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Research article

Stability for a 3D Ladyzhenskaya fluid model with unbounded variable delay

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Abstract: This paper is concerned with the stability of solutions to a Ladyzhenskaya fluid model with unbounded variable delay. We first prove the existence, uniqueness and regularity of global weak solutions to the Ladyzhenskaya model by using Galerkin approximations and the energy method based on some suitable assumptions about external forces. Then we obtain that the stationary solution is locally stable. Finally, we establish that the stationary solution has polynomial stability in a particular case of unbounded variable delay.

Keywords: Ladyzhenskaya model; unbounded variable delays; asymptotic stability

1. Introduction

The 3D incompressible Navier-Stokes model is expressed as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu_0 \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

which was proposed by Navier and Stokes, respectively, for the motion of incompressible viscous fluids with a very small velocity gradient. Since the last century, there have been many interesting results on the existence, uniqueness, regularity and long-time behaviour of the Navier-Stokes equations [1–5]. However, for the 3D Navier-Stokes model, the uniqueness of weak solutions and the existence of strong solutions have not been solved.

Mathematicians and physicists have proposed many modified Navier-Stokes models to overcome the difficulty brought about by the nonlinear convection $(u \cdot \nabla)u$, such as the Navier-Stokes-α model, Navier-α model, Navier-Stokes-Voigt model and the globally modified Navier-Stokes model [6]. In particular, Ladyzhenskaya [7] and Ladyzhenskaya et al. [8] also gave a modified Navier-Stokes model. For the Navier-Stokes equations, the velocity derivative should not be very large. Therefore, Ladyzhenskaya et al. [8] replaced the Navier-Stokes equations with the following equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\Big[(\nu_0 + \nu_1|Du|^{q-2})Du\Big] + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \quad Du = \dfrac{1}{2}(\nabla u + \nabla u^T). \end{cases} \tag{1.2}$$

The model (1.2) can be used to describe the flow of fluid around objects placed in the fluid. Guo and Zhu [9] studied the partial regularity of the generalized solutions to model (1.2), and proved that the singular points are concentrated on a closed set whose $5 - 2q$ dimensional Hausdorff measure is zero when $2 < q \leq \frac{5}{2}$. Furthermore, the solution is a regular one when $q > \frac{5}{2}$. Recently, da Veiga and Yang [10] obtained that the singular points are concentrated on a closed set whose one-dimensional Hausdorff measure is zero when $q < 2$. Regarding the model (1.2), da Veiga has done a series of meaningful works [10–12].

This model stipulates that for $q > 2$, the stress tensor

$$T = -pI + (\nu_0 + \nu_1|Du|^{q-2})Du$$

is dependent on the symmetrical part Du of the velocity gradient in a polynomial growth manner. As $q = n = 3$ (n is the spatial dimension), (1.2) reduces to the classical Smagorinsky model, which is a turbulence model introduced by Smagorinsky [13]. One of the characteristics of the above q growth rate is that one can use the increased coercivity to obtain the existence and uniqueness of solutions. Assuming that q is large enough, the nonlinearity caused by $(u \cdot \nabla)u$ is no longer critical. This overcomes the problem of the Navier-Stokes equation lacking corresponding solvability. In addition, the peculiar features of certain fluids can be better described by using the polynomial growth of the stress tensors (refer to the monograph [14]). In order to study some specific types of fluids, and especially to describe shear thickening ($q > 2$) and shear thinning ($q < 2$) phenomena, Belloutet et al. [15] and Málek et al. [16, 17] have conducted in-depth research on such models.

However, the uniqueness and stability of solutions for model (1.2) are still open when the Reynolds number is large. With the establishment of the monotonicity method, Lions replaced Du with ∇u and reduced the model (1.2) to

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu_0 \Delta u - \nu_1 \displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}(|\nabla u|^{q-1} \dfrac{\partial u}{\partial x_i}) + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0 \end{cases} \tag{1.3}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu_0 \displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}(|\nabla u|^{q-1} \dfrac{\partial u}{\partial x_i}) + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0. \end{cases} \tag{1.4}$$

For the system (1.3) or (1.4), Lions [18] proved the existence of weak solutions when $q \geq 1+2n/(n+2)$. Lions derived the uniqueness of solutions to the system (1.3) when $q \geq (n + 2)/2$. However, the uniqueness of the system (1.4) remains an open problem. In particular, Lions also considered a variant of the model (1.3), namely

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\nu_0 + \nu_1 \|\nabla u\|^2_{L^2(\Omega)})\Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0. \end{cases} \tag{1.5}$$

In this paper, we focus on the model (1.5). Note that it reduces to the classical Navier-Stokes equations comprising (1.1) when $\nu_1 = 0$. Lions [18] proved the existence and uniqueness of global weak solutions for the initial-boundary value problem of the model (1.5). Under some assumptions on the external force, the existence of uniform attractors of the model (1.5) was proved in [19]. Recently, Yang et al. [20] considered the pullback dynamics of (1.5), and presented the finite fractal dimension of pullback attractors. Moreover, the upper semi-continuity of pullback attractors was also studied in [20].

The Navier-Stokes equation with hereditary terms was first considered by Caraballo and Real [21], and there are many significant results on this model [22–25]. Because the current state may be affected by the distant historical state, the delay time is quite large at this time. For the case of infinite delays, it can be seen from the literature [26–31] that the initial value of the delay is usually considered in the following space

$$C_\gamma(H) = \{\varphi \in C((-\infty, 0]; H) | \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \text{ exists in } H\}(\gamma > 0).$$

Here H is the square-integrable function space satisfying the incompressible condition. At this time, the delayed external force is a distributed delay. That is, it only includes the case of infinite distributed delays. Marín-Rubio et al. [27] considered the globally modified Navier-Stokes model with infinite delays, and established the global well-posedness of solutions when the initial value of the delay was in the space $C_\gamma(H)$. Moreover, they proved the exponential stability of stationary solutions. For the 3D Navier-Stokes-Voigt equations in the space $C_\gamma(H)$, Anh and Thanh [30] obtained the exponential stability of solutions and proved the existence of global attractors.

The positive constant γ plays an important role in the above works. If γ disappears, can we get similar results? Liu et al. [32] considered the 2D Navier-Stokes models with unbounded variable delays, which is in the following phase space

$$BCL_{-\infty}(H) = \{\varphi \in C((-\infty, 0]; H) | \lim_{s\to-\infty} \varphi(s) \text{ exists in } H\}.$$

They established the existence and uniqueness of solutions and analyzed the stability of stationary solutions by using several different methods. $BCL_{-\infty}(H)$ seems to be a natural relaxation of $C_\gamma(H)$. Recently, Toi [33] studied the 3D Navier-Stokes-Voigt equations in $BCL_{-\infty}(H)$ and gave a sufficient condition for the polynomial stability of stationary solutions.

Inspired by the results [27, 32, 33], this paper is concerned with the stability of solutions to a Ladyzhenskaya model with unbounded variable delays defined on $(0, \infty) \times \Omega$, as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\nu_0 + \nu_1 \|\nabla u\|^2_{L^2(\Omega)})\Delta u + (u \cdot \nabla)u + \nabla p = f(t) + g(t, u_t), \quad (t, x) \in (0, \infty) \times \Omega, \\ \nabla \cdot u = 0, \quad (t, x) \in (0, \infty) \times \Omega, \\ u(t, x)|_{\partial\Omega} = 0, \quad t \in (0, \infty), \\ u_0(s, x) = u(s, x) = \phi(s, x), \quad s \in (-\infty, 0], \ x \in \Omega, \end{cases} \tag{1.6}$$

where $\Omega \in \mathbb{R}^3$ is a bounded open domain with the sufficiently smooth boundary $\partial\Omega$, $u = (u_1, u_2, u_3)$ is the velocity field, p is pressure, $\nu_0 > 0$ and $\nu_1 > 0$ are the kinematic viscosities, $f(t)$ is a non-delayed external force and $g(t, u_t)$ is an external force with some hereditary characteristics. The function u_t in the delay term $g(t, u_t)$ is defined on $(-\infty, 0]$ by $u_t(s) = u(t+s),\ s \in (-\infty, 0]$. ϕ is the prescribed initial condition. More conditions on f and ϕ will be specified later.

The main results of the present paper are summarized as follows:

(I) First, the existence and uniqueness of global weak solutions to the model (1.6) are established by combining Galerkin approximations and the energy method; then, the regularity of weak solutions is obtained. See Theorem 3.1.

(II) Furthermore, we prove the existence and uniqueness of stationary solutions to the model (1.6) by employing a corollary of Schauder's fixed point theorem. For the stability of stationary solutions, the local stability is obtained. Then, by using the Lyapunov function method, we derive that the stationary solution is exponentially stable when the delay length is bounded. See Theorems 4.1, 4.3, and 4.4.

(III) Finally, by virtue of a lemma on the proportional equation, we give a sufficient condition of parameters for the polynomial stability of the stationary solution in a special case of unbounded variable delay. See Theorem 4.6.

2. Preliminaries

2.1. Functional spaces, operators and lemmas

Denote

$$E := \{u | u \in (C_0^\infty(\Omega))^3, \nabla \cdot u = 0\}.$$

Let H and V be the closures of E in $(L^2(\Omega))^3$ and $(H_0^1(\Omega))^3$, respectively. The inner products in H and V are represented by $(\cdot,\cdot)$ and $((\cdot,\cdot))$, respectively, which are defined as follows:

$$(u, v) = \sum_{i=1}^{3} \int_\Omega u_i(x) v_i(x) dx,\ \ ((u, v)) = \sum_{i=1}^{3} \int_\Omega \nabla u_i(x) \nabla v_i(x) dx.$$

The associated norms in H and V are represented by $|\cdot|_2$ and $\|\cdot\|$, respectively, which are defined as follows:

$$|u|_2 = (u, u)^{\frac{1}{2}},\ \ \|u\| = ((u, u))^{\frac{1}{2}}.$$

Then, we have that $\|u\| = |\nabla u|_2$ for all $u \in V$. It is easy to verify that H and V are Hilbert spaces. Let H' and V' be dual spaces of H and V, respectively. We have that $V \hookrightarrow\hookrightarrow H \equiv H' \hookrightarrow V'$, where the injections are dense and continuous. We use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the dual pairing between V and V', where $\|\cdot\|_*$ is defined as follows:

$$\|f\|_* = \sup_{v \in V, \|v\|=1} |\langle f, v\rangle|, \quad \forall f \in V'. \tag{2.1}$$

P denotes the Helmholz-Leray orthogonal projection from $(L^2(\Omega))^3$ onto the space H (see [34, 35]). We define $A := -P\Delta$ as the Stokes operator on $D(A) = (H^2(\Omega))^3 \cap V$; then, $A : V \to V'$ satisfies that

$\langle Au, v\rangle = ((u, v))$, and A is an isomorphism from V into V'. It holds that $\|Au\|_* = \sup_{v\in V, \|v\|=1} |\langle Au, v\rangle| = \sup_{v\in V, \|v\|=1} |((u, v))| \leq \|u\|$, i.e., $\|A\| \leq 1$. Let $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues of the operator A with the Dirichlet boundary condition, which satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \to +\infty \text{ as } i \to +\infty.$$

From the property of the Stokes operator, the corresponding eigenfunctions given by $\{\omega_i\}_{i=1}^{\infty}$ form an orthonormal complete basis in H. Moreover, we have the following Poincaré inequality

$$|v|_2^2 \leq \frac{1}{\lambda_1}\|v\|^2, \quad \forall v \in V. \tag{2.2}$$

In order to deal with the nonlinear term $\nu_1\|u\|^2$, we define the operator $A_1 : V \to V'$ as $A_1u := -\nu_1\|u\|^2\Delta u$, which satisfies

$$\langle A_1u, v\rangle = \nu_1\|u\|^2\langle -\Delta u, v\rangle = \nu_1\|u\|^2((u, v)), \quad \forall u, v \in V. \tag{2.3}$$

Obviously, $\langle A_1u - A_1v, u - v\rangle \geq 0$ for any $u, v \in V$. That is, A_1 is a monotone operator. We can obtain from (2.1) and (2.3) that

$$\|A_1u\|_* = \sup_{v\in V, \|v\|=1} |\langle A_1u, v\rangle| = \sup_{v\in V, \|v\|=1} \nu_1\|u\|^2((u, v)) \leq \nu_1\|u\|^3, \quad \forall u \in V. \tag{2.4}$$

We also introduce the bilinear operator

$$B(u, v) = P((u \cdot \nabla)v), \quad \forall u, v \in V$$

and the trilinear operator

$$b(u, v, \omega) = (B(u, v), \omega) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i}\omega_j dx, \quad \forall u, v, \omega \in V.$$

Furthermore, the bilinear operator $B(u, v)$ and the trilinear operator $b(u, v, \omega)$ satisfy the following conditions (see [5, 34, 35]):

$$\begin{cases} \|B(u, v)\|_* \leq c_1\|u\|\|v\|, \quad \forall u, v \in V, \\ b(u, v, v) = 0, \quad \forall u, v \in V, \\ |b(u, v, w)| \leq c_1\|u\|^{\frac{1}{2}}|Au|_2^{\frac{1}{2}}\|v\||w|_2, \quad \forall u \in D(A), v \in V, w \in H, \\ |b(u, v, w)| \leq c_1|u|_2^{\frac{1}{4}}\|u\|^{\frac{3}{4}}\|v\||w|_2^{\frac{1}{4}}\|w\|^{\frac{3}{4}}, \quad \forall u, v, w \in V, \end{cases} \tag{2.5}$$

where c_1 is a constant greater than zero. The following lemma is a refined version of Dini's theorem that has been developed to deal with the delay effect.

Lemma 2.1. *([29]) (**Dini theorem**) Suppose that a function sequence $\{S_n(x)\}_{n=1}^{\infty}$ on the finite interval $[a, b]$ satisfies*

$$\begin{cases} S_n(x) \to S(x), \ \ a.e.\ a \leq x \leq b, \ \ as\ n \to \infty; \\ S_n(a) \to S(a) \text{ and } S_n(b) \to S(b), \ \ as\ n \to \infty. \end{cases} \tag{2.6}$$

If $\{S_n(x)\}_{n=1}^{\infty}$ is monotonic on $[a, b]$ for any $n \in \mathbb{N}^+$, and $S(x)$ is continuous, then $\{S_n(x)\}_{n=1}^{\infty}$ uniformly converges to $S(x)$ on $[a, b]$.

For the fixed-field case, the monotone operator has properties similar to the evolution case (see [18]). There is a new type of operator as follows.

Definition 2.2. *([18]) The operator $K : V \to V'$ is called the operator of (M) type if K satisfies: supposing the sequence $\{u_m\} \subset V$ such that*

$$u_m \rightharpoonup u \text{ weakly in } V,$$
$$Ku_m \rightharpoonup \chi \text{ weakly in } V'$$

and

$$\limsup_{m\to\infty} \langle K(u_m), u_m\rangle \leq \langle \chi, u\rangle;$$

then, $\chi = K(u)$.

The monotone operators and the operators of (M) type have the following inclusion relation.

Lemma 2.3. *([18]) Let A be an operator mapping V to V'; if A is a bounded semi-continuous monotone operator, then A is an operator of (M) type.*

The next lemma is a corollary of Schauder's fixed point theorem to prove the existence of stationary solutions to the problem (1.6).

Lemma 2.4. *([36]) Let X be a finite-dimensional Hilbert space equipped with scalar product $[\cdot,\cdot]$ and norm $[\cdot]$. $F : X \to X$ is a continuous mapping. If there exists $k > 0$, such that*

$$[F(\xi), \xi] \geq 0, \quad \text{as } [\xi] = k,$$

then there exist $\xi \in X$ and $[\xi] \leq k$, such that $F(\xi) = 0$.

2.2. Statement of the problem

In virtue of the above operators P, A, and A_1, we can transform the problem (1.6) into the following equivalent abstract form

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nu_0 Au + P(A_1 u) + B(u,u) = Pf(x,t) + Pg(t,u_t), \quad (t,x) \in (0,\infty)\times\Omega, \\ \nabla\cdot u = 0, \quad (t,x)\in(0,\infty)\times\Omega, \\ u(t,x)|_{\partial\Omega} = 0, \quad t\in(0,\infty), \\ u_0(s,x) = u(s,x) = \phi(s,x), \quad s\in(-\infty,0], \quad x\in\Omega. \end{cases} \tag{2.7}$$

We will establish the well-posedness and stability results for the problem (2.7) with infinite delay operators in the following phase:

$$BCL_{-\infty}(H) = \{\varphi \in C((-\infty,0];H) | \lim_{s\to-\infty} \varphi(s) \text{ exists in } H\},$$

which is a Banach space endowed with the norm

$$\|\varphi\|_{BCL_{-\infty}(H)} = \sup_{s\in(-\infty,0]} |\varphi(s)|_2, \quad \forall \varphi \in BCL_{-\infty}(H).$$

In order to obtain the existence and uniqueness of solutions to the problem (2.7), we impose some appropriate assumptions on delay term g.

(**H-g**) Assume that $g : [0,T] \times BCL_{-\infty}(H) \to (L^2(\Omega))^3$ satisfies the following:

($g1$) $g(t,0) = 0, \quad \forall t \in [0,T]$.

($g2$) For all $\xi \in BCL_{-\infty}(H)$, the mapping $[0,T] \ni t \to g(t,\xi) \in (L^2(\Omega))^3$ is measurable.

($g3$) There exists $L_g > 0$ such that for all $\xi, \eta \in BCL_{-\infty}(H)$,

$$|g(t,\xi) - g(t,\eta)|_2 \le L_g \|\xi - \eta\|_{BCL_{-\infty}(H)}, \quad \forall t \in [0,T].$$

Remark 1. **(i)** As pointed out in [28, 32], ($g1$) is not a restriction. Indeed, if $g(t,0) \ne 0$, we could redefine $\bar{f}(t) = f(t) + g(t,0)$ and $\bar{g}(t,\cdot) = g(t,\cdot) - g(t,0)$. At this time, $\bar{g}$ meets the condition ($g1$), and the problem is not changed in this way.

(ii) The conditions ($g1$) and ($g3$) indicate that for any $\xi \in BCL_{-\infty}(H)$,

$$|g(t,\xi)|_2 \le L_g \|\xi\|_{BCL_{-\infty}(H)}, \quad \forall t \in [0,T].$$

Thus, it holds that $|g(\cdot,\xi)|_2 \in L^\infty(0,T)$.

(iii) In particular, examples of a distributed delay operator and a variable delay operator satisfying conditions ($g1$)–($g3$) can be given respectively (refer to Examples 2.2 and 2.4 in [32]).

Above all, we establish the following definition of global weak solutions to the problem (2.7).

Definition 2.5. *Let $T > 0$ and the initial datum $\phi \in BCL_{-\infty}(H)$. A weak solution to the problem* (2.7) *in $(-\infty, T]$ is a function $u = u(t,x) \in C((-\infty,T];H) \cap L^4(0,T;V)$ such that $u_0 = \phi$ and*

$$\frac{d}{dt}(u,v) + \nu_0\langle Au, v\rangle + \langle A_1 u, v\rangle + b(u,u,v) = \langle f(t), v\rangle + (g(t,u_t), v)$$

holds for all $v \in V$ in the sense of $D'(0,T)$. $D'(0,T)$ denotes the distribution space composed of functions, and it is defined on $(0,T)$ and valued in $\mathbb{R}$.

Remark 2. If u is a weak solution to the problem (2.7), then u satisfies the following energy equality

$$\begin{aligned} &\frac{1}{2}|u(t)|_2^2 + \nu_0\int_s^t \|u(r)\|^2 dr + \nu_1 \int_s^t \|u(r)\|^4 dr \\ &= \frac{1}{2}|u(s)|_2^2 + \int_s^t \langle f(r), u(r)\rangle dr + \int_s^t (g(r,u_t), u(r)) dr, \quad \forall 0 \le s \le t \le T. \end{aligned} \tag{2.8}$$

3. Well-posedness and regularity of global solutions

In this section, we prove the existence, uniqueness and regularity of weak solutions to the problem (2.7). The following theorem is one of the main results.

Theorem 3.1. **1)** *Consider that g satisfies the assumptions encompassed by (*__H-g__*). If $f \in L^{\frac{4}{3}}(0,T;V')$ and $\phi \in BCL_{-\infty}(H)$, then the problem* (2.7) *possesses a unique weak solution $u \in C((-\infty,T];H) \cap L^4(0,T;V)$.*

2) *Moreover, if $f \in L^2(0,T;(L^2(\Omega))^3)$ and $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$, the weak solution u is strong. That is, $u \in C([0,T];V) \cap L^2(0,T;D(A))$.*

Proof. We split the proof into several steps.

Step 1: Galerkin scheme and a priori estimates

By applying the classical spectral theory of the elliptic operators, let $\{\omega_i\}_{i=1}^{\infty} \in V$ be the basis of all eigenfunctions for the Stokes operator A, which is also a complete orthonormal basis in H and V. Denote $V_m = \text{span}\{\omega_1, \cdots, \omega_m\}$ and consider the projector $P_m : H \to V_m$ given by

$$P_m u = \sum_{i=1}^{m} (u, \omega_i)\omega_i, \quad \forall u \in H.$$

We define the approximated solution $u_m(t) = \sum_{i=1}^{m} h_{im}(t)\omega_i$ that satisfies following Cauchy problem:

$$\begin{cases} \frac{d}{dt}(u_m, \omega_i) + \nu_0\langle Au_m, \omega_i\rangle + \langle A_1 u_m, \omega_i\rangle + b(u_m, u_m, \omega_i) \\ \qquad\qquad = \langle f(t), \omega_i\rangle + (g(t, u_{mt}), \omega_i), \quad 1 \le i \le m, \\ u_m(s) = P_m\phi(s), \quad s \in (-\infty, 0]. \end{cases} \tag{3.1}$$

The problem (3.1) is equivalent to a set of functional differential equations with infinite delay and the unknown variable $\{h_{1m}(t), h_{2m}(t), \cdots, h_{mm}(t)\}$. From Theorem 1.1 in [37], it can be obtained that the problem (3.1) has a unique local solution u_m on the interval $[0, t_m]$.

Next, we will obtain that u_m exists globally by proving a prior estimate. Multiplying $(3.1)_1$ by $h_{im}(t)$, and then summing i from 1 to m, with the help of the Cauchy-Schwartz inequality and Young's inequality, we have

$$\begin{aligned} &\frac{d}{dt}|u_m(t)|_2^2 + 2\nu_0\|u_m(t)\|^2 + 2\nu_1\|u_m(t)\|^4 \\ &= 2\langle f(t), u_m(t)\rangle + 2(g(t, u_{mt}), u_m(t)) \\ &\le 2\|f(t)\|_*\|u_m(t)\| + 2|u_m(t)|_2|g(t, u_{mt})|_2 \\ &\le (\frac{27}{16\nu_1})^{\frac{1}{3}}\|f(t)\|_*^{\frac{4}{3}} + \nu_1\|u_m(t)\|^4 + 2L_g\|u_{mt}\|^2_{BCL_{-\infty}(H)}. \end{aligned} \tag{3.2}$$

For convenience, we use μ to denote $(\frac{27}{16\nu_1})^{\frac{1}{3}}$. Integrating (3.2) with respect to the time variable t from 0 to t, we derive

$$\begin{aligned} &|u_m(t)|_2^2 + \nu_1\int_0^t \|u_m(s)\|^4 ds \\ &\le |u_m(0)|_2^2 + \mu\int_0^t \|f(s)\|_*^{\frac{4}{3}} ds + 2L_g\int_0^t \|u_{ms}\|^2_{BCL_{-\infty}(H)} ds, \end{aligned} \tag{3.3}$$

which implies that

$$\|u_{mt}\|^2_{BCL_{-\infty}(H)} \le \|\phi\|^2_{BCL_{-\infty}(H)} + \mu\int_0^t \|f(s)\|_*^{\frac{4}{3}} ds + 2L_g\int_0^t \|u_{ms}\|^2_{BCL_{-\infty}(H)} ds. \tag{3.4}$$

Applying the Gronwall inequality to (3.4), we get

$$\|u_{mt}\|^2_{BCL_{-\infty}(H)} \le (\|\phi\|^2_{BCL_{-\infty}(H)} + \mu\int_0^t \|f(s)\|_*^{\frac{4}{3}} ds)e^{2L_g t},$$

i.e.

$$\|u_{mt}\|^2_{BCL_{-\infty}(H)} \le C(R,T), \quad \forall t \in [0,T], \forall \|\phi\|_{BCL_{-\infty}(H)} \le R \text{ and } \forall m \ge 1, \tag{3.5}$$

where $C(R,T)$ is a constant that is dependent on $\nu_1, L_g, T, R > 0$ and f.

Further, from (3.3) and (3.5), it follows that

$$\nu_1 \int_0^t \|u_m(s)\|^4 ds \le |u_m(0)|_2^2 + \int_0^t (\mu\|f(s)\|_*^{\frac{4}{3}} + 2L_g C(R,T))ds.$$

Considering that $f \in L^{\frac{4}{3}}(0,T;V')$, there exists another constant $C(R,T)$ (relabeled to be the same) such that

$$\|u_m\|_{L^4(0,T;V)} \le C(R,T), \ \ \forall m \ge 1. \tag{3.6}$$

Hence, (3.5) and (3.6) indicate that

$$\{u_m\} \text{ is uniformly bounded in } L^\infty(0,T;H) \cap L^4(0,T;V). \tag{3.7}$$

Observe that $(3.1)_1$ is equivalent to the following:

$$\frac{\partial u_m}{\partial t} = P_m f(t) + P_m g(t,u_{mt}) - P_m(A_1 u_m) - \nu_0 A u_m - P_m B(u_m,u_m). \tag{3.8}$$

Combining Remark **1(ii)** and (3.5), one has

$$\int_0^T |g(t,u_{mt})|_2^2 dt \le L_g^2 \int_0^T \|u_{mt}\|^2_{BCL_{-\infty}(H)} dt \le C. \tag{3.9}$$

Notice that $\|P_m\|_{\mathscr{L}(V,V)} \le 1$ and $P_m^* = P_m$, so $\|P_m\|_{\mathscr{L}(V',V')} \le 1$. Thus, it follows from (2.4), $(2.5)_1$ and (3.7)–(3.9) that

$$\{\frac{\partial u_m}{\partial t}\} \text{ is uniformly bounded in } L^{\frac{4}{3}}(0,T;V'). \tag{3.10}$$

Step 2: Compactness results and approximations in $BCL_{-\infty}(H)$ for the initial datum

By (3.1), (3.6)–(3.10) and the Aubin-Lions compactness lemma, we deduce that there exists a subsequence (still denoted by $\{u_m\}$) and $u \in L^\infty(0,T;H) \cap L^4(0,T;V)$ such that, when $m \to +\infty$,

$$\begin{cases} u_m \to u \ \text{ strongly in } L^4(0,T;H), \\ u_m \rightharpoonup u \ \text{ weakly * in } L^\infty(0,T;H), \\ u_m \rightharpoonup u \ \text{ weakly in } L^4(0,T;V), \\ \dfrac{\partial u_m}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \ \text{ weakly in } L^{\frac{4}{3}}(0,T;V'), \\ A_1(u_m) \rightharpoonup \psi \ \text{ weakly in } L^{\frac{4}{3}}(0,T;V'), \\ g(t,u_{mt}) \rightharpoonup \xi \ \text{ weakly in } L^2(0,T;(L^2(\Omega))^3). \end{cases} \tag{3.11}$$

Thanks to $(3.11)_3$ and $(3.11)_4$, we infer that $u_m, u \in C([0,T];H)$. Next, we verify that the initial value sequence satisfies

$$P_m\phi \to \phi \text{ strongly in } BCL_{-\infty}(H), \quad m \to \infty. \tag{3.12}$$

In consideration of $\phi \in BCL_{-\infty}(H)$, we set $\lim_{\theta \to -\infty} \phi(\theta) = \tilde{\phi} \in H$. That is to say, for any $\varepsilon > 0$, there is a large enough $N > 0$ such that

$$|\phi(\theta) - \tilde{\phi}|_2 < \frac{\varepsilon}{6}, \quad \text{as } \theta \leq -N. \tag{3.13}$$

Since P_m is a projection operator, we can find an $M_1 > N$ such that

$$\begin{aligned} &\sup_{\theta \in (-\infty, -N]} |P_m \phi(\theta) - \phi(\theta)|_2 \\ &\leq \sup_{\theta \in (-\infty, -N]} |P_m \phi(\theta) - P_m \tilde{\phi}|_2 \\ &\quad + |P_m \tilde{\phi} - \tilde{\phi}|_2 + \sup_{\theta \in (-\infty, -N]} |\tilde{\phi} - \phi(\theta)|_2 \\ &< \frac{\varepsilon}{2}, \quad \forall m > M_1. \end{aligned} \tag{3.14}$$

Observe that, for any $\theta \in [-N, 0]$, $|P_m \phi(\theta) - \phi(\theta)|_2$ is non-increasing with respect to m, and, for almost everywhere where $-N \leq \theta \leq 0$,

$$|P_m \phi(\theta) - \phi(\theta)|_2 \to 0, \quad m \to \infty.$$

Also, it holds that

$$|P_m \phi(-N) - \phi(-N)|_2 \to 0, \quad |P_m \phi(0) - \phi(0)|_2 \to 0.$$

Based on one Dini's theorem (i.e. Lemma 2.1), there exists an $M_2 > 0$ such that

$$\sup_{\theta \in [-N, 0]} |P_m \phi(\theta) - \phi(\theta)|_2 < \frac{\varepsilon}{2}, \quad \forall m > M_2. \tag{3.15}$$

Now, we conclude from (3.14) and (3.15) that

$$\begin{aligned} \|P_m \phi - \phi\|_{BCL_{-\infty}(H)} \leq &\sup_{\theta \in (-\infty, -N]} |P_m \phi(\theta) - \phi(\theta)|_2 \\ &+ \sup_{\theta \in [-N, 0]} |P_m \phi(\theta) - \phi(\theta)|_2 \\ &< \varepsilon, \quad m > \max\{M_1, M_2\}. \end{aligned}$$

Thus, we complete the proof of (3.12).

Step 3: Energy method and the existence of solutions

To take the limit of the approximate system (3.1), we will prove that the nonlinear term can exceed the limit. For the nonlinear convection term $b(u_m, u_m, \omega_i)$, the same procedure as in [38] can yield

$$\int_0^T |b(u_m, u_m, \omega_i) - b(u, u, \omega_i)| ds \to 0, \quad m \to \infty. \tag{3.16}$$

Regarding the nonlinear viscosity term $\langle A_1 u_m, \omega_i \rangle$, we use a similar technique as in [39]. Notice that here we can only obtain the weak convergence of the infinite delay term $g(t, u_{mt})$, that is,

$$g(t, u_{mt}) \rightharpoonup \xi \text{ weakly in } L^2(0, T; (L^2(\Omega))^3). \tag{3.17}$$

However, by using the Hölder inequality, we have

$$\begin{aligned}
&\int_0^T (g(t,u_{mt}),u_m)-(\xi,u)dt\\
&=\int_0^T (g(t,u_{mt}),u_m-u)+(g(t,u_{mt})-\xi,u)dt\\
&\le \int_0^T |g(t,u_{mt})|_2|u_m-u|_2dt+\int_0^T (g(t,u_{mt})-\xi,u)dt\\
&\le \|g(t,u_{mt})\|_{L^2(0,T;(L^2(\Omega))^3)}\|u_m-u\|_{L^2(0,T;H)}+\int_0^T (g(t,u_{mt}),u)dt-\int_0^T(\xi,u)dt,
\end{aligned} \tag{3.18}$$

which, along with $(3.11)_1$ and (3.17), yields

$$\int_0^T (g(t,u_{mt}),u_m)dt\to\int_0^T(\xi,u)dt,\quad m\to\infty.$$

Then, following the same proof process as equation (28) in our previous work [38], we have

$$\psi=A_1(u). \tag{3.19}$$

Because the delay $g(t,u_{mt})$ is infinite here, we also need to prove that the limit can exceed it. Next, we will use the energy method to obtain following limit process:

$$g(t,u_{mt})\to g(t,u_t) \text{ strongly in } L^2(0,T;(L^2(\Omega))^3),\quad m\to\infty. \tag{3.20}$$

On account of the condition ($g3$), we have

$$|g(t,u_{mt})-g(t,u_t)|_2\le L_g\|u_{mt}-u_t\|_{BCL_{-\infty}(H)},\quad \forall t\in[0,T]. \tag{3.21}$$

Thus, in order to prove (3.20), we only need to prove that, for all $0\le t\le T$,

$$u_{mt}\to u_t \text{ strongly in } BCL_{-\infty}(H). \tag{3.22}$$

From the definition of the space $BCL_{-\infty}(H)$, we can derive

$$\begin{aligned}
&\|u_{mt}-u_t\|_{BCL_{-\infty}(H)}\\
&\le \max\Big\{\sup_{\theta\in(-\infty,-t]}|P_m\phi(\theta+t)-\phi(\theta+t)|_2,\\
&\qquad \sup_{\theta\in[-t,0]}|u_m(\theta+t)-u(\theta+t)|_2\Big\}\\
&\le \max\Big\{\|P_m\phi-\phi\|_{BCL_{-\infty}(H)},\sup_{t\in[0,T]}|u_m(t)-u(t)|_2\Big\},\quad \forall 0\le t\le T.
\end{aligned} \tag{3.23}$$

Noting the convergence of (3.12), we only need to discuss the second term on the right-hand side of (3.23), i.e.,

$$u_m\to u \text{ strongly in } C([0,T];H). \tag{3.24}$$

First of all, taking into account $(3.11)_1$, it follows that

$$u_m(t) \to u(t) \text{ in } H \text{ for a.e. } t \in [0,T]. \tag{3.25}$$

However, this is not enough to prove (3.24). By applying the Hölder inequality, we can infer the following:

$$\begin{aligned} \|u_m(t) - u_m(s)\|_* &= \int_s^t \|u_m'(\tau)\|_* d\tau \\ &\le (t-s)^{\frac{1}{4}} \|u_m'\|_{L^{\frac{4}{3}}(\tau,T;V')}, \quad \forall 0 \le s \le t \le T. \end{aligned} \tag{3.26}$$

Hence, (3.26) and (3.10) show that $\{u_m(t)\}$ is uniformly equicontinuous on $[0,T]$ in V'. Given the fact that $H \hookrightarrow V'$ is compact and (3.5), we can see that for any $t \in [0,T]$, $\{u_m(t)\}$ is a precompact set in V'. Then by the Arzelà-Ascoli theorem, one has

$$u_m \to u \text{ strongly in } C([0,T];V'). \tag{3.27}$$

Again from (3.7), we obtain that for any sequence $\{t_m\} \in [0,T]$ with $\lim_{m\to\infty} t_m = t$,

$$u_m(t_m) \rightharpoonup u(t) \text{ weakly in } H, \tag{3.28}$$

where we have used (3.27) to identify which is the weak limit.

Next, we are going to prove (3.24) by a contradiction argument. If (3.24) does not hold, considering $u \in C([0,T];H)$, then we have that $\varepsilon > 0$ and $t_0 \in [0,T]$, as well as the subsequences $\{u_m(t)\}$ (still relabelled with the same notation) and $\{t_m\} \subset [0,T]$ with $\lim_{m\to\infty} t_m = t_0$, such that

$$|u_m(t_m) - u(t_0)|_2 \ge \varepsilon, \quad m \ge 1. \tag{3.29}$$

We will prove that this is incorrect by using the energy method. From (3.2), it can be seen that for all u_m, there is the following energy inequality:

$$\begin{aligned} &\frac{1}{2}|u_m(t)|_2^2 + \frac{\nu_0}{2}\int_s^t \|u_m(r)\|^2 dr + \nu_1 \int_s^t \|u_m(r)\|^4 dr \\ &\le \frac{1}{2}|u_m(s)|_2^2 + \int_s^t \langle f(r), u_m(r)\rangle dr + \frac{1}{2\nu_0\lambda_1}\int_s^t |g(r,u_{mr})|_2^2 dr \\ &\le \frac{1}{2}|u_m(s)|_2^2 + \int_s^t \langle f(r), u_m(r)\rangle dr + C''(t-s), \quad \forall 0 \le s \le t \le T, \end{aligned} \tag{3.30}$$

where $C'' = \frac{D}{2\nu_0\lambda_1}$ and D corresponds to the upper bound given by

$$\int_s^t |g(r,u_{mr})|_2^2 dr \le D(t-s), \quad \forall 0 \le s \le t \le T.$$

Given (3.11), (3.16) and (3.19), taking the limit of (3.1), we deduce that

$$\frac{d}{dt}(u,v) + \nu_0(Au,v) + (A_1u,v) + b(u,u,v) = \langle f(t), v\rangle + (\xi, v), \quad \forall v \in V, \tag{3.31}$$

and $u(0) = \phi(0)$. If $v = u$ in (3.31), the following energy equality holds

$$\begin{aligned}&\frac{1}{2}|u(t)|_2^2 + \nu_0 \int_s^t \|u(r)\|^2 dr + \nu_1 \int_s^t \|u(r)\|^4 dr \\ &= \frac{1}{2}|u(s)|_2^2 + \int_s^t \langle f(r), u(r)\rangle dr + \int_s^t (\xi, u(r)) dr, \quad \forall 0 \le s \le t \le T.\end{aligned} \tag{3.32}$$

On the other hand, by $(3.11)_6$ and the weak lower semicontinuity of norms, it can be concluded that

$$\int_s^t |\xi(r)|_2^2 dr \le \liminf_{m\to\infty} \int_s^t |g(r, u_{mr})|_2^2 dr \le D(t-s), \quad \forall 0 \le s \le t \le T. \tag{3.33}$$

Then, by using (3.32), (3.33), combined with the Cauchy-Schwartz inequality and (2.2), it follows that for any $0 \le s \le t \le T$,

$$\begin{aligned}&\frac{1}{2}|u(t)|_2^2 + \nu_0 \int_s^t \|u(r)\|^2 dr + \nu_1 \int_s^t \|u(r)\|^4 dr \\ &\le \frac{1}{2}|u(s)|_2^2 + \int_s^t \langle f(r), u(r)\rangle dr + \int_s^t |\xi(r)|_2 |u(r)|_2 dr \\ &\le \frac{1}{2}|u(s)|_2^2 + \int_s^t \langle f(r), u(r)\rangle dr + \int_s^t (\frac{1}{2\nu_0\lambda_1}|\xi(r)|_2^2 + \frac{\nu_0\lambda_1}{2}|u(r)|_2^2) dr \\ &\le \frac{1}{2}|u(s)|_2^2 + \frac{\nu_0}{2}\int_s^t \|u(r)\|^2 dr + \int_s^t \langle f(r), u(r)\rangle dr + C''(t-s).\end{aligned} \tag{3.34}$$

So, we see that u also satisfies the energy inequality (3.30) with the same constant C''.

Now, we consider the functions $J, J_m : [0, T] \to \mathbb{R}$ respectively defined by

$$\begin{aligned}J(t) &= \frac{1}{2}|u(t)|_2^2 - \int_0^t \langle f(r), u(r)\rangle dr - C''t, \\ J_m(t) &= \frac{1}{2}|u_m(t)|_2^2 - \int_0^t \langle f(r), u_m(r)\rangle dr - C''t.\end{aligned}$$

On account of (3.30) and (3.34), it is clear that J and J_m are continuous and non-increasing functions on $[0, T]$. We infer from $(3.11)_3$ and (3.25) that $J_m(t) \to J(t)$ alomst everywhere that $t \in [0, T]$. Thanks to (3.28), we have that

$$u_m(t_m) \rightharpoonup u(t_0) \text{ weakly in } H. \tag{3.35}$$

Therefore, it holds that

$$\limsup_{m\to\infty} |u_m(t_m)|_2 \le |u(t_0)|_2. \tag{3.36}$$

We obtain from (3.35) and (3.36) that

$$\lim_{m\to\infty} |u_m(t_m)|_2 = |u(t_0)|_2. \tag{3.37}$$

Thanks to (3.35) and (3.37), one has

$$u_m(t_m) \to u(t_0) \text{ strongly in } H,$$

which contradicts (3.29); so, we get (3.24). Further, (3.22) and (3.20) are also true. Now, based on the limit of (3.1), by combining (3.16), (3.19) and (3.20), we can infer that u is indeed a weak solution to the problem (2.7).

Finally, it is necessary to prove that (3.36) is valid. We discuss t_0 in the following two cases:

1) If $t_0 = 0$, let $s = 0$ and $t = t_m$ in (3.30); then, taking the upper limit of both ends, by $\lim_{m\to\infty} t_m = 0$ and $u_m(0) = P_m\phi(0) = P_m u(0)$, we obtain

$$\limsup_{m\to\infty} |u_m(t_m)|_2 \le \limsup_{m\to\infty} |P_m u(0)|_2 = |u(0)|_2,$$

namely, (3.36) is true. Only when $0 < t_0 \le T$ can we take the approximation sequence from the left so that its limit is t_0; so, a separate discussion of $t_0 = 0$ is necessary here.

2) If $0 < t_0 \le T$, we can take the sequence $\{t'_k\} \subset (0, t_0)$ such that $\lim_{k\to\infty} t'_k = t_0$ and $\lim_{m\to\infty} J_m(t'_k) = J(t'_k)$ for all k. From the continuity of $J(s)$, it can be deduced that for any $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$|J(t'_k) - J(t_0)| < \frac{\varepsilon}{2}, \quad \forall k \ge k_\varepsilon. \tag{3.38}$$

Because $\lim_{m\to\infty} t_m = t_0$ and the sequence $\{t'_k\}$ tends to t_0 from the left, we can take $m(k_\varepsilon)$ such that, for all $m \ge m(k_\varepsilon)$,

$$t'_{k_\epsilon} \le t_m \ \text{ and } \ |J_m(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| < \frac{\varepsilon}{2}. \tag{3.39}$$

According to the non-increasing property of J_m, for any $m \ge m(k_\varepsilon)$, we can obtain from (3.38) and (3.39) that

$$\begin{aligned} J_m(t_m) - J(t_0) &\le J_m(t'_{k_\epsilon}) - J(t_0) \\ &\le |J_m(t'_{k_\epsilon}) - J(t_0)| \\ &\le |J_m(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| + |J(t'_{k_\epsilon}) - J(t_0)| < \varepsilon. \end{aligned} \tag{3.40}$$

By the arbitrariness of ε, (3.40) yields that $\limsup_{m\to\infty} J_m(t_m) \le J(t_0)$, i.e.,

$$\limsup_{m\to\infty}\Big(\frac{1}{2}|u_m(t_m)|_2^2 + \int_0^{t_m} \langle f(r), u_m(r)\rangle dr + C''t_m\Big) \le \frac{1}{2}|u(t_0)|_2^2 + \int_0^{t_0} \langle f(r), u(r)\rangle dr + C''t_0. \tag{3.41}$$

Thanks to $(3.11)_3$, we know that

$$\int_0^{t_m} \langle f(r), u_m(r)\rangle dr \to \int_0^{t_0} \langle f(r), u(r)\rangle dr. \tag{3.42}$$

Then, by using (3.41) and (3.42), we have

$$\limsup_{m\to\infty} |u_m(t_m)|_2^2 \le |u(t_0)|_2^2.$$

So (3.36) is proved. From the above two cases, we deduce that (3.36) is valid for any $0 \le t_0 \le t$. Thus, according to the above analysis, we have proved the existence of weak solutions.

Step 4: Uniqueness of solution

If u and v are two solutions of problem (2.7) with the same initial value ϕ, and if $\omega(t) = u(t) - v(t)$, then we have

$$\frac{\partial \omega}{\partial t} + \nu_0 A\omega + A_1 u - A_1 v + B(u,u) - B(v,v) = P(g(t,u_t) - g(t,v_t)). \tag{3.43}$$

Note that

$$B(u,u) - B(v,v) = B(\omega,u) + B(v,\omega). \tag{3.44}$$

Taking the inner product of (3.43) and $\omega(t)$, we can derive from (3.44) and $(2.5)_2$ that

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \nu_0\|\omega\|^2 + \langle A_1 u - A_1 v, u - v\rangle + b(\omega,u,\omega) = (g(t,u_t) - g(t,v_t),\omega).$$

Because of conditions (g_3) and the monotonicity of A_1, with the help of $(2.5)_4$ and the Young inequality, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \nu_0\|\omega\|^2 &\le |b(\omega,u,\omega)| + (g(t,u_t) - g(t,v_t),\omega)\\ &\le c_1|\omega|_2^{\frac{1}{2}}\|\omega\|^{\frac{3}{2}}\|u\| + |g(t,u_t) - g(t,v_t)|_2|\omega|_2\\ &\le \frac{\nu_0}{2}\|\omega\|^2 + \frac{27c_1^4}{32\nu_0^3}\|u\|^4|\omega|_2^2 + L_g\|\omega_t\|_{BCL_{-\infty}(H)}|\omega|_2.\end{aligned} \tag{3.45}$$

Denote $\mu_1 = \frac{27c_1^4}{32\nu_0^3}$. From $\omega(0) = 0$ and the definition of the norm of $BCL_{-\infty}(H)$, one has

$$|\omega(t)|_2^2 \le 2\mu_1\int_0^t \|u\|^4|\omega|_2^2 ds + 2L_g\int_0^t \|\omega_s\|_{BCL_{-\infty}(H)}^2 ds. \tag{3.46}$$

Since $\omega(\theta) = 0$ for any $\theta \le 0$, (3.46) indicates that

$$\|\omega_t\|_{BCL_{-\infty}(H)}^2 \le 2(\mu_1 + L_g)\int_0^t (\|u\|^4 + 1)\|\omega_s\|_{BCL_{-\infty}(H)}^2 ds,$$

which, together with the Gronwall inequality, yields that $\omega \equiv 0$, i.e., the solution is unique.

Step 5: Regularity of the weak solution

At this time, we assume that the non-delay external force $f \in L^2(0,T;(L^2(\Omega))^3)$ and the initial value satisfies that $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$. Multiplying $(3.1)_1$ by $\lambda_i h_{im}(t)$, and then summing from $i = 1$ to $i = m$, in view of $A\omega_i = \lambda_i\omega_i$ and Young's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\|u_m(t)\|^2 + 2\nu_0|Au_m(t)|_2^2 + 2\nu_1\|u_m(t)\|^2|Au_m(t)|_2^2\\ &\quad + 2b(u_m(t),u_m(t),Au_m(t))\\ &= 2(f(t),Au_m(t)) + 2(g(t,u_{mt}),Au_m(t))\\ &\le (\frac{1}{\sqrt{2}}|Au_m(t)|_2)(2\sqrt{2}|f(t)|_2) + (\frac{1}{\sqrt{2}}|Au_m(t)|_2)(2\sqrt{2}|g(t,u_{mt})|_2)\\ &\le \frac{\nu_0}{2}|Au_m(t)|_2^2 + \frac{4}{\nu_0}|g(t,u_{mt})|_2^2 + \frac{4}{\nu_0}|f(t)|_2^2.\end{aligned} \tag{3.47}$$

In particular, it can be derived from $(2.5)_4$ and Young's inequality that

$$\begin{aligned}
&2|b(u_m(t), u_m(t), Au_m(t))| \\
&\leq 2c_1|u_m(t)|_2^{\frac{1}{4}}\|u_m(t)\|^{\frac{3}{4}}\|u_m(t)\|^{\frac{1}{4}}|Au_m(t)|_2^{\frac{3}{4}}|Au_m(t)|_2 \\
&= (2|u_m(t)|_2^{\frac{1}{4}}|Au_m(t)|_2^{\frac{3}{4}})(c_1\|u_m(t)\||Au_m(t)|_2) \\
&\leq 2\nu_1\|u_m(t)\|^2|Au_m(t)|_2^2 + \frac{c_1^2}{2\nu_1}|u_m(t)|_2^{\frac{1}{2}}|Au_m(t)|_2^{\frac{3}{2}} \\
&\leq 2\nu_1\|u_m(t)\|^2|Au_m(t)|_2^2 + \mu_2|u_m(t)|_2^2 + \frac{\nu_0}{2}|Au_m(t)|_2^2,
\end{aligned} \tag{3.48}$$

where $\mu_2 = \frac{27c_1^8}{32\nu_0^3\nu_1^4}$. Integrating (3.47) from 0 to t, we obtain from (3.48) that

$$\begin{aligned}
&\|u_m(t)\|^2 + \nu_0\int_0^t |Au_m(s)|_2^2 ds \\
&\leq \|u_m(0)\| + \frac{4}{\nu_0}\int_0^t (|f(s)|_2^2 + |g(t, u_{mt})|_2^2) ds + C\mu_2\|u_m\|^2_{L^\infty(0,T;H)}, \quad \forall 0 \leq t \leq T.
\end{aligned} \tag{3.49}$$

By $\phi(0) \in V$, $f \in L^2(0, T; (L^2(\Omega))^3)$, (3.9) and the fact that $\|u_m(0)\| = \|P_m\phi(0)\| \leq \|\phi(0)\|$, we conclude that

$$\{u_m\} \text{ is uniformly bounded in } L^\infty(0, T; V) \cap L^2(0, T; D(A)). \tag{3.50}$$

Moreover, applying Agmon's inequality ($\|u\|_{L^\infty(\Omega)} \leq c_2\|u\|^{\frac{1}{2}}|Au|_2^{\frac{1}{2}}$, $\forall u \in D(A)$) to the convection term, one has

$$\begin{aligned}
\int_0^T |u_m\nabla u_m|_2^2 dt &\leq \int_0^T \|u_m\|^2_{L^\infty(\Omega)}|\nabla u_m|_2^2 dt \\
&\leq c_2^2\int_0^T \|u_m\|^3|Au_m|_2 dt \\
&\leq \frac{c_2^2}{2}\int_0^T (\|u_m\|^6 + |Au_m|_2^2) dt \leq C,
\end{aligned} \tag{3.51}$$

where the last inequality uses (3.50). Thus, $B(u_m, u_m) \in L^2(0, T; H)$. For the nonlinear viscosity, we can get

$$\int_0^T \|u_m\|^4|Au_m|_2^2 dt \leq \|u_m\|^4_{L^\infty(0,T;V)}\int_0^T |Au_m|_2^2 dt \leq C. \tag{3.52}$$

Then, using a similar process as in Step 1, we deduce that

$$\{\frac{\partial u_m}{\partial t}\} \text{ is uniformly bounded in } L^2(0, T; H). \tag{3.53}$$

Combining (3.50), (3.53) and the Aubin-Lions compactness lemma, it follows that $u \in C([0, T]; V) \cap L^2(0, T; D(A))$. Therefore, the weak solution obtained above is indeed strong. □

4. Stability of stationary solutions

In this section, we shall consider the existence and stability of stationary solutions to the problem (2.7) under some appropriate assumptions.

4.1. Existence and uniqueness of stationary solutions

In this subsection, by using a corollary of Schauder's fixed point theorem, we establish the existence and uniqueness of the stationary solution to the problem (2.7).

In order to investigate the existence and properties of stationary solutions to (2.7), we assume that f is independent of time, i.e., $f \in V'$. Assuming that $\rho \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\sup_{t\geq 0}\rho'(t) = \rho_* < 1$, $g(t, u_t) = G(u(t-\rho(t))) : H \to (L^2(\Omega))^3$ satisfies the following conditions (see [32, 33]):

$(G1)$ $G(0) = 0$;

$(G2)$ There exists $L_G > 0$, such that for any $\xi, \eta \in H$,

$$|G(\xi) - G(\eta)|_2 \leq L_G|\xi - \eta|_2.$$

Under the above assumptions, the stationary equation of the problem (2.7) has the following form:

$$\nu_0 Au + A_1 u + B(u, u) = Pf + PG(u). \tag{4.1}$$

A function $u^* \in V$ is called a weak solution to (4.1) if it satisfies that

$$\nu_0((u^*, v)) + \nu_1\|u^*\|^2((u^*, v)) + b(u^*, u^*, v) = \langle f, v\rangle + (G(u^*), v), \quad \forall v \in V. \tag{4.2}$$

And, u^* is also called the stationary solution of the problem (2.7).

Theorem 4.1. *Assume that $f \in V'$, $g(t, u_t) = G(u(t-\rho(t)))$ satisfies conditions (G_1) and (G_2) and $\nu_0 > \frac{L_G}{\lambda_1}$; then, there exists at least one weak solution u^* to (4.1) satisfying*

$$\|u^*\| \leq \frac{\|f\|_*}{\nu_0 - L_G\lambda_1^{-1}}. \tag{4.3}$$

In addition, if $(\nu_0 - \lambda_1^{-1}L_G)^2 > c_1\lambda_1^{-\frac{1}{4}}\|f\|_*$ holds, the solution is unique.

Proof. First, we prove the existence of weak solutions. Let $\{v_j\}_{j=1}^{\infty}$ be the orthonormal basis of V, composed of the characteristic function of Stokes operator A. Set $V_m = \text{span}\{v_1, \cdots, v_m\}$; then we respectively construct the approximate solution and the approximate system as follows:

$$u_m = \sum_{j=1}^{m} h_{jm}v_j, \quad h_{jm} \in \mathbb{R}, \tag{4.4}$$

$$(\nu_0 + \nu_1\|u_m\|^2)((u_m, v_j)) + b(u_m, u_m, v_j) = \langle f, v_j\rangle + (G(u_m), v_j), \quad j = 1, ..., m. \tag{4.5}$$

Since system (4.5) is nonlinear, the existence of $\{h_{1m}, \cdots h_{mm}\}$ is not obvious, that is, we need to prove the existence of the approximate solution u_m. We define $R_m : V_m \to V_m$ as follows: for all $u, v \in V_m$,

$$((R_m u, v)) = (\nu_0 + \nu_1\|u\|^2)\langle Au, v\rangle + b(u, u, v) - \langle f, v\rangle - (G(u), v). \tag{4.6}$$

Obviously, R_m is continuous. Furthermore, we have the following for any $u \in V_m$ and $u \neq 0$:

$$\begin{aligned}((R_m u, u)) &= (\nu_0 + \nu_1\|u\|^2)\langle Au, u\rangle - \langle f, u\rangle - (G(u), u)\\ &\geq \nu_0\|u\|^2 + \nu_1\|u\|^4 - \|f\|_*\|u\| - \frac{L_G}{\lambda_1}\|u\|^2\\ &\geq (\nu_0 - \frac{L_G}{\lambda_1})\|u\|^2 - \|f\|_*\|u\|.\end{aligned}$$

By taking $\|u\| = \beta = \frac{\|f\|_*}{\nu_0 - L_G\lambda_1^{-1}}$, we obtain that $((R_m u, u)) \geq 0$. Thus, according to a corollary of Schauder's fixed point theorem (namely, Lemma 2.4), for each $m \geq 1$, there exists $u_m \in V_m$ such that $R_m(u_m) = 0$ and $\|u_m\| \leq \beta$, i.e.,

$$\nu_0 A u_m + A_1 u_m + B(u_m, u_m) = f + G(u_m). \tag{4.7}$$

So sequence $\{u_m\}$ is uniformly bounded in V. Because the embedding $V \hookrightarrow H$ is compact, there exists a subsequence (still recorded as $\{u_m\}$), which converges weakly in V and strongly converges to an element $u^* \in V$ in H.

In order to exceed the limit of (4.5), it is necessary to discuss the nonlinear term. Using similar methods in Theorem 3.1, it can be proved that the convection term can exceed the limit. For the nonlinear term $A_1 u_m$, because A_1 is a bounded semi-continuous monotone operator and Lemma 2.3 holds, A_1 is an operator of (M) type. In addition, the above results show that

$$u_m \to u^* \text{ weakly in } V, \tag{4.8}$$

$$A_1 u_m \to \chi \text{ weakly in } V'. \tag{4.9}$$

By taking the limit of Eq (4.7), $\chi = f + G(u^*) - \nu_0 A u^* - B(u^*, u^*)$. We multiply (4.5) by $h_{jm}(t)$, and sum j to obtain

$$\langle A_1 u_m, u_m\rangle = \langle f, u_m\rangle + (G(u_m), u_m) - \nu_0\langle A u_m, u_m\rangle. \tag{4.10}$$

By virtue of (4.8), one has

$$\langle f, u_m\rangle \to \langle f, u^*\rangle, \quad m \to \infty. \tag{4.11}$$

Using the weak lower semi-continuity of norms, one has

$$\langle Au^*, u^*\rangle = \|u^*\|^2 \leq \liminf_{m\to\infty}\|u_m\|^2 \leq \limsup_{m\to\infty}\langle Au_m, u_m\rangle. \tag{4.12}$$

It follows from the condition $(G2)$ that

$$\begin{aligned}&(G(u_m), u_m) - (G(u^*), u^*)\\ &= (G(u_m) - G(u^*), u_m) + (G(u^*), u_m - u^*)\\ &\leq |G(u_m) - G(u^*)|_2|u_m|_2 + |G(u^*)|_2|u_m - u^*|_2\\ &\leq L_G(|u_m - u^*|_2|u_m|_2 + |u^*|_2|u_m - u^*|_2),\end{aligned}$$

which, together with

$$u_m \to u^* \text{ strongly in } H, \tag{4.13}$$

gives

$$(G(u_m), u_m) \to (G(u^*), u^*), \quad m \to \infty. \tag{4.14}$$

Combining (4.10)–(4.12) with (4.14), we get

$$\begin{aligned}\limsup_{m\to\infty}\langle A_1(u_m), u_m\rangle &= \limsup_{m\to\infty}\left[\langle f, u_m\rangle + (G(u_m), u_m) - \nu_0\langle Au_m, u_m\rangle\right]\\ &\le \langle f + G(u^*) - \nu_0 Au^*, u^*\rangle\\ &= \langle f + G(u^*) - \nu_0 Au^* - B(u^*, u^*), u^*\rangle\\ &= \langle \chi, u^*\rangle.\end{aligned} \tag{4.15}$$

Since A_1 is an operator of type (M), we infer from (4.8), (4.9) and (4.15) that

$$\chi = A_1(u^*). \tag{4.16}$$

Now, taking the limit of (4.5) and combining the convergences given by (4.8), (4.13) and (4.16), we know that u^* is indeed a weak solution to problem (4.1), and it satisfies that $\|u^*\| \le \frac{\|f\|_*}{\nu_0 - L_G\lambda_1^{-1}}$.

Next, we prove the uniqueness of the solutions. Let u^* and v^* be the two solutions to (4.1). Set $w = u^* - v^*$; then,

$$\nu_0 A(u^* - v^*) + A_1 u^* - A_1 v^* + B(u^*, u^*) - B(v^*, v^*) = PG(u^*) - PG(v^*). \tag{4.17}$$

Taking the inner product of (4.17) by w, by the monotonicity of A_1 and $(G2)$, we deduce that

$$\nu_0\|w\|^2 \le |b(w, u^*, w)| + (G(u^*) - G(v^*), w) \le c_1\lambda_1^{-\frac{1}{4}}\|w\|^2\|u^*\| + \frac{L_G}{\lambda_1}\|w\|^2. \tag{4.18}$$

Because of $u^* \le \beta$,

$$(\nu_0 - \frac{L_G}{\lambda_1})\|w\|^2 \le c_1\lambda_1^{-\frac{1}{4}}\|w\|^2\|u^*\| \le \frac{c_1\lambda_1^{-\frac{1}{4}}}{\nu_0 - L_G\lambda_1^{-1}}\|w\|^2\|f\|_*, \tag{4.19}$$

which, together with $(\nu_0 - \lambda_1^{-1}L_G)^2 > c_1\lambda_1^{-\frac{1}{4}}\|f\|_*$, implies that $w = 0$, that is, the solution is unique. Thus, we have completed the proof of the theorem. □

4.2. Local stability of stationary solutions

In this subsection, we prove the local stability of the stationary solution obtained in Theorem 4.1. In order to analyze the stability of stationary solutions to problem (2.7), we first review the definition of stability as follows (see [32, 33] for details).

Definition 4.2. *A stationary solution u_∞ to* (2.7) *is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\varphi \in BCL_{-\infty}(H)$ and $\|\varphi - u_\infty\|_{BCL_{-\infty}(H)} < \delta$, then the solution $u(\cdot, \varphi)$ to* (2.7) *exists for all $t \ge 0$ and satisfies*

$$|u(t, \varphi) - u_\infty|_2 < \varepsilon, \quad \forall t \ge 0.$$

A stationary solution u_∞ to (2.7) *is said to be asymptotically stable if it is stable and the solution $u(\cdot, \varphi)$ to* (2.7) *exists for all $t \ge 0$ and satisfies*

$$\lim_{t\to\infty} |u(t, \varphi) - u_\infty|_2 = 0.$$

Theorem 4.3. *Consider that* $f \in V'$, $g(t, u_t) = G(u(t-\rho(t)))$ *satisfies the assumptions* (G_1) *and* (G_2), *and* ν_0 *satisfies that* $\nu_0 > \frac{L_G}{\lambda_1}$; *then,* (4.1) *has a weak solution* u_∞ *satisfying* (4.3). *Moreover, if*

$$2\nu_0 > (1 + \frac{1}{1-\rho^*})L_G\lambda_1{}^{-1} + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G}, \tag{4.20}$$

then the solution u_∞ *is unique and, for any* $\phi \in BCL_{-\infty}(H)$, *the solution* $u(t)$ *to* (2.7) *with* $f(t) \equiv f$ *satisfies*

$$|u(t) - u_\infty|_2 \le C_1(|\phi(0) - u_\infty|_2 + |\phi - u_\infty|_{L^2(-\rho(0),0;H)}), \quad \forall t \ge 0,$$

where $C_1 = \max\{1, (\frac{L_G}{1-\rho^*})^{\frac{1}{2}}\}$. *Namely, the stationary solution* u_∞ *to the problem* (2.7) *is stable.*

Proof. First, it is easy to verify that $(\nu_0 - \lambda_1^{-1}L_G)^2 > c_1\lambda_1^{-\frac{1}{4}}\|f\|_*$ by using (4.20). Thus, from Theorem 4.1, we can know that problem (4.1) has a unique weak solution u_∞ satisfying

$$\|u_\infty\| \le \frac{\|f\|_*}{\nu_0 - L_G\lambda_1^{-1}}. \tag{4.21}$$

Using Theorem 3.1, under the above assumptions, the problem (2.7) has a unique weak solution, which is recorded as $u(t)$. Set $w = u(t) - u_\infty$; then, it is immediate that

$$\frac{\partial w}{\partial t} + \nu_0 Aw + A_1 u - A_1 u_\infty + B(u,u) - B(u_\infty, u_\infty) = P(G(u(t-\rho(t))) - G(u_\infty)). \tag{4.22}$$

Note the following fact

$$B(u,u) - B(u_\infty, u_\infty) = B(w,w) + B(u_\infty, w) + B(w, u_\infty).$$

Taking the inner product of (4.22) and w, it is easy to obtain the following from the Young inequality and (4.21):

$$\begin{aligned}\frac{d}{dt}|w(t)|_2^2 \le &-2\nu_0\|w\|^2 + 2|b(w, u_\infty, w)| + 2(G(u(t-\rho(t))) - G(u_\infty), w)\\ &\le -2\nu_0\|w\|^2 + 2c_1\lambda_1^{-\frac{1}{4}}\|w\|^2\|u_\infty\| + 2L_G(|w(t-\rho(t))|_2|w|_2)\\ &\le -2\nu_0\|w\|^2 + 2c_1\lambda_1^{-\frac{1}{4}}\|w\|^2\|u_\infty\| + L_G|w(t-\rho(t))|_2^2 + L_G|w|_2^2\\ &\le (L_G\lambda_1^{-1} - 2\nu_0 + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G})\|w\|^2 + L_G|w(t-\rho(t))|_2^2.\end{aligned} \tag{4.23}$$

Integrating (4.23) with respect to the time, with the help of a change of variable in the integral of ρ

and (4.20), we deduce that

$$
\begin{aligned}
|w(t)|_2^2 &\leq (L_G\lambda_1^{-1} - 2\nu_0 + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G})\int_0^t \|w(s)\|^2 ds \\
&+ \frac{L_G}{1-\rho^*}(\int_{-\rho(0)}^0 |w(s)|_2^2 ds + \int_0^t |w(s)|_2^2 ds) + |w(0)|_2^2 \\
&\leq (L_G\lambda_1^{-1} + \frac{L_G\lambda_1^{-1}}{1-\rho^*} - 2\nu_0 + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G})\int_0^t \|w(s)\|^2 ds \\
&+ \frac{L_G}{1-\rho^*}\int_{-\rho(0)}^0 |w(s)|_2^2 ds + |w(0)|_2^2 \\
&\leq |w(0)|_2^2 + \frac{L_G}{1-\rho^*}\int_{-\rho(0)}^0 |w(s)|_2^2 ds.
\end{aligned}
\tag{4.24}
$$

Therefore, by taking $C_1 = \max\{1, (\frac{L_G}{1-\rho^*})^{\frac{1}{2}}\}$, one has

$$
|w(t)|_2 \leq C_1(|\phi(0) - u_\infty|_2 + |\phi - u_\infty|_{L^2(-\rho(0),0;H)}).
$$

Thus, the proof is completed. □

4.3. Asymptotic stability of stationary solutions

Theorem 4.3 only gives the local stability of stationary solutions to the problem (2.7), and we have not demonstrated the asymptotic stability of stationary solutions. In this subsection, the asymptotic stability of stationary solutions will be proved under two specific conditions for delayed forces.

1). Under the assumption of Theorem 4.3, we additionally require that the delay interval of external force G is bounded. That is, $\rho(t) \in [0, h]$, where $h > 0$ is a constant. At this time, using the Lyapunov function method, we obtain that the stationary solution is exponentially stable.

Theorem 4.4. *Under the assumption of Theorem 4.3, if $\rho(t)$ satisfies that $\rho(t) \in [0, h]$, then there exists a sufficiently small positive constant λ such that for any $t \geq 0$,*

$$
|u(t) - u_\infty|_2 \leq C_2 e^{-\frac{\lambda t}{2}}(|\phi(0) - u_\infty|_2 + \|\phi - u_\infty\|_{L^2(-\rho(0),0;H)}),
$$

where $C_2 = \max\{1, (\frac{L_G e^{\lambda h}}{1-\rho^*})^{\frac{1}{2}}\}$.

Proof. By a process similar to the proof of Theorem 4.3, multiplying both sides of (4.23) by $e^{\lambda t}$, we obtain

$$
\begin{aligned}
\frac{d}{dt}(e^{\lambda t}|w(t)|_2^2) &\leq \left(L_G\lambda_1^{-1} - 2\nu_0 + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G}\right) e^{\lambda t}\|w\|^2 \\
&+ \lambda e^{\lambda t}|w(t)|_2^2 + L_G e^{\lambda t}|w(t-\rho(t))|_2^2 \\
&\leq \left(\frac{L_G + \lambda}{\lambda_1} - 2\nu_0 + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G}\right) e^{\lambda t}\|w\|^2 + L_G e^{\lambda t}|w(t-\rho(t))|_2^2.
\end{aligned}
\tag{4.25}
$$

Integrating (4.25) in time, we get

$$\begin{aligned} e^{\lambda t}|w(t)|_2^2 \le &\left((L_G+\lambda)\lambda_1^{-1} - 2\nu_0 + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G}\right)\\ &\times \int_0^t \|w(s)\|^2 e^{\lambda s}ds + |w(0)|_2^2 + L_G\int_0^t e^{\lambda s}|w(s-\rho(s))|_2^2 ds. \end{aligned} \tag{4.26}$$

In order to estimate the delay term, by transforming $\tau = s - \rho(s) = \theta(s)$, it is easy to deduce that

$$\int_0^t e^{\lambda s}|w(s-\rho(s))|_2^2 ds \le \frac{1}{1-\rho^*}\int_{-\rho(0)}^{t-\rho(t)} e^{\lambda\theta^{-1}(\tau)}|w(\tau)|_2^2 d\tau.$$

Because θ is a monotonic increasing function and $\rho(t) \in [0, h]$, $\theta^{-1}(\tau) \le \tau + h$ for any $\tau \ge -\rho(0)$; then,

$$\int_0^t e^{\lambda s}|w(s-\rho(s))|_2^2 ds \le \frac{e^{\lambda h}}{1-\rho^*}\int_{-\rho(0)}^{t} e^{\lambda\tau}|w(\tau)|_2^2 d\tau. \tag{4.27}$$

By combining (4.26) and (4.27), we conclude that

$$\begin{aligned} e^{\lambda t}|w(t)|_2^2 \le &\left((L_G+\lambda)\lambda_1^{-1} - 2\nu_0 + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G}\right)\times \int_0^t \|w(s)\|^2 e^{\lambda s}ds + |w(0)|_2^2\\ &+ \frac{L_G e^{\lambda h}}{1-\rho^*}\left(\int_{-\rho(0)}^0 e^{\lambda\tau}|w(\tau)|_2^2 d\tau + \int_0^t e^{\lambda\tau}|w(\tau)|_2^2 d\tau\right)\\ \le &-\left(2\nu_0 - (L_G + \lambda + \frac{L_G e^{\lambda h}}{1-\rho^*})\lambda_1{}^{-1} - \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G}\right)\\ &\times \int_0^t \|w(s)\|^2 e^{\lambda s}ds + |w(0)|_2^2 + \frac{L_G e^{\lambda h}}{1-\rho^*}\int_{-\rho(0)}^0 e^{\lambda\tau}|w(\tau)|_2^2 d\tau. \end{aligned} \tag{4.28}$$

In view of the assumption (4.20), there exists a sufficiently small $\lambda > 0$ such that

$$2\nu_0 - L_G\lambda_1{}^{-1} - \lambda\lambda_1{}^{-1} - \frac{L_G\lambda_1{}^{-1}}{1-\rho^*}e^{\lambda h} + \frac{2c_1\lambda_1{}^{-\frac{1}{4}}\|f\|_*}{\nu_0 - \lambda_1^{-1}L_G} > 0. \tag{4.29}$$

Hence, we obtain

$$\begin{aligned} |w(t)|_2 &\le e^{-\frac{\lambda t}{2}}\left(|w(0)|_2^2 + \frac{L_G e^{\lambda h}}{1-\rho^*}\int_{-\rho(0)}^0 |w(\tau)|_2^2 d\tau\right)^{\frac{1}{2}}\\ &\le C_2 e^{-\frac{\lambda t}{2}}(|\phi(0) - u_\infty|_2 + \|\phi - u_\infty\|_{L^2(-\rho(0),0;H)}), \end{aligned}$$

where $C_2 = \max\{1, (\frac{L_G e^{\lambda h}}{1-\rho^*})^{\frac{1}{2}}\}$. Hence, we complete the proof of Theorem 4.4. □

2). Under the assumption of Theorem 4.3, we additionally require that external force G be a special case of proportional delay. That is, $\rho(t) = (1-q)t$ for $0 < q < 1$. At this point, we will give a sufficient condition for the trivial stationary solution to have polynomial stability. Here, we use a lemma of the proportional equation

$$y'(t) = ay(t) + by(qt), \quad \forall t \ge 0, q \in (0, 1). \tag{4.30}$$

Lemma 4.5. *([40], Lemma 3.6) Let $a < 0, b > 0$ and $q \in (0, 1)$. If $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies that for any $t \geq 0$, $D^+h(t) \leq ah(t) + bh(qt)$ and $h(0) > 0$, here,*

$$D^+h = \limsup_{\delta\to 0^+} \frac{h(t+\delta) - h(t)}{\delta};$$

then, there exists a constant $C = C(a, b, q) > 0$ such that

$$h(t) \leq Ch(0)(1+t)^{\gamma}, \quad \forall t \geq 0,$$

where γ satisfies that $a + bq^{\gamma} = 0$.

In particular, using the lemma above, we can obtain a polynomial stability result for the trivial stationary solution to the problem (2.7).

Theorem 4.6. *Consider (2.7) with $f \equiv 0, g(t, u_t) := L_G u(qt)$ with $0 < q < 1$, $L_G \in \mathbb{R}$ and $\nu_0 > \frac{|L_G|}{\lambda_1}$. Then the origin is the only stationary solution to the problem. Moreover, for any solution to (2.7) given the initial data $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \neq 0$, there exists a positive constant C_3 that id dependent on L_G, ν_0, λ_1 and q such that*

$$|u(t)|_2 \leq \sqrt{C_3}|u(0)|_2(1+t)^{\frac{\gamma}{2}}, \quad \forall t \geq 0,$$

where

$$\gamma = \log_q\left(\frac{2\lambda_1\nu_0 - |L_G|}{|L_G|}\right) < 0. \tag{4.31}$$

Proof. The existence and uniqueness of u_∞ can be obtained from Theorem 4.1. In addition, it is easy to know that the origin satisfies the conditions of the stationary equation, so $u_\infty \equiv 0$. Taking the inner product of (2.7) and u in the space H, we get

$$\frac{d}{dt}|u(t)|_2^2 + 2\nu_0\|u(t)\|^2 + 2\nu_1\|u(t)\|^4 = 2(L_G u(qt), u). \tag{4.32}$$

By virtue of the Poincaré inequality and Young's inequality, it is immediate that

$$\begin{aligned}\frac{d}{dt}|u(t)|_2^2 + 2\lambda_1\nu_0|u(t)|_2^2 &\leq 2(L_G u(qt), u)\\ &\leq |L_G||u(t)|_2^2 + |L_G||u(qt)|_2^2.\end{aligned} \tag{4.33}$$

In order to use the Lemma 4.5, set $h(t) = |u(t)|_2^2$. It holds from (4.33) that

$$h'(t) \leq (|L_G| - 2\lambda_1\nu_0)|u(t)|_2^2 + |L_G||u(qt)|_2^2, \tag{4.34}$$

and $h(0) = |u(0)|_2^2 = |\phi(0)|_2^2 > 0$. From $\nu_0 > \frac{|L_G|}{\lambda_1}$, it is evident that $|L_G| - 2\lambda_1\nu_0 < 0$ and $|L_G| > 0$. Therefore, according to Lemma 4.5, there exists a constant $C_3(L_G, \nu_0, \lambda_1, q) > 0$ such that

$$h(t) \leq C_3 h(0)(1+t)^{\gamma}, \quad \forall t \geq 0,$$

i.e.,

$$|u(t)|_2 \leq \sqrt{C_3}|u(0)|_2(1+t)^{\frac{\gamma}{2}}, \quad \forall t \geq 0,$$

where γ satisfies that $|L_G| - 2\lambda_1\nu_0 + |L_G|q^{\gamma} = 0$. That is, γ is given by $\gamma = \log_q(\frac{2\lambda_1\nu_0 - |L_G|}{|L_G|})$. Since $q \in (0, 1)$, one has that $\gamma < 0$. This completes the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflicts of interest.

References

1. H. Bae, Existence and analyticity of Lei-Lin solution to the Navier-Stokes equations, *Proc. Am. Math. Soc.*, **143** (2015), 2887–2892. https://doi.org/10.1090/S0002-9939-2015-12266-6
2. T. Buckmaster, V. Vicol, Nonuniqueness of weak solutions to the Navier-Stokes equation, *Ann. Math.*, **189** (2019), 101–144. https://doi.org/10.4007/annals.2019.189.1.3
3. H. Koch, D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.*, **157** (2001), 22–35. https://doi.org/10.1006/aima.2000.1937
4. P. L. Lions, *Mathematical Topics in Fluid Dynamics*, Oxford University Press, Oxford, 1996.
5. R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, 1984. https://doi.org/10.1090/chel/343
6. T. Caraballo, J. Real, P. E. Kloeden, Unique strong solutions and V-attractors of a three dimensional system of globally modified Navier-Stokes equations, *Adv. Nonlinear Stud.*, **6** (2006), 411–436. https://doi.org/10.1515/ans-2006-0304
7. O. A. Ladyzhenskaya, On some nonlinear problems in the theory of continuous media, in *Thirty-One Invited Addresses (Eight in Abstract) at the International Congress of Mathematicians in Moscow, 1966*, **70** (1968), 73–89. https://doi.org/10.1090/trans2/070/15
8. O. A. Ladyzhenskaya, R. A. Silverman, J. T. Schwartz, J. E. Romain, The mathematical theory of viscous incompressible flow, *Phys. Today*, **17** (1964), 57–58.
9. B. Guo, P. Zhu, Partial regularity of suitable weak solution to the system of the incompressible non-Newtonian fluids, *J. Differ. Equations*, **178** (2002), 281–297. https://doi.org/10.1006/jdeq.2000.3958
10. H. B. da Veiga, J. Yang, On the partial regularity of suitable weak solutions in the non-Newtonian shear-thinning case, *Nonlinearity*, **34** (2021), 562. https://doi.org/10.1088/1361-6544/abcd06
11. H. B. da Veiga, On the regularity of flows with Ladyzhenskaya shear dependent viscosity and slip and non-slip boundary conditions, *Commun. Pure Appl. Math.*, **58** (2005), 552–577. https://doi.org/10.1002/cpa.20036

12. H. B. da Veiga, Navier–Stokes equations with shear thinning viscosity. Regularity up to the boundary, *J. Math. Fluid Mech.*, **11** (2009), 258–273. https://doi.org/10.1007/s00021-008-0258-1

13. J. Smagorinsky, General circulation experiments with the primitive equations, *Mon. Weather Rev.*, **91** (1963), 99–164.

14. J. Necas, J. Malek, M. Rokyta, M. Ruzicka, *Weak and Measure-Valued Solutions to Evolutionary PDEs*, Chapman and Hall/CRC, New York, 1996. https://doi.org/10.1201/9780367810771

15. H. Bellout, F. Bloom, J. Nečas, Young measure-valued solutions for non-Newtonian incompressible fluids, *Commun. Partial Differ. Equations*, **19** (1994), 1763–1803. https://doi.org/10.1080/03605309408821073

16. J. Málek, J. Nečas, K. R. Rajagopal, Global Analysis of the Flows of Fluids with Pressure-Dependent Viscosities, *Arch. Rational Mech. Anal.*, **165** (2002), 243–269. https://doi.org/10.1007/s00205-002-0219-4

17. J. Málek, J. Nečas, M. Ružička, On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$, *Adv. Differ. Equations*, **6** (2001), 257–302. https://doi.org/10.57262/ade/1357141212

18. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

19. Y. Chen, X. Yang, M. Si, The long-time dynamics of 3D non-autonomous Navier-Stokes equations with variable viscosity, *ScienceAsia*, **44** (2018), 18–26. https://doi.org/10.2306/scienceasia1513-1874.2018.44.018

20. X. Yang, B. Feng, S. Wang, Y. Lu, T. F. Ma, Pullback dynamics of 3D Navier-Stokes equations with nonlinear viscosity, *Nonlinear Anal.: Real World Appl.*, **48** (2019), 337–361. https://doi.org/10.1016/j.nonrwa.2019.01.013

21. T. Caraballo, J. Real, Navier-Stokes equations with delays, *Proc. R. Soc. London*, **457** (2001), 2441–2453. https://doi.org/10.1098/rspa.2001.0807

22. T. Caraballo, X. Han, A survey on Navier-Stokes models with delays: Existence, uniqueness and asymptotic behavior of solutions, *Discrete Contin. Dyn. Syst.-Ser. S*, **8** (2015), 1079–1101. https://doi.org/10.3934/dcdss.2015.8.1079

23. T. Caraballo, J. Real, Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays, *Proc. R. Soc. London*, **459** (2003), 3181–3194. https://doi.org/10.1098/rspa.2003.1166

24. T. Caraballo, J. Real, Attractors for 2D Navier-Stokes models with delays, *J. Differ. Equations*, **205** (2004), 271–297. https://doi.org/10.1016/j.jde.2004.04.012

25. J. García-Luengo, P. Marín-Rubio, José Real, Pullback attractors for 2D Navier-Stokes equations with delays and their regularity, *Adv. Nonlinear Stud.*, **13** (2013), 331–357. https://doi.org/10.1515/ans-2013-0205

26. T. Caraballo, P. Marín-Rubio, J. Valero, Attractors for differential equations with unbounded delays, *J. Differ. Equations*, **239** (2007), 311–342. https://doi.org/10.1016/j.jde.2007.05.015

27. P. Marín-Rubio, A. M. Márquez-Durán, J. Real, Three dimensional system of globally modified Navier-Stokes equations with infinite delays, *Discrete Contin. Dyn. Syst.-Ser. B*, **14** (2010), 655–673. https://doi.org/10.3934/dcdsb.2010.14.655
28. P. Marín-Rubio, J. Real, J. Valero, Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case. *Nonlinear Anal.*, **74** (2011), 2012–2030. https://doi.org/10.1016/j.na.2010.11.008
29. W. Liu, R. Yang, X. Yang, Dynamics of a 3D Brinkman-Forchheimer equation with infinite delay, *Commun. Pure Appl. Anal.*, **20** (2021), 1907–1930. https://doi.org/10.3934/cpaa.2021052
30. C. T. Anh, D. T. Thanh, Existence and long-time behavior of solutions to Navier-Stokes-Voigt equations with infinite delay, *Bull. Korean Math. Soc.*, **55** (2018), 379–403. https://doi.org/10.4134/BKMS.b170044
31. J. Wang, C. Zhao, T. Caraballo, Invariant measures for the 3D globally modified Navier-Stokes equations with unbounded variable delays, *Commun. Nonlinear Sci. Numer. Simul.*, **9** (2020), 105459. https://doi.org/10.1016/j.cnsns.2020.105459
32. L. Liu, T. Caraballo, P. Marín-Rubio, Stability results for 2D Navier-Stokes equations with unbounded delay, *J. Differ. Equations*, **265** (2018), 5685–5708. https://doi.org/10.1016/j.jde.2018.07.008
33. V. M. Toi, Stability and stabilization for the three-dimensional Navier-Stokes-Voigt equations with unbounded variable delay, *Evol. Equations Control Theory*, **10** (2021), 1007–1023. https://doi.org/10.3934/eect.2020099
34. C. Foias, O. Manley, R. Rosa, R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001. https://doi.org/10.1017/CBO9780511546754
35. R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2^{nd} Edition, Springer, New York, 1997. https://doi.org/10.1007/978-1-4612-0645-3
36. G. Łukaszewicz, P. Kalita, *Navier-Stokes Equations: An Introduction with Applications*, Springer International Publishing, Switzerland, 2016. https://doi.org/10.1007/978-3-319-27760-8
37. Y. Hino, S. Murakami, T. Naito, *Functional-Differential Equations with Infinite Delay*, Springer, Berlin, 1991. https://doi.org/10.1007/BFb0084432
38. P. Zhang, L. Huang, R. Lu, X Yang, Pullback dynamics of a 3D modified Navier-Stokes equations with double delays, *Electron. Res. Arch.*, **29** (2021), 4137–4157. https://doi.org/10.3934/era.2021076
39. B. Wang, B. Guo, Asymptotic behavior of non-autonomous stochastic parabolic equations with nonlinear Laplacian principal part, *Electron. J. Differ. Equations*, **191** (2013), 1–25.
40. J. A. D. Appleby, E. Buckwar, Sufficient conditions for polynomial asymptotic behaviour of the stochastic pantograph equation, *Electron. J. Qual. Theory Differ. Equations*, **2** (2016), 1–32. https://doi.org/10.14232/ejqtde.2016.8.2

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