.21}$$

Then using the Young's inequality and the fact that $0\le\overline{s}-\overline{s-a}\le a$, we derive that

$$\begin{aligned}&\int_0^{T-a}\|v_m(t+a)-v_m(t)\|_H^2dt\\&\le 2a^{\frac12}\|v_m\|_{L^2(0,T;V)}\int_0^T G_m(s)ds+2\beta a^{\frac23}\|v_m\|_{L^3(0,T;\mathbf{L}^3(O))}\int_0^T\|v_m(s)+z(\theta_s\omega)\|_3^2ds\\&\quad+2\gamma a^{\frac34}\|v_m\|_{L^4(0,T;\mathbf{L}^4(O))}\int_0^T\|v_m(s)+z(\theta_s\omega)\|_4^3ds\\&\le 2a^{\frac12}\|v_m\|_{L^2(0,T;V)}\int_0^T G_m(s)ds+2\beta a^{\frac23}\|v_m\|_{L^3(0,T;\mathbf{L}^3(O))}T^{\frac13}\|v_m(s)+z(\theta_s\omega)\|^3_{L^3(0,T;\mathbf{L}^3(O))}\\&\quad+2\gamma a^{\frac34}\|v_m\|_{L^4(0,T;\mathbf{L}^4(O))}T^{\frac14}\|v_m(s)+z(\theta_s\omega)\|^4_{L^4(0,T;\mathbf{L}^4(O))}.\end{aligned}\tag{3.22}$$

By simple computation shows that

$$\begin{aligned}\int_0^T G_m(s)ds=&\int_0^T[\nu\|v_m(s)\|_V+\nu\|z(\theta_s\omega)\|_V+\frac{\alpha}{\sqrt{\lambda_1}}\|v_m(s)+z(\theta_s\omega)\|_H+\frac{\mu}{\sqrt{\lambda_1}}\|z(\theta_s\omega)\|_H\\&\quad+\|g\|_*]ds\\\le&\, T^{\frac12}[\nu\|v_m\|_{L^2(0,T;V)}+\nu\|z(\theta_t\omega)\|_{L^2(0,T;V)}+\frac{\alpha}{\sqrt{\lambda_1}}\|v_m+z(\theta_t\omega)\|_{L^2(0,T;H)}\\&+\frac{\mu}{\sqrt{\lambda_1}}\|z(\theta_t\omega)\|_{L^2(0,T;H)}]+T\|g\|_*.\end{aligned}\tag{3.23}$$

Combining (3.13), (3.15), (3.22) and (3.23), one achieves (3.16).

Step 3: Passing to limit for deriving the global solution of (3.6) and (3.7) by a truncation argument.

Combining the preceding uniform estimates (3.13) and (3.15), we can deduce that there exists a subsequence v_m (without relabeling) such that, when $m\to\infty$,

$$v_m\rightharpoonup v \text{ weakly } * \text{ in } L^\infty(0,T;H);\tag{3.24}$$

$$v_m\rightharpoonup v \text{ weakly in } L^2(0,T;V);\tag{3.25}$$

$$v_m + z \rightharpoonup v + z \text{ weakly in } L^2(0,T;H); \tag{3.26}$$

$$|v_m + z|(v_m + z) \rightharpoonup \chi \text{ weakly in } L^{\frac{3}{2}}(0,T;\mathbf{L}^{\frac{3}{2}}(\mathcal{O})); \tag{3.27}$$

$$|v_m + z|^2(v_m + z) \rightharpoonup \zeta \text{ weakly in } L^{\frac{4}{3}}(0,T;\mathbf{L}^{\frac{4}{3}}(\mathcal{O})), \tag{3.28}$$

with $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $v + z \in L^2(0,T;H)$.

Next, we split into four steps to obtain the weak solution.

(I): Using a truncation argument analogously to [13, 19, 22], we prove that

$$\{v_m\} \text{ is relatively compact in } L^2(0,T;\mathbf{L}^2(\mathcal{K})) \text{ for any bounded open subsets } \mathcal{K} \subset \mathcal{O}. \tag{3.29}$$

For any bounded subset $\mathcal{K} \subset \mathcal{O}$, there exists a bounded open ball B_R such that $\mathcal{K} \subset B_R$. Denote $\widetilde{\mathcal{K}} = \mathcal{O} \cap B_{2R}$ and then the compact injection $(H_0^1(\widetilde{\mathcal{K}}))^3 \subset \mathbf{L}^2(\widetilde{\mathcal{K}})$ holds. Define a blob function $\rho \in C^\infty(\mathbb{R}^+)$ with

$$\rho(v) = \begin{cases} 1, & \text{if } 0 \le v \le 1, \\ 0, & \text{if } v \ge 3. \end{cases} \tag{3.30}$$

Define $v_m^R(x) = v_m(x)\rho(\frac{|x|^2}{R^2})$, then by (3.15) and (3.16), one has

$$\{v_m^R\} \text{ is bounded in } L^\infty(0,T;\mathbf{L}^2(\widetilde{\mathcal{K}})) \cap L^2(0,T;(H_0^1(\widetilde{\mathcal{K}}))^3), \tag{3.31}$$

$$\lim_{a\to 0} \sup_m \int_0^{T-a} \|v_m^R(t+a) - v_m^R(t)\|^2_{\mathbf{L}^2(\widetilde{\mathcal{K}})} dt \to 0 \text{ as } a \to 0. \tag{3.32}$$

From Corollary 2.34 in [19], one obtains that

$$\{v_m^R\} \text{ is relatively compact in } L^2(0,T;\mathbf{L}^2(\mathcal{K})). \tag{3.33}$$

Notice that $v_m^R(x) = v_m(x)$ for $x \in \mathcal{K}$, one achieves (3.29) immediately.

(II): Passing to the limit of (3.10).

By Lemma 1.3 in [23] and (3.29), we can extract a subsequence of $\{v_m\}$ such that the limit of (3.27) and (3.28) satisfy

$$\chi = |v + z|(v + z), \ \zeta = |v + z|^2(v + z), \text{ on any bounded subsets } \mathcal{K} \subset \mathcal{O}. \tag{3.34}$$

Let $\psi \in C^1([0,T])$ with $\psi(T) = 0$. From (3.10), one has

$$\begin{aligned} &-\int_0^T (v_m, w_j)\psi' dt + \nu \int_0^T ((v_m, w_j))\psi dt + \nu \int_0^T ((z, w_j))\psi dt + \alpha \int_0^T (v_m + z, w_j)\psi dt \\ &+ \beta \int_0^T (|v_m + z|(v_m + z), w_j)\psi dt + \gamma \int_0^T (|v_m + z|^2(v_m + z), w_j)\psi dt \\ &= (v_m(0), w_j)\psi(0) + \int_0^T \langle g, w_j \rangle \psi dt + \int_0^T (\mu z, w_j)\psi dt. \end{aligned} \tag{3.35}$$

Collecting (3.24)–(3.28) and (3.34) together, and then taking the limit $m \to \infty$, we have

$$\begin{aligned}
&-\int_0^T (v,w)\psi' dt + \nu\int_0^T ((v,w))\psi dt + \nu\int_0^T ((z,w))\psi dt + \alpha\int_0^T (v+z,w)\psi dt\\
&+\beta\int_0^T (|v+z|(v+z),w)\psi dt + \gamma\int_0^T (|v+z|^2(v+z),w)\psi dt\\
&= (v_0,w)\psi(0) + \int_0^T \langle g,w\rangle \psi dt + \int_0^T (\mu z,w)\psi dt,
\end{aligned} \tag{3.36}$$

for any $w \in \{w_j\}_{j=1}^\infty$. Since $\{w_j\}_{j=1}^\infty$ is dense in V, then v satisfies the first equation of (3.8) by taking $\psi \in C_0^\infty(0,T)$ in (3.36).

(III): Proving that $v \in C([0,T];H)$.

For all $\xi \in V$, we have

$$\begin{aligned}
(\frac{\partial v}{\partial t},\xi) &= -\nu((v,\xi)) - \nu((z,\xi)) - \alpha(v+z,\xi) - \beta(|v+z|(v+z),\xi) - \gamma(|v+z|^2(v+z),\xi)\\
&\quad + \langle g,\xi\rangle + (\mu z,\xi)\\
&\le \nu\|v\|_V\|\xi\|_V + \nu\|z\|_V\|\xi\|_V + \alpha\|v+z\|_2\|\xi\|_2 + \beta\|v+z\|_3^2\|\xi\|_3 + \gamma\|v+z\|_4^3\|\xi\|_4\\
&\quad + \|g\|_*\|\xi\|_V + \mu\|z\|_2\|\xi\|_2.
\end{aligned} \tag{3.37}$$

Then we have $\frac{\partial v}{\partial t} \in L^2(0,T;V') + L^{\frac{3}{2}}(0,T;\mathbf{L}^{\frac{3}{2}}(O)) + L^{\frac{4}{3}}(0,T;\mathbf{L}^{\frac{4}{3}}(O))$. Combining the fact that $v \in L^2(0,T;V) \cap L^3(0,T;\mathbf{L}^3(O)) \cap L^4(0,T;\mathbf{L}^4(O))$, it follows from the similar calculation process of Theorem 3.6 in [24] that v satisfies energy equality and hence $v \in C([0,T];H)$.

(IV): Checking the initial data $v(0) = v_0$.

For any $\psi \in C^\infty([0,T])$ with $\psi(T) = 0$ and $w \in V$, since v satisfies the first equation of (3.8), then

$$\begin{aligned}
&\int_0^T (\frac{\partial v}{\partial t},w)\psi dt + \nu\int_0^T ((v,w))\psi dt + \nu\int_0^T ((z,w))\psi dt + \alpha\int_0^T (v+z,w)\psi dt\\
&+\beta\int_0^T (|v+z|(v+z),w)\psi dt + \gamma\int_0^T (|v+z|^2(v+z),w)\psi dt\\
&= \int_0^T \langle g,w\rangle \psi dt + \int_0^T (\mu z,w)\psi dt.
\end{aligned} \tag{3.38}$$

After integrating by parts, one has

$$\begin{aligned}
&-\int_0^T (v,w)\psi' dt + \nu\int_0^T ((v,w))\psi dt + \nu\int_0^T ((z,w))\psi dt + \alpha\int_0^T (v+z,w)\psi dt\\
&+\beta\int_0^T (|v+z|(v+z),w)\psi dt + \gamma\int_0^T (|v+z|^2(v+z),w)\psi dt\\
&= (v(0),w)\psi(0) + \int_0^T \langle g,w\rangle \psi dt + \int_0^T (\mu z,w)\psi dt.
\end{aligned} \tag{3.39}$$

Comparing (3.36) and (3.39), we obtain that

$$(v(0),w)\psi(0) = (v_0,w)\psi(0), \quad \forall\ w \in V,\ \psi \in C^\infty([0,T]) \text{ with } \psi(T) = 0, \tag{3.40}$$

which means that $v(0) = v_0$.

Furthermore, we also obtain the following theorem about the stability of (3.6) and (3.7).

Theorem 3.2. *For any $T \geq 0$ and given functions $(v_{0i}, g_i) \in H \times V'$ for $i = 1, 2$, then problems* (3.6) *and* (3.7) *possesses two weak solutions $\{v_i\}_{i=1,2} \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^3(0, T; \mathbf{L}^3(\mathcal{O})) \cap L^4(0, T; \mathbf{L}^4(\mathcal{O}))$ with respect to $\{(v_{0i}, g_i)\}_{i=1,2}$, and the following stability estimate holds:*

$$\max_{r\in[0,t]} \|v_1(r) - v_2(r)\|_H^2 \leq \|v_{01} - v_{02}\|_H^2 + \frac{1}{\nu}\int_0^t \|g_1 - g_2\|_*^2 ds. \tag{3.41}$$

Proof. Setting $w = v_1 - v_2$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w\|_H^2 + \nu(Aw, w) + \alpha(w, w) + \beta(|v_1 + z|(v_1 + z) - |v_2 + z|(v_2 + z), w) \\ &+ \gamma(|v_1 + z|^2(v_1 + z) - |v_2 + z|^2(v_2 + z), w) \\ =\ &\|g_1 - g_2\|_*\|w\|_V \leq \frac{1}{2\nu}\|g_1 - g_2\|_*^2 + \frac{\nu}{2}\|w\|_V^2. \end{aligned} \tag{3.42}$$

By Lemma 4.4 in [25], we derive that

$$\frac{d}{dt}\|w\|_H^2 + \nu\|w\|_V^2 \leq \frac{1}{\nu}\|g_1 - g_2\|_*^2. \tag{3.43}$$

Integrating the above inequality from 0 to t, we get

$$\begin{aligned} \|w(t)\|_H^2 + \nu\int_0^t \|w\|_V^2 ds &\leq \|w(0)\|_H^2 + \frac{1}{\nu}\int_0^t \|g_1 - g_2\|_*^2 ds. \\ &= \|v_{01} - v_{02}\|_H^2 + \frac{1}{\nu}\int_0^t \|g_1 - g_2\|_*^2 ds. \end{aligned} \tag{3.44}$$

Thus,

$$\max_{r\in[0,t]} \|v_1(r) - v_2(r)\|_H^2 \leq \|v_{01} - v_{02}\|_H^2 + \frac{1}{\nu}\int_0^t \|g_1 - g_2\|_*^2 ds. \tag{3.45}$$

Since $u(t, \omega, u_0) = v(t, \omega, v_0) + z(\theta_t\omega)$, one can easily obtain that $u(t, \omega)$ is a unique solution to problem (3.1). We now define a mapping $\phi : \mathbb{R}^+ \times \Omega \times H \to H$ by

$$\phi(t, \omega, u_0) = u(t, \omega, u_0) = v(t, \omega, v_0) + z(\theta_t\omega), \tag{3.46}$$

where $v_0 = u_0 - z(\omega)$. Then ϕ satisfies conditions (i)–(iii) in Definition 2.2. Therefore, ϕ is a continuous random dynamical system associated with problem (3.1).

4. Existence of a unique $\mathcal{D}$-random attractor

Let $\mathcal{D}$ be the collection of all tempered families of subsets $\{D(\omega)\}_{\omega\in\Omega}$ of H, i.e., for every $\omega \in \Omega$

$$\lim_{t\to+\infty} e^{-\frac{\lambda_1\nu}{4}t}\|D(\theta_{-t}\omega)\|_H = 0, \tag{4.1}$$

where λ_1 is Poincaré constant in (1.2) and $\|D(\theta_{-t}\omega)\|_H = sup_{x\in D(\theta_{-t}\omega)}\|x\|_H$.

Lemma 4.1. *Assume that $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for every $\omega \in \Omega$, there exist $T = T(D,\omega) > 0$ and a tempered function $r : \Omega \to \mathbb{R}^+$ such that*

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_H \le r(\omega),$$

for all $t \ge T$ and $v_0(\theta_{-t}\omega) \in D(\theta_{-t}\omega)$.

Proof. From (3.6), for all $\varphi \in V$, we have

$$\begin{aligned}&(v_t, \varphi) + \nu((v, \varphi)) + \nu(Az(\theta_t\omega), \varphi) + \alpha(v + z(\theta_t\omega), \varphi) + \beta(|v + z(\theta_t\omega)|(v + z(\theta_t\omega)), \varphi)\\ &+ \gamma(|v + z(\theta_t\omega)|^2(v + z(\theta_t\omega)), \varphi) = \langle g(x), \varphi\rangle + (\mu z(\theta_t\omega), \varphi).\end{aligned}$$

Choosing $\varphi = v$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|_H^2 + \nu\|v\|_V^2 + \nu(Az(\theta_t\omega), v) + \alpha(v + z(\theta_t\omega), v) + \beta(|v + z(\theta_t\omega)|(v + z(\theta_t\omega)), v)\\ &+ \gamma(|v + z(\theta_t\omega)|^2(v + z(\theta_t\omega)), v) = \langle g(x), v\rangle + (\mu z(\theta_t\omega), v).\end{aligned} \tag{4.2}$$

Using the Young's inequality, we get

$$\begin{aligned}\frac{d}{dt}\|v\|_H^2 \le& -2\nu\|v\|_V^2 + 2\nu(\frac{1}{\lambda_1}\|Az(\theta_t\omega)\|_H^2 + \frac{\lambda_1}{4}\|v\|_H^2) - 2\alpha(v + z(\theta_t\omega), v + z(\theta_t\omega))\\ &+ 2\alpha(v + z(\theta_t\omega), z(\theta_t\omega)) - 2\beta(|v + z(\theta_t\omega)|(v + z(\theta_t\omega)), v + z(\theta_t\omega))\\ &+ 2\beta(|v + z(\theta_t\omega)|(v + z(\theta_t\omega)), z(\theta_t\omega)) - 2\gamma(|v + z(\theta_t\omega)|^2(v + z(\theta_t\omega)), v + z(\theta_t\omega))\\ &+ 2\gamma(|v + z(\theta_t\omega)|^2(v + z(\theta_t\omega)), z(\theta_t\omega)) + \frac{2}{\nu}\|g(x)\|_*^2 + \frac{\nu}{2}\|v\|_V^2\\ &+ 2\mu(\frac{\mu}{\lambda_1\nu}\|z(\theta_t\omega)\|_H^2 + \frac{\lambda_1\nu}{4\mu}\|v\|_H^2)\\ \le& -\frac{3\nu}{2}\|v\|_V^2 + \lambda_1\nu\|v\|_H^2 + \frac{2\nu}{\lambda_1}\|Az(\theta_t\omega)\|_H^2 - 2\alpha\|v + z(\theta_t\omega)\|_H^2\\ &+ 2\alpha(\frac{1}{2}\|v + z(\theta_t\omega)\|_H^2 + \frac{1}{2}\|z(\theta_t\omega)\|_H^2) - 2\beta\|v + z(\theta_t\omega)\|_3^3\\ &+ 2\beta(\frac{1}{2}\|v + z(\theta_t\omega)\|_3^3 + \frac{16}{27}\|z(\theta_t\omega)\|_3^3) - 2\gamma\|v + z(\theta_t\omega)\|_4^4\\ &+ 2\gamma(\frac{1}{2}\|v + z(\theta_t\omega)\|_4^4 + \frac{27}{32}\|z(\theta_t\omega)\|_4^4) + \frac{2}{\nu}\|g(x)\|_*^2 + \frac{2\mu^2}{\lambda_1\nu}\|z(\theta_t\omega)\|_H^2\\ \le& -\frac{3\nu}{2}\|v\|_V^2 + \lambda_1\nu\|v\|_H^2 - \alpha\|v + z(\theta_t\omega)\|_H^2 - \beta\|v + z(\theta_t\omega)\|_3^3 - \gamma\|v + z(\theta_t\omega)\|_4^4\\ &+ c(\|Az(\theta_t\omega)\|_H^2 + \|z(\theta_t\omega)\|_H^2 + \|z(\theta_t\omega)\|_3^3 + \|z(\theta_t\omega)\|_4^4 + \|g(x)\|_*^2)\\ \le& -\frac{3\nu}{2}\|v\|_V^2 + \lambda_1\nu\|v\|_H^2 - \alpha\|v + z(\theta_t\omega)\|_H^2 - \beta\|v + z(\theta_t\omega)\|_3^3 - \gamma\|v + z(\theta_t\omega)\|_4^4\\ &+ c(1 + |y(\theta_t\omega)|^4)\end{aligned}$$

where the last inequality is obtained by $\|z(\theta_t\omega)\|_p^p = \|h(x)\|_p^p|y(\theta_t\omega)|^p \le c|y(\theta_t\omega)|^p$.

By Poincaré inequality, we get

$$\begin{aligned}&\frac{d}{dt}\|v\|_H^2 + \frac{\nu}{4}\|v\|_V^2 + \alpha\|v + z(\theta_t\omega)\|_H^2 + \beta\|v + z(\theta_t\omega)\|_3^3 + \gamma\|v + z(\theta_t\omega)\|_4^4\\ \le& -\frac{\lambda_1\nu}{4}\|v\|_H^2 + c(1 + |y(\theta_t\omega)|^4).\end{aligned} \tag{4.3}$$

Multiplying both sides of (4.3) by $e^{\frac{\lambda_1\nu}{4}t}$ and integrating over $(0, s)$, we obtain

$$\|v(s,\omega,v_0(\omega))\|_H^2 \le e^{-\frac{\lambda_1\nu}{4}s}\|v_0(\omega)\|_H^2 + c\int_0^s e^{\frac{\lambda_1\nu}{4}(\tau-s)}(1+|y(\theta_\tau\omega)|^4)d\tau. \tag{4.4}$$

Replacing s and ω by t and $\theta_{-t}\omega$, then we obtain

$$\begin{aligned}\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_H^2 &\le e^{-\frac{\lambda_1\nu}{4}t}\|v_0(\theta_{-t}\omega)\|_H^2 + c\int_0^t e^{\frac{\lambda_1\nu}{4}(\tau-t)}(1+|y(\theta_{\tau-t}\omega)|^4)d\tau \\ &= e^{-\frac{\lambda_1\nu}{4}t}\|v_0(\theta_{-t}\omega)\|_H^2 + c\int_{-t}^0 e^{\frac{\lambda_1\nu}{4}\tau}(1+|y(\theta_\tau\omega)|^4)d\tau.\end{aligned} \tag{4.5}$$

Since $v_0(\theta_{-t}\omega) \in D(\theta_{-t}\omega)$ and $\{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, we get

$$\lim_{t\to+\infty} e^{-\frac{\lambda_1\nu}{4}t}\|v_0(\theta_{-t}\omega)\|_H^2 = 0. \tag{4.6}$$

Since $|y(\theta_\tau\omega)|$ is tempered, then by (3.5), we have

$$\lim_{\tau\to-\infty} e^{\frac{\lambda_1\nu}{4}\tau}(1+|y(\theta_\tau\omega)|^4) = 0.$$

It implies that

$$r_0(\omega) = c\int_{-\infty}^0 e^{\frac{\lambda_1\nu}{4}\tau}(1+|y(\theta_\tau\omega)|^4)d\tau < +\infty. \tag{4.7}$$

Taking into account (4.5)–(4.7), then there exists $T(D,\omega) > 0$ such that, for $t \ge T$

$$\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_H^2 \le 2r_0(\omega), \tag{4.8}$$

and

$$\begin{aligned} r_0(\theta_{-t}\omega) &= c\int_{-\infty}^0 e^{\frac{\lambda_1\nu}{4}\tau}(1+|y(\theta_{\tau-t}\omega)|^4)d\tau \\ &= c\int_{-\infty}^{-t} e^{\frac{\lambda_1\nu}{4}(\tau+t)}(1+|y(\theta_\tau\omega)|^4)d\tau \\ &\le ce^{\frac{\lambda_1\nu}{4}t}\int_{-\infty}^0 e^{\frac{\lambda_1\nu}{4}\tau}(1+|y(\theta_\tau\omega)|^4)d\tau \\ &\le ce^{\frac{\lambda_1\nu}{4}t}r_1(\omega),\end{aligned} \tag{4.9}$$

where

$$r_1(\omega) = \int_{-\infty}^0 e^{\frac{\lambda_1\nu}{4}\tau}(1+|y(\theta_\tau\omega)|^4)d\tau < +\infty. \tag{4.10}$$

Then, we have

$$e^{-\frac{\lambda_1\nu}{4}t}\sqrt{2r_0(\theta_{-t}\omega)} \le e^{-\frac{\lambda_1\nu}{8}t}\sqrt{2cr_1(w)} \to 0 \ as\ t \to +\infty. \tag{4.11}$$

Thus, we can choose $r(\omega) = \sqrt{2r_0(w)}$ and $r(\omega)$ is tempered from (4.11). This completes the proof.

Proposition 4.1. *Assume that $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then the random dynamical system ϕ associated with problem* (3.1) *has a random absorbing set $K \in \mathcal{D}$.*

Proof. By (3.46), we get

$$\phi(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)) = v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) + z(\omega). \tag{4.12}$$

and $v_0(\omega) = u_0(\omega) - z(\omega)$, then

$$\begin{aligned}\|v_0(\omega)\|_H &= \|u_0(\omega) - z(\omega)\|_H \\ &\le \|u_0(\omega)\|_H + \|z(\omega)\|_H \\ &\le \|D(\omega)\|_H + \|z(\omega)\|_H.\end{aligned}$$

Since $D \in \mathcal{D}$ and $|z(\omega)|$ is tempered, we can easily get $v_0(\omega) \in D^*(\omega)$ for some $D^* \in \mathcal{D}$. Then by Lemma 4.1, there exists $T = T(D^*, \omega) > 0$ such that

$$\|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_H \le r(\omega), \tag{4.13}$$

for all $t \geq$T and $v_0(\theta_{-t}\omega) \in D^*(\theta_{-t}\omega)$. Combining (4.12) and (4.13), we obtain

$$\begin{aligned}\|\phi(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|_H &\le \|v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_H + \|z(\omega)\|_H \\ &\le r(\omega) + \|z(\omega)\|_H,\end{aligned} \tag{4.14}$$

for all $t \geq$T and $u_0(\theta_{-t}\omega) \in D(\theta_{-t}\omega)$. It implies that there exists a random absorbing set of ϕ in $\mathcal{D}$.

In order to show that ϕ is $\mathcal{D}$-pullback asymptotically compact in H, we need the following lemma.

Lemma 4.2. *For any sequence $\{x_n\} \subset H$ such that $x_n \rightharpoonup x_0$ in H, then for P-a.e. $\omega \in \Omega$,*

$$v(t, \omega, x_n) \rightharpoonup v(t, \omega, x_0) \quad \text{in } H, \ \forall\, t \ge 0, \tag{4.15}$$

$$v(\cdot, \omega, x_n) \rightharpoonup v(\cdot, \omega, x_0) \quad \text{in } L^2(0, T; V), \ \forall\, T > 0, \tag{4.16}$$

$$\begin{gathered} v(\cdot, \omega, x_n) + z(\theta.\omega) \rightharpoonup v(\cdot, \omega, x_0) + z(\theta.\omega) \\ \text{in } L^2(0, T; H) \cap L^3(0, T; \mathbf{L}^3(\mathcal{O})) \cap L^4(0, T; \mathbf{L}^4(\mathcal{O})), \ \forall\, T > 0, \end{gathered} \tag{4.17}$$

$$\begin{gathered} |v(\cdot, \omega, x_n) + z(\theta.\omega)|(v(\cdot, \omega, x_n) + z(\theta.\omega)) \rightharpoonup |v(\cdot, \omega, x_0) + z(\theta.\omega)|(v(\cdot, \omega, x_0) + z(\theta.\omega)) \\ \text{in } L^{\frac{3}{2}}(0, T; \mathbf{L}^{\frac{3}{2}}(\mathcal{O})), \ \forall\, T > 0, \end{gathered} \tag{4.18}$$

and

$$\begin{gathered} |v(\cdot, \omega, x_n) + z(\theta.\omega)|^2(v(\cdot, \omega, x_n) + z(\theta.\omega)) \rightharpoonup |v(\cdot, \omega, x_0) + z(\theta.\omega)|^2(v(\cdot, \omega, x_0) + z(\theta.\omega)) \\ \text{in } L^{\frac{4}{3}}(0, T; \mathbf{L}^{\frac{4}{3}}(\mathcal{O})), \ \forall\, T > 0. \end{gathered} \tag{4.19}$$

Proof. Denote by $v_n(t) = v(t, \omega, x_n)$ and $v(t) = v(t, \omega, x_0)$ the corresponding solutions to problem (3.6) and (3.7). Observe that by Theorem 3.1, one has uniform bounds of v_n ans v in $L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^3(0, T; \mathbf{L}^3(\mathcal{O})) \cap L^4(0, T; \mathbf{L}^4(\mathcal{O}))$, then $v_n + z$ and $v + z$ are uniformly bounded in $L^2(0, T; H) \cap L^3(0, T; \mathbf{L}^3(\mathcal{O})) \cap L^4(0, T; \mathbf{L}^4(\mathcal{O}))$, and v_n belongs to $C([0, T]; H)$. Then there exists a subsequence $\{n\}$ (without relabeling) such that, when $n \to \infty$,

$$v_n(t) \rightharpoonup v \quad \text{in } H,$$
$$v_n(t) \rightharpoonup v \quad \text{in } L^2(0,T;V),$$
$$v_n(t) + z(\theta_t\omega) \rightharpoonup v + z(\theta_t\omega) \quad \text{in } L^2(0,T;H) \cap L^3(0,T;\mathbf{L}^3(\mathcal{O})) \cap L^4(0,T;\mathbf{L}^4(\mathcal{O})),$$
$$|v_n(t) + z(\theta_t\omega)|(v_n(t) + z(\theta_t\omega)) \rightharpoonup \chi \quad \text{in } L^{\frac{3}{2}}(0,T;\mathbf{L}^{\frac{3}{2}}(\mathcal{O})),$$

and

$$|v_n(t) + z(\theta_t\omega)|^2(v_n(t) + z(\theta_t\omega)) \rightharpoonup \zeta \quad \text{in } L^{\frac{4}{3}}(0,T;\mathbf{L}^{\frac{4}{3}}(\mathcal{O})).$$

Using a truncation argument analogously to step 3 (I) in Theorem 3.1, we obtain that

$v_n(t)$ is relatively compact in $L^2(0,T;\mathbf{L}^2(\mathcal{K}))$ for any bounded open subsets $\mathcal{K} \subset \mathcal{O}$.

Proceeding similarly to Theorem 3.1, we can prove that $\chi = |v(t) + z(\theta_t\omega)|(v(t) + z(\theta_t\omega))$ and $\zeta = |v(t) + z(\theta_t\omega)|^2(v(t) + z(\theta_t\omega))$. Hence, (4.18) and (4.19) hold.

Lemma 4.3. *Assume that $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for P-a.e. $\omega \in \Omega$, any sequence $t_n \to +\infty$ and $x_n \in D(\theta_{-t_n}\omega)$, the sequence $\{v(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ is precompact in H.*

Proof. By Lemma 4.1, one knows that $\{v(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ is bounded in H, thus we can get a subsequence of $\{n\}$ (without relabeling) and a $w_0 \in H$ such that

$$v(t_n, \theta_{-t_n}\omega, x_n) \rightharpoonup w_0 \quad \text{in } H. \tag{4.20}$$

By the lower semi-continuity of the norm, we have

$$\liminf_{n\to\infty} \|v(t_n, \theta_{-t_n}\omega, x_n)\|_H \geq \|w_0\|_H. \tag{4.21}$$

In order to prove that (4.20) is actually strong convergence, we need to show that

$$\limsup_{n\to\infty} \|v(t_n, \theta_{-t_n}\omega, x_n)\|_H \leq \|w_0\|_H. \tag{4.22}$$

Replacing s and ω by $t_n - m$ and $\theta_{-t_n}\omega$ in (4.4) for any fixed $m > 0$, we get

$$\begin{aligned}
&\|v(t_n - m, \theta_{-t_n}\omega, x_n)\|_H^2 \\
&\leq e^{\frac{\lambda_1\nu}{4}(m-t_n)}\|x_n\|_H^2 + \int_0^{t_n-m} e^{\frac{\lambda_1\nu}{4}(\tau-t_n+m)}(1 + |y(\theta_{\tau-t_n}\omega)|^4)d\tau \\
&= e^{\frac{\lambda_1\nu}{4}m}\left[e^{-\frac{\lambda_1\nu}{4}t_n}\|x_n\|_H^2 + \int_{-t_n}^{-m} e^{\frac{\lambda_1\nu}{4}\tau}(1 + |y(\theta_\tau\omega)|^4)d\tau\right] \\
&\leq e^{\frac{\lambda_1\nu}{4}m}\left[e^{-\frac{\lambda_1\nu}{4}t_n}\|x_n\|_H^2 + \int_{-\infty}^{0} e^{\frac{\lambda_1\nu}{4}\tau}(1 + |y(\theta_\tau\omega)|^4)d\tau\right].
\end{aligned} \tag{4.23}$$

Since $t_n \to +\infty$ and $x_n \in D(\theta_{-t_n}\omega)$, there exists $N_m > 0$ such that for all $n \geq N_m$,

$$\|v(t_n - m, \theta_{-t_n}\omega, x_n)\|_H^2 \leq 2e^{\frac{\lambda_1\nu}{4}m}r_0(\omega). \tag{4.24}$$

Then $\{v(t_n - m, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ is bounded in H. That means there exists a subsequence of $\{n\}$ (without relabeling) such that

$$v(t_n - m, \theta_{-t_n}\omega, x_n) \rightharpoonup w_m, \quad \text{for all } m > 0. \tag{4.25}$$

By (4.25) and Lemma 4.2, we obtain

$$v(t_n, \theta_{-t_n}\omega, x_n) = v(m, \theta_{-m}\omega, v(t_n - m, \theta_{-t_n}\omega, x_n)) \rightharpoonup v(m, \theta_{-m}\omega, w_m). \tag{4.26}$$

From (4.20) and (4.26), we obtain

$$v(m, \theta_{-m}\omega, w_m) = w_0, \quad \text{for all } m > 0. \tag{4.27}$$

Choosing $\xi = e^{\frac{\lambda_1 \nu}{2}(s-t)} v$ in (3.8), we get

$$\begin{aligned}
&\frac{d}{ds}\left(e^{\frac{\lambda_1 \nu}{2}(s-t)}\|v\|_H^2\right) + 2\nu e^{\frac{\lambda_1 \nu}{2}(s-t)}\|v\|_V^2 + 2\nu e^{\frac{\lambda_1 \nu}{2}(s-t)}((z(\theta_s\omega), v)) + 2\alpha e^{\frac{\lambda_1 \nu}{2}(s-t)}(v + z(\theta_s\omega), v)\\
&+ 2\beta e^{\frac{\lambda_1 \nu}{2}(s-t)}(|v + z(\theta_s\omega)|(v + z(\theta_s\omega)), v) + 2\gamma e^{\frac{\lambda_1 \nu}{2}(s-t)}(|v + z(\theta_s\omega)|^2(v + z(\theta_s\omega)), v)\\
&= \frac{\lambda_1 \nu}{2} e^{\frac{\lambda_1 \nu}{2}(s-t)}\|v\|_H^2 + 2e^{\frac{\lambda_1 \nu}{2}(s-t)}(g + \mu z(\theta_s\omega), v).
\end{aligned} \tag{4.28}$$

Moreover, we integrate (4.28) on $[0, t]$ and deduce that

$$\begin{aligned}
&\|v(t, \omega, v_0(\omega))\|_H^2 + 2\int_0^t e^{\frac{\lambda_1 \nu}{2}(s-t)}\left[\nu\|v(s, \omega, v_0(\omega))\|_V^2 - \frac{\lambda_1 \nu}{4}\|v(s, \omega, v_0(\omega))\|_H^2\right] ds\\
&= e^{-\frac{\lambda_1 \nu}{2}t}\|v_0(\omega)\|_H^2 - 2\int_0^t e^{\frac{\lambda_1 \nu}{2}(s-t)}\nu((z(\theta_s\omega), v(s, \omega, v_0(\omega))))ds\\
&\quad - 2\int_0^t e^{\frac{\lambda_1 \nu}{2}(s-t)}\Big[\alpha\|v(s, \omega, v_0(\omega)) + z(\theta_s\omega)\|_2^2 + \beta\|v(s, \omega, v_0(\omega)) + z(\theta_s\omega)\|_3^3\\
&\qquad\qquad + \gamma\|v(s, \omega, v_0(\omega)) + z(\theta_s\omega)\|_4^4\Big] ds\\
&\quad + 2\int_0^t e^{\frac{\lambda_1 \nu}{2}(s-t)}\Big[\alpha(v(s, \omega, v_0(\omega)) + z(\theta_s\omega), z(\theta_s\omega))\\
&\qquad\qquad + \beta(|v(s, \omega, v_0(\omega)) + z(\theta_s\omega)|(v(s, \omega, v_0(\omega)) + z(\theta_s\omega)), z(\theta_s\omega))\\
&\qquad\qquad + \gamma(|v(s, \omega, v_0(\omega)) + z(\theta_s\omega)|^2(v(s, \omega, v_0(\omega)) + z(\theta_s\omega)), z(\theta_s\omega))\Big] ds\\
&\quad + 2\int_0^t e^{\frac{\lambda_1 \nu}{2}(s-t)}(g + \mu z(\theta_s\omega), v(s, \omega, v_0(\omega)))ds.
\end{aligned} \tag{4.29}$$

After replacing ω by $\theta_{-t}\omega$, we can easily obtain

$$
\begin{aligned}
&\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_H^2\\
&+2\int_0^t e^{\frac{\lambda_1\nu}{2}(s-t)}\left[\nu\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_V^2-\frac{\lambda_1\nu}{4}\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|_H^2\right]ds\\
=&\,e^{-\frac{\lambda_1\nu}{2}t}\|v_0(\theta_{-t}\omega)\|_H^2-2\int_0^t e^{\frac{\lambda_1\nu}{2}(s-t)}\nu((z(\theta_{s-t}\omega),v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))))ds\\
&-2\int_0^t e^{\frac{\lambda_1\nu}{2}(s-t)}\Big[\alpha\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega)\|_2^2+\beta\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega)\|_3^3\\
&\qquad+\gamma\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega)\|_4^4\Big]ds\\
&+2\int_0^t e^{\frac{\lambda_1\nu}{2}(s-t)}\Big[\alpha(v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega),z(\theta_{s-t}\omega))\\
&\qquad+\beta(|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega)|(v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega)),z(\theta_{s-t}\omega))\\
&\qquad+\gamma(|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega)|^2(v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))+z(\theta_{s-t}\omega)),z(\theta_{s-t}\omega))\Big]ds\\
&+2\int_0^t e^{\frac{\lambda_1\nu}{2}(s-t)}(g+\mu z(\theta_{s-t}\omega),v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)))ds.
\end{aligned}
\tag{4.30}
$$

Applying (4.30) for $t=m$, $v_0(\theta_{-t}\omega)=v_{n,m}:=v(t_n-m,\theta_{-t_n}\omega,x_n)$ and (4.24), we have

$$
\begin{aligned}
&\|v(m,\theta_{-m}\omega,v_{n,m})\|_H^2+2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\left[\nu\|v(s,\theta_{-m}\omega,v_{n,m})\|_V^2-\frac{\lambda_1\nu}{4}\|v(s,\theta_{-m}\omega,v_{n,m})\|_H^2\right]ds\\
=&\,e^{-\frac{\lambda_1\nu}{2}m}\|v_{n,m}\|_H^2-2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\nu((z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,v_{n,m})))ds\\
&-2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_2^2+\beta\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_3^3\\
&\qquad+\gamma\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_4^4\Big]ds\\
&+2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha(v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega),z(\theta_{s-m}\omega))\\
&\qquad+\beta(|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)|(v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)),z(\theta_{s-m}\omega))\\
&\qquad+\gamma(|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)|^2(v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)),z(\theta_{s-m}\omega))\Big]ds\\
&+2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(g+\mu z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,v_{n,m}))ds\\
\le&\,2e^{-\frac{\lambda_1\nu}{4}m}r_0(\omega)-2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\nu((z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,v_{n,m})))ds
\end{aligned}
$$

$$
\begin{aligned}
&- 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_2^2+\beta\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_3^3\\
&\qquad+\gamma\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_4^4\Big]ds\\
&+ 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha(v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega),z(\theta_{s-m}\omega))\\
&\qquad+\beta(|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)|(v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)),z(\theta_{s-m}\omega))\\
&\qquad+\gamma(|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)|^2(v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)),z(\theta_{s-m}\omega))\Big]ds\\
&+ 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(g+\mu z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,v_{n,m}))ds.
\end{aligned}
\tag{4.31}
$$

From (4.25), we have $v_{n,m} \rightharpoonup w_m$ in H. Then, by Lemma 4.2 we get

$$
v(\cdot,\theta_{-m}\omega,v_{n,m}) \rightharpoonup v(\cdot,\theta_{-m}\omega,w_m) \quad \text{in } L^2(0,m;V). \tag{4.32}
$$

Since $\lambda_1\nu - \frac{\lambda_1\nu}{4} > 0$, we get that $\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(\nu\|v(s,\theta_{-m}\omega,v_{n,m})\|_V^2 - \frac{\lambda_1\nu}{4}\|v(s,\theta_{-m}\omega,v_{n,m})\|_H^2)ds$ defines a norm in $L^2(0,m;V)$, thus

$$
\begin{aligned}
&\liminf_{n\to\infty}\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(\nu\|v(s,\theta_{-m}\omega,v_{n,m})\|_V^2-\frac{\lambda_1\nu}{4}\|v(s,\theta_{-m}\omega,v_{n,m})\|_H^2)ds\\
&\geq \int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(\nu\|v(s,\theta_{-m}\omega,w_m)\|_V^2-\frac{\lambda_1\nu}{4}\|v(s,\theta_{-m}\omega,w_m)\|_H^2)ds.
\end{aligned}
\tag{4.33}
$$

Similarly, since $e^{\frac{\lambda_1\nu}{2}(s-m)}\nu z(\theta_{s-m}\omega) \in L^2(0,m;V)$, we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\nu((z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,v_{n,m})))ds\\
&=\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\nu((z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,w_m)))ds.
\end{aligned}
\tag{4.34}
$$

Moreover, from (4.17), since $(\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_{L^p}^p ds)^{\frac{1}{p}}$ defines an equivalent norm in $L^p(0,m;\mathbf{L}^p(\mathcal{O}))$, we have

$$
\begin{aligned}
&\liminf_{n\to\infty}\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_2^2+\beta\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_3^3\\
&+\gamma\|v(s,\theta_{-m}\omega,v_{n,m})+z(\theta_{s-m}\omega)\|_4^4\Big]ds\\
&\geq\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha\|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)\|_2^2+\beta\|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)\|_3^3\\
&+\gamma\|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)\|_4^4\Big]ds.
\end{aligned}
\tag{4.35}
$$

Since $z(\theta_{s-m}\omega) \in L^2(0,m;H) \cap L^3(0,m;\mathbf{L}^3(\mathcal{O})) \cap L^4(0,m;\mathbf{L}^4(\mathcal{O}))$, (4.18) and (4.19), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)} \Big[\alpha(v(s,\theta_{-m}\omega,v_{n,m}) + z(\theta_{s-m}\omega), z(\theta_{s-m}\omega)) \\
&+ \beta(|v(s,\theta_{-m}\omega,v_{n,m}) + z(\theta_{s-m}\omega)|(v(s,\theta_{-m}\omega,v_{n,m}) + z(\theta_{s-m}\omega)), z(\theta_{s-m}\omega)) \\
&+ \gamma(|v(s,\theta_{-m}\omega,v_{n,m}) + z(\theta_{s-m}\omega)|^2(v(s,\theta_{-m}\omega,v_{n,m}) + z(\theta_{s-m}\omega)), z(\theta_{s-m}\omega))\Big] ds \\
= &\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)} \Big[\alpha(v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega), z(\theta_{s-m}\omega)) \\
&+ \beta(|v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)|(v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)), z(\theta_{s-m}\omega)) \\
&+ \gamma(|v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)|^2(v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)), z(\theta_{s-m}\omega))\Big] ds.
\end{aligned} \tag{4.36}$$

Since $e^{\frac{\lambda_1\nu}{2}(s-m)}(g + \mu z(\theta_{s-m}\omega) \in L^2(0,m;V')$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(g + \mu z(\theta_{s-m}\omega), v(s,\theta_{-m}\omega,v_{n,m}))ds \\
= &\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(g + \mu z(\theta_{s-m}\omega), v(s,\theta_{-m}\omega,w_m))ds.
\end{aligned} \tag{4.37}$$

Letting $n \to \infty$ in (4.31) and applying (4.33)–(4.37), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \|v(m,\theta_{-m}\omega,v_{n,m})\|_H^2 \\
&+ 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)} \Big[\nu\|v(s,\theta_{-m}\omega,w_m)\|_V^2 - \frac{\lambda_1\nu}{4}\|v(s,\theta_{-m}\omega,w_m)\|_H^2\Big] ds \\
\le\ &2e^{-\frac{\lambda_1\nu}{4}m} r_0(\omega) - 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)} \nu((z(\theta_{s-m}\omega), v(s,\theta_{-m}\omega,w_m)))ds \\
&- 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)} \Big[\alpha\|v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)\|_2^2 + \beta\|v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)\|_3^3 \\
&\qquad + \gamma\|v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)\|_4^4\Big] ds \\
&+ 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)} \Big[\alpha(v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega), z(\theta_{s-m}\omega)) \\
&\qquad + \beta(|v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)|(v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)), z(\theta_{s-m}\omega)) \\
&\qquad + \gamma(|v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)|^2(v(s,\theta_{-m}\omega,w_m) + z(\theta_{s-m}\omega)), z(\theta_{s-m}\omega))\Big] ds \\
&+ 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(g + \mu z(\theta_{s-m}\omega), v(s,\theta_{-m}\omega,w_m))ds.
\end{aligned} \tag{4.38}$$

Applying (4.30) for $t = m$ and $v_0(\theta_{-t}\omega) = w_m$, we have

$$
\begin{aligned}
&\|v(m,\theta_{-m}\omega,w_m)\|_H^2 + 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\left[\nu\|v(s,\theta_{-m}\omega,w_m)\|_V^2 - \frac{\lambda_1\nu}{4}\|v(s,\theta_{-m}\omega,w_m)\|_H^2\right]ds\\
&= e^{-\frac{\lambda_1\nu}{2}m}\|w_m\|_H^2 - 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\nu((z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,w_m)))ds\\
&\quad - 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha\|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)\|_2^2 + \beta\|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)\|_3^3\\
&\qquad\qquad + \gamma\|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)\|_4^4\Big]ds\\
&\quad + 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}\Big[\alpha(v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega),z(\theta_{s-m}\omega))\\
&\qquad\qquad + \beta(|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)|(v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)),z(\theta_{s-m}\omega))\\
&\qquad\qquad + \gamma(|v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)|^2(v(s,\theta_{-m}\omega,w_m)+z(\theta_{s-m}\omega)),z(\theta_{s-m}\omega))\Big]ds\\
&\quad + 2\int_0^m e^{\frac{\lambda_1\nu}{2}(s-m)}(g+\mu z(\theta_{s-m}\omega),v(s,\theta_{-m}\omega,w_m))ds.
\end{aligned}
\tag{4.39}
$$

Combining (4.38) and (4.39), we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\|v(m,\theta_{-m}\omega,v_{n,m})\|_H^2 - \|v(m,\theta_{-m}\omega,w_m)\|_H^2\\
&\le 2e^{-\frac{\lambda_1\nu}{4}m}r_0(\omega) - e^{-\frac{\lambda_1\nu}{2}m}\|w_m\|_H^2\\
&\le 2e^{-\frac{\lambda_1\nu}{4}m}r_0(\omega).
\end{aligned}
\tag{4.40}
$$

Letting $m \to \infty$ in (4.40), and noticing that $v(t_n,\theta_{-t_n}\omega,x_n) = v(m,\theta_{-m}\omega,v_{n,m})$ and $w_0 = v(m,\theta_{-m}\omega,w_m)$, we have

$$
\limsup_{n\to\infty}\|v(t_n,\theta_{-t_n}\omega,x_n)\|_H^2 \le \|w_0\|_H^2,
\tag{4.41}
$$

which implies (4.22).

Lemma 4.4. *Assume that $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then the random dynamical system ϕ is $\mathcal{D}$-pullback asymptotically compact in H.*

Proof. For P-*a.e.* $\omega \in \Omega$, $t_n \to \infty$ and $x_n \in D(\theta_{-t_n}\omega)$, we will show that $\{\phi(t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^{\infty}$ is precompact. It follows from (3.46) that

$$
\phi(t_n,\theta_{-t_n}\omega,x_n) = v(t_n,\theta_{-t_n}\omega,x_n - z(\theta_{-t_n}\omega)) + z(\omega).
\tag{4.42}
$$

and

$$
\begin{aligned}
\|x_n - z(\theta_{-t_n}\omega)\|_H &\le \|x_n\|_H + \|z(\theta_{-t_n}\omega)\|_H\\
&\le \|B(\theta_{-t_n}\omega)\|_H + \|z(\theta_{-t_n}\omega)\|_H.
\end{aligned}
\tag{4.43}
$$

Since $D \in \mathcal{D}$ and $|z(\omega)|$ is tempered, there exists a $D^* \in \mathcal{D}$ such that $x_n - z(\theta_{-t_n}\omega) \in D^*(\theta_{-t_n}\omega)$. Then by Lemma 4.3, it is easily to get that $\{v(t_n, \theta_{-t_n}\omega, y_n)\}_{n=1}^{\infty}$ has a subsequence $\{n'\} \subset \{n\}$, which satisfying

$$\lim_{m',n' \to \infty} \|v(t_{m'}, \theta_{-t_{m'}}\omega, y_{m'}) - v(t_{n'}, \theta_{-t_{n'}}\omega, y_{n'})\|_H = 0. \tag{4.44}$$

Combining (4.42) and (4.44), we have

$$\begin{aligned} &\lim_{m',n' \to \infty} \|\phi(t_{m'}, \theta_{-t_{m'}}\omega, x_{m'}) - \phi(t_{n'}, \theta_{-t_{n'}}\omega, x_{n'})\|_H \\ =& \lim_{m',n' \to \infty} \|v(t_{m'}, \theta_{-t_{m'}}\omega, y_{m'}) - v(t_{n'}, \theta_{-t_{n'}}\omega, y_{n'})\|_H \\ =& \, 0. \end{aligned} \tag{4.45}$$

It means that ϕ is $\mathcal{D}$-pullback asymptotically compact in H.

Theorem 4.1. *The random dynamical system ϕ corresponding to problem* (3.1) *has a unique $\mathcal{D}$-random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ in H.*

Proof. By Proposition 4.1, we can get ϕ has a family of random absorbing set $\{K(\omega)\}_{\omega \in \Omega}$ in $\mathcal{D}$. Moreover, ϕ is $\mathcal{D}$-pullback asymptotically compact in H by Lemma 4.4. Hence, by Theorem 2.1, it is easily to get that the existence of a unique $\mathcal{D}$-random attractor for ϕ.

5. Conclusions

In the previous sections, we mainly studied the long time behavior of the stochastic Brinkman-Forchheimer equations driven by additive noise on unbounded domains, and have obtained the existence of a unique pullback random attractor. This provides a new result for the study of Brinkman-Forchheimer equations, which has important significance for the study of porous media fluids in the future.

From a practical point of view, it is also common for a fluid to be affected by a nonlinear random disturbance. Therefore, it is of great significance to study the long time behavior of the stochastic differential equations driven by nonlinear color noise. To obtain more research results for the study of Brinkman-Forchheimer equations, in the next research, we may consider that the dynamics for the stochastic Brinkman-Forchheimer equations driven by nonlinear color noise on the unbounded domains.

Acknowledgments

The work is supported by the NSFC (11601021, 11831003, 11771031, 12171111) and Beijing Municipal Education Commission Foundation (KZ202110005011).

Conflict of interest

The authors declare there is no conflicts of interest.

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