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Research article

Existence and asymptotical behavior of the ground state solution for the Choquard equation on lattice graphs

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Abstract: In this paper, we study the nonlinear Choquard equation

$$-\Delta u + V(x)u = \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^{p-2}u$$

on lattice graph $\mathbb{Z}^N$. Under some suitable assumptions, we prove the existence of a ground state solution of the equation on the graph when the function V is periodic or confining. Moreover, when the potential function $V(x) = \lambda a(x) + 1$ is confining, we obtain the asymptotic properties of the solution u_λ which converges to a solution of a corresponding Dirichlet problem as $\lambda \to \infty$.

Keywords: ground state solution; Brézis-Lieb Lemma; lattice graph; choquard equation

1. Introduction

In the present paper, we proved the existence of a ground state solution of the following nonlinear Choquard equation

$$\begin{cases} -\Delta u + V(x)u = \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^{p-2}u \quad x \in \mathbb{Z}^N, \\ u \in H^1(\mathbb{Z}^N), \end{cases} \tag{1.1}$$

on lattice graph $\mathbb{Z}^N$. This equation can be viewed as a discrete version of the following Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\alpha \in (0, N)$, $p > 1$ and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined at $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{and} \quad A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{N/2}2^\alpha},$$

with Γ being the Euler gamma function.

In the past few decades, many mathematicians have been devoted to studying the Eq (1.2), for example, see [1–6]. In particular, if $N = 3$, $V = 1$ and $p = 2$, i.e., $-\Delta u + u = (I_2 * |u|^2)u$, appeared in the literature at least as early as in 1954's work by Pekar on quantum theory of a Polaron at rest [7]. Later in the 1970s, Choquard utilized model (1.2) to describe an electron caught in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [1]. Particularly, the equation is also knows as the Schrödinger-Newton equation, which was used to a model of self-gravitating matter [8]. Also, the article [9] used this system to study the pseudo-relativistic boson stars. In a pioneering work, Lieb [1] proved the existence and uniqueness of the ground state to the Eq (1.2) in $\mathbb{R}^3$ with $V = 1$, $\alpha = 2$ and $p = 2$. In the paper [3], Moroz and Van Schaftingen first obtained the sharp range of the parameter for the existence of solutions of the Eq (1.2) with $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. If V is the periodic function, since the nonlocal term is invariant under translation, the paper [10] got the existence results. Furthermore, Alves [11] proved the existence and convergence of nontrivial solutions of the nonlocal Choquard equation. There are tremendous results on this direction in [12–18] and the references therein.

On the other hand, the analysis on the graph has become more and more popular, for example, see [19–27]. In a series of work of Grigor'yan et al. [19–21], they studied the Yamabe type equations, Kazdan-Warner equation and some other nonlinear equations on graph by using the variational methods. In [27], Zhang and Zhao investigated the existence of nontrivial solution of the equation $-\Delta u + (\lambda a(x) + 1)u = |u|^{p-1}u$ on the locally finite graphs by using Nehari methods (see [28]) and the asymptotic properties of the solution. Later, the paper [22] generalized the results of [27] to higher order. Furthermore, Hua and Xu [24] obtained the existence results of nonlinear equation $-\Delta u + V(x)u = f$ on the lattice graph $\mathbb{Z}^N$. Recently, Huang et al. investigated extensively the Mean field equation and the relativistic Abelian Chern-Simons equations on the finite graphs by using the variational method in [23]. For other related results about the graph, we refer the reader to [29–34] and references therein.

Inspired by the poineering works, in this paper we study the existence and asymptotical behavior of solution for the Choquard equation (1.2) on the lattice graph $\mathbb{Z}^N$. For clarity, let us introduce the basic setting on the lattice graph $\mathbb{Z}^N$. The graph $\mathbb{Z}^N$ consists of the set of vertices

$$V = \{x = (x_1, \cdots, x_N) : x_i \in \mathbb{Z}, 1 \le i \le N\},$$

and the set of edges

$$E = \left\{ \{x, y\} : x, y \in \mathbb{Z}^N, \sum_{i=1}^{N} |x_i - y_i| = 1 \right\}.$$

For any two vertices $x, y \in \mathbb{Z}^N$, the distance $d(x, y)$ between them is defined by

$$d(x, y) := inf\{k : x = x_1 \sim x_2 \sim \ldots \sim x_k = y\},$$

where we write $y \sim x$ if and only if the edge $\{x, y\} \in E$. Assume $\Omega \subset \mathbb{Z}^N$, we say Ω is bounded if $d(x, y)$ is uniformly bounded for any $x, y \in \Omega$. It is easy for us to see that a bounded domain of $\mathbb{Z}^N$ can contain only finite vertices. We denote the boundary of Ω is

$$\partial\Omega := \{y \notin \Omega : \exists x \in \Omega \text{ such that } xy \in E\}.$$

$C(\mathbb{Z}^N)$ denotes the set of real-valued functions on $\mathbb{Z}^N$. For any $u \in C(\mathbb{Z}^N)$, its support set is defined as $\text{supp}(u) = \{x \in \mathbb{Z}^N, u(x) \neq 0\}$. Let $C_c(\mathbb{Z}^N)$ denote the set of all functions of finite support. We can define the associated gradient for any function $u, v \in C(\mathbb{Z}^N)$ by

$$\Gamma(u,v)(x) := \sum_{y\sim x} \frac{1}{2}(u(y) - u(x))(v(y) - v(x)).$$

In particular, let $\Gamma(u) = \Gamma(u,u)$ for simplicity. The length of the gradient of u is written by

$$|\nabla u|(x) := \sqrt{\Gamma(u)(x)} = \left(\sum_{y\sim x} \frac{1}{2}(u(y) - u(x))^2\right)^{1/2}.$$

Let μ be the counting measure on $\mathbb{Z}^N$, i.e., for any subset $A \subset \mathbb{Z}^N$, $\mu(A) := \#\{x : x \in A\}$. For any function f on $\mathbb{Z}^N$, we write

$$\int_{\mathbb{Z}^N} f d\mu := \sum_{x\in\mathbb{Z}^N} f(x),$$

whenever it makes sense. ℓ^p is a space endowed with the norm

$$\|u\|_{\ell^p(\mathbb{Z}^N)} := \begin{cases} \left(\sum_{x\in\mathbb{Z}^N} |u(x)|^p\right)^{\frac{1}{p}} & 1 \leq p < \infty. \\ \sup_{x\in\mathbb{Z}^N} |u(x)| & p = \infty. \end{cases}$$

Assume $u \in C(\mathbb{Z}^N)$, the Laplacican on $\mathbb{Z}^N$ is defined as

$$\Delta u = \sum_{y\sim x} (u(y) - u(x)).$$

The inner product of the Hilbert space $H^1(\mathbb{Z}^N)$ is given by

$$\langle u, v\rangle := \int_{\mathbb{Z}^N} (\Gamma(u,v) + uv) d\mu = \int_{\mathbb{Z}^N} (\nabla u \nabla v + uv) d\mu.$$

Therefore, the corresponding norm reads

$$\|u\|_{H^1(\mathbb{Z}^N)} = \left(\int_{\mathbb{Z}^N} \left(|\nabla u|^2 + u^2\right) d\mu\right)^{\frac{1}{2}}.$$

For a bounded uniformly positive function $V : \mathbb{Z}^N \to \mathbb{R}$, it is natural for us to consider the equivalent norm in $H^1(\mathbb{Z}^N)$ as

$$\|u\|^2 := \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2\right) d\mu.$$

Then we have the conclusions for the Eq (1.1).

Theorem 1.1. *Let $N \in \mathbb{N}^*$, $\alpha \in (0, N)$ and $p \in (\frac{N+\alpha}{N}, \infty)$. Suppose that $V(x) : \mathbb{Z}^N \to \mathbb{R}$ satisfies the conditions:*

(*i*) *V is bounded uniformly positive, i.e. there exist constant $C_1, C_2 > 0$ satisfying $C_1 < V(x) < C_2$ for any $x \in \mathbb{Z}^N$.*

(*ii*) *V is T-periodic, i.e. for the positive integer T, we have $V(x + Te_i) = V(x)$, $\forall x \in \mathbb{Z}^N, 1 \le i \le N$, where e_i is the unit vector in the i-th coordinate.*

Then there exists a ground state solution of (1.1).

Remark 1.2. *The preceding theorem is a discrete version of the results in [3]. As in the paper [24], we use the Concentration-Compactness Principle (P. L. Lions [35,36]) to recover the compactness and prove the existence of ground state solution of* (1.1). *Interestingly, since the discreteness of the graph, the Sobolev embedding on the lattice graph is different from that in the continuous setting, which allows us to remove the upper critical exponents $\frac{N+\alpha}{N-\alpha}$ in the continuous case.*

Next we turn to studying the convergence of the solution for the nonlinear Choquard equation. The results of Schrödinger type equation is already considered in the Euclidean space (see [11, 37]). We may expect that the nonlocal Choquard equation on lattice graphs has some similar results. As the paper [22, 27], we also consider the confining potential $V = \lambda a(x) + 1$, i.e.,

$$-\Delta u + (\lambda a(x) + 1)u = \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x - y|^{N-\alpha}} \right) |u|^{p-2}u. \tag{1.3}$$

To study the problem (1.3), we introduce the following subspace of $H^1(\mathbb{Z}^N)$:

$$E_\lambda(\mathbb{Z}^N) = \left\{ u \in H^1(\mathbb{Z}^N) : \int_{\mathbb{Z}^N} \lambda a(x) u^2 d\mu < +\infty \right\}.$$

It is easy to recognize that the scalar product of $E_\lambda(\mathbb{Z}^N)$ is

$$\langle u, v \rangle_{E_\lambda(\mathbb{Z}^N)} := \int_{\mathbb{Z}^N} (\Gamma(u, v) + (\lambda a(x) + 1)uv) d\mu = \int_{\mathbb{Z}^N} (\nabla u \nabla v + (\lambda a(x) + 1)uv) d\mu.$$

Then we have the following conclusions.

Theorem 1.3. *Let $N \in \mathbb{N}^*$, $\alpha \in (0, N)$ and $p \in [\frac{N+\alpha}{N}, \infty)$. Suppose that $a(x) : \mathbb{Z}^N \to \mathbb{R}$ satisfying*

(*A*1) *$a(x) \ge 0$ and the potential well $\Omega = \{x \in \mathbb{Z}^N : a(x) = 0\}$ is a non-empty, connected and bounded domain in $\mathbb{Z}^N$.*

(*A*2) *There exists a point x_0 satisfying $a(x) \to \infty$ when $d(x, x_0) \to \infty$.*

Then (1.3) *has a ground state solution u_λ for any constant $\lambda > 1$.*

In order to observe the asymptotical properties of u_λ as $\lambda \to \infty$, we first study the following Dirichlet problem.

$$\begin{cases} -\Delta u + u = \left(\sum_{\substack{y \neq x \\ y \in \Omega}} \frac{|u(y)|^p}{|x - y|^{N-\alpha}} \right) |u|^{p-2}u & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We study the Eq (1.4) in the $H_0^1(\Omega)$ with the norm:

$$\|u\|_{H_0^1(\Omega)}^2 := \int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu.$$

Similarly, the Eq (1.4) also possess a ground state solution.

Theorem 1.4. *Let $N \in \mathbb{N}^*$, $\alpha \in (0, N)$ and $p \in (1, \infty)$. Suppose Ω is a non-empty, connected and bounded domain in $\mathbb{Z}^N$. Then the Eq* (1.4) *has a ground state solution $u \in H_0^1(\Omega)$.*

Finally, we show that the solutions u_λ of (1.3) converge to a solution of (1.4) as $\lambda \to \infty$ when the domain in (1.4) is the set of satisfying $a(x) = 0$. On the other words, we obtain the following conclusions.

Theorem 1.5. *Let $N \in \mathbb{N}^*$, $\alpha \in (0, N)$ and $p \in [2, \infty)$. Assume that $a(x)$ satisfies* $(A1)$ *and* $(A2)$*, then for any sequence $\lambda_k \to \infty$, up to a subsequence, the corresponding ground state solutions u_{λ_k} of* (1.3) *converge in $H^1(\mathbb{Z}^N)$ to a ground state solution of* (1.4).

The remaining parts of this paper are organized as follows. In Section 2, we give basic definitions and Lemmas on the lattice graph. In Section 3, we establish the discrete Brézis-Lieb Lemma for the nonlocal term and some important conclusions. Section 4 is devoted to proving Theorem 1.1. Then we complete the proof of Theorems 1.3 and 1.4 in Section 5. Finally, we prove Theorem 1.5 in Section 6.

2. Preliminary results

In this section we give some basic results on the lattice graph. Firstly, we present the formula of integration by parts on lattice graph, which is the basic conclusion when we apply variational methods. Here we omit the concrete proofs and one can refer to [22] for more details.

Lemma 2.1. *Suppose that $u \in H^1(\mathbb{Z}^N)$. Then for any $v \in C_c(\mathbb{Z}^N)$, we obtain*

$$\int_{\mathbb{Z}^N} \nabla u \cdot \nabla v d\mu = \int_{\mathbb{Z}^N} \Gamma(u, v) d\mu = -\int_{\mathbb{Z}^N} \Delta u \cdot v d\mu. \tag{2.1}$$

Lemma 2.2. *Suppose $\Omega \subset \mathbb{Z}^N$ is a bounded domain and $u \in H_0^1(\Omega)$. Then for any $v \in C_c(\Omega)$, we have*

$$\int_{\Omega\cup\partial\Omega} \nabla u \cdot \nabla v d\mu = \int_{\Omega\cup\partial\Omega} \Gamma(u, v) d\mu = -\int_\Omega \Delta u \cdot v d\mu. \tag{2.2}$$

Now we are ready to define the weak solution as follows.

Definition 1. *Assume $u \in H^1(\mathbb{Z}^N)$. A function u is called a weak solution of* (1.1) *if for any $\varphi \in H^1(\mathbb{Z}^N)$,*

$$\int_{\mathbb{Z}^N} \nabla u \nabla \varphi d\mu + \int_{\mathbb{Z}^N} V(x) u\varphi d\mu = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^{p-2} u\varphi d\mu. \tag{2.3}$$

Definition 2. *Assume $u \in E_\lambda(\mathbb{Z}^N)$. A function u is called a weak solution of* (1.3) *if for any $\varphi \in E_\lambda(\mathbb{Z}^N)$,*

$$\int_{\mathbb{Z}^N} \nabla u \nabla \varphi d\mu + \int_{\mathbb{Z}^N} (\lambda a(x) + 1) u\varphi d\mu = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^{p-2} u\varphi d\mu. \tag{2.4}$$

Definition 3. *Assume $u \in H_0^1(\Omega)$. A function u is called a weak solution of* (1.4) *if for any $\varphi \in H_0^1(\Omega)$,*

$$\int_{\Omega\cup\partial\Omega} \nabla u \nabla \varphi d\mu + \int_{\Omega} u\varphi d\mu = \int_{\Omega} \left(\sum_{\substack{y\neq x \\ y\in\Omega}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^{p-2} u\varphi d\mu. \tag{2.5}$$

Notice that if u is a weak solution of (1.1), we infer from Lemma 2.1 that for any test function $\varphi \in H^1(\mathbb{Z}^N)$,

$$\int_{\mathbb{Z}^N} (-\Delta u\varphi d\mu + V(x)u\varphi)\, d\mu = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^{p-2} u\varphi d\mu. \tag{2.6}$$

For any fixed $x_0 \in \mathbb{Z}^N$, choosing a test function $\varphi : \mathbb{Z}^N \to \mathbb{R}$ in (2.6) which is defined as

$$\varphi(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0, \end{cases}$$

we obtain

$$-\Delta u(x_0) + V(x_0)u(x_0) = \left(\sum_{\substack{y\neq x_0 \\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x_0-y|^{N-\alpha}} \right) |u(x_0)|^{p-2} u(x_0),$$

which implies that u is a point wise solution of (1.1). Thus, we have the following conclusion for the relationship between the weak solution and the point wise solution.

Proposition 2.3. *If u is a weak solution of* (1.1)*, then u is a point wise solution. Similarly, if u is a weak solution of* (1.3) *or* (1.4)*, then u is also a point wise solution of the corresponding equation.*

Finally, we state the following conclusions for the Sobolev embedding.

Lemma 2.4. *([38]) $H^1(\mathbb{Z}^N)$ is continuously embedded into $\ell^q(\mathbb{Z}^N)$ for any $q \in [2,\infty]$. Namely, for any $u \in H^1(\mathbb{Z}^N)$, there exists a constant C_q depending only on q such that*

$$\|u\|_{\ell^q(\mathbb{Z}^N)} \leq C_q \|u\|_{H^1(\mathbb{Z}^N)}. \tag{2.7}$$

Lemma 2.5. *([27, Lemma 2.6]) Assume that $\lambda > 1$ and $a(x)$ satisfies* (A1) *and* (A2)*. Then $E_\lambda(\mathbb{Z}^N)$ is continuously embedded into $\ell^q(\mathbb{Z}^N)$ for any $q \in [2,\infty]$ and the embedding is independent of λ. Namely, there exists a constant C_q depending only on q such that for any $u \in E_\lambda(\mathbb{Z}^N)$, $\|u\|_{\ell^q(\mathbb{Z}^N)} \leq C_q\|u\|_{E_\lambda(\mathbb{Z}^N)}$. Moreover, for any bounded sequence $\{u_k\} \in E_\lambda(\mathbb{Z}^N)$, there exists $u \in E_\lambda(\mathbb{Z}^N)$ such that, up to a subsequence,*

$$\begin{cases} u_k \rightharpoonup u & in\ E_\lambda(\mathbb{Z}^N). \\ u_k(x) \to u(x) & \forall x \in \mathbb{Z}^N. \\ u_k \to u & in\ \ell^q(\mathbb{Z}^N). \end{cases}$$

Lemma 2.6. *([27, Lemma 2.7]) Assume that Ω is a bounded domain in $\mathbb{Z}^N$. Then $H_0^1(\Omega)$ is continuously embedded into $\ell^q(\Omega)$ for any $q \in [1,\infty]$. Namely, there exists a constant C_q depending only on q* *such that for any $u \in H_0^1(\Omega)$, $\|u\|_{\ell^q(\Omega)} \le C_q\|u\|_{H_0^1(\Omega)}$. Moreover, for any bounded sequence $\{u_k\} \in H_0^1(\Omega)$, there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence,*

$$\begin{cases} u_k \rightharpoonup u & \text{in } H_0^1(\Omega). \\ u_k(x) \to u(x) & \forall x \in \Omega. \\ u_k \to u & \text{in } \ell^q(\Omega). \end{cases}$$

3. Discrete Brézis-Lieb Lemma

In this section, we give a proof of the discrete Brézis-Lieb Lemma(see [3,39,40] for the continuous case) for the nonlocal term on the lattice graph. First, let us recall the discrete Brézis-Lieb Lemma [38] for the local case.

Lemma 3.1. *([38, Lemma 9]) Let $\Omega \subset \mathbb{Z}^N$ be a domain and $\{u_n\} \subset \ell^q(\Omega)$ with $0 < q < \infty$. If $\{u_n\}$ is bounded in $\ell^q(\Omega)$ and $u_n \to u$ pointwise on Ω as $n \to \infty$, then*

$$\lim_{n\to\infty} \left(\|u_n\|^q_{\ell^q(\Omega)} - \|u_n - u\|^q_{\ell^q(\Omega)}\right) = \|u\|^q_{\ell^q(\Omega)}. \tag{3.1}$$

From Lemma 3.1 and [38, Corollary 10], it is not hard for us to get the following corollary.

Corollary 3.2. *Assume V is a uniformly bounded positive function. If $\{u_n\}$ is bounded in $H^1(\mathbb{Z}^N)$ and $u_n \to u$ pointwise on $\mathbb{Z}^N$, then*

$$\begin{aligned} &\lim_{n\to\infty} \left(\int_{\mathbb{Z}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) d\mu - \int_{\mathbb{Z}^N} \left(|\nabla (u_n - u)|^2 + V(x)(u_n - u)^2\right) d\mu \right) \\ &= \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2\right) d\mu. \end{aligned} \tag{3.2}$$

Next, we prove a variant of the discrete Brézis-Lieb Lemma.

Lemma 3.3. *Let $\Omega \subset \mathbb{Z}^N$ be a domain, $1 \le q < \infty$. If the sequence $\{u_n\}$ is bounded in $\ell^r(\Omega)$ and $u_n \to u$ pointwise on Ω as $n \to \infty$, then for every $q \in [1, r]$,*

$$\lim_{n\to\infty} \int_\Omega \left| |u_n|^q - |u_n - u|^q - |u|^q \right|^{\frac{r}{q}} d\mu = 0. \tag{3.3}$$

Proof. Applying the Fatou's Lemma, we obtain

$$\|u\|_{\ell^r(\Omega)} \le \varliminf_{n\to\infty} \|u_n\|_{\ell^r(\Omega)} < \infty. \tag{3.4}$$

Fix $\varepsilon > 0$ and for all $a, b \in \mathbb{R}$, there exists C_ε satisfying

$$\left| |a + b|^q - |a|^q \right| \le \varepsilon |a|^q + C_\varepsilon |b|^q.$$

Hence we obtain

$$\begin{aligned} f_n^\varepsilon &:= \left(\left| |u_n|^q - |u_n - u|^q - |u|^q \right| - \varepsilon |u_n - u|^q \right)^+ \\ &\le \left(\left| |u_n|^q - |u_n - u|^q \right| + |u|^q - \varepsilon |u_n - u|^q \right)^+ \\ &\le (\varepsilon |u_n - u|^q + C_\varepsilon |u|^q + |u|^q - \varepsilon |u_n - u|^q)^+ \\ &= (1 + C_\varepsilon)\, |u|^q. \end{aligned}$$

Thus

$$(f_n{}^{\varepsilon})^{\frac{r}{q}} \le (1 + C_\varepsilon)^{\frac{r}{q}}|u|^r. \tag{3.5}$$

It follows from the Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_\Omega (f_n{}^{\varepsilon})^{\frac{r}{q}} d\mu = \int_\Omega \lim_{n\to\infty} (f_n{}^{\varepsilon})^{\frac{r}{q}} d\mu = 0. \tag{3.6}$$

From the definition of f_n^ε, we obtain

$$\left||u_n|^q - |u_n - u|^q - |u|^q\right| \le f_n^\varepsilon + \varepsilon |u_n - u|^q.$$

Moreover, one deduces from the basic inequality $(a + b)^p \le C_p(a^p + b^p)(\forall a, b, p > 0)$ that

$$\left||u_n|^q - |u_n - u|^q - |u|^q\right|^{\frac{r}{q}} \le \left(f_n^\varepsilon + \varepsilon|u_n - u|^q\right)^{\frac{r}{q}} \le C_{q,r}\left((f_n^\varepsilon)^{\frac{r}{q}} + \varepsilon^{\frac{r}{q}}|u_n - u|^r\right). \tag{3.7}$$

Therefore, from (3.6) and (3.7), we get

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \int_\Omega \left||u_n|^q - |u_n - u|^q - |u|^q\right|^{\frac{r}{q}} d\mu \\
&\le \overline{\lim_{n\to\infty}}\, C_{q,r}\left(\int_\Omega (f_n{}^{\varepsilon})^{\frac{r}{q}} d\mu + \int_{\mathbb{Z}^N} \varepsilon^{\frac{r}{q}} |u_n - u|^r d\mu\right) \\
&\le C_{q,r}\varepsilon^{\frac{r}{q}} \sup_{n\in\mathbb{N}} \|u_n - u\|^r_{\ell^r(\Omega)}.
\end{aligned}$$

Then let $\varepsilon \to 0$,

$$\overline{\lim_{n\to\infty}} \int_\Omega \left||u_n|^q - |u_n - u|^q - |u|^q\right|^{\frac{r}{q}} d\mu = 0.$$

This finishes the proof.

Next, we state the discrete Brézis-Lieb type Lemma.

Lemma 3.4. *Suppose $\Omega \subset \mathbb{Z}^N$ and $1 \le p < \infty$. If the sequence $\{u_n\}$ is bounded in $\ell^p(\Omega)$ and $u_n \to u$ pointwise on Ω as $n \to \infty$, then for every $x \in \mathbb{Z}^N$, we have*

$$\lim_{n\to\infty}\left(\sum_{\substack{y\neq x\\ y\in\Omega}} \frac{|u_n(y)|^p}{|x - y|^{N-\alpha}} - \sum_{\substack{y\neq x\\ y\in\Omega}} \frac{|u_n(y) - u(y)|^p}{|x - y|^{N-\alpha}}\right) = \sum_{\substack{y\neq x\\ y\in\Omega}} \frac{|u(y)|^p}{|x - y|^{N-\alpha}}. \tag{3.8}$$

Proof. Since $x \neq y$ and $x, y \in \mathbb{Z}^N$, we obtain $|x - y| \ge 1$ and it follows that

$$\sum_{\substack{y\neq x\\ y\in\Omega}} \frac{\left||u_n(y)|^p - |u_n(y) - u(y)|^p - |u(y)|^p\right|}{|x - y|^{N-\alpha}} \le \sum_{y\in\Omega} \left||u_n(y)|^p - |u_n(y) - u(y)|^p - |u(y)|^p\right|.$$

Thus the proof is complete as $n \to \infty$ from Lemma 3.3.

Now we are in position to establish the discrete Brézis-Lieb Lemma for the nonlocal term of the functional. To this purpose we first present an important inequality on the lattice graph which is studied by many authors in the continuous setting.

Lemma 3.5. *([41]) (Discrete Hardy-Littlewood-Sobolev Inequality) Let* $0 < \alpha < N$, $1 < r, s < \infty$ *and* $\frac{1}{r} + \frac{1}{s} + \frac{N-\alpha}{N} \geq 2$. *Assume* $f \in \ell^r(\mathbb{Z}^N)$ *and* $g \in \ell^s(\mathbb{Z}^N)$. *Then there exists a positive constant* $C_{r,s,\alpha}$ *depending only on* r, s, α *such that*

$$\sum_{\substack{x,y\in\mathbb{Z}^N \\ y\neq x}} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} \leq C_{r,s,\alpha}\|f\|_{\ell^r(\mathbb{Z}^N)}\|g\|_{\ell^s(\mathbb{Z}^N)}. \tag{3.9}$$

The paper [38] also give the following equivalent form of (3.9).

Lemma 3.6. *Let* $0 < \alpha < N$, $1 < r, t < \infty$ *and* $\frac{1}{t} + \frac{\alpha}{N} \leq \frac{1}{r}$. *Assume* $f \in \ell^r(\mathbb{Z}^N)$, *then there exists a positive constant* $C_{r,t,\alpha}$ *depending only on* r, t, α *such that*

$$\left\| \sum_{\substack{y\in\mathbb{Z}^N \\ y\neq x}} \frac{f(y)}{|x-y|^{N-\alpha}} \right\|_{\ell^t(\mathbb{Z}^N)} \leq C_{r,t,\alpha}\|f\|_{\ell^r(\mathbb{Z}^N)}. \tag{3.10}$$

The next lemma states the discrete Brézis-Lieb Lemma for the nonlocal term.

Lemma 3.7. *Let* $1 \leq p < \infty$ *and the sequence* $\{u_n\}$ *is bounded in* $\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)$. *Suppose* $u_n \to u$ *pointwise on* $\mathbb{Z}^N$ *as* $n \to \infty$, *then*

$$\begin{aligned} &\lim_{n\to\infty}\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_n(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n|^p d\mu - \int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n-u|^p d\mu\right) \\ &= \int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu. \end{aligned} \tag{3.11}$$

Proof. For every n, we can divide the left-hand side of (3.11) into two parts,

$$\begin{aligned} &\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_n(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n|^p d\mu - \int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n-u|^p d\mu \\ &= \int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_n(y)|^p - |u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)(|u_n|^p - |u_n-u|^p) d\mu \\ &\quad + 2\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_n(y)|^p - |u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n-u|^p d\mu \\ &=: J_1 + 2J_2, \end{aligned} \tag{3.12}$$

where

$$J_1 = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u_n(y)|^p - |u_n(y) - u(y)|^p}{|x-y|^{N-\alpha}} \right) (|u_n|^p - |u_n - u|^p) d\mu,$$

$$J_2 = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u_n(y)|^p - |u_n(y) - u(y)|^p}{|x-y|^{N-\alpha}} \right) |u_n - u|^p d\mu.$$

By Lemma 3.3, taking $q = p$, $r = \frac{2Np}{N+\alpha}$, one has

$$\lim_{n\to\infty} \int_{\mathbb{Z}^N} \left| |u_n|^p - |u_n - u|^p - |u|^p \right|^{\frac{2N}{N+\alpha}} d\mu = 0. \tag{3.13}$$

We first give the estimate for the term J_1. From the Hardy-Littlewood-Sobolev inequality (Eq 3.9), one deduces that

$$\begin{aligned}
&\left| J_1 - \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu \right| \\
\leq& \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{\left| |u_n(y)|^p - |u_n(y) - u(y)|^p - |u(y)|^p \right|}{|x-y|^{N-\alpha}} \right) \left| |u_n|^p - |u_n - u|^p - |u|^p \right| d\mu \\
&+ 2 \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{\left| |u_n(y)|^p - |u_n(y) - u(y)|^p - |u(y)|^p \right|}{|x-y|^{N-\alpha}} \right) |u|^p d\mu \\
\leq& \left\| |u_n|^p - |u_n - u|^p - |u|^p \right\|^2_{\ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)} + 2 \left\| |u_n|^p - |u_n - u|^p - |u|^p \right\|_{\ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)} \left\| |u|^p \right\|_{\ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)}.
\end{aligned}$$

From (3.13) and $\|u\|_{\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)} \leq \liminf_{n\to\infty} \|u_n\|_{\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)} < \infty$, it gives that

$$\lim_{n\to\infty} J_1 = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu. \tag{3.14}$$

Now we give the estimate for J_2. From the Banach-Alaoglu theorem, $|u_n - u|^p \rightharpoonup 0$ weakly in

$\ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)$ as $n \to \infty$ and (3.9), we deduce that

$$\begin{aligned} J_2 = &\int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u_n(y)|^p - |u_n(y) - u(y)|^p - |u(y)|^p}{|x-y|^{N-\alpha}} \right) |u_n - u|^p d\mu \\ &+ \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u_n - u|^p d\mu \\ \leq & \left\| |u_n|^p - |u_n - u|^p - |u|^p \right\|_{\ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)} \left\| |u_n - u|^p \right\|_{\ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)} \\ &+ \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u_n - u|^p d\mu. \end{aligned}$$

We infer from (3.10) that

$$\left\| \sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right\|_{\ell^{\frac{2N}{N-\alpha}}(\mathbb{Z}^N)} \leq C_{N,p,\alpha} \|u\|^p_{\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)}.$$

Moreover, $|u_n - u|^p \rightharpoonup 0$ in $\ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)$. Hence we know that

$$\lim_{n\to\infty} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u_n - u|^p d\mu = 0.$$

Then one deduces from (3.13) that $\lim_{n\to\infty} J_2 = 0$. This together with (3.14), we get the results.

4. Proof of the Theorem 1.1

In the present section we are devoted to the proof of Theorem 1.1. Obviously, for any function $u : \mathbb{Z}^N \to \mathbb{R}$, the energy functional related to (1.1) is given by

$$J(u) = \frac{1}{2} \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x) u^2 \right) d\mu - \frac{1}{2p} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu. \tag{4.1}$$

Notice that the functional J is well defined in $H^1(\mathbb{Z}^N)$. Indeed, assume that $u \in \ell^{\frac{2NP}{N+\alpha}}(\mathbb{Z}^N)$, then by applying the Hardy-littlewood-Sobolev inequality (Eq 3.9) to the function $f = |u|^p \in \ell^{\frac{2N}{N+\alpha}}(\mathbb{Z}^N)$, we obtain

$$\int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu \leq C_{N,p,\alpha} \left(\int_{\mathbb{Z}^N} |u|^{\frac{2Np}{N+\alpha}} d\mu \right)^{\frac{N+\alpha}{N}}. \tag{4.2}$$

It sufficient for us to confirm when the condition $u \in \ell^{\frac{2NP}{N+\alpha}}(\mathbb{Z}^N)$ is satisfied. According to the Lemma 2.4, $H^1(\mathbb{Z}^N)$ is continuously embedded into $\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)$ if and only if $p \geq \frac{N+\alpha}{N}$. Moreover, we infer from the inequality (Eq 3.9) that

$$\int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu \leq C_{N,p,\alpha} \|u\|^{2p}_{H^1(\mathbb{Z}^N)}, \tag{4.3}$$

where the constant $C_{N,p,\alpha}$ depends only on N, α and p. Based on the previous argument, the function J is meaningful.

Next, we define the Nehari manifold related to (4.1) by

$$\begin{aligned} \mathcal{N} :&= \left\{ u \in H^1(\mathbb{Z}^N) \setminus \{0\} : J'(u)u = 0 \right\} \\ &= \left\{ u \in H^1(\mathbb{Z}^N) \setminus \{0\} : \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2 \right) d\mu = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu \right\}. \end{aligned}$$

Let

$$m = \inf_{u \in \mathcal{N}} J(u).$$

If there exists a function $u \in \mathcal{N}$ satisfying $J(u) = m$, then the function u is called a ground state solution. Obviously, u is a critical point of J.

Next, we shall find the critical point of the functional (4.1).

Proposition 4.1. *Let $N \in \mathbb{N}^*$, $\alpha \in (0, N)$ and $p \in (1, \infty)$. If $u \in H^1(\mathbb{Z}^N) \cap \ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N) \setminus \{0\}$ and V is a uniformly bounded positive function, there holds*

$$\max_{t>0} J(tu) = \left(\frac{1}{2} - \frac{1}{2p} \right) S(u)^{\frac{p}{p-1}},$$

where

$$S(u) = \frac{\displaystyle\int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2 \right) d\mu}{\left(\displaystyle\int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu \right)^{\frac{1}{p}}}.$$

Proof. For any $t > 0$, we set

$$s(t) := J(tu) = \frac{t^2}{2} \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2 \right) d\mu - \frac{t^{2p}}{2p} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu.$$

By a direct computation,

$$s'(t) = t \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2 \right) d\mu - t^{2p-1} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu.$$

When $s'(t) = 0$, we can obtain a unique t_u such that $s'(t_u) = 0$. Moreover, one has

$$t_u = \left(\frac{\int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2\right) d\mu}{\int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu} \right)^{\frac{1}{2p-2}}.$$

Since as $0 < t < t_u$, $s'(t) > 0$ and as $t > t_u$, $s'(t) < 0$, thus

$$\max_{t>0} J(tu) = J(t_u u) = \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\frac{\int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2\right) d\mu}{\left(\int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu\right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}}.$$

This finishes the proof.

Note that the ground state energy of J can be characterized as

$$m = \inf_{u \in \mathcal{N}} J(u) = \inf_{u \in H^1(\mathbb{Z}^N)\backslash\{0\}} \max_{t>0} J(tu) = \inf_{u \in H^1(\mathbb{Z}^N)\backslash\{0\}} (\frac{1}{2} - \frac{1}{2p}) S(u)^{\frac{p}{p-1}}.$$

In the next conclusion we show the infirmum of $S(u)$ can be achieved by some nontrivial function.

Proposition 4.2. *Let $N \in \mathbb{N}^*$, $\alpha \in (0, N)$ and $p \in \left(\frac{N+\alpha}{N}, \infty\right)$. Suppose that V is a uniformly bounded positive function, then there exists $u \in H^1(\mathbb{Z}^N)$ satisfying*

$$S(u) = \inf\{S(v) : v \in H^1(\mathbb{Z}^N)\backslash\{0\}\}.$$

Combining with Propositions 4.1 and 4.2, we complete the proof of Theorem 1.1. Then we only need to focus on the proof Proposition 4.2 in the next. In the Euclidean space, we are familiar with the different kinds of the proof of Proposition 4.2. For example, a strategy consists in minimizing among radial functions and then prove with the symmetrization by rearrangement that a radial minimizer is a global minimizer. In our setting, the main difficulty for the analysis is that there is no proper counterpart for radial functions on $\mathbb{Z}^N$ and moreover we do not have the compactness in this problem. To overcome the difficulty we borrow an idea of [42, Section 4](also see [24]) and use the constraint method to prove Proposition 4.2.

Proof of Proposition 4.2. Set

$$m = \inf\{S(u) : u \in H^1(\mathbb{Z}^N)\backslash\{0\}\},$$

then we can get

$$\frac{1}{m} = \sup \left\{ \frac{1}{S(u)} : u \in H^1(\mathbb{Z}^N) \ and \ \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2\right) d\mu = 1 \right\}.$$

Let $\{u_n\}$ be a minimizing sequence in $H^1(\mathbb{Z}^N)$ such that

$$\int_{\mathbb{Z}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) d\mu = 1,$$

and $\lim\limits_{n\to\infty} \frac{1}{S(u_n)} = \frac{1}{m}$. By the discrete Hardy-Littlewood-Sobolev inequality (Eq 3.9), we obtain

$$\begin{aligned} & C_{N,p,\alpha} \left(\int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u_n(y)|^p}{|x-y|^{N-\alpha}} \right) |u_n|^p d\mu \right)^{\frac{1}{p}} \\ \leq & ||u_n||_{\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)} \\ \leq & ||u_n||_{\ell^2(\mathbb{Z}^N)}^{\frac{N+\alpha}{Np}} ||u_n||_{\ell^\infty(\mathbb{Z}^N)}^{1-\frac{N+\alpha}{Np}} \\ \leq & ||u_n||_{H^1(\mathbb{Z}^N)}^{\frac{N+\alpha}{Np}} ||u_n||_{\ell^\infty(\mathbb{Z}^N)}^{1-\frac{N+\alpha}{Np}}. \end{aligned} \tag{4.4}$$

Taking the limit from both sides, one can see

$$C_{N,p,\alpha}\left(\frac{1}{m}\right)^{\frac{1}{p}} \leq \varliminf_{n\to\infty} ||u_n||_{\ell^\infty(\mathbb{Z}^N)}^{1-\frac{N+\alpha}{Np}}. \tag{4.5}$$

Since $p > \frac{N+\alpha}{N}$, we obtain

$$\varliminf_{n\to\infty} ||u_n||_{\ell^\infty(\mathbb{Z}^N)} \geq C > 0. \tag{4.6}$$

Hence, there exists a subsequence $\{u_n\}$ and a sequence $\{y_n\} \subset \mathbb{Z}^N$ such that $|u_n(y_n)| \geq C$ for each n. By translations, we define $\widetilde{u}_n =: u_n(y + k_nT)$ with $k_n = (k_n^1, \cdots k_n^N)$ to ensure that $(y_n - k_nT) \subset \Omega$ where $\Omega = [0, T)^N \cap \mathbb{Z}^N$ is a bounded domain in $\mathbb{Z}^N$. Then for each $\widetilde{u}_n$,

$$||\widetilde{u}_n||_{\ell^\infty(\Omega)} \geq |u_n(y_n)| \geq C > 0.$$

Moreover, by translation invariance, we infer from $V(x)$ is T-periodic in x that

$$1 = \int_{\mathbb{Z}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) d\mu = \int_{\mathbb{Z}^N} \left(|\nabla \widetilde{u}_n|^2 + V(x)\widetilde{u}_n^2\right) d\mu$$

and

$$S(u_n) = S(\widetilde{u}_n).$$

Without loss of generality, we can get a minimizing sequence $\{u_n\}$ satisfying $||u_n||_{\ell^\infty(\Omega)} \geq C > 0$. Since Ω is bounded, there exists at least one point, say x_0, such that $u_n(x_0) \to u(x_0) \geq C > 0$. Since the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{Z}^N)$, it follows that $u_n \rightharpoonup u$ in $H^1(\mathbb{Z}^N)$ and $u_n \to u \neq 0$ pointwise on $\mathbb{Z}^N$. Then it follows from Corollary 3.2 and Lemma 3.7 that

$$\begin{aligned}
\frac{1}{m} &= \lim_{n\to\infty} \frac{\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_n(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n|^p d\mu\right)^{\frac{1}{p}}}{\int_{\mathbb{Z}^N}\left(|\nabla u_n|^2 + V(x)u_n^2\right)d\mu} \\
&= \overline{\lim_{n\to\infty}} \frac{\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu + \int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n-u|^p d\mu\right)^{\frac{1}{p}}}{\int_{\mathbb{Z}^N}\left(|\nabla u|^2 + V(x)u^2\right)d\mu + \int_{\mathbb{Z}^N}\left(|\nabla (u_n-u)|^2 + V(x)(u_n-u)^2\right)d\mu} \\
&\leq \overline{\lim_{n\to\infty}} \frac{\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu\right)^{\frac{1}{p}} + \left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n-u|^p d\mu\right)^{\frac{1}{p}}}{\int_{\mathbb{Z}^N}\left(|\nabla u|^2 + V(x)u^2\right)d\mu + \int_{\mathbb{Z}^N}\left(|\nabla (u_n-u)|^2 + V(x)(u_n-u)^2\right)d\mu}.
\end{aligned} \tag{4.7}$$

For every n, we have

$$\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n-u|^p d\mu\right)^{\frac{1}{p}} \leq \frac{1}{m}\int_{\mathbb{Z}^N}\left(|\nabla (u_n-u)|^2 + V(x)(u_n-u)^2\right)d\mu.$$

Since $u \neq 0$, one has

$$\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu\right)^{\frac{1}{p}} \geq \frac{1}{m}\int_{\mathbb{Z}^N}\left(|\nabla (u)|^2 + V(x)(u)^2\right)d\mu,$$

which yields

$$\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu.\right)^{\frac{1}{p}} = \frac{1}{m}\int_{\mathbb{Z}^N}\left(|\nabla (u)|^2 + V(x)(u)^2\right)d\mu.$$

By (4.7), one has

$$\lim_{n\to\infty}\left(\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_n(y)-u(y)|^p}{|x-y|^{N-\alpha}}\right)|u_n-u|^p d\mu\right)^{\frac{1}{p}} = \lim_{n\to\infty}\frac{1}{m}\int_{\mathbb{Z}^N}\left(|\nabla (u_n-u)|^2 + V(x)(u_n-u)^2\right)d\mu.$$

By Fatou's Lemma, one gets

$$\int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2\right) d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{Z}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) d\mu \leq 1.$$

Then it is enough for us to prove that $\int_{\mathbb{Z}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) d\mu = 1$. Using a contradiction argument, suppose that

$$0 < \int_{\mathbb{Z}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) d\mu = K < 1.$$

then by

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{Z}^N} \left(|\nabla (u_n - u)|^2 + V(x)(u_n - u)^2\right) d\mu \\ =&\lim_{n\to\infty} \int_{\mathbb{Z}^N} \left(|\nabla u_n|^2 + V(x)u_n^2\right) d\mu - \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + V(x)u^2\right) d\mu \\ =&1 - K > 0. \end{aligned}$$

However, $(a + b)^p > a^p + b^p$ if $a, b > 0$. This yields a contradiction by (4.7).

5. Proof of Theorems 1.3 and 1.4

In this section we shall prove the existence result for (1.3) and (1.4) by using the standard variational methods. Obviously, the functional associated with the problem (1.3) is given by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{Z}^N} \left(|\nabla u|^2 + (\lambda a(x) + 1)u^2\right) d\mu - \frac{1}{2p}\int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right) |u|^p d\mu,$$

where $p \geq \frac{N+\alpha}{N}$. The corresponding Nehari manifold is defined as

$$\begin{aligned} \mathcal{N}_\lambda :&= \left\{u \in E_\lambda(\mathbb{Z}^N) \setminus \{0\} : J'_\lambda(u)u = 0\right\} \\ &= \left\{ u \in E_\lambda(\mathbb{Z}^N) \setminus \{0\} : \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + (\lambda a(x) + 1)u^2\right) d\mu = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right) |u|^p d\mu \right\}. \end{aligned}$$

We define the least energy level m_λ by

$$m_\lambda := \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u).$$

Then we first prove the Nehari manifold $\mathcal{N}_\lambda$ is nonempty.

Lemma 5.1. *The Nehari manifold $\mathcal{N}_\lambda$ is non-empty.*

Proof. For $t \in \mathbb{R}$ and fix a function $u \in E_\lambda(\mathbb{Z}^N)\backslash\{0\}$ and, we define

$$\gamma(t) := J'(tu)tu = t^2 \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + (\lambda a(x) + 1)u^2\right) d\mu - t^{2p} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu.$$

Since $p > 1$ and $u \neq 0$, it is obvious that $\gamma(t) > 0$ for small $t > 0$ and that $\lim_{t \to \infty} \gamma(t) = -\infty$. Then there exists $t_0 \in (0, \infty)$ such that $\gamma(t_0) = 0$, which implies that $t_0 u \in \mathcal{N}_\lambda$.

Next, we prove the least energy level m_λ is positive.

Lemma 5.2. *We have* $m_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u) > 0$.

Proof. Since $u \in \mathcal{N}_\lambda$, then

$$\int_{\mathbb{Z}^N} \left(|\nabla u|^2 + (\lambda a(x) + 1)u^2\right) d\mu = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu.$$

By Lemma 2.5 and (3.9), we obtain

$$\|u\|^2_{E_\lambda(\mathbb{Z}^N)} = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y \neq x \\ y \in \mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} \right) |u|^p d\mu \leq C\|u\|^{2p}_{\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)} \leq C\|u\|^{2p}_{E_\lambda(\mathbb{Z}^N)},$$

where C is independent of λ. It follows from $p > 1$ that

$$\|u\|_{E_\lambda(\mathbb{Z}^N)} \geq \left(\frac{1}{C}\right)^{\frac{1}{2(p-1)}} > 0. \tag{5.1}$$

This gives

$$m_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2p}\right) \inf_{u \in \mathcal{N}_\lambda} \|u\|^2_{E_\lambda(\mathbb{Z}^N)} \geq \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\frac{1}{C}\right)^{\frac{1}{2(p-1)}} > 0.$$

The next lemma states that the least energy m_λ can be achieved.

Lemma 5.3. *The value m_λ can be achieved by some $u_\lambda \in \mathcal{N}_\lambda$. Namely, there exists some $u_\lambda \in \mathcal{N}_\lambda$ such that $J_\lambda(u_\lambda) = m_\lambda$.*

Proof. Take a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda$ such that $\lim_{k \to \infty} J_\lambda(u_k) = m_\lambda$. Since

$$o_k(1) + m_\lambda = J_\lambda(u_k) = \frac{p-1}{2p} \|u_k\|^2_{E_\lambda(\mathbb{Z}^N)},$$

we have that $\{u_k\}$ is bounded in $E_\lambda(\mathbb{Z}^N)$, where $\lim\limits_{k\to\infty} o_k(1) = 0$. By Lemma 2.5, we can assume that there exists some $u_\lambda \in E_\lambda(\mathbb{Z}^N)$ such that

$$\begin{cases} u_k \rightharpoonup u_\lambda & \text{in } E_\lambda(\mathbb{Z}^N). \\ u_k(x) \to u_\lambda(x) & \forall x \in \mathbb{Z}^N. \\ u_k \to u_\lambda & \text{in } \ell^q(\mathbb{Z}^N). \end{cases}$$

From the discrete Hardy-Littlewood-Sobolev inequality (Eq 3.9), we infer that

$$\int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_k(y) - u_\lambda(y)|^p}{|x-y|^{N-\alpha}} \right) |u_k - u_\lambda|^p d\mu \leq C\|u_k - u_\lambda\|^{2p}_{\ell^{\frac{2Np}{N+\alpha}}(\mathbb{Z}^N)}.$$

Therefore, one has

$$\lim_{k\to\infty} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_k(y) - u_\lambda(y)|^p}{|x-y|^{N-\alpha}} \right) |u_k - u_\lambda|^p d\mu = 0.$$

Then from the Lemma 3.7, we get

$$\lim_{k\to\infty} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_k(y)|^p}{|x-y|^{N-\alpha}} \right) |u_k|^p d\mu = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_\lambda(y)|^p}{|x-y|^{N-\alpha}} \right) |u_\lambda|^p d\mu. \tag{5.2}$$

Since the E_λ norm is weakly lower semi-continuous, one has

$$\begin{aligned} J_\lambda(u_\lambda) &= \frac{1}{2}\|u_\lambda\|^2_{E_\lambda(\mathbb{Z}^N)} - \frac{1}{2p}\int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_\lambda(y)|^p}{|x-y|^{N-\alpha}} \right) |u_\lambda|^p d\mu \\ &\leq \liminf_{k\to\infty} \left(\frac{1}{2}\|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} - \frac{1}{2p}\int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_k(y)|^p}{|x-y|^{N-\alpha}} \right) |u_k|^p d\mu \right) \\ &= \liminf_{k\to\infty} J_\lambda(u_k) = m_\lambda. \end{aligned} \tag{5.3}$$

Next it suffices to show that $u_\lambda \in \mathcal{N}_\lambda$. We infer from (5.1) that

$$0 < c \leq \|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} = \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_k(y)|^p}{|x-y|^{N-\alpha}} \right) |u_k|^p d\mu.$$

This together with (5.2) which implies that

$$0 < c \leq \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_\lambda(y)|^p}{|x-y|^{N-\alpha}} \right) |u_\lambda|^p d\mu. \tag{5.4}$$

Therefore $u_\lambda \neq 0$. Since $u_k \in \mathcal{N}_\lambda$, we infer that

$$\|u_\lambda\|^2_{E_\lambda(\mathbb{Z}^N)} \leq \liminf_{k\to\infty} \|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} = \liminf_{k\to\infty} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_k(y)|^p}{|x-y|^{N-\alpha}} \right) |u_k|^p d\mu$$
$$= \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_\lambda(y)|^p}{|x-y|^{N-\alpha}} \right) |u_\lambda|^p d\mu.$$

We use the contradiction argument to obtain our results. Assume that

$$\|u_\lambda\|^2_{E_\lambda(\mathbb{Z}^N)} < \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_\lambda(y)|^p}{|x-y|^{N-\alpha}} \right) |u_\lambda|^p d\mu.$$

Similar as the proof of Lemma 5.1, there would exist a $t \in (0, 1)$ such that $tu_\lambda \in \mathcal{N}_\lambda$. This implies that

$$\begin{aligned} 0 < m_\lambda \leq J_\lambda(tu_\lambda) &= (\frac{1}{2} - \frac{1}{2p})\|tu_\lambda\|^2_{E_\lambda(\mathbb{Z}^N)} \\ &\leq t^2 \liminf_{k\to\infty}(\frac{1}{2} - \frac{1}{2p})\|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} \\ &= t^2 \liminf_{k\to\infty} J_\lambda(u_k) \\ &= t^2 m_\lambda < m_\lambda. \end{aligned}$$

This contradicts the fact that $m_\lambda = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u)$. Therefore we have $u_\lambda \in \mathcal{N}_\lambda$. Moreover, we infer from (5.3) that m_λ is achieved by u_λ.

The following Lemma finishes the proof of Theorem 1.3.

Lemma 5.4. *$u_\lambda \in \mathcal{N}_\lambda$ is a critical point for J_λ.*

Proof. It is enough for us to prove that for any $\phi \in E_\lambda(\mathbb{Z}^N)$, there holds

$$J'_\lambda(u_\lambda)\phi = 0.$$

Since $u_\lambda \not\equiv 0$, we can choose a constant $\varepsilon > 0$ such that $u_\lambda + s\phi \not\equiv 0$ for all $s \in (-\varepsilon, \varepsilon)$. Furthermore, for every given $s \in (-\varepsilon, \varepsilon)$, we can find some $t(s) \in (0, \infty)$ satisfying $t(s)(u_\lambda + s\phi) \in \mathcal{N}_\lambda$. Indeed, $t(s)$ can be taken as

$$t(s) = \left(\frac{\|u_\lambda + s\phi\|^2_{E_\lambda(\mathbb{Z}^N)}}{\int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|(u_\lambda+s\phi)(y)|^p}{|x-y|^{N-\alpha}} \right) |u_\lambda + s\phi|^p d\mu} \right)^{\frac{1}{2p-2}}.$$

Obviously, we can get $t(0) = 1$. Take a function $\gamma(s) : (-\varepsilon, \varepsilon) \to \mathbb{R}$ which is defined as

$$\gamma(s) := J_\lambda(t(s)(u_\lambda + s\phi)).$$

For $t(s)(u_\lambda + s\phi) \in \mathcal{N}_\lambda$ and $J_\lambda(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u)$, $\gamma(s)$ achieves its minimum at $s = 0$. Together with $u_\lambda \in \mathcal{N}_\lambda$ and $J'_\lambda(u_\lambda)u_\lambda = 0$, it follows that

$$\begin{aligned} 0 = \gamma'(0) &= J'_\lambda(t(0)u_\lambda)[t'(0)u_\lambda + t(0)\phi] \\ &= J'_\lambda(u_\lambda)t'(0)u_\lambda + J'_\lambda(u_\lambda)\phi \\ &= J'_\lambda(u_\lambda)\phi. \end{aligned}$$

Next we focus on the proof of Theorem 1.4. The functional associated with the Eq (1.4) is given by

$$J_\Omega(u) = \frac{1}{2}\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu - \frac{1}{2p}\int_\Omega \left(\sum_{\substack{y\neq x\\ y\in\Omega}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu. \tag{5.5}$$

We remark that $\|u\|_{\ell^q(\Omega)} \le C\|u\|_{H^1_0(\Omega)}$ for $q \in [1, \infty]$ by Lemma 2.6. Therefore, the functional $J_\Omega(u)$ is well defined as $p \ge \frac{N+\alpha}{2N}$. The corresponding Nehari manifold is defined as

$$\begin{aligned} \mathcal{N}_\Omega &= \left\{u \in H^1_0(\Omega)\setminus\{0\} : J'_\lambda(u)u = 0\right\} \\ &= \left\{u \in H^1_0(\Omega)\setminus\{0\} : \int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu = \int_\Omega \left(\sum_{\substack{y\neq x\\ y\in\Omega}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu\right\}. \end{aligned} \tag{5.6}$$

Let m_Ω be

$$m_\Omega := \inf_{u\in\mathcal{N}_\Omega} J_\Omega(u).$$

Since Ω contains only finite vertices, the proofs of the previous results can be easily applied to the Eq (1.4). Moreover, $p > 1$ is enough for us to prove Theorem 1.4. Here we omit the details of the proofs.

6. Convergence of the ground state solution

In the current section, we mainly focus on the asymptotical properties of the solution. That is, we show that the ground state solutions u_λ of (1.3) converge to a ground state solution of (1.4) as $\lambda \to \infty$. To accomplish this we first prove that any solution of (1.3) is bounded away from zero.

Lemma 6.1. *There exists a constant $\sigma > 0$ which is independent of λ, such that for any critical point $u \in E_\lambda(\mathbb{Z}^N)$ of J_λ, we have $\|u\|_{E_\lambda(\mathbb{Z}^N)} \ge \sigma$.*

Proof. From Lemma 2.5 and the inequality (Eq 3.9), one has

$$\begin{aligned} 0 = J'(u)u &= \int_{\mathbb{Z}^N} \left(|\nabla u|^2 + (\lambda a(x) + 1)u^2\right) d\mu - \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu \\ &\ge \|u\|^2_{E_\lambda(\mathbb{Z}^N)} - C^{2p}\|u\|^{2p}_{E_\lambda(\mathbb{Z}^N)}, \end{aligned}$$

where C is independent of λ. Then we can choose $\sigma = (\frac{1}{C})^{\frac{p}{p-1}}$ and Lemma 6.1 is proved.

The next lemma studies the property of $(PS)_c$ sequence of J_λ.

Lemma 6.2. *For any $(PS)_c$ sequence $\{u_k\}$ of J_λ, there holds*

$$\lim_{k\to\infty} \|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} = \frac{2p}{p-1}c. \tag{6.1}$$

Furthermore, there would exist a constant $C_1 > 0$ independent of λ, such that either $c \geq C_1$ or $c = 0$.

Proof. Since $J_\lambda(u_k) \to c$ and $J'_\lambda(u_k) \to 0$ as $k \to \infty$, we have

$$c = \lim_{k\to\infty}\left(J_\lambda(u_k) - \frac{1}{2p}J'_\lambda(u_k)u_k\right) = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{1}{2p}\right)\|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} = \frac{p-1}{2p}\lim_{k\to\infty}\|u_k\|^2_{E_\lambda(\mathbb{Z}^N)},$$

which gives (6.1). By Lemma 2.5 and (3.9), for any $u \in E_\lambda(\mathbb{Z}^N)$, we obtain

$$J'_\lambda(u)u = \|u\|^2_{E_\lambda(\mathbb{Z}^N)} - \int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}}\frac{|u(y)|^p}{|x-y|^{N-\alpha}}\right)|u|^p d\mu \geq \|u\|^2_{E_\lambda(\mathbb{Z}^N)} - C^{2p}\|u\|^{2p}_{E_\lambda(\mathbb{Z}^N)}. \tag{6.2}$$

Take $\rho = (\frac{1}{2C^{2p}})^{\frac{1}{2p-2}}$. If $\|u\|_{E_\lambda(\mathbb{Z}^N)} \leq \rho$, we get

$$J'_\lambda(u)u \geq \frac{1}{2}\|u\|^2_{E_\lambda(\mathbb{Z}^N)}.$$

Take $C_1 = \frac{p-1}{2p}\rho^2$ and suppose $c < C_1$. Since $\{u_k\}$ is a $(PS)_c$ sequence, it yields

$$\lim_{k\to\infty}\|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} = \frac{2p}{p-1}c < \frac{2p}{p-1}C_1 = \rho^2.$$

Hence, for large k, we have

$$\frac{1}{2}\|u_k\|^2_{E_\lambda(\mathbb{Z}^N)} \leq J'_\lambda(u_k)u_k = o_k(1)\|u_k\|_{E_\lambda(\mathbb{Z}^N)},$$

which implies that $\|u_k\|_{E_\lambda(\mathbb{Z}^N)} \to 0$ as $k \to \infty$. It follows immediately that $J_\lambda(u_k) \to c = 0$ and the positive constant can be taken as $C_1 = \frac{p-1}{2p}\rho^2 = (\frac{1}{2C^{2p}})^{\frac{1}{p-1}}$.

Remark 6.3. *If we take $c = m_\lambda$, then there would exist a $(PS)_c$ sequence u_k such that $u_k \rightharpoonup u_\lambda$ when proving the existence of a ground state solutions u_λ. Since the E_{λ_k} norm of u_{λ_k} is weakly lower semi-continuous, then $\|u_\lambda\|_{E_\lambda(\mathbb{Z}^N)}$ is bounded by $\frac{2p}{p-1}m_\lambda$.*

Next, we study the relationship between the ground states m_λ and m_Ω.

Lemma 6.4. *$m_\lambda \to m_\Omega$ as $\lambda \to \infty$.*

Proof. Notice that $m_\lambda \leq m_\Omega$ for every positive λ owing to $\mathcal{N}_\Omega \subset \mathcal{N}_\lambda$. Take a sequence $\lambda_k \to \infty$ satisfying

$$\lim_{k\to\infty} m_{\lambda_k} = M \leq m_\Omega, \tag{6.3}$$

where m_{λ_k} is the ground state and $u_{\lambda_k} \in \mathcal{N}_{\lambda_k}$ is the corresponding ground state solution of (1.3). Then it follows $M > 0$ from Lemma 6.2. According to Remark 6.3, we know that the E_{λ_k} norm of u_{λ_k} is controlled by the constant $\frac{2p}{p-1}m_\Omega$, which is independent of λ_k. Up to a subsequence, we can assume that $u_{\lambda_k}(x) \to u_0(x)$ on $\mathbb{Z}^N$ and for any $q \in [2, +\infty)$, $u_{\lambda_k} \to u_0$ in $\ell^q(\mathbb{Z}^N)$. Moreover, we get that $u_0 \not\equiv 0$ from Lemma 6.1.

We first claim that $u_0|_{\Omega^c} = 0$. If it is not true, we can find a point x_0 satisfying $u_0(x_0) \neq 0$. Since $u_{\lambda_k} \in \mathcal{N}_{\lambda_k}$, then

$$J_{\lambda_k}(u_{\lambda_k}) = \frac{p-1}{2p}\|u_{\lambda_k}\|^2_{E_{\lambda_k}(\mathbb{Z}^N)} \geq \frac{p-1}{2p}\lambda_k \int_{\mathbb{Z}^N} a(x)u^2_{\lambda_k} d\mu \geq \frac{p-1}{2p}\lambda_k a(x_0)u^2_{\lambda_k}(x_0).$$

Since $a(x_0) > 0, u_{\lambda_k}(x_0) \to u_0(x_0) \neq 0$ and $\lambda_k \to \infty$, we get

$$\lim_{k\to\infty} J_{\lambda_k}(u_{\lambda_k}) = \infty,$$

which contradicts with the conclusion $m_{\lambda_k} \leq m_\Omega$. Since the norm $\|\cdot\|_{H^1(\mathbb{Z}^N)}$ is weakly lower semi-continuous and (5.2), we get

$$\begin{aligned}
\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2 d\mu + \int_\Omega u_0^2 d\mu &\leq \int_{\mathbb{Z}^N} \left(|\nabla u_0|^2 + u_0^2\right) d\mu \\
&\leq \liminf_{k\to\infty} \int_{\mathbb{Z}^N} \left(|\nabla u_{\lambda_k}|^2 + u^2_{\lambda_k}\right) d\mu \\
&\leq \liminf_{k\to\infty} \int_{\mathbb{Z}^N} \left(|\nabla u_{\lambda_k}|^2 + (\lambda_k a(x) + 1)u^2_{\lambda_k}\right) d\mu \\
&= \liminf_{k\to\infty} \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_{\lambda_k}(y)|^p}{|x-y|^{N-\alpha}}\right) |u_{\lambda_k}|^p d\mu \\
&= \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}} \frac{|u_0(y)|^p}{|x-y|^{N-\alpha}}\right) |u_0|^p d\mu.
\end{aligned}$$

Noticing that $u_0|_{\Omega^c} = 0$, we get

$$\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2 d\mu + \int_\Omega u_0^2 d\mu \leq \int_\Omega \left(\sum_{\substack{y\neq x\\ y\in\Omega}} \frac{|u_0(y)|^p}{|x-y|^{N-\alpha}}\right) |u_0|^p d\mu. \tag{6.4}$$

Then there exists $\alpha \in (0, 1]$ such that $\alpha u_0 \in \mathcal{N}_\Omega$, i.e.,

$$\int_{\Omega\cup\partial\Omega} |\alpha\nabla u_0|^2 d\mu + \int_\Omega |\alpha u_0|^2 d\mu = \int_\Omega \left(\sum_{\substack{y\neq x\\ y\in\Omega}} \frac{|\alpha u_0(y)|^p}{|x-y|^{N-\alpha}}\right) |\alpha u_0|^p d\mu.$$

This implies that

$$\begin{aligned}J_\Omega(\alpha u_0) &= \frac{p-1}{2p}\left(\int_{\Omega\cup\partial\Omega}|\alpha\nabla u_0|^2 d\mu + \int_\Omega |\alpha u_0|^2 d\mu\right)\\ &\le \frac{p-1}{2p}\int_{\mathbb{Z}^N}\left(|\alpha\nabla u_0|^2 + |\alpha u_0|^2\right) d\mu\\ &\le \frac{p-1}{2p}\int_{\mathbb{Z}^N}\left(|\nabla u_0|^2 + |u_0|^2\right) d\mu\\ &\le \liminf_{k\to\infty}\frac{p-1}{2p}\int_{\mathbb{Z}^N}\left(|\nabla u_{\lambda_k}|^2 + (\lambda_k a(x)+1)u_{\lambda_k}^2\right) d\mu\\ &= \liminf_{k\to\infty} J_{\lambda_k}(u_{\lambda_k}) = M.\end{aligned}$$

Consequently, $M \ge m_\Omega$. Combining with (6.3), we get that

$$\lim_{\lambda\to\infty} m_\lambda = m_\Omega.$$

Next, we are devoted to proving Theorem 1.5.

Proof of Theorem 1.5. We need to prove that for any sequence $\lambda_k \to \infty$, the corresponding $u_{\lambda_k} \in \mathcal{N}_{\lambda_k}$ satisfying $J_{\lambda_k}(u_{\lambda_k}) = m_{\lambda_k}$ converges in $H^1(\mathbb{Z}^N)$ to a ground state solution u_Ω of (1.4) along a subsequence. According to Remark 6.3, the E_{λ_k} norm of u_{λ_k} is uniformly bounded by the constant $\frac{2p}{p-1}m_\Omega$, which is independent of λ_k. Consequently, we can assume that there would exist some u_0 satisfying $u_{\lambda_k}(x) \to u_0(x)$ in $\mathbb{Z}^N$and for any $q \in [2, +\infty)$, $u_{\lambda_k} \to u_0$ in $\ell^q(\mathbb{Z}^N)$. Moreover, we get that $u_0 \not\equiv 0$ from Lemma 6.1. As what we have done in Lemma 6.4, we can prove that $u_0|_{\Omega^c} = 0$.

First, we claim that

$$\lambda_k \int_{\mathbb{Z}^N} a(x)u_{\lambda_k}^2 d\mu \to 0, \quad \text{as } k \to \infty \tag{6.5}$$

and

$$\int_{\mathbb{Z}^N}|\nabla u_{\lambda_k}|^2 d\mu \to \int_{\mathbb{Z}^N}|\nabla u_0|^2 d\mu. \tag{6.6}$$

If for some $\delta > 0$, there holds

$$\lim_{k\to\infty}\lambda_k\int_{\mathbb{Z}^N} a(x)u_{\lambda_k}^2 d\mu = \delta > 0,$$

we have

$$\begin{aligned}\int_{\Omega\cup\partial\Omega}\left(|\nabla u_0|^2+u_0^2\right)d\mu &< \int_{\mathbb{Z}^N}\left(|\nabla u_0|^2+u_0^2\right)d\mu+\delta\\ &\le \liminf_{k\to\infty}\int_{\mathbb{Z}^N}\left(|\nabla u_{\lambda_k}|^2+(\lambda_k a(x)+1)u_{\lambda_k}^2\right)d\mu\\ &= \liminf_{k\to\infty}\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}}\frac{|u_{\lambda_k}(y)|^p}{|x-y|^{N-\alpha}}\right)|u_{\lambda_k}|^p d\mu\\ &= \int_{\Omega}\left(\sum_{\substack{y\neq x\\ y\in\Omega}}\frac{|u_0(y)|^p}{|x-y|^{N-\alpha}}\right)|u_0|^p d\mu.\end{aligned}$$

Then there exists $\alpha\in(0,1)$ such that $\alpha u_0\in\mathcal{N}_\Omega$. On the other hand, if

$$\liminf_{k\to\infty}\int_{\mathbb{Z}^N}|\nabla u_{\lambda_k}|^2 d\mu > \int_{\mathbb{Z}^N}|\nabla u_0|^2 d\mu,$$

we also have $\int_{\Omega\cup\partial\Omega}\left(|\nabla u_0|^2+u_0^2\right)d\mu < \int_{\Omega}\left(\sum_{\substack{y\neq x\\ y\in\Omega}}\frac{|u_0(y)|^p}{|x-y|^{N-\alpha}}\right)|u_0|^p d\mu$. Then in both cases, we can find $\alpha\in(0,1)$ such that $\alpha u_0\in\mathcal{N}_\Omega$. Consequently, we have

$$\begin{aligned}m_\Omega\le J_\Omega(\alpha u_0) &= \frac{p-1}{2p}\left(\int_{\Omega\cup\partial\Omega}|\alpha\nabla u_0|^2 d\mu+\int_\Omega|\alpha u_0|^2 d\mu\right)\\ &= \frac{p-1}{2p}\alpha^2\left(\int_{\Omega\cup\partial\Omega}|\nabla u_0|^2 d\mu+\int_\Omega|u_0|^2 d\mu\right)\\ &< \frac{p-1}{2p}\int_{\mathbb{Z}^N}\left(|\nabla u_0|^2+|u_0|^2\right)d\mu\\ &\le \liminf_{k\to\infty}\frac{p-1}{2p}\int_{\mathbb{Z}^N}\left(|\nabla u_{\lambda_k}|^2+(\lambda_k a(x)+1)u_{\lambda_k}^2\right)d\mu\\ &= \liminf_{k\to\infty}J_{\lambda_k}(u_{\lambda_k})=m_\Omega,\end{aligned}$$

which arrives at a contradiction.

To prove Theorem 1.5, we also need verify that u_0 is a ground state solution of (1.4). The first step is to prove that u_0 is a critical point of J_Ω. Since $J'_{\lambda_k}(u_{\lambda_k})\phi=0$, for any $\phi\in H_0^1(\Omega)\subset H^1(\mathbb{Z}^N)$, we have

$$\int_{\mathbb{Z}^N}\nabla u_{\lambda_k}\nabla\phi d\mu+\int_{\mathbb{Z}^N}(\lambda_k a(x)+1)u_{\lambda_k}\phi d\mu=\int_{\mathbb{Z}^N}\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}}\frac{|u_{\lambda_k}(y)|^p}{|x-y|^{N-\alpha}}\right)|u_{\lambda_k}|^{p-2}u_{\lambda_k}\phi d\mu. \tag{6.7}$$

Since $a(x)=0$ in Ω and $\phi=0$ in Ω^c, there holds

$$\int_{\Omega\cup\partial\Omega}\nabla u_{\lambda_k}\nabla\phi d\mu+\int_\Omega u_{\lambda_k}\phi d\mu=\int_\Omega\left(\sum_{\substack{y\neq x\\ y\in\mathbb{Z}^N}}\frac{|u_{\lambda_k}(y)|^p}{|x-y|^{N-\alpha}}\right)|u_{\lambda_k}|^{p-2}u_{\lambda_k}\phi d\mu. \tag{6.8}$$

Let $k \to \infty$, the above equality becomes

$$\int_{\Omega\cup\partial\Omega} \nabla u_0 \nabla \phi d\mu + \int_{\Omega} u_0 \phi d\mu = \int_{\Omega} \lim_{k\to\infty} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_{\lambda_k}(y)|^p}{|x-y|^{N-\alpha}} \right) |u_0|^{p-2} u_0 \phi d\mu. \tag{6.9}$$

Since $u_{\lambda_k} \to u_0$ in $\ell^p(\mathbb{Z}^N)$ with $p \geq 2$ and Lemma 3.4, we obtain

$$\int_{\Omega\cup\partial\Omega} \nabla u_0 \nabla \phi d\mu + \int_{\Omega} u_0 \phi d\mu = \int_{\Omega} \left(\sum_{\substack{y\neq x \\ y\in\Omega}} \frac{|u_0(y)|^p}{|x-y|^{N-\alpha}} \right) |u_0|^{p-2} u_0 \phi d\mu, \tag{6.10}$$

which yields $u_0 \in \mathcal{N}_\Omega$, and u_0 is a solution of (1.4).

Finally, we prove that u_0 achieves the infimum of J_Ω in $\mathcal{N}_\Omega$.

$$\begin{aligned} J_{\lambda_k}(u_{\lambda_k}) &= \frac{1}{2}\int_{\mathbb{Z}^N} \left(|\nabla u_{\lambda_k}|^2 + (\lambda_k a(x) + 1) u_{\lambda_k}^2 \right) d\mu - \frac{1}{2p}\int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_{\lambda_k}(y)|^p}{|x-y|^{N-\alpha}} \right) |u_{\lambda_k}|^p d\mu \\ &= \frac{1}{2}\int_{\mathbb{Z}^N} \left(|\nabla u_0|^2 + u_0^2 \right) d\mu - \frac{1}{2p}\int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_0(y)|^p}{|x-y|^{N-\alpha}} \right) |u_0|^p d\mu + o_k(1) \\ &= \frac{1}{2}\int_{\Omega\cup\partial\Omega} |\nabla u_0|^2 d\mu + \int_{\Omega} u_0^2 d\mu - \frac{1}{2p}\int_{\Omega} \left(\sum_{\substack{y\neq x \\ y\in\Omega}} \frac{|u_0(y)|^p}{|x-y|^{N-\alpha}} \right) |u_0|^p d\mu + o_k(1) \\ &= J_\Omega(u_0) + o_k(1). \end{aligned} \tag{6.11}$$

Since $J_{\lambda_k}(u_{\lambda_k}) = m_{\lambda_k}$, we get $J_\Omega(u_0) = m_\Omega$ by Lemma 6.4. Hence the function u_0 is a ground state solution of (1.4).

Finally, we have the following lemma for the convergence of the sequence $\{u_{\lambda_k}\}$.

Corollary 6.5. *Furthermore, we have* $\lim_{k\to\infty} \|u_{\lambda_k} - u_0\|_{E_{\lambda_k}(\mathbb{Z}^N)} = 0$.

Proof. Indeed, since $u_{\lambda_k} \in \mathcal{N}_{\lambda_k}$ and $u_0|_{\Omega^c} = 0$, we have

$$\begin{aligned} \|u_{\lambda_k} - u_0\|^2_{E_{\lambda_k}(\mathbb{Z}^N)} &= \int_{\mathbb{Z}^N} \left(|\nabla(u_{\lambda_k} - u_0)|^2 + (\lambda_k a(x) + 1)(u_{\lambda_k} - u_0)^2 \right) d\mu \\ &= \|u_{\lambda_k}\|^2_{E_{\lambda_k}(\mathbb{Z}^N)} + \|u_0\|^2_{E_{\lambda_k}(\mathbb{Z}^N)} - 2\int_{\mathbb{Z}^N} \nabla u_{\lambda_k} \nabla u_0 d\mu - 2\int_{\mathbb{Z}^N} u_{\lambda_k} u_0 d\mu \\ &= \|u_{\lambda_k}\|^2_{E_{\lambda_k}(\mathbb{Z}^N)} + \|u_0\|^2_{H_0^1(\Omega)} - 2\int_{\Omega\cup\partial\Omega} \nabla u_{\lambda_k} \nabla u_0 d\mu - 2\int_{\Omega} u_{\lambda_k} u_0 d\mu \\ &= \|u_{\lambda_k}\|^2_{E_{\lambda_k}(\mathbb{Z}^N)} + \|u_0\|^2_{H_0^1(\Omega)} - 2\|u_0\|^2_{H_0^1(\Omega)} + o_k(1) \\ &= \|u_{\lambda_k}\|^2_{E_{\lambda_k}(\mathbb{Z}^N)} - \|u_0\|^2_{H_0^1(\Omega)} + o_k(1) \\ &= \int_{\mathbb{Z}^N} \left(\sum_{\substack{y\neq x \\ y\in\mathbb{Z}^N}} \frac{|u_k(y)|^p}{|x-y|^{N-\alpha}} \right) |u_k|^p d\mu - \int_{\Omega} \left(\sum_{\substack{y\neq x \\ y\in\Omega}} \frac{|u_0(y)|^p}{|x-y|^{N-\alpha}} \right) |u_0|^p d\mu + o_k(1), \end{aligned}$$

which finishes the proof.

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Conflict of interest

The authors declare no conflict of interest.

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