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Research article

A priori bounds and existence of smooth solutions to a L_p Aleksandrov problem for Codazzi tensor with log-convex measure

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Abstract: In the present paper, we prove the existence of smooth solutions to a L_p Aleksandrov problem for Codazzi tensor with a log-convex measure in compact Riemannian manifolds (M, g) with positive constant sectional curvature under suitable conditions. Our proof is based on the solvability of a Monge-Ampère equation on (M, g) via the method of continuity whose crucial factor is the a priori bounds of smooth solutions to the Monge-Ampère equation mentioned above. It is worth mentioning that our result can be seen as an extension of the classical L_p Aleksandrov problem in Euclidian space to the frame of Riemannian manifolds with weighted measures and that our result can also be seen as some attempts to get some new results on geometric analysis for Codazzi tensor.

Keywords: Codazzi tensor; log-convex measures; L_p Aleksandrov problem; the continuous method; a priori bounds

1. Introduction

Let (M, g) be a n-dimensional Riemannian manifolds. In the present paper, we focus on the following Monge-Ampère equation on compact Riemannian manifolds (M, g) without boundary:

$$S^{1-p}e^{-\varphi(\rho^2)}\frac{\det(S_{ij}+g_{ij}S)}{\sqrt{\det g_{ij}}} = \phi(z), \tag{1.1}$$

for any $z \in M$ where $\rho^2 = g^{ij}S_iS_j + S^2$ and $S_{ij} + g_{ij}S$ is a $(0, 2)$ type Codazzi tensor.

An equivalent form of (1.1) is

$$S^{1-p}e^{-\varphi(\rho^2)}\sigma_n\sqrt{\det g_{ij}}dz = \phi(z)dz, \tag{1.2}$$

where $\sigma_n = \lambda_1\lambda_2\cdots\lambda_n$ and $\{\lambda_l\}_{l=1}^n$ is a solution sequence to the following algebraic equation,

$$det(S_{ij} + g_{ij}S - \lambda g_{ij}) = 0. \tag{1.3}$$

Noting that the volume element of Riemannian manifold is $dV = \sqrt{\det g_{ij}}dz$, we get

$$S^{1-p}e^{-\varphi(\rho^2)}\sigma_n dV = \phi(z)dz. \tag{1.4}$$

The left-side of (1.2) is called the density of p-integral Gaussian curvature measure for the log-convex measure $e^{-\varphi(\rho^2)}dx$ for Codazzi tensor in the present paper.

In particular, if $M = \mathbb{S}^2$, $p = 1$, $\varphi \equiv 0$ and S is the support function of hypersurface $N \subseteq \mathbb{R}^3$, the second fundamental form $(h_{ij})_{2\times 2}$ of the hypersurface N is given as

$$h_{ij} = S_{ij} + g_{ij}S \tag{1.5}$$

for any fixed $i, j \in \{1, 2\}$ where S_{ij} is the covariant derivation of S of second order. Then the left hand side of (1.4) becomes

$$\sigma_2 dV \tag{1.6}$$

which is associated with the well-known Gauss-Bonnet formula for 2-dimensional Riemannian submanifold without boundary, see pp 358 of Kobayashi and Nomizu [1].

Lemma A.1(Gauss-Bonnet formula([1])). *Let $N \subseteq \mathbb{R}^3$ be an orientable, compact, smooth 2-dimensional submanifold without boundary. Then,*

$$\int_N \sigma_2 dV = 2\pi\chi(N) \tag{1.7}$$

where $\chi(N) \in \mathbb{Z}$ denotes the Euler characteristic number of N.

For any fixed $n \geq 3$, one of the geometric interests for the measure $\sigma_n dV$ is the generalization of classical Gauss-Bonnet formula in higher dimensional space, see Fenchel [2], Allendoerfer [3], Allendoerfer and Weil [4], Chern [5, 6], Chern and Lasf [7] and so on.

Another geometric interest is to the measure $\sigma_n dV$ in the well-known Steiner-Weyl formula, see Weyl [8], Federer [9], Chern [6], Schneider [10] and so on.

Since the measure $\sigma_n dV$ has its geometric origin, and therefore it is natural to get an intrinsic construction of the measure $\sigma_n dV$. In polar coordinates, one can formulate the measure $\sigma_n dV$ as follows:

$$\int_\omega \sigma_n dV = \int_{\nu_N(r_N(\omega))} d\xi \tag{1.8}$$

for any Borel set $\omega \subseteq \mathbb{S}^n$, where ν_N and r_N are the normal mapping and radial mapping of the hypersurface N,

$$\rho_N(\xi) = \max\{\lambda \geq 0 : \lambda\xi \in N\}, \quad \forall \xi \in \mathbb{S}^n, \tag{1.9}$$

and

$$r_N(\xi) = \rho_N(\xi)\xi, \quad \forall \xi \in \mathbb{S}^n, \tag{1.10}$$

see Oliker [11–13] or Schneider [10]. This observation led Aleksandrov to pose the following classical Aleksandrov problem, see Aleksandrov [14, 15], Bakelman [16] or Guan, Li and Li [17].

Problem A.2 (The classical Aleksandrov problem). *For any fixed $n \geq 1$, given a Borel measures μ which is supported on the unit sphere $\mathbb{S}^n$, finds a convex hypersurface $N \subseteq \mathbb{R}^{n+1}$ such that*

$$\int_{\nu_N(r_N(\omega))} d\xi = \mu(\omega) \tag{1.11}$$

for any Borel set $\omega \subseteq \mathbb{S}^n$ where $d\xi$, ν_N and r_N are the standard n-dimensional spherical Lebesgue measure, normal mapping and radial mapping of the hypersurface N.

Aleksandrov [14, 15] solved **Problem A.2.** via the mapping argument which is a kind of method of continuity, see also Bakelman [16] or Pogorelov [18]. Later, from the point of view of nonlinear analysis or PDEs theory, Oliker [11–13] resolved **Problem A.2.** Moreover, concerning the regularity or curvature bounds of the hypersurface, Treibergs [19] and Guan and Li [20] also analyzed the **Problem A.2.** Recently, Huang, Lutwak, Yang and Zhang [21] introduced the so-called p-integral Gaussian curvature measure and posed the L_p Aleksandrov problem which can be stated as follows.

Problem A.3 (L_p Aleksandrov problem ([21]). *For any fixed $n \geq 1$ and $p \in \mathbb{R}$, given a Borel measures μ which is supported on the unit sphere $\mathbb{S}^n$, find a convex hypersurface $N \subseteq \mathbb{R}^{n+1}$ such that*

$$\int_{\nu_N(r_N(\omega))} u^{1-p} d\xi = \mu(\omega)) \tag{1.12}$$

for any Borel set $\omega \subseteq \mathbb{S}^n$ where $d\xi$, ν_N, u and r_N are the standard spherical Lebesgue measure, normal mapping, support function and radial mapping of the hypersurface N.

Recently, more and more interesting geometric analysis has been focused on the weighted measure $e^{-\varphi(|x|^2)}dx$, see [22–27] and so on [28–34]. It may be interesting to mention that the convexity of φ can deduce some interesting geometric inequalities for the measure $e^{-\varphi(|x|^2)}dx$, such as Brunn-Minkowski inequality, Prékopa-Leindler inequalities or Blaschke-Santaló inequalities, see [10, 35–38].

It is interesting to focus on *the geometry of weighted measure $e^{-\varphi(|x|^2)}dx$ without the assumption of convexity of φ.*

If φ is concave, we call the measure $e^{-\varphi(|x|^2)}dx$ a log-convex measure.

One interest in *the geometry of log-convex measure $e^{-\varphi(|x|^2)}dx$* is the so-called log-convex density conjecture in geometric measure theory which can be stated as follows.

Problem A.4 (Log-convex density conjecture). *In $\mathbb{R}^{n+1}$ with a smooth, radial, log-convex density, balls about the origin provide isoperimetric regions of any given volume.*

The Log-convex density conjecture was posed by Brakke and solved by Chambers [39]. More interesting comments on this topic can be referred to [40–43].

Motivated by these beautiful results mentioned above, the main focus of the present paper is on L_p Aleksandrov problem for log-convex measure $e^{-\varphi(|x|^2)}dx$ in the frame of Riemannian Geometry.

It is well-known that the main language of Riemannian Geometry is the so-called tensor, see Bishop and Goldberg [44] or Gerretsen [45]. This leads to our consideration on the problem in tensor spaces. By the analysis mentioned above, the core concept is the concept of Gaussian curvature. It is easy to see that the Gaussian curvature of a hypersurface can be calculated by means of the metric and second fundamental form of the hypersurface, see Kobayashi and Nomizu [1]. Therefore, to formulate a natural generalization of the classical L_p Aleksandrov problem in tensor spaces, we need to replace the second fundamental form by some interesting symmetric tensors and pose a natural generalization of Gauss curvature. Since the second fundamental form of any hypersurface in a space of constant curvature satisfies the Codazzi equation, we may say a natural generalization of the second fundamental form of the hypersurface is the so-called Codazzi tensor of Riemannian manifolds in higher dimensional tensor space which is defined as follows.

For any connected smooth n-dimensional Riemannian manifolds (M, g), we let ST_2 be the bundle of smooth symmetric $(0, 2)$ type tensor field over M, the covariant differential in the metric g is denoted by ∇_X where X is a vector field from the tangle bundle TM. The so-called Codazzi tensor is defined as follows:

Definition A.5 (Cadazzi tensor [46]). *Let $A : M \to ST_2$ be a smooth section. It is called a Codazzi tensor if A satisfies Codazzi equation,*

$$\nabla_X A(Y, Z) = \nabla_Y A(X, Z) \tag{1.13}$$

for any $X, Y, Z \in TM$. The set of Codazzi tensors on M is denoted by $Cod(M, g)$.

In particular, the second fundamental form of any hypersurface in a space of constant curvature is a Codazzi tensor, see pp. 26 of Kobayashi and Nomizu [1].

Some basic differential geometric theories about the Codazzi tensor are listed as follows, see [46].

Let $x : M \mapsto \mathbb{R}^{n+1}$ be an isometric immersion and assume also that rank $A = n$ at z. Then rank $A = n$ in some neighborhood U of z. Suppose that ξ is the unit normal vector field over $x(U)$.

Lemma A.6 ([46]).

(i) Let $x : M \mapsto \mathbb{R}^{n+1}$ be an isometric immersion and ξ be the unit normal vector field over $x(U)$. Then "support" function $f(\xi) = -(x, \xi)$ where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^{n+1}$.

(ii) the second order covariant differential of f and the coefficient of the second fundamental form $(b_{ij})_{n\times n}$ satisfies

$$b_{ij} = f_{ij} + g_{ij} f. \tag{1.14}$$

Lemma A.7 ([46]). *Let (M, g) be a Riemannian manifold of constant sectional curvature K_{sec} (possibly zero) and $A \in Cod(M, g)$. Then for every point on M, there exists a neighborhood V and a smooth function $f : M \to \mathbb{R}$ such that in V*

$$(A)_{ij}(f) = f_{ij} + K_{sec} g_{ij} f \tag{1.15}$$

where $(A)_{ij}(f)$ is the coefficient of $A = A(f)$. In addition, if M is simply connected then such representation is available on the entire M. Conversely, on a manifold of constant curvature K_{sec}, any smooth function f generates a Codazzi tensor $A(f)$ via Eq (1.15).

For any $f \in C^\infty(M)$, we let $A = (A_{ij})_{n\times n}$ be the Codazzi tensor generated by f, that is, A_{ij} is given by (1.15). Let

$$P_n^0(f) = \frac{\det(f_{ij}(z) + K_{sec} g_{ij} f)}{\det g_{ij}} = \frac{\det A}{\det g} \tag{1.16}$$

Remark A.8. *It is easy to see that P_n^0 satisfies the following equation*

$$P_n^0(f) = \lambda_1 \lambda_2 \cdots \lambda_n \tag{1.17}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are n solutions to the following equation

$$\det(A - \lambda g) = 0. \tag{1.18}$$

This means that the geometric meaning of P_n^0 is that P_n^0 is a concept of "Gaussian" curvature for the quadratic form A.

Oliker and Simon [46] proved the following interesting prescribed Gaussian curvature problem for Codazzi tensor:

Theorem A.9 ([46]). *Let (M, g) be a closed Riemannian manifold with constant sectional curvature $K_{sec} \neq 0$, and $\phi : M \to (0, \infty)$ is a strictly positive C^∞ function. If $K_{sec} > 0$ suppose also that M is not isometrically diffeomorphic to the sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Then there exists a (unique) function $f \in C^\infty(M)$ such that is a positive definite Codazzi tensor on M and*

$$P_n^0(f) = \phi. \tag{1.19}$$

Suppose the sectional curvature K_{sec} of M is positive constant, with loss of generality, we may assume that $K_{sec} = 1$. In (1.5), if we replace the standard Lebesgue dx with $e^{-\varphi(|x|^2)}dx$, the p-integral Gaussian curvature function of Codazzi tensor with log-convex measure $e^{-\varphi(|x|^2)}dx$ can be defined as follows:

$$f^{1-p}e^{-\varphi(\rho^2)}\sigma_n\sqrt{\det g_{ij}}. \tag{1.20}$$

Therefore, we let

$$P_{n,p}(f) = P_n^0 f^{1-p}e^{-\varphi(\rho^2)}\sqrt{\det g_{ij}} = f^{1-p}e^{-\varphi(\rho^2)}\frac{\det(f_{ij}(z) + g_{ij}f)}{\sqrt{\det g_{ij}}} \tag{1.21}$$

and call $P_{n,p}(f)$ as p-integral Gaussian curvature function for Codazzi tensor with log-convex measure $e^{-\varphi(|x|^2)}dx$.

In the present paper, we focus on L_p Aleksandrov problem for Codazzi tensor with log-convex measures $e^{-\varphi(|x|^2)}dx$ which is stated as follows:

Problem A.10 (L_p Minkowski problem for Codazzi tensor with log-convex measure). *For any fixed $n \geq 1$ and $p \in \mathbb{R}$, does there exist a Codazzi tensor A whose sectional curvature is 1 and is generated by f such that*

$$P_{n,p}(f) = \phi? \tag{1.22}$$

The main result of the present paper can be stated as follows.

Theorem 1.1. *For any fixed $n \geq 1$ and $p > n+1$, there exist positive constants c, τ and a positive solution $S \in C^{2,\tau}(M)$ to the Eq (1.1) satisfying*

$$0 < c^{-1} \leq ||S||_{C^{2,\tau}(M)} \leq c < \infty \tag{1.23}$$

where $\tau \in (0, 1)$, c is independent of S provided the following conditions hold.

(A.1.) $0 < \phi \in C^4(M)$, φ is a non-negative, radially symmetric, increasing, smooth and concave function in $\mathbb{R}$, $0 < \phi \in C^4(M)$ and

$$||\phi||_{C^4(M)} + ||\varphi||_{C^4(0,\infty)} < \infty.$$

(A.2.)

$$\lim_{t\to\infty}\frac{t^{n+p-1}}{e^{\varphi(t^2)}} = 0, \lim_{t\to 0}\frac{t^{n+p-1}}{e^{\varphi(t^2)}} = \infty.$$

(A.3.) There exists $\delta_4 > 0$ such that

$$\min_{t\in[a,b]}\varphi'(t^2) + 2\varphi''(t^2)t^2 \geq \delta_4 > 0.$$

for any compact $[a, b] \subseteq (0, \infty)$.

$(A.4.)$

$$\max_{t\in[a,b]} 2\varphi'(t^2)t^2 < p - n - 1.$$

for any compact $[a, b] \subseteq (0, \infty)$.

Remark 1.2. The main result of the present paper can be seen as an attempt at some new results on the integral geometry of differential forms which are associated with some interesting invariants arising in geometry and topology. Apart from the beautiful Gauss-Bonnet Theorem, there are many famous theorems which link the analysis, topology and geometry, such as Euler-Poincaré characteristic formula, Riemann-Roch-Hirzebruch-Grothendieck Theorem, Atiyah-Singer index Theorem, Chern-Simons invariants, see Palais [47], Chern-Simons [48], Atiyah [49], Shanahan [50], Mukherjee [51], Moore [52], Gilkey [53], Freed [54] and so on. Following some classical ideas of Steiner, Federer and Chern, it may be interesting to focus on some kinematic formulas for these invariants and consider analogous geometric problems which prescribe differential forms for these invariants. In particular, in the frame of Kähler Manifolds or Symplectic Manifolds, some great results on this topic can be referred to the great works of Patodi [55], Duistermaat [56], Atiyah and Bott [57], Karshon and Tolman [58], Donaldson [59–61], Abreu [62], Grossberg and Karshonand [63], Boyer, Calderbank and Tønnesen-Friedman [64] and so on. In our forthcoming study, we will focus on these topics.

Our proof of Theorem 1.1 is based on the well-known continuous method. We let the set of the positive continuous function on M be $C_+(M)$ and

$$C = \{S \in C^{2,\tau}(M) : (S_{ij} + g_{ij}S)_{n\times n} \text{ is positive definite}\}$$

The main ingredient is the a priori bounds of solutions to the following auxiliary problem for any $S \in C$:

$$S^{1-p}e^{-\varphi(\rho^2)}\frac{\det(S_{ij}(z) + S(z)g_{ij})}{\sqrt{\det g_{ij}}} = t\phi(\xi) + (1-t)e^{-\varphi(1)}\sqrt{\det g_{ij}} \tag{1.24}$$

for $t \in [0, 1]$.

Remark 1.3. It is worth mentioning that without the assumption of convexity of φ, some necessary geometric inequalities have not been established which does not guarantee the validity of the classical variational framework for **Problem A.10**. Therefore, we adopt the well-known continuous method to solve this problem. Moreover, by the a priori bounds (1.23) of S, we get the compactness of the solution set and the *curvature estimate* of the Codazzi tensor which have their independent interests.

Remark 1.4. Just like the case of concavity, log-convexity may be defined in the form of Prékopa-Leindler inequality: a function (or functional) $f : I \mapsto \mathbb{R}$ is called log-convex if f satisfies the following condition,

$$f(tx + (1-t)y) \leq f^t(x)f^{1-t}(y) \tag{1.25}$$

for any $t \in [0, 1]$ and $x, y \in I$. The former case can be referred to pp. 369–374 of Schneider [10]. Some inequalities for log-convex functions (or functionals) have been analyzed in [65–68] and so on. It may be worth mentioning that Klartag [22] and Rotem [28] introduced some geometric notions for log-concave or more general α-concave functions and measures, such as the support function and mean width. Motivated by these interesting results, it may be interesting to introduce similar notions and get more geometric analysis for log-convex measures and this will also be a topic of our future study.

The remaining part of this paper is arranged as follows: In Section 2, we prove the a priori bounds of S. In Section 3, we prove Theorem 1.1.

2. A priori bounds of S

In this section, we consider the a priori bounds of solutions to the following equation on Riemannian manifolds (M, g):

$$S^{1-p}e^{-\varphi(\rho^2)}\frac{\det(S_{ij}+g_{ij}S)}{\sqrt{\det g_{ij}}}=\phi(\xi) \tag{2.1}$$

where $\rho^2 = |\nabla S|^2 + S^2$ and the following conditions hold.

$(A.1.)$ $0 < \phi \in C^4(M)$, φ is a non-negative, radially symmetric, increasing, smooth and convex function in $\mathbb{R}$, $0 < \phi \in C^4(M)$ and

$$\|\phi\|_{C^4(M)} + \|\varphi\|_{C^4(0,\infty)} < \infty.$$

$(A.2.)$

$$\lim_{t\to\infty}\frac{t^{n+p-1}}{e^{\varphi(t^2)}}=0, \lim_{t\to 0}\frac{t^{n+p-1}}{e^{\varphi(t^2)}}=\infty.$$

$(A.3.)$ There exist $\delta_4 > 0$ such that

$$\min_{t\in[a,b]}\varphi'(t^2)+2\varphi''(t^2)t^2\geq\delta_4>0.$$

for any compact $[a, b] \subseteq (0, \infty)$.

We let the set of the positive continuous function on M be $C_+(M)$,

$$C = \{S \in C^{2,\tau}(M) : (S_{ij} + g_{ij}S)_{n\times n} \text{ is positive definite}\}$$

and

$$\widetilde{C} = \{u \in C^{2,\tau}(M) : (u_{ij})_{n\times n} \text{ is positive definite}\}$$

This main result of this section can be stated as follows,

Theorem 2.0. *For any fixed $n \geq 1$ and $p > n + 1$, we let $S \in C \cap C_+(M)$ be a solution to (2.1). Suppose that $(A.1) \sim (A.3)$ hold. Then there exists a positive constant c, independent of S, such that*

$$0 < c^{-1} \leq \|S\|_{C^{2,\tau}(M)} \leq c < \infty, \tag{2.2}$$

where $\tau \in (0, 1)$.

Now, we divide the proof of Theorem 2.0 into following four steps.

Step (a). *For any fixed $n \geq 1$ and $p > n + 1$, we let $S \in C \cap C_+(M)$ be a solution to (2.1). Suppose that $(A.1) \sim (A.3)$ hold. Then there exists a positive constant c such that*

$$0 < c^{-1} \leq S(\xi) \leq c < \infty \tag{2.3}$$

for any $z \in M$.

Proof of Step (a). We consider the following extremal problem,

$$R = \max_{z\in M} S(z). \tag{2.4}$$

It follows from the compactness of M and the continuity of S that there exists $z_1 \in M$ such that

$$R = S(z_1). \tag{2.5}$$

It follows from the Eq (2.1) that at the point $z = z_1$,

$$0 < \sqrt{\det g_{ij}} \min_{z\in M} \phi(z) \le \sqrt{\det g_{ij}} \phi(z_1) \le \frac{R^{n+1-p}}{e^{\varphi(R^2)}}. \tag{2.6}$$

Combining this and condition $(A.1)$, we can see that there exists a positive constant $c > 0$ such that

$$R \le c < \infty. \tag{2.7}$$

We next consider the following extremal problem,

$$r = \min_{z\in M} S(z). \tag{2.8}$$

Adopting a similar argument mentoned above, we also can see that there exists a positive constant $c > 0$ such that

$$r \ge c > 0. \tag{2.9}$$

(2.7) and (2.9) yield the desired conclusion of **Step** (a).

Step (b)**.** *For any fixed $n \ge 1$ and $p > n+1$, we let $S \in C \cap C_+(M)$ be a solution to (2.1). Suppose that $(A.1) \sim (A.3)$ hold. Then there exists a positive constant c such that*

$$0 \le |\nabla S(z)|^2 \le c, \forall z \in M. \tag{2.10}$$

Proof of Step (b)**.** The proof is based on Maximum Principle. We let

$$v = \frac{S^2 + |\nabla S|^2}{2} = \frac{1}{2}(S^2 + \Sigma_{ij} g^{ij} S_i S_j). \tag{2.11}$$

Suppose that there exists $z_0 \in M$ such that

$$v(z_0) = \max_{z\in M} v(z). \tag{2.12}$$

Then,

$$0 = \nabla_l v = 2(S S_l + \Sigma_{ij} g^{ij} S_{li} S_j) = 2\Sigma_{ij} g^{ji} (S_{il} + S g_{il}) S_j \tag{2.13}$$

for any fixed $l \in \{1, 2, \cdots, n\}$ at the point z_0. It follows from Lemma 2.1 that there exists a positive constant c,

$$\det(S_{il} + S g_{il}) = S^{p-1} \phi(z) e^{\varphi(\rho^2)} \det g_{il} \ge S^{p-1} \phi(z) e^{\varphi(0)} \det g_{il} \ge c_0 \det g_{il} \tag{2.14}$$

at the point $z = z_0$ where

$$c_0 = e^{\varphi(0)} \min_{z\in M} S^{p-1}(z) \phi(z) > 0. \tag{2.15}$$

Noting that $g = (g_{ij})_{n\times n}$ is strictly positive, it follows from (2.14) and (2.15) that the matric $(S_{il} + S g_{il})_{n\times n}$ is reversible at the point $z = z_0$ and therefore,

$$S_l = 0 \tag{2.16}$$

at the point $z = z_0$ for any fixed $l \in \{1, 2, \cdots, n\}$ due to (2.13) and the positivity of g. From (2.16), we can see that

$$|\nabla S|^2 = g^{ij} S_i S_j = 0 \tag{2.17}$$

at the point $z = z_0$. Therefore, it follows from Lemma 2.1 that there exists a positive constant c, independent of S,

$$\frac{1}{2}|\nabla S|^2(z) \leq v(z) \leq v(z_0) \leq \frac{1}{2}\max_{z\in M} S(z) \leq c, \tag{2.18}$$

for any $z \in M$. This completes the proof of **Step** (b).

Before getting the higher order estimates of S, we let

$$u_{ij} = S_{ij} + g_{ij}S, \quad \mathcal{G}(u_{ij}) = (\det u_{ij})^{\frac{1}{n}} \tag{2.19}$$

and

$$\Psi(z) = (\psi(z) S^{p-1} e^{\varphi(\rho^2)} \det g_{ij})^{\frac{1}{n}}. \tag{2.20}$$

Then, Eq (2.1) becomes

$$\mathcal{G}(u_{ij}) = \Psi. \tag{2.21}$$

Step (c)**.** *For any fixed $n \geq 1$ and $p > n + 1$, we let $S \in C \cap C_+(M)$ be a solution to* (2.1) *and $u \in \widetilde{C} \cap C_+(M)$ be a solution to* (2.21). *Suppose that* (A.1) $\sim$ (A.3) *hold. Then there exists a positive constant c, independent of S, such that*

$$\Delta u \leq c. \tag{2.22}$$

Proof of Step (c)**.** We let $H = \sum_i u_{ii}$. Suppose that H achieves it maximum at the point $z = z_0$. Without loss of generality, we may $(H_{ij})_{n\times n}$ is diagonal at the point $z = z_0$. Therefore, at the point $z = z_0$,

$$\nabla H = 0, \tag{2.23}$$

and $(H_{ij})_{n\times n}$ is non-positive. We let

$$G^{ij} = \frac{\partial \mathcal{G}}{\partial u_{ij}}, G^{ij,rs} = \frac{\partial^2 \mathcal{G}}{\partial u_{ij} \partial u_{rs}}. \tag{2.24}$$

for any fixed $i, j, s, t \in \{1, 2, \cdots, n\}$. Therefore, at the point $z = z_0$,

$$0 \geq \Sigma_{ij} G^{ij} H_{ij} = \Sigma_{i\alpha} G^{ii} H_{ii}. \tag{2.25}$$

By the commutator identity, we have,

$$H_{ii} = \Delta u_{ii} - n u_{ii} + H. \tag{2.26}$$

Putting (2.25) into (2.26), we get

$$0 \geq \Sigma_i G^{ii} \Delta u_{ii} - n \Sigma_i G^{ii} u_{ii} + H \Sigma_{i\alpha} G^{ii}. \tag{2.27}$$

Taking the α-th partial derivatives on both sides of (2.21) twice for any fixed $\alpha \in \{1, 2, \cdots, n\}$, we have

$$\Sigma_{ij} G^{ij} u_{ij\alpha} = \Psi_\alpha, \Sigma_{ijst} G^{ij,rs} u_{ij\alpha} u_{rs\alpha} + \Sigma_{ij} G^{ij} (u_{\alpha\alpha})_{ij} = \Psi_{\alpha\alpha} \tag{2.28}$$

for any fixed $\alpha \in \{1, 2, \cdots, n\}$. By the concavity of $\mathcal{G}$, we have

$$\Sigma_{ijst\alpha} G^{ij,rs} u_{ij\alpha} u_{rs\alpha} \leq 0. \tag{2.29}$$

This implies that

$$\Sigma_i G^{ii} \Delta u_{ii} \geq \Sigma_{ij} G^{ij} \Delta u_{ij} + \Sigma_{ijst\alpha} G^{ij,rs} u_{ij\alpha} u_{rs\alpha} = \Delta\Psi. \tag{2.30}$$

at the point $z = z_0$. Therefore,

$$\Sigma_i G^{ii} \Delta u_{ii} \geq \Delta\Psi. \tag{2.31}$$

at the point $z = z_0$. It follows from Newton-MacLaurin inequality that

$$\Sigma_i G^{ii} \geq 1, \tag{2.32}$$

see Guan and Ma [69]. Putting (2.31), (2.32) into (2.27), we have, at the point $z = z_0$,

$$0 \geq \Delta\Psi - n\Psi + H\Sigma_i G^{ii} \geq \Delta\Psi - n\Psi + H \geq \Delta\Psi - n\Psi. \tag{2.33}$$

We let

$$r_1 = \min_{z \in M} \rho^2(z), R_1 = \max_{z \in M} \rho^2(z). \tag{2.34}$$

It follows from Lemmas 2.1 and 2.2 that

$$0 < r_1 \leq R_1 < \infty. \tag{2.35}$$

Now, we claim that at the point $z = z_0$,

$$\frac{\Delta\Psi}{\Psi} - n \geq \frac{2\delta_4}{n} \Sigma_{ij} S_{ij}^2 - c\sqrt{n}\sqrt{\Sigma_{ij} S_{ij}^2} - c \tag{2.36}$$

where $\delta_4 > 0$ to be chosen. Indeed, it follows from the definition of Ψ that

$$\log \Psi = \frac{\log \phi(\xi)}{n} + \frac{p-1}{n} \log S \frac{\varphi(\rho^2)}{n} + \frac{1}{n} \log \det g_{ij}. \tag{2.37}$$

Noting $\rho^2 = |\nabla S|^2 + S^2$, for any fixed $\alpha \in \{1, 2, \cdots, n\}$, taking α-th partial derivatives on both sides of (2.37) twice, we have

$$\frac{\Psi_\alpha}{\Psi} = \frac{1}{n}(\log \phi)' + \frac{p-1}{n} S_\alpha + \frac{2}{n}\varphi'(\rho^2)(\sum_j S_j S_{j\alpha} + S S_\alpha) \tag{2.38}$$

and

$$\begin{aligned} \frac{\Delta\Psi}{\Psi} \geq \Sigma_\alpha\left(\frac{\Psi_{\alpha\alpha}}{\Psi} - \frac{\Psi_\alpha^2}{\Psi^2}\right) = \Sigma_\alpha\Big(&\frac{1}{n}(\log \phi)'' + \frac{p-1}{n} S_{\alpha\alpha} \\ &+ \frac{2\varphi'(\rho^2)}{n}(\Sigma_j S_{j\alpha}^2 + S_j S_{j\alpha\alpha} + S S_{\alpha\alpha} + S_\alpha^2) \\ &+ \frac{4}{n}(\varphi''(\rho^2)(\Sigma_j S_j S_{j\alpha} + S S_\alpha)^2) \\ \triangleq {}& I_1 + I_2. \end{aligned} \tag{2.39}$$

where

$$I_1 = \frac{2\varphi'(\rho^2)}{n}\Sigma_{jl}S_{jl}^2 + \frac{4}{n}\varphi''(\rho^2)\Sigma_l(\Sigma_j S_j S_{jl})^2, \tag{2.40}$$

and

$$I_2 = (\log\phi)'' + \frac{2}{n}(\varphi'(\rho^2) + 2\varphi''(\rho^2)S^2)|\nabla S|^2 + \frac{1}{n}(2\varphi'(\rho^2) + (p-1))\Delta S + \frac{2\varphi'(\rho^2)}{n}\nabla S\cdot\nabla\Delta S. \tag{2.41}$$

We now get some estimates of I_2. Since $\phi \in C^4(M)$, it follows from Lemmas 2.1 and 2.2 that

$$(\log\phi)'' + \frac{2}{n}(\varphi'(\rho^2) + 2\varphi''(\rho^2)S^2)|\nabla S|^2 \geq -c. \tag{2.42}$$

By the definition of H, we have,

$$H = \Delta S + nS. \tag{2.43}$$

Therefore, it follows from Lemma 2.1 and Hölder inequality that

$$\begin{aligned} |\frac{1}{n}(2\varphi'(\rho^2)S + (p-1))\Delta S| &= |\frac{1}{n}(2\varphi'(\rho^2)S + (p-1))(H - nS)| \\ &\leq cH + c \leq c\sqrt{n}\sqrt{\Sigma_i S_{ii}^2} + c \leq c\sqrt{n}\sqrt{\Sigma_{ij}S_{ij}^2} + c \end{aligned} \tag{2.44}$$

for some c which means that

$$\frac{1}{n}(2\varphi'(\rho^2)S + (p-1))\Delta S \geq -c\sqrt{n}\sqrt{\Sigma_{ij}S_{ij}^2} - c. \tag{2.45}$$

Moreover, it follows from (2.43), (2.23) and Lemma 2.2 that

$$\begin{aligned} \frac{2\varphi'(\rho^2)}{n}\nabla S\cdot\nabla\Delta S &= \frac{2\varphi'(\rho^2)}{n}\nabla S\cdot\nabla(H - nS) \\ &= \frac{2\varphi'(\rho^2)}{n}\nabla S\cdot\nabla H - 2\varphi'(\rho^2)|\nabla S|^2 = -2\varphi'(\rho^2)|\nabla S|^2 \geq -c \end{aligned} \tag{2.46}$$

at the point $z = z_0$. Therefore, combining (2.42), (2.45) and (2.46), we have,

$$I_2 \geq -c\sqrt{n}\sqrt{\Sigma_{ij}S_{ij}^2} - c \tag{2.47}$$

at the point $z = z_0$.

Since $\varphi \in C^2$ is concave, we have,

$$\varphi''(\rho^2) \leq 0 \tag{2.48}$$

for any $z \in M$. Noting that

$$2\Sigma_l(\Sigma_j S_j S_{jl})^2 \leq 2\Sigma_l(\Sigma_j S_j^2 \Sigma_j S_{jl}^2) = 2|\nabla S|^2\Sigma_{jl}S_{jl}^2 \leq 2\rho^2\Sigma_{jl}S_{jl}^2. \tag{2.49}$$

Therefore,

$$I_1 = \frac{2\varphi'(\rho^2)}{n}\Sigma_{jl}S_{jl}^2 + \frac{4}{n}\varphi''(\rho^2)\Sigma_l(\Sigma_j S_j S_{jl})^2 \geq \frac{2}{n}(\varphi'(\rho^2) + 2\rho^2\varphi''(\rho^2))\Sigma_{jl}S_{jl}^2. \tag{2.50}$$

We let

$$r_1 = \min_{z\in M}\rho(z), R_1 = \max_{z\in M}\rho(z). \tag{2.51}$$

It follows from **Step** (a) and **Step** (b) that

$$0 < r_1 \le R_1 < \infty. \tag{2.52}$$

Therefore, it follows from (A.3) that there exists $\delta_4 > 0$ such that

$$\min_{z\in M}\varphi'(\rho^2(z)) + 2\varphi''(\rho^2(z))\rho^2(z) \ge \delta_4 > 0. \tag{2.53}$$

Therefore,

$$I_1 \ge \frac{2\delta_4}{n}\Sigma_{ij}S_{ij}^2. \tag{2.54}$$

Therefore, (2.47) and (2.54) yield

$$I_1 + I_2 \ge \frac{2\delta_4}{n}\Sigma_{j,\alpha}S_{j\alpha}^2 - c\sqrt{n}\sqrt{\Sigma_{ij}S_{ij}^2} - c \tag{2.55}$$

at the point $z = z_0$. This is the desired inequality (2.36).

It follows from (2.33) and (2.36) that

$$\Sigma_{ij}S_{ij}^2 \le c \tag{2.56}$$

at the point $z = z_0$. It follows from Hölder inequality that

$$\Delta S = \sqrt{n}\sqrt{\Sigma_i S_{ii}^2} \le \sqrt{n}\sqrt{\Sigma_{ij}S_{ij}^2} \le c \tag{2.57}$$

at the point $z = z_0$. Combining (2.57), the definition of u and **Step** (a), it is easy to get the inequality (2.22) which completes the proof of **Step** (c).

Step (d)**.** It follows from (2.21) that Eq (2.1) becomes

$$\mathcal{F}(u_{ij}) = 0 \tag{2.58}$$

provided $\mathcal{F}(u_{ij}) = \mathcal{G}(u_{ij}) - \Psi$. We let $\mathcal{F}_{ij} = \frac{\partial\mathcal{F}}{\partial u_{ij}}$. It follows from **Step** (a)**, Step** (b) and **Step** (c) that there exist positive constants λ and Λ, independent of S, such that

$$1 \le \frac{\Lambda}{\lambda} < \infty, \tag{2.59}$$

and

$$0 < \lambda|\zeta|^2 \le \mathcal{F}_{ij}\zeta_i\zeta_j \le \Lambda|\zeta|^2, \tag{2.60}$$

for any $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \mathbb{R}^n$. That is,

(d.i) *(2.58) is elliptic uniformly.*

Now, we claim that

(d.ii) *$\mathcal{F}$ is concave with respect to $(u_{ij})_{n\times n}$.*

It follows from the definition of $\mathcal{F}$ that it suffices to prove that $\mathcal{G} = \det^{\frac{1}{n}}$ is concave with respect to $(u_{ij})_{n\times n}$. Indeed, For any $u, v \in C$ and $t \in [0, 1]$, we let $\{\delta_i^1\}_{i=1}^n$ and $\{\delta_i^2\}_{i=1}^n$ be the eigenvalue sequence of

$(u_{ij})_{n\times n}$ and $(v_{ij})_{n\times n}$ respectively. Then $\{t\delta_i^1+(1-t)\delta_i^2\}_{i=1}^n$ is a eigenvalue sequence of $(tu_{ij}+(1-t)v_{ij})_{n\times n}$. Moreover, since $(u_{ij})_{n\times n}$ and $(v_{ij})_{n\times n}$ are convex, we have

$$\delta_i^j \geq 0 \tag{2.61}$$

for any fixed $i \in \{1,2,\cdots,n\}$ and $j \in \{1,2\}$ and therefore,

$$(\prod_{i=1}^n (t\delta_i^1 + (1-t)t\delta_i^2))^{\frac{1}{n}} \geq t(\prod_{i=1}^n (\delta_i^1)^{\frac{1}{n}} + (1-t)(\prod_{i=1}^n (\delta_i^2))^{\frac{1}{n}} \tag{2.62}$$

for any $t \in [0,1]$. Combining the definition of $\mathcal{G}$ and (2.62), we get

$$\mathcal{G}(tu_{ij} + (1-t)v_{ij}) \geq t\mathcal{G}(u_{ij}) + (1-t)\mathcal{G}(v_{ij}) \tag{2.63}$$

which proves that $\mathcal{G} = \det^{\frac{1}{n}}$ is concave with respect to $(u_{ij})_{n\times n}$ and this completes the proof of the claim.

Then, it follows from **(d.i)**, **(d.ii)** and **Theorem 17.14** of Gilbarg and Trudinger [70] that there exist $\tau_1 \in (0,1)$ and positive constant c, independent of S, such that

$$\|u\|_{C^{2,\tau_1}(M)} \leq c, \tag{2.64}$$

(see pp. 457–461 of Gilbarg and Trudinger [70]). Therefore there exist $\tau \in (0,1)$ and positive constant c, independent of S, such that

$$\|S\|_{C^{2,\tau}(M)} \leq c, \tag{2.65}$$

This is the desired conclusion of Theorem 2.0.

3. Existence

This section is devoted to the proof of Theorem 1.1.

Motivated by [69, 71] and so on, we consider the following auxiliary problem with a parameter $t \in [0,1]$,

$$S^{1-p}e^{-\varphi(\rho^2)}\frac{\det(S_{ij}(z) + S(z)g_{ij})}{\sqrt{\det g_{ij}}} = t\phi(z) + (1-t)e^{-\varphi(1)}\sqrt{\det g_{ij}} \triangleq f_t \tag{3.1}$$

for any $z \in M$ where $\rho^2 = |\nabla S|^2 + S^2 = g^{ij}S_iS_j + S^2$, $0 < \phi \in C^4(M)$ and the following conditions hold.

$(A.1.)$ $0 < \phi \in C^4(M)$, φ is a non-negative, radially symmetric, increasing, smooth and convex function in $\mathbb{R}$, $0 < \phi \in C^4(M)$ and

$$\|\phi\|_{C^4(M)} + \|\varphi\|_{C^4(0,\infty)} < \infty.$$

$(A.2.)$

$$\lim_{t\to\infty}\frac{t^{n+p-1}}{e^{\varphi(t^2)}} = 0, \lim_{t\to 0}\frac{t^{n+p-1}}{e^{\varphi(t^2)}} = \infty.$$

$(A.3.)$ There exist $\delta_4 > 0$ such that

$$\min_{t\in[a,b]} \varphi'(t^2) + 2\varphi''(t^2)t^2 \geq \delta_4 > 0.$$

for any compact $[a, b] \subseteq (0, \infty)$.

$(A.4.)$

$$\max_{t\in[a,b]} 2\varphi'(t^2)t^2 < p - n - 1.$$

for any compact $[a, b] \subseteq (0, \infty)$.

We let the set of the positive continuous function on the Riemannian manifolds (M, g) be $C_+(M)$ and

$$C = \{S \in C^{2,\tau}(M) : (S_{ij} + S g_{ij})_{n\times n} \text{ is positive definite}\}$$

$$\mathcal{I} = \{t \in [0, 1] : S \in C \cap C_+(M),\ (3.1) \text{ is solvable.}\} \tag{3.2}$$

Since f_t is a contious function, independent of z, satisfying

$$0 < \min\{e^{\varphi(1)}, \min_{z\in M} \phi(z)\} \leq f_t(z) \leq \max\{e^{\varphi(1)}, \max_{z\in M} \phi(z)\} < \infty,$$

for any $t \in [0, 1]$ and $z \in M$, adopting some similar arguments in Section 2, we get

Lemma 3.1. *For any fixed $n \geq 1$, $p > n + 1$ and $t \in [0, 1]$, we let $S_t \in C \cap C_+(M)$ be a solution to (3.1). Suppose that $(A.1) \sim (A.3)$ hold. Then there exists a constant c, independent of t, such that*

$$0 < c^{-1} \leq \|S_t\|_{C^{2,\tau}(M)} \leq c,$$

for any $t \in [0, 1]$ and some $\tau \in (0, 1)$.

As a corollary of Lemma 3.1, we have,

Corollary 3.2. *For any fixed $n \geq 1$, $p > n + 1$, we let $\mathcal{I}$ is the set defined in* (3.2). *Suppose that* $(A.1) \sim (A.3)$ *hold. Then $\mathcal{I}$ is closed.*

Proof. It suffices to show that for any sequence $\{t_j\}_{j=1}^{\infty} \subseteq \mathcal{I}$ satisfying

$$t_j \to t_0,$$

as $j \to \infty$ for some $t_0 \in [0, 1]$, we need to prove $t_0 \in \mathcal{I}$.

We let S^j be a solutions of problem (3.1) at $t = t_j$. It follows from the conclusion of Lemma 3.1 that there exists a positive constant c, independent of j such that

$$\|S^j\|_{C^{2,\tau}(M)} \leq c,$$

it follows from Ascoli-Arzela Theorem that up to a subsequence, there exists a $S^0 \in C^2(M)$, such that

$$\|S^j - S^0\|_{C^2(M)} \to 0$$

as $j \to \infty$. It is easy to see that

$$(S^j)^{1-p} \to (S^0)^{1-p}, \rho_j \to \rho_0 \tag{3.3}$$

uniformly on M as $j \to \infty$ where $(\rho^j)^2 = (S^j)^2 + |\nabla S^j|^2$ for any $j \in \{0, \cdots\}$. Letting $j \to \infty$, we can see that (t_0, S^0) is a solution to the following problem:

$$S^{1-p} e^{-\varphi(\rho^2)} \frac{\det(S_{ij}(\xi) + S g_{ij})}{\sqrt{\det g_{ij}}} = t\phi(z) + (1 - t)e^{-\varphi(1)} \sqrt{\det g_{ij}} \tag{3.4}$$

for any $z \in M$. (3.4) implies that $t_0 \in \mathcal{I}$. This is the desired conclusion of Corollary 3.2.

Lemma 3.3. *For any fixed $n \geq 1$, $p > n+1$, we let $\mathcal{I}$ is the set defined in* (3.2). *Suppose that* $(A.1) \sim (A.4)$ *hold. Then $\mathcal{I}$ is open.*

Proof. Suppose that there exists a $\bar{t} \in \mathcal{I}$, it suffices to prove $t \in \mathcal{I}$ for any $t \in B_\delta(\bar{t}) \cap [0,1]$. To achieve this goal, joint with Implicit Function Theorem, we need to analyze the kernel of linearized equation associated to (3.1). We assume that $\bar{S}$ is a solution to (3.1) at $t = \bar{t}$. For any $\zeta \in M$, we let

$$M(S) = S^{1-p} e^{-\varphi(\rho^2)} \rho^{n+1} \frac{\det(S_{ij}(\xi) + S g_{ij})}{\det g_{ij}}, f_t = t\phi(\xi) + (1-t)e^{-\varphi(1)} \sqrt{\det g_{ij}}, \tag{3.5}$$

$$G_t(S) = M(S) - f_t, M[\bar{S}](\zeta) = \frac{d}{d\varepsilon} M(\bar{S} + \varepsilon\zeta)|_{\varepsilon=0}, \tag{3.6}$$

and

$$G_t[\bar{S}](\zeta) = \frac{d}{d\varepsilon} G_t(\bar{S} + \varepsilon\zeta)|_{\varepsilon=0} = \frac{d}{d\varepsilon} M(\bar{S} + \varepsilon\zeta)|_{\varepsilon=0}. \tag{3.7}$$

By the Eq (3.1), we have

$$M(\bar{S}) = f_t. \tag{3.8}$$

Taking logarithm on both sides of (3.8), since f_t is independent of $\bar{S}$, we get,

$$\frac{M'[\bar{S}](\zeta)}{M(\bar{S})} = \frac{1-p}{\bar{S}} \zeta + 2\varphi'(\bar{\rho}^2)(\bar{S}\zeta + \nabla\bar{S} \cdot \nabla\zeta) + \bar{P}_{ij} B(\zeta) \tag{3.9}$$

where $(\bar{P}_{ij})_{n\times n}$ is the inverse of the matrix $(\bar{S}_{ij} + \bar{S} g_{ij})_{n\times n}$ and

$$B(\zeta) = \zeta_{ij} + \zeta g_{ij}. \tag{3.10}$$

We let $\zeta = \bar{S} v$. Direct Calculation shows that

$$\zeta_i = \bar{S} v_i + \bar{S}_i v \tag{3.11}$$

and

$$\zeta_{ij} = \bar{S} v_{ij} + (\bar{S}_i v_j + \bar{S}_j v_i) + \bar{S}_{ij} v. \tag{3.12}$$

Therefore, we get

$$\bar{S}\zeta + \nabla\bar{S} \cdot \nabla\zeta = (\bar{S}^2 + |\nabla\bar{S}|^2) v + \bar{S}\nabla\bar{S} \cdot \nabla v = \bar{\rho}^2 v + \bar{S}\nabla\bar{S} \cdot \nabla v \tag{3.13}$$

which implies that

$$\frac{1-p}{\bar{S}} \zeta + (2\varphi'(\bar{\rho}^2))(\bar{S}\zeta + \nabla\bar{S} \cdot \nabla\zeta) = (1 - p + (2\varphi'(\bar{\rho}^2)\bar{\rho}^2) v + 2\varphi'(\bar{\rho}^2)\bar{S}\nabla\bar{S} \cdot \nabla v. \tag{3.14}$$

It follows from (3.12) and (3.10) that

$$\begin{aligned} B(\zeta) &= \bar{S} v_{ij} + (\bar{S}_i v_j + \bar{S}_j v_i) + (\bar{S}_{ij} + \bar{S} g_{ij}) v \\ &= \bar{S}(v_{ij} + g_{ij} v) + (\bar{S}_i v_j + \bar{S}_j v_i) + (\bar{S}_{ij} + \bar{S} g_{ij}) v - \bar{S} g_{ij} v \end{aligned} \tag{3.15}$$

and thus,

$$\bar{P}_{ij} B(\zeta) = \bar{S}\bar{P}_{ij} v_{ij} + 2\bar{P}_{ij}\bar{S}_i v_j + nv - \bar{S}\Sigma_i P_i v \tag{3.16}$$

due to the symmetry of $(\bar{P}_{ij})_{n\times n}$. Putting (3.14) and (3.16) into (3.9), we have,

$$\begin{aligned} G[\bar{S}](v) = M[\bar{S}](v) &= \bar{S}M(\bar{S})\bar{P}_{ij}v_{ij} + 2M(\bar{S})\bar{P}_{ij}\bar{S}_j v_i + (2\varphi'(\bar{\rho}^2) + (n+1)\rho^{n-1})M(\bar{S})\bar{S}\nabla\bar{S}\cdot\nabla v \\ &+ (n+1-p+(2\varphi'(\bar{\rho}^2)\bar{\rho}^2) - \bar{S}\Sigma_i P_{ij}g_{ij})Mv \triangleq a_{ij}v_{ij} + b_i v_i + Nv \end{aligned} \tag{3.17}$$

where

$$a_{ij} = \bar{S}M(\bar{S})\bar{P}_{ij}, b_i = 2M(\bar{S})\bar{P}_{ij}\bar{S}_j - 2\varphi'(\bar{\rho}^2)M(\bar{S})\bar{S}\bar{S}_i \tag{3.18}$$

and

$$N = (n+1-p+(2\varphi'(\bar{\rho}^2)\bar{\rho}^2) - \bar{S}\Sigma_i P_{ij}g_{ij})M(\bar{S}). \tag{3.19}$$

Since $\bar{S}, M(\bar{S}) > 0$, $(\bar{P}_{ij})_{n\times n}$ is positive, we see that $(a_{ij})_{n\times n}$ is positive. It follows from Lemma 3.1 that b_i is bounded. By the condition $(A.4.)$, we have

$$n+1-p+2\varphi'(\bar{\rho}^2)\bar{\rho}^2 < 0. \tag{3.20}$$

Since $M(\bar{S})$ is positive, we have,

$$-\bar{S}\Sigma_{ij}P_{ij}g_{ij})M(\bar{S}) < 0. \tag{3.21}$$

Therefore, it follows from (3.20) and (3.21) we get $N < 0$. By Strong Maximum Principle for elliptic equations of second order, we see that

$$v \equiv 0 \tag{3.22}$$

(see pp. 35 of Gilbarg and Trudinger [70]) and thus,

$$\zeta \equiv 0 \tag{3.23}$$

since $\bar{S} > 0$. Then by the standard Implicit Function Theorem, for any $t \in B_\delta(\bar{t}) \cap [0,1]$, there exists a $S \in C^{2,\tau}(M)$, such that $G_t(S) = 0$. This means that $t \in \mathcal{I}$ and completes the proof of Lemma 3.3.

Final proof of Theorem 1.1. It is easy to see that $S \equiv 1$ is a solution of (3.1) at $t = 0$. This means that $\mathcal{I}$ is not-empty. This, together with Corollary 3.2 and Lemma 3.3, implies that $\mathcal{I} = [0,1]$. Taking $t = 1$, we get the proof of Theorem 1.1.

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Conflict of interest

There is no conflict of interest in this work.

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