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Research article

On σ-subnormal subgroups and products of finite groups

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Abstract: Suppose that $\sigma = \{\sigma_i : i \in I\}$ is a partition of the set $\mathbb{P}$ of all primes. A subgroup A of a finite group G is said to be σ-*subnormal* in G if A can be joined to G by a chain of subgroups $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ such that either A_{i-1} normal in A_i or $A_i/Core_{A_i}(A_{i-1})$ is a σ_j-group for some $j \in I$, for every $1 \leq i \leq n$. A σ-subnormality criterion related to products of subgroups of finite σ-soluble groups is proved in the paper. As a consequence, a characterisation of the σ-Fitting subgroup of a finite σ-soluble group naturally emerges.

Keywords: finite group; σ-soluble group; σ-subnormal subgroup; products of groups

1. Introduction

We consider only finite groups.

The starting point for this note is the following nice connection between the subnormality of a subgroup A of a group G and the number of elements of the product AB for any subgroup B of G showed by Levi in [1].

Theorem 1. *Let A be a subgroup of a group G. Then the following are equivalent.*

1. *A is a subnormal subgroup of G.*
2. *$|AB|$ divides $|G|$ for every subgroup B of G.*
3. *$|AP|$ divides $|G|$ for every Sylow p-subgroup P of G and all primes p.*

This result is a consequence of the known Kegel-Wielandt conjecture proved by Kleidman [2] making use of the classification of finite simple groups.

Theorem 2. *A subgroup A of a group G is subnormal in G if and only if $A \cap P$ is a Sylow p-subgroup of A for each Sylow p-subgroup P of G and each prime p.*

Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set $\mathbb{P}$ of all prime numbers. Following Skiba [3–5], a subgroup A of a finite group G is said to be *σ-subnormal* in G if A can be joined to G by a chain of subgroups

$$A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$$

such that either A_{i-1} normal in A_i or $A_i/\operatorname{Core}_{A_i}(A_{i-1})$ is a σ_j-group for some $j \in I$, for every $1 \leq i \leq n$.

It is abundantly clear that the embedding property of σ-subnormality coincides with the subnormality when σ is the partition of $\mathbb{P}$ into sets containing exactly one prime each.

A group $G \neq 1$ is called *σ-primary* if all the primes dividing $|G|$ belong to the same member of the partition σ. We stipulate that the trivial group is σ-primary.

Definition 1. *A group G is called σ-soluble if all chief factors of G are σ-primary. G is called σ-*nilpotent *if it is a direct product of σ-primary groups.*

If $\pi = \{p_1, \cdots, p_r\}$, and $\sigma = \{\{p_1\}, \cdots, \{p_r\}, \pi'\}$, then the class of all σ-soluble groups is just the class of all π-soluble groups, and the class of all σ-nilpotent groups is just the class of all groups having a normal Hall π'-subgroup and a normal Sylow p_i-subgroup, for all i. In particular, soluble and nilpotent groups are exactly the σ-soluble and σ-nilpotent groups for the partition $\sigma = \{\{2\}, \{3\}, \{5\}, ...\}$.

Skiba [6, 7] proved that σ-soluble groups have a nice arithmetic structure.

Theorem 3. *Assume that G is a σ-soluble group. Then G has a Hall σ_i-subgroup E and every σ_i-subgroup is contained in a conjugate of E for all $i \in I$. In particular, the Hall σ_i-subgroups are conjugate for all $i \in I$. Furthermore, G has Hall σ_i'-subgroups.*

A non-σ-nilpotent group has a non-trivial proper σ-subnormal subgroup if and only if it si not simple Therefore criteria for the σ-subnormality of a subgroup is important in the study of the normal structure of a group [8]. The significance of the σ-subnormal subgroups in σ-soluble groups is also apparent since they are precisely the $K - \mathcal{N}_\sigma$-subnormal subgroups. In particular, they form a distinguished sublattice of the subgroup lattice of G [9].

It is worth mentioning that σ-subnormality has been recently studied in the locally finite case by Ferrara and Trombetti in [10].

Definition 2. *Let A be a subgroup of a σ-soluble group G. We say that A satisfies* property C_{σ_i} *in G if $|AB|$ divides $|G|$ for every Hall σ_i-subgroup B of G.*

Taking the close relationship between σ-subnormal subgroups and Hall subgroups of σ-soluble groups into account, it seems natural to think about an extension of Theorem 1 in the σ-soluble universe. Then main result here is the following σ-subnormality criterion.

Theorem A. *Let A be a subgroup of a σ-soluble group G. Then the following are equivalent.*

1. *A is σ-subnormal in G.*
2. *A satisfies C_{σ_i} in G for all $i \in I$.*

2. Proof of Theorem A

The proof of Theorem A depends on the following lemma.

Lemma 1 ([4]). *Let A, B and N be subgroups of a group G. Suppose that A is σ-subnormal in G and N is normal in G. Then:*

1. *$A \cap B$ is a σ-subnormal subgroup of B.*
2. *If B is σ-subnormal in A, then B is σ-subnormal in G.*
3. *If B is a σ-subnormal subgroup of G, then $A \cap B$ is σ-subnormal in G.*
4. *AN/N is σ-subnormal in G/N.*
5. *If $N \subseteq B$ and B/N is a σ-subnormal subgroup of G/N, then B is σ-subnormal in G.*
6. *If $L \leq B$ and B is a σ-nilpotent group, then L is σ-subnormal in B.*
7. *If the primes dividing $|G : A|$ belong to σ_i, then $O^{\sigma_i}(A) = O^{\sigma_i}(G)$.*

Proof of Theorem A. Assume that A is σ-subnormal in G, and let B be a Hall σ_i-subgroup of G for some $i \in I$. We show that $|AB|$ divides $|G|$ by induction on the order of G. If A is normal in G, then AB is a subgroup of G and the result follows. Suppose that A is a maximal subgroup of G. Then $G/\operatorname{Core}_G(A)$ is a σ_j-group for some $j \in I$. If $i \neq j$, then B is contained in $\operatorname{Core}_G(A)$, $AB = A$ and $|AB| = |A|$ divides $|G|$. If $i = j$, then $G = \operatorname{Core}_G(A)B$ and the result also follows.

Assume that A is not a maximal subgroup of G, and let M be a σ-subnormal maximal subgroup of G containing A. Then A is σ-subnormal in M. By the above argument, $B \leq M$ or $G = \operatorname{Core}_G(M)B$. In both cases, $B \cap M$ is a Hall σ_i-subgroup of M. By induction, $|A(B \cap M)|$ divides $|M|$. Then there exists a positive integer a such that

$$|M| = a \cdot |A(B \cap M)| = a \cdot \frac{|A||B \cap M|}{|A \cap B|}$$

If $B \leq M$, then $|AB|$ divides $|M|$ and the result follows. Assume that $G = \operatorname{Core}_G(M)B = MB$. Then

$$|G| = \frac{|M||B|}{|B \cap M|} = a \cdot \frac{|A||B \cap M|}{|A \cap B|} \cdot \frac{|B|}{|B \cap M|} = a \cdot \frac{|A||B|}{|A \cap B|} = a \cdot |AB|.$$

Therefore the condition is necessary.

Conversely, assume that A satisfies property C_{σ_i} in G for all $i \in I$, but A is not σ-subnormal in G. We argue by induction on the order of G. Let N be a minimal normal subgroup of G. Since G is σ-soluble, it follows that N is a σ_j-group for some $j \in I$. If T/N is a Hall σ_i-subgroup of G/N for some $i \in I$, then there exists a Hall σ_i-subgroup B of G such that $T/N = BN/N$ by Theorem 3. Moreover, either $N \leq B$ or $N \cap B = 1$. Since $|AB|$ divides $|G| = |B||G : B|$, we conclude that $|A : A \cap B|$ divides $|G : B|$ and then $A \cap B$ is a Hall σ_i-subgroup of A. Hence if $N \cap B = 1$, then $AN \cap B = A \cap B$. Thus $|(AN/N)(T/N)|$ divides $|G/N|$ and AN/N satisfies the C_{σ_i} in G/N for all $i \in I$. By induction, AN/N is σ-subnormal in G/N. From Lemma 1(5), we have that AN is σ-subnormal in G. Suppose that AN is a proper subgroup of G. Let C be a Hall σ_i-subgroup of AN. If $i = j$, then N is contained in C and $C = (A \cap C)N$. In this case, $AC = AN$ and so $|AC|$ divides $|AN|$. Suppose that $i \neq j$. By Theorem 3, there exists a Hall σ_i-subgroup B of G such that $C \leq B$. Since $|AB|$ divides $|G| = |B||G : B|$, it follows that $|A : A \cap B|$ divides $|G : B|$ and so $A \cap B$ is a Hall σ_i-subgroup of A. Hence $C = A \cap B$ and $AC = A$. In particular, $|AC|$ divides $|AN|$. Consequently, A satisfies property C_{σ_i} in AN for all $i \in I$. Then the induction hypothesis again applies and gives that A is σ-subnormal in AN. From Lemma 1(2), we conclude that A is σ-subnormal in G. Therefore we may assume that $G = AN$. From Lemma 1(7), we conclude that $O^{\sigma_i}(A) = O^{\sigma_i}(G)$. This yields $O^{\sigma_i}(G) \leq \operatorname{Core}_G(A)$. If $O^{\sigma_i}(G) \neq 1$, we can take $N \leq O^{\sigma_i}(G)$ and conclude $A = AN$ is

σ-subnormal in G and if $O^{\sigma_i}(G) = 1$, then G is a σ_i-group and then A is obviously a σ-subnormal subgroup of G, as desired. □

3. Corollaries

We now derive some consequences of Theorem A, the first being a particular case of the theorem.

Corollary 1. *Let A be a σ_i-subgroup of a σ-soluble group G. Then A is σ-subnormal in G if and only if A satisfies property C_{σ_i} in G.*

Proof. Only the necessity of the condition is in doubt. Assume that B is a Hall σ_i-subgroup of G. Since $|AB|$ divides $|G|$, it follows that every prime dividing $|AB|$ belongs to σ_i. Therefore, $AB = B$ and so $A \leq B$. Since the Hall σ_i-subgroups are conjugate, we have that $A \leq O_{\sigma_i}(G)$ and so A is σ-subnormal in $O_{\sigma_i}(G)$. Since $O_{\sigma_i}(G)$ is normal in G, we have that A is σ-subnormal in G by Lemma 1(2). □

A nice consequence of Theorem A is the following extension of a classical result of Kegel due to Skiba [6].

Corollary 2. *A subgroup A of a σ-soluble group G is σ-subnormal in G if and only if $A \cap B$ is a Hall σ_i-subgroup of A for every Hall σ_i-subgroup B of G for all $i \in I$.*

Proof. Assume that A is σ-subnormal in G and let B be a Hall σ_i-subgroup of G. Then $|AB|$ divides $|G|$ and so $|A : A \cap B|$ is a σ_i'-number. Hence $A \cap B$ is a Hall σ_i-subgroup of A.

Conversely, if B is a Hall σ_i-subgroup of G such that $A \cap B$ is a Hall σ_i-subgroup of A, then $|A : A \cap B|$ divides the order of a Hall σ_i'-subgroup of G. Consequently, $|AB|$ divides $|G|$ and hence A satisfies property C_{σ_i} in G for all $i \in I$. By Theorem A, A is σ-subnormal in G. □

It is clear that the class $\mathcal{N}_\sigma$ of all σ-nilpotent groups behaves in the class of all σ-soluble groups like nilpotent groups in the class of all soluble groups. In fact, $\mathcal{N}_\sigma$ is a subgroup-closed saturated Fitting formation [4].

The $\mathcal{N}_\sigma$-radical of a group G is called the *σ-Fitting subgroup* of G and it is denoted by $F_\sigma(G)$. By [9], $F_\sigma(G)$ contains every σ-subnormal σ-nilpotent subgroup of G. Consequently, a group G is σ-nilpotent if and only if every subgroup of G is σ-subnormal in G. Therefore the following σ-version of [1] holds.

Corollary 3. *Let G be a σ-soluble group. Then G is σ-nilpotent if and only if every σ_i-subgroup of G satisfies property C_{σ_i} in G for all $i \in I$.*

Proof. If G is σ-nilpotent and $i \in I$, then every σ_i-subgroup A of G is σ-subnormal in G. By Theorem A, A satisfies property C_{σ_i} in G. Conversely, let $i \in I$ and let A be a Hall σ_i-subgroup of G. Then $|AB|$ divides $|G|$ for every Hall σ_i-subgroup B of G. Since $|AB|$ is a σ_i-number, it follows that $A = B$. Therefore G has a normal Hall σ_i-subgroup for all $i \in I$ and hence G is σ-nilpotent. □

Our last result can be regarded as an extension of [1].

Corollary 4. *Let G be a σ-soluble group. Then*

1. *All σ_i-subgroups of $F_\sigma(G)$ satisfy property C_{σ_i} in G for all $i \in I$.*
2. *$F_\sigma(G)$ contains every subgroup F of G such that, for every $i \in I$, all σ_i-subgroups of F satisfy property C_{σ_i} in G.*

Proof. Let $i \in I$ and let A be a σ_i-subgroup of a σ-soluble group G contained in $F_\sigma(G)$. Then A is a σ-subnormal subgroup of $F_\sigma(G)$ and $F_\sigma(G)$ is normal in G, it follows that A is σ-subnormal in G by Lemma 1(2). Therefore A satisfies property C_{σ_i} in G.

Assume that F is a subgroup of G with all its σ_i-subgroups satisfying property C_{σ_i} for every $i \in I$. By Corollary 1, every Hall σ_i-subgroup F_i of F is σ-subnormal in G. Then F_i is contained in $F_\sigma(G)$ by [9]. Since F is generated by its Hall σ_i-subgroups for all $i \in I$, it follows that $F \leq F_\sigma(G)$.

□

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Conflict of interest

The authors declare that there is no conflict of interest.

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