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Research article

Existence of nontrivial solutions for a poly-Laplacian system involving concave-convex nonlinearities on locally finite graphs

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Abstract: We discuss a poly-Laplacian system involving concave-convex nonlinearities and parameters subject to the Dirichlet boundary condition on locally finite graphs. It is obtained that the system admits at least one nontrivial solution of positive energy and one nontrivial solution of negative energy based on the mountain pass theorem and the Ekeland's variational principle. We also obtain an estimate about semi-trivial solutions. Moreover, by using a result due to Brown et al., which is based on the fibering method and the Nehari manifold, we get the existence of the ground-state solution to the single equation corresponding to the poly-Laplacian system. Especially, we present some ranges of parameters for all of the results.

Keywords: poly-Laplacian system; concave-convex nonlinearities; mountain pass theorem; Ekeland's variational principle; locally finite graph

1. Introduction

The research on the existence of nontrivial solutions of elliptic partial differential equations and systems involving concave-convex nonlinearities in Euclidean space have attracted some attentions. In [1], Ambrosetti et al. studied the second order Laplacian equation involving concave-convex nonlinearities with a constant coefficient. With the help of the sub- and supersolutions, as well as variational arguments, they obtained some existence and multiplicity results of solutions. In [2], Brown and Wu also studied the second order Laplacian equation involving concave-convex nonlinearities with weight functions. With the help of the fibering method and the Nehari manifold which was introduced by Pohozaev in [3], they obtained that the equation has at least two nontrivial solutions. Moreover, in [4], Brown and Wu studied a potential operator equation. By using methods similar to those in [2], they obtained that the equation has at least two nontrivial solutions when the functionals related to potential operators satisfy some appropriate conditions. In [5], Chen et al. studied a class of second order Kirchhoff equations involving concave-convex nonlinearities and parameters. Their result was that the equation has multiple positive solutions basd on the fibering method and the Nehari manifold. In [6], Chen et al. studied the nonhomogeneous p-Kirchhoff equation involving concave-convex nonlinearities with weight functions and a perturbation. Their result was the existence of two nontrivial solutions of the equation basd on the mountain pass theorem and Ekeland's variational principle. In [7], Wu investigated the following second order Laplacian elliptic system:

$$\begin{cases}-\Delta u = \lambda f(x)|u|^{\gamma-2}u + \frac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega,\\ -\Delta v = \mu g(x)|v|^{\gamma-2}v + \frac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega,\\ u(x) = v(x) = 0, & x \in \partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain and $1 < \gamma < 2 < \alpha+\beta < \frac{2N}{N-2}$. With the help of the fibering method and the Nehari manifold, he obtained that when the parameters λ and μ belong to the appropriate range, the system has two nontrivial nonnegative solutions. In [8], Echarghaoui and Sersif investigated a class of second order semilinear elliptic systems involving critical Sobolev growth and concave nonlinearities. By using the fountain theorem, they obtained that the system has two completely different but infinitely many radial solution sets. Moreover, the energy functional of one solution set is positive, and the energy functional of the other solution set is negative. In [9], Bozhkova and Mitidieri considered the (p, q)-Laplacian elliptic system with the Diriclet boundary condition. Their results were on the existence and multiplicity of solutions of the system basd on the fibering method and the Nehari manifold. Moreover, by using the Pokhozhaev identity, they obtained the nonexistence result of solutions. In [10], Liu and Ou investigated the following (p, q)-Laplacian elliptic system:

$$\begin{cases}-\Delta_p u = \lambda\alpha a(x)|u|^{\alpha-2}u|v|^{\beta} + \gamma b(x)|u|^{\gamma-2}u|v|^{\eta}, & x \in \Omega,\\ -\Delta_q v = \lambda\beta a(x)|u|^{\alpha}u|v|^{\beta-2} + \eta b(x)|u|^{\gamma}u|v|^{\eta-2}, & x \in \Omega,\\ u(x) = v(x) = 0, & x \in \partial\Omega,\end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $1 < p,q < N$, $\alpha,\beta,\gamma,\eta > 0$, $1 < \alpha + \beta < \min\{p, q\}$ and $\max\{p, q\} < \gamma + \eta < \min\{p^*, q^*\}$. Their result was on the existence of two nontrivial solutions basd on the fibering method and the Nehari manifold. We refer the reader to [11–14] for more results about the elliptic systems involving concave-convex nonlinearities.

Moreover, in recent years, the research about equations on graphs have also attracted some attentions. For example, see [15–17]. In [15], Grigor'yan et al. considered the second order Laplacian equation with the nonlinear term satisfying the superquadratic condition and some additional assumptions on finite graphs and locally finite graphs. With the help of the mountain pass theorem, the conclusion they reached was on the existence of a nontrivial solution for the equation. Furthermore, they also investigated the p-Laplacian equation and poly-Laplacian equation on finite graphs and locally finite graphs and obtained some similar results. In [16], Han and Shao studied the p-Laplacian equation with the Dirichlet boundary condition on the locally finite graph. With the help of the mountain pass theorem and the Nehari manifold, their result was the existence of a positive solution and a ground-state solution for the equation. In addition, in [17], Han et al. studied a nonlinear biharmonic equation with a parameter λ and the Dirichlet boundary condition on the locally finite graph. With the help of the mountain pass theorem and the method of the Nehari manifold, they obtained that when the parameter λ is small enough, the equation has a ground-state solution. Moreover, when $\lambda \to +\infty$, the ground-state solutions converge.

Next, we recall some basic knowledge and notations of locally finite graphs, which were taken from [15–17]. Suppose that $G = (V, E)$ is a graph, where V is a vertex set and E is a edge set. xy denotes the edge connecting x with y. Assume that for any $x \in V$, there are only finite edges $xy \in E$; then, (V, E) is called a locally finite graph. Moreover, assume that both the vertex set V and edge set E are finite sets; then, (V, E) is called a finite graph. $\omega_{xy} > 0$ is defined as the weight of the edge $xy \in E$, and it is assumed that $\omega_{xy} = \omega_{yx}$. Furthermore, for any $x \in V$, the degree is defined as $\deg(x) = \sum_{y\sim x} \omega_{xy}$, where $y \sim x$ denotes that $y \in V$ and $xy \in E$. $d(x, y)$ is the distance of two vertices x and y, which is the minimal number of edges that connect x with y. Let $\Omega \subset V$. Assume that for any $x, y \in \Omega$, there exists a positive constant c such that $d(x, y) \leq c$; then, Ω is a bounded domain in V. The definition of the boundary of Ω is as follows:

$$\partial\Omega = \{y \in V,\ y \notin \Omega \big|\ \exists\ x \in \Omega \text{ such that } xy \in E\}.$$

Assume that $\mu : V \to \mathbb{R}^+$ is a finite measure and it is assumed that $\mu(x) \geq \mu_0 > 0$. For any function $u : V \to \mathbb{R}$, one denotes

$$\int_V u(x)d\mu = \sum_{x\in V} u(x)\mu(x). \tag{1.2}$$

Let $C(V) = \{u|u : V \to \mathbb{R}\}$. Define the Laplacian operator $\Delta : C(V) \to C(V)$ by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x)) \tag{1.3}$$

and define the associate gradient $\Gamma(u_1, u_2)$ as

$$\Gamma(u_1, u_2)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u_1(y) - u_1(x))(u_2(y) - u_2(x)). \tag{1.4}$$

Denote $\Gamma(u) := \Gamma(u, u)$. The definition of the length of the gradient is as follows:

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))^2\right)^{\frac{1}{2}}, \tag{1.5}$$

and the definition of the length of the m-order gradient is as follows:

$$|\nabla^m u| = \begin{cases} |\nabla \Delta^{\frac{m-1}{2}} u|, & \text{if } m \text{ is odd,} \\ |\Delta^{\frac{m}{2}} u|, & \text{if } m \text{ is even.} \end{cases} \tag{1.6}$$

For any given $s > 1$, define the s-Laplacian operator $\Delta_s : C(V) \to C(V)$ by

$$\Delta_s u(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \left(|\nabla u|^{s-2}(y) + |\nabla u|^{s-2}(x)\right) \omega_{xy}(u(y) - u(x)). \tag{1.7}$$

Let $C_c(\Omega) := \{u : V \to \mathbb{R} \big| supp\ u \subset \Omega \text{ and } \forall x \in V\backslash\Omega, u(x) = 0\}$. For any function $\phi \in C_c(\Omega)$, the following equality holds:

$$\int_{\Omega} \Delta_s u \phi d\mu = -\int_{\Omega\cup\partial\Omega} |\nabla u|^{s-2}\Gamma(u,\phi)d\mu. \tag{1.8}$$

For any $1 \le r < +\infty$, assume that the completion space of $C_c(\Omega)$ is $L^r(\Omega)$ under the norm

$$||u||_{L^r(\Omega)} = \left(\int_{\Omega} |u(x)|^r d\mu\right)^{\frac{1}{r}}.$$

Moreover, the completion space of $C_c(\Omega)$ is $W_0^{m,s}(\Omega)$ under the norm

$$||u||_{W_0^{m,s}(\Omega)} = \left(\int_{\Omega\cup\partial\Omega} |\nabla^m u(x)|^s d\mu\right)^{\frac{1}{s}},$$

where m is a positive integer and $s > 1$. For any $u \in W_0^{m,s}(\Omega)$, we also define the following norm:

$$||u||_{\infty} = \max_{x\in\Omega} |u(x)|.$$

$W_0^{m,s}(\Omega)$ is of finite dimension. See [15, 16] for more details.

In this paper, our work was mainly inspired by [4, 6, 7, 10, 15]. We shall employ the mountain pass theorem and Ekeland's variational principle as in [6] to investigate the multiplicity of solutions for a class of poly-Laplacian systems on graphs, which can be seen as a discrete version of (1.1) on graphs in some sense, and we also obtain that a poly-Laplacian equation on a locally finite graph has a ground-state solution basd on an abstract result in [4], which was essentially obtained by using the fibering method and the Nehari manifold as in [2, 4, 7, 10]. To be specific, we discuss the following poly-Laplacian system on a locally finite graph $G = (V, E)$:

$$\begin{cases} \pounds_{m_1,p} u = \lambda_1 h_1(x)|u|^{\gamma_1-2}u + \frac{\alpha}{\alpha+\beta}c(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega, \\ \pounds_{m_2,q} v = \lambda_2 h_2(x)|v|^{\gamma_2-2}v + \frac{\beta}{\alpha+\beta}c(x)|u|^{\alpha}|v|^{\beta-2}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.9}$$

where $\Omega \cup \partial\Omega \subset V$ is a bounded domain, $m_i, i = 1, 2$ denotes positive integers, $p, q, \gamma_1, \gamma_2 > 1$, $\lambda_1, \lambda_2, \alpha, \beta > 0$, $\max\{\gamma_1, \gamma_2\} < \min\{p, q\} \le \max\{p, q\} < \alpha + \beta$, $h_1(x), h_2(x), c(x) : \Omega \to \mathbb{R}^+$ and the definitions of $\pounds_{m_i,s}$ $(i = 1, 2, s = p, q)$ is expressed as follows for any function $\phi : \Omega \cup \partial\Omega \to \mathbb{R}$:

$$\int_{\Omega} (\pounds_{m_i,s} u)\phi d\mu = \begin{cases} \int_{\Omega\cup\partial\Omega} |\nabla^{m_i} u|^{s-2}\Gamma(\Delta^{\frac{m_i-1}{2}}u, \Delta^{\frac{m_i-1}{2}}\phi), & \text{if } m_i \text{ is odd}, \\ \int_{\Omega\cup\partial\Omega} |\nabla^{m_i} u|^{s-2}\Delta^{\frac{m_i}{2}}u\Delta^{\frac{m_i}{2}}\phi, & \text{if } m_i \text{ is even}. \end{cases} \tag{1.10}$$

When $m = 1$, $\pounds_{m,p} u = -\Delta_p u$, and when $p = 2$, $\pounds_{m,p} u = (-\Delta^m)u$, which is called a poly-Laplacian operator of u. More details can be seen in [15] for the definition of $\pounds_{m,p}$. Obviously, system (1.9) with $m_1 = m_2 = 1$, $p = q = 2$ and $\gamma_1 = \gamma_2 = \gamma$ is a generalization of (1.1) from the Euclidean setting to a locally finite graph.

In this paper, when (u, v) is a solution of system (1.9) with either $(u, v) = (u, 0)$ or $(u, v) = (0, v)$, (u, v) is called a semi-trivial solution of system (1.9). Moreover, when (u, v) is a solution of system (1.9) and $(u, v) \neq (0, 0)$, (u, v) is called a nontrivial solution of system (1.9).

Denote

$$\begin{aligned} M_{(\lambda_1,\lambda_2)} &= 2^{1-\max\{p,q\}} \min\left\{\frac{1-\lambda_1 C^p_{m_1,p}(\Omega)}{p}, \frac{1-\lambda_2 C^q_{m_2,q}(\Omega)}{q}\right\}, \\ M_2 &= \frac{C_0}{(\alpha+\beta)^2}\left(\alpha C^{\alpha+\beta}_{m_1,p}(\Omega) + \beta C^{\alpha+\beta}_{m_2,q}(\Omega)\right), \end{aligned}$$

where $C_0 = \max_{x\in\Omega} c(x)$ and $C_{m_1,p}(\Omega)$ and $C_{m_2,q}(\Omega)$ are embedding constants given in Lemma 2.1 below. Especially, we present concrete values of $C_{m_1,p}(\Omega)$ and $C_{m_2,q}(\Omega)$ if $m_1 = m_2 = 1$, $p, q \geq 2$ and for each $x \in \Omega$, there is at least one $y \in \partial\Omega$ satisfying that $y \sim x$ (see Lemma 2.2 below).

Our main results are as follows. We suppose that λ_1 and λ_2 satisfy the following inequalities:

$$\begin{cases} 0 < \lambda_1 < C^{-p}_{m_1,p}(\Omega), \\ 0 < \lambda_2 < C^{-q}_{m_2,q}(\Omega), \\ M_{(\lambda_1,\lambda_2)} \leq \frac{\alpha+\beta}{\max\{p,q\}} M_2, \\ \frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|^{\frac{p}{p-\gamma_1}}_{L^{\frac{p}{p-\gamma_1}}(\Omega)} + \frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|^{\frac{q}{q-\gamma_2}}_{L^{\frac{q}{q-\gamma_2}}(\Omega)} < \frac{\alpha+\beta-\max\{p,q\}}{\alpha+\beta} M^{\frac{\alpha+\beta}{\alpha+\beta-\max\{p,q\}}}_{(\lambda_1,\lambda_2)} \left(\frac{\max\{p,q\}}{(\alpha+\beta)M_2}\right)^{\frac{\max\{p,q\}}{\alpha+\beta-\max\{p,q\}}}. \end{cases} \tag{1.11}$$

Theorem 1.1. *Suppose that $G = (V, E)$ is a locally finite graph, $\Omega \neq \emptyset$ and $\partial\Omega \neq \emptyset$. If (λ_1, λ_2) satisfies (1.11), system (1.9) admits one nontrivial solution of positive energy and one nontrivial solution of negative energy .*

Remark 1.1. *There exist λ_1 and λ_2 satisfying (1.11). For example, let $m_1 = 2$, $m_2 = 3$, $\gamma_1 = 2$, $\gamma_2 = 3$, $p = 4$, $q = 5$, $\alpha = 2$, $\beta = 4$ and*

$$C_0 = \frac{1}{C^6_{2,4}(\Omega) + 2C^6_{3,5}(\Omega)}, \quad \|h_1\|^2_{L^2(\Omega)} = \frac{5^6}{9\cdot 2^{31}} C^4_{2,4}(\Omega), \quad \|h_2\|^{\frac{5}{2}}_{L^{\frac{5}{2}}(\Omega)} = \frac{5^6}{2^{32}} C^5_{3,5}(\Omega).$$

When $\lambda_1 = \frac{1}{5}C^{-4}_{2,4}(\Omega)$ and $\lambda_2 = \frac{1}{6}C^{-5}_{3,5}(\Omega)$, we can obtain that

$$M_{(\lambda_1,\lambda_2)} = \frac{1}{3\cdot 2^5} = \frac{1}{96}, \quad M_2 = \frac{1}{18}.$$

Evidently,

$$M_{(\lambda_1,\lambda_2)} < \frac{6}{5} M_2.$$

Moreover,

$$\frac{1}{20} C^{-4}_{2,4}(\Omega)\|h_1\|^2_{L^2(\Omega)} + \frac{1}{45} C^{-5}_{3,5}(\Omega)\|h_2\|^{\frac{5}{2}}_{L^{\frac{5}{2}}(\Omega)} = \frac{5^5}{9\cdot 2^{33}} + \frac{5^5}{9\cdot 2^{32}} < \frac{5^5}{9\cdot 2^{31}}.$$

Hence, (1.11) holds for $\lambda_1 = \frac{1}{5}C^{-4}_{2,4}(\Omega)$ and $\lambda_2 = \frac{1}{6}C^{-5}_{3,5}(\Omega)$.

Theorem 1.2. *Suppose that $G = (V, E)$ is a locally finite graph, $\Omega \neq \emptyset$ and $\partial\Omega \neq \emptyset$. For each $\lambda_1 > 0$, assume that $(u, 0)$ is a semi-trivial solution of system (1.9). Then*

$$\|u\|_{W^{m_1,p}_0(\Omega)} \leq \left(\lambda_1 H_1 C^{\gamma_1}_{m_1,p}(\Omega)\right)^{\frac{1}{p-\gamma_1}},$$

where $H_1 = \max_{x\in\Omega} h_1(x)$. Similarly, for each $\lambda_2 > 0$, assume that $(0, v)$ is a semi-trivial solution of system (1.9). Then

$$\|v\|_{W^{m_2,q}_0(\Omega)} \leq \left(\lambda_2 H_2 C^{\gamma_2}_{m_2,q}(\Omega)\right)^{\frac{1}{q-\gamma_2}},$$

where $H_2 = \max_{x\in\Omega} h_2(x)$.

Moreover, we also investigate the existence of a ground-state solution for the following poly-Laplacian equation on $G = (V, E)$ by applying Theorem 3.3 in [4]:

$$\begin{cases} \pounds_{m,p} u = \lambda h(x)|u|^{\gamma-2}u + c(x)|u|^{\alpha-2}u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.12}$$

where $\Omega \cup \partial\Omega \subset V$ is a bounded domain, m is a positive integer, $p, \gamma > 1$, $\lambda > 0$, $\gamma < p < \alpha$ and $h(x), c(x) : \Omega \to \mathbb{R}^+$. Denote

$$\lambda_0 = \frac{p-\gamma}{H_0} C_{m,p}^{-\gamma}(\Omega) \left(\left(C_0 C_{m,p}^{\alpha}(\Omega) \right)^{p-\alpha} (\alpha - p)^{\alpha - p} (\alpha - \gamma)^{\gamma - \alpha} \right)^{\frac{1}{p-\gamma}},$$

$$\lambda_\star = \frac{\gamma(\alpha - p)}{p\alpha H_0 C_{m,p}^{\gamma}(\Omega)} \left(C_0 C_{m,p}^{\alpha}(\Omega) \right)^{\frac{p-\gamma}{p-\alpha}}, \quad \lambda_{\star\star} = \min\{\lambda_0, \lambda_\star\}, \tag{1.13}$$

where $H_0 = \max_{x\in\Omega} h(x)$ and $C_0 = \max_{x\in\Omega} c(x)$. We obtain the following result.

Theorem 1.3. *Suppose that $G = (V, E)$ is a locally finite graph, $\Omega \neq \emptyset$ and $\partial\Omega \neq \emptyset$. If $\lambda \in (0, \lambda_0)$, then (1.12) admits one nontrivial solution of positive energy and one nontrivial solution of negative energy. Furthermore, if $\lambda \in (0, \lambda_{\star\star})$, the negative energy solution is the ground-state solution of (1.12).*

Remark 1.2. *Similar to the arguments of Theorem 1.1, applying the mountain pass theorem and Ekeland's variational principle, we can also get one nontrivial solution of positive energy and one nontrivial solution of negative energy for (1.12). We do not know whether these two solutions are different from those two solutions in Theorem 1.3 which were obtained, essentially, by using the fibering method and the Nehari manifold.*

2. Preliminaries

Define the space $W = W_0^{m_1,p}(\Omega) \times W_0^{m_2,q}(\Omega)$ with the norm

$$\|(u, v)\|_W = \|u\|_{W_0^{m_1,p}(\Omega)} + \|v\|_{W_0^{m_2,q}(\Omega)}.$$

Then, W is a finite dimensional Banach space. The energy functional $\psi : W \to \mathbb{R}$ of system (1.9) is defined as follows:

$$\begin{aligned} \psi(u, v) = & \frac{1}{p} \int_{\Omega\cup\partial\Omega} |\nabla^{m_1} u|^p d\mu - \frac{\lambda_1}{\gamma_1} \int_\Omega h_1(x)|u|^{\gamma_1} d\mu \\ & + \frac{1}{q} \int_{\Omega\cup\partial\Omega} |\nabla^{m_2} v|^q d\mu - \frac{\lambda_2}{\gamma_2} \int_\Omega h_2(x)|v|^{\gamma_2} d\mu - \frac{1}{\alpha+\beta} \int_\Omega c(x)|u|^\alpha |v|^\beta d\mu. \end{aligned} \tag{2.1}$$

Then, $\psi(u, v) \in C^1(W, \mathbb{R})$. Moreover,

$$\begin{aligned} \langle \psi'(u, v), (\phi, \varphi) \rangle = & \int_{\Omega\cup\partial\Omega} (\pounds_{m_1,p} u)\phi d\mu - \lambda_1 \int_\Omega h_1(x)|u|^{\gamma_1 - 2} u\phi d\mu \\ & + \int_{\Omega\cup\partial\Omega} (\pounds_{m_2,q} v)\varphi d\mu - \lambda_2 \int_\Omega h_2(x)|v|^{\gamma_2 - 2}\varphi d\mu \\ & - \frac{\alpha}{\alpha+\beta} \int_\Omega c(x)|u|^{\alpha-2} u|v|^\beta \phi d\mu - \frac{\beta}{\alpha+\beta} \int_\Omega c(x)|u|^\alpha |v|^{\beta-2} v\varphi d\mu. \end{aligned} \tag{2.2}$$

Definition 2.1. *$(u,v)\in W$ is called a weak solution of system (1.9) whenever, for all $(\phi,\varphi)\in W$, the following equalities are true:*

$$\int_{\Omega\cup\partial\Omega}(\pounds_{m_1,p}u)\phi d\mu=\lambda_1\int_\Omega h_1(x)|u|^{\gamma_1-2}u\phi d\mu+\frac{\alpha}{\alpha+\beta}\int_\Omega c(x)|u|^{\alpha-2}u|v|^\beta\phi d\mu, \tag{2.3}$$

$$\int_{\Omega\cup\partial\Omega}(\pounds_{m_2,q}v)\varphi d\mu=\lambda_2\int_\Omega h_2(x)|v|^{\gamma_2-2}v\varphi d\mu+\frac{\beta}{\alpha+\beta}\int_\Omega c(x)|u|^{\alpha}|v|^{\beta-2}v\varphi d\mu. \tag{2.4}$$

Evidently, $(u,v)\in W$ is a weak solution of system (1.9) if and only if (u,v) is a critical point of ψ.

Proposition 2.1. *Assume that $(u,v)\in W$ is a weak solution of system (1.9). Then, $(u,v)\in W$ is also a point-wise solution of (1.9).*

Proof. We define two functions $\phi,\varphi:V\to\mathbb{R}$ as follows for any fixed $y\in V$:

$$\varphi(x)=\phi(x)=\begin{cases}1, & x=y,\\ 0, & x\neq y.\end{cases}$$

Hence, by (2.3) and (2.4), the following holds:

$$\pounds_{m_1,p}u(y)=\lambda_1h_1(y)|u(y)|^{\gamma_1-2}u(y)+\frac{\alpha}{\alpha+\beta}c(y)|u(y)|^{\alpha-2}u(y)|v(y)|^\beta,$$
$$\pounds_{m_2,q}v(y)=\lambda_2h_2(y)|v(y)|^{\gamma_2-2}v(y)+\frac{\beta}{\alpha+\beta}c(y)|u(y)|^{\alpha}|v(y)|^{\beta-2}v(y).$$

By the arbitrariness of y, we come to the conclusion.

Lemma 2.1. *([15]). Assume that $G=(V,E)$ is a locally finite graph: $\Omega\cup\partial\Omega\subset V$ is a bounded domain with $\Omega\neq\emptyset$. For any given m and s with $m\in\mathbb{N}^+$ and $s>1$, $W_0^{m,s}(\Omega)$ is embedded in $L^r(\Omega)$ for each $1\le r\le+\infty$. Especially, there exists a positive constant $C_{m,s}(\Omega)$ such that*

$$\left(\int_\Omega|u(x)|^rd\mu\right)^{\frac{1}{r}}\le C_{m,s}(\Omega)\left(\int_{\Omega\cup\partial\Omega}|\nabla^m u(x)|^sd\mu\right)^{\frac{1}{s}}, \tag{2.5}$$

where

$$C_{m,s}(\Omega)=\frac{C}{\mu_{min}}(1+|\Omega|)\ \text{with}\ C\ \text{satisfying that}\ \|u\|_{L^r(\Omega)}\le C\|u\|_{W_0^{m,r}(\Omega)}, \tag{2.6}$$

and $|\Omega|=\sum_{x\in\Omega}\mu(x)$. Furthermore, $W_0^{m,s}(\Omega)$ is pre-compact.

Moreover, if for each $x\in\Omega$, there is at least one $y\in\partial\Omega$ satisfying that $y\sim x$, we can present a specific value of $C_{m,s}(\Omega)$ with $m=1$ and $s\ge 2$. The details are as follows.

Lemma 2.2. *Suppose that $G=(V,E)$ is a locally finite graph, $\Omega\cup\partial\Omega\subset V$ is a bounded domain with $\Omega\neq\emptyset$ and $\partial\Omega\neq\emptyset$, and for each $x\in\Omega$, there is at least one $y\in\partial\Omega$ satisfying that $y\sim x$. Then, for any $s\ge2$ and $1\le r<+\infty$,*

$$\left(\int_\Omega|u(x)|^rd\mu\right)^{\frac{1}{r}}\le C_{1,s}(\Omega)\left(\int_{\Omega\cup\partial\Omega}|\nabla u(x)|^sd\mu\right)^{\frac{1}{s}}, \tag{2.7}$$

where

$$C_{1,s}(\Omega) = (1 + |\Omega|)\hat{\mu}_{\min}^{-\frac{1}{s}} \left(\frac{2\mu_{\max}}{w_{\min}}\right)^{\frac{1}{2}},$$

$\hat{\mu}_{\min} = \min_{x\in\Omega} \mu(x)$, $\mu_{\max} = \max_{x\in\Omega\cup\partial\Omega} \mu(x)$ *and* $w_{\min} = \min_{x\in\Omega\cup\partial\Omega} w_{xy}$.

Proof. The following holds:

$$\begin{aligned}
\|u\|^s_{W_0^{1,s}(\Omega)} &= \int_{\Omega\cup\partial\Omega} |\nabla u(x)|^s d\mu \\
&= \sum_{x\in\Omega\cup\partial\Omega} \left(\frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))^2\right)^{\frac{s}{2}} \mu(x) \\
&\geq \left(\frac{w_{\min}}{2\mu_{\max}}\right)^{\frac{s}{2}} \sum_{x\in\Omega\cup\partial\Omega} \sum_{y\sim x} |u(y) - u(x)|^s \mu(x) \\
&= \left(\frac{w_{\min}}{2\mu_{\max}}\right)^{\frac{s}{2}} \left(\sum_{x\in\Omega} \sum_{y\sim x} |u(y) - u(x)|^s + \sum_{x\in\partial\Omega} \sum_{y\sim x} |u(y) - u(x)|^s\right) \mu(x) \\
&\geq \left(\frac{w_{\min}}{2\mu_{\max}}\right)^{\frac{s}{2}} \left(\sum_{x\in\Omega} \sum_{y\sim x, y\in\Omega} |u(y) - u(x)|^s + \sum_{x\in\Omega} \sum_{y\sim x, y\in\partial\Omega} |u(x)|^s\right) \mu(x) \\
&\geq \left(\frac{w_{\min}}{2\mu_{\max}}\right)^{\frac{s}{2}} \sum_{x\in\Omega} \sum_{y\sim x, y\in\partial\Omega} |u(x)|^s \mu(x) \\
&\geq \left(\frac{w_{\min}}{2\mu_{\max}}\right)^{\frac{s}{2}} \sum_{x\in\Omega} |u(x)|^s \mu(x).
\end{aligned}$$

Hence,

$$\|u\|_{L^s(\Omega)} \leq \left(\frac{2\mu_{\max}}{w_{\min}}\right)^{\frac{1}{2}} \|u\|_{W_0^{1,s}(\Omega)}. \tag{2.8}$$

Moreover, by (2.8), we have

$$\|u\|_\infty \leq \hat{\mu}_{\min}^{-\frac{1}{s}} \|u\|_{L^s(\Omega)} \leq \hat{\mu}_{\min}^{-\frac{1}{s}} \left(\frac{2\mu_{\max}}{w_{\min}}\right)^{\frac{1}{2}} \|u\|_{W_0^{1,s}(\Omega)}. \tag{2.9}$$

It follows from (2.9) that for any $1 \leq r < +\infty$, the following holds:

$$\begin{aligned}
\|u\|_{L^r(\Omega)} &= \left(\sum_{x\in\Omega} |u(x)|^r \mu(x)\right)^{\frac{1}{r}} \\
&\leq |\Omega|^{\frac{1}{r}} \|u\|_\infty \\
&\leq |\Omega|^{\frac{1}{r}} \hat{\mu}_{\min}^{-\frac{1}{s}} \left(\frac{2\mu_{\max}}{w_{\min}}\right)^{\frac{1}{2}} \|u\|_{W_0^{1,s}(\Omega)}
\end{aligned}$$

$$\leq \quad (1+|\Omega|)\hat{\mu}_{\min}^{-\frac{1}{s}}\left(\frac{2\mu_{\max}}{w_{\min}}\right)^{\frac{1}{2}}\|u\|_{W_0^{1,s}(\Omega)}.$$

Assume that B is a real Banach space and $f \in C^1(B,\mathbb{R})$. We say that f satisfies the Palais-Smale condition if any Palais-Smale sequence $\{u_n\} \subseteq B$ has a convergent subsequence, where $\{u_n\}$ is called the Palais-Smale sequence if for all $n \in \mathbb{N}$, there exists a positive constant c such that $|f(u_n)| \leq c$ and $f'(u_n) \to 0$ as $n \to \infty$.

Lemma 2.3. *([18]) Assume that B is a real Banach space and $f \in C^1(B,\mathbb{R})$, where f satisfies the Palais-Smale condition and $f(0) = 0$. Moreover, if f satisfies the following conditions:*
(i) there exist two constants r and m with $r, m \in \mathbb{R}^+$ such that $f_{\partial B_r(0)} \geq m$, where $B_r = \{x \in B : \|x\|_B < r\}$;
(ii) there is $x \in B\backslash\bar{B}_r(0)$ satisfying that $f(x) \leq 0$,
then f admits a critical value $m_* \geq m$ and

$$m_* := \inf_{\pi\in\Pi}\max_{t\in[0,1]} f(\pi(t)),$$

where

$$\Pi := \{\pi \in C([0,1],B) : \pi(0) = 0, \pi(1) = x\}.$$

Lemma 2.4. *([19]) Assume that (B,ρ) is a complete metric space and $f : B \to \mathbb{R}$, which is lower-semicontinuous and bounded from below. Moreover, there exist $\delta > 0$ and $x \in B$ such that*

$$f(x) \leq \inf_B f + \delta.$$

On that occasion, there exists $y \in B$ such that

$$f(y) \leq f(x),\ \rho(x,y) \leq 1.$$

Furthermore, for all $w \in B$, the following holds:

$$f(y) \leq f(w) + \delta\rho(y,w).$$

3. Proofs for Theorems 1.1 and 1.2

Lemma 3.1. *For each (λ_1,λ_2) satisfying (1.11), there exists a positive constant $r_{(\lambda_1,\lambda_2)}$ such that $\psi(u,v) > 0$ whenever $\|(u,v)\|_W = r_{(\lambda_1,\lambda_2)}$.*

Proof. Note that

$$c(x) \leq \max_{x\in\Omega} c(x) := C_0,\ \ h_i^\star := \min_{x\in\Omega} h_i(x) \leq h_i(x) \leq \max_{x\in\Omega} h_i(x) := H_i,\ \ i = 1,2,\ \text{for all } x \in \Omega. \tag{3.1}$$

Then, by using Lemma 2.1 and Young's inequality, for all $u \in W_0^{m_1,p}(\Omega)$, the following holds:

$$\int_\Omega h_1(x)|u|^{\gamma_1}d\mu$$

$$\begin{aligned} &\le \frac{p-\gamma_1}{p}\int_\Omega h_1(x)^{\frac{p}{p-\gamma_1}}d\mu + \frac{\gamma_1}{p}\int_\Omega |u|^p d\mu \\ &\le \frac{p-\gamma_1}{p}\|h_1\|_{L^{\frac{p}{p-\gamma_1}}(\Omega)}^{\frac{p}{p-\gamma_1}} + \frac{\gamma_1}{p}C_{m_1,p}^p(\Omega)\|u\|_{W_0^{m_1,p}(\Omega)}^p. \end{aligned} \tag{3.2}$$

Similarly, for all $v \in W_0^{m_2,q}(\Omega)$, the following holds:

$$\int_\Omega h_2(x)|v|^{\gamma_2}d\mu \le \frac{q-\gamma_2}{q}\|h_2\|_{L^{\frac{q}{q-\gamma_2}}(\Omega)}^{\frac{q}{q-\gamma_2}} + \frac{\gamma_2}{q}C_{m_2,q}^q(\Omega)\|v\|_{W_0^{m_2,q}(\Omega)}^q. \tag{3.3}$$

Furthermore, for all $(u, v) \in W$, Lemma 2.1 and Young's inequality imply that

$$\begin{aligned} &\int_\Omega c(x)|u|^\alpha|v|^\beta d\mu \\ \le\ & C_0\int_\Omega |u|^\alpha|v|^\beta d\mu \\ \le\ & C_0\left(\frac{\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha+\beta}d\mu + \frac{\beta}{\alpha+\beta}\int_\Omega |v|^{\alpha+\beta}d\mu\right) \\ \le\ & C_0\left(\frac{\alpha C_{m_1,p}^{\alpha+\beta}(\Omega)}{\alpha+\beta}\|u\|_{W_0^{m_1,p}(\Omega)}^{\alpha+\beta} + \frac{\beta C_{m_2,q}^{\alpha+\beta}(\Omega)}{\alpha+\beta}\|v\|_{W_0^{m_2,q}(\Omega)}^{\alpha+\beta}\right). \end{aligned} \tag{3.4}$$

Thus, (2.1) and (3.2)–(3.4) imply that when $(\lambda_1, \lambda_2) \in (0, C_{m_1,p}^{-p}(\Omega)) \times (0, C_{m_2,q}^{-q}(\Omega))$, for any $(u, v) \in W$ with $\|(u, v)\|_W \le 1$, the following holds:

$$\begin{aligned} \psi(u,v) =\ & \frac{1}{p}\|u\|_{W_0^{m_1,p}(\Omega)}^p - \frac{\lambda_1}{\gamma_1}\int_\Omega h_1(x)|u|^{\gamma_1}d\mu \\ & + \frac{1}{q}\|v\|_{W_0^{m_2,q}(\Omega)}^q - \frac{\lambda_2}{\gamma_2}\int_\Omega h_2(x)|v|^{\gamma_2}d\mu - \frac{1}{\alpha+\beta}\int_\Omega c(x)|u|^\alpha|v|^\beta d\mu \\ \ge\ & \frac{1}{p}\left(1-\lambda_1 C_{m_1,p}^p(\Omega)\right)\|u\|_{W_0^{m_1,p}(\Omega)}^p - \frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|_{L^{\frac{p}{p-\gamma_1}}(\Omega)}^{\frac{p}{p-\gamma_1}} \\ & + \frac{1}{q}\left(1-\lambda_2 C_{m_2,q}^q(\Omega)\right)\|v\|_{W_0^{m_2,q}(\Omega)}^q - \frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|_{L^{\frac{q}{q-\gamma_2}}(\Omega)}^{\frac{q}{q-\gamma_2}} \\ & - \frac{C_0}{\alpha+\beta}\left(\frac{\alpha C_{m_1,p}^{\alpha+\beta}(\Omega)}{\alpha+\beta}\|u\|_{W_0^{m_1,p}(\Omega)}^{\alpha+\beta} + \frac{\beta C_{m_2,q}^{\alpha+\beta}(\Omega)}{\alpha+\beta}\|v\|_{W_0^{m_2,q}(\Omega)}^{\alpha+\beta}\right) \\ \ge\ & \min\left\{\frac{1-\lambda_1 C_{m_1,p}^p(\Omega)}{p}, \frac{1-\lambda_2 C_{m_2,q}^q(\Omega)}{q}\right\}\left(\|u\|_{W_0^{m_1,p}(\Omega)}^{\max\{p,q\}} + \|v\|_{W_0^{m_2,q}(\Omega)}^{\max\{p,q\}}\right) \\ & - \frac{C_0}{(\alpha+\beta)^2}\left(\alpha C_{m_1,p}^{\alpha+\beta}(\Omega) + \beta C_{m_2,q}^{\alpha+\beta}(\Omega)\right)\|(u,v)\|_W^{\alpha+\beta} \\ & - \frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|_{L^{\frac{p}{p-\gamma_1}}(\Omega)}^{\frac{p}{p-\gamma_1}} - \frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|_{L^{\frac{q}{q-\gamma_2}}(\Omega)}^{\frac{q}{q-\gamma_2}} \\ \ge\ & 2^{1-\max\{p,q\}}\min\left\{\frac{1-\lambda_1 C_{m_1,p}^p(\Omega)}{p}, \frac{1-\lambda_2 C_{m_2,q}^q(\Omega)}{q}\right\}\|(u,v)\|_W^{\max\{p,q\}} \\ & - \frac{C_0}{(\alpha+\beta)^2}\left(\alpha C_{m_1,p}^{\alpha+\beta}(\Omega) + \beta C_{m_2,q}^{\alpha+\beta}(\Omega)\right)\|(u,v)\|_W^{\alpha+\beta} \end{aligned}$$

$$-\frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|^{\frac{p}{p-\gamma_1}}_{L^{\frac{p}{p-\gamma_1}}(\Omega)}-\frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|^{\frac{q}{q-\gamma_2}}_{L^{\frac{q}{q-\gamma_2}}(\Omega)}. \tag{3.5}$$

Note that

$$\begin{aligned} M_{(\lambda_1,\lambda_2)} &= 2^{1-\max\{p,q\}}\min\left\{\frac{1-\lambda_1 C^p_{m_1,p}(\Omega)}{p},\frac{1-\lambda_2 C^q_{m_2,q}(\Omega)}{q}\right\},\\ M_2 &= \frac{C_0}{(\alpha+\beta)^2}\left(\alpha C^{\alpha+\beta}_{m_1,p}(\Omega)+\beta C^{\alpha+\beta}_{m_2,q}(\Omega)\right). \end{aligned}$$

Define

$$f(t)=M_{(\lambda_1,\lambda_2)}t^{\max\{p,q\}}-M_2t^{\alpha+\beta}-\frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|^{\frac{p}{p-\gamma_1}}_{L^{\frac{p}{p-\gamma_1}}(\Omega)}-\frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|^{\frac{q}{q-\gamma_2}}_{L^{\frac{q}{q-\gamma_2}}(\Omega)},\quad t\in[0,1]. \tag{3.6}$$

To find $r_{(\lambda_1,\lambda_2)}$ satisfying that $\psi(u,v)>0$ whenever $\|(u,v)\|_W=r_{(\lambda_1,\lambda_2)}$, it suffices to prove that there is $t^\star_{(\lambda_1,\lambda_2)}\in(0,1]$ satisfying that $f(t^\star_{(\lambda_1,\lambda_2)})>0$. In fact, by (3.6), we have

$$\begin{aligned} f'(t) &= \max\{p,q\}M_{(\lambda_1,\lambda_2)}t^{\max\{p,q\}-1}-(\alpha+\beta)M_2t^{\alpha+\beta-1},\\ f''(t) &= \max\{p,q\}(\max\{p,q\}-1)M_{(\lambda_1,\lambda_2)}t^{\max\{p,q\}-2}-(\alpha+\beta)(\alpha+\beta-1)M_2t^{\alpha+\beta-2}. \end{aligned}$$

Let $f'(t^\star_{(\lambda_1,\lambda_2)})=0$. We obtain that

$$t^\star_{(\lambda_1,\lambda_2)}=\left(\frac{\max\{p,q\}M_{(\lambda_1,\lambda_2)}}{(\alpha+\beta)M_2}\right)^{\frac{1}{\alpha+\beta-\max\{p,q\}}}.$$

Since λ_1 and λ_2 satisfy (1.11), we have that $0<t^\star_{(\lambda_1,\lambda_2)}\le 1$. Moreover,

$$\begin{aligned} f''(t^\star_{(\lambda_1,\lambda_2)}) &= \max\{p,q\}(\max\{p,q\}-1)M_{(\lambda_1,\lambda_2)}\left(\frac{\max\{p,q\}M_{(\lambda_1,\lambda_2)}}{(\alpha+\beta)M_2}\right)^{\max\{p,q\}-2}\\ &\quad -(\alpha+\beta)(\alpha+\beta-1)M_2\left(\frac{\max\{p,q\}M_{(\lambda_1,\lambda_2)}}{(\alpha+\beta)M_2}\right)^{\alpha+\beta-2}\\ &= (\max\{p,q\}-\alpha-\beta)M_2\left(\frac{\max\{p,q\}M_{(\lambda_1,\lambda_2)}}{(\alpha+\beta)M_2}\right)^{\alpha+\beta-2}\\ &< 0. \end{aligned}$$

Hence, by (1.11), we have

$$\begin{aligned} \max_{t\in[0,1]}f(t) &= f(t^\star_{(\lambda_1,\lambda_2)})\\ &= M_{(\lambda_1,\lambda_2)}\left(\frac{\max\{p,q\}M_{(\lambda_1,\lambda_2)}}{(\alpha+\beta)M_2}\right)^{\frac{\max\{p,q\}}{\alpha+\beta-\max\{p,q\}}}-M_2\left(\frac{\max\{p,q\}M_{(\lambda_1,\lambda_2)}}{(\alpha+\beta)M_2}\right)^{\frac{\alpha+\beta}{\alpha+\beta-\max\{p,q\}}}\\ &\quad -\frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|^{\frac{p}{p-\gamma_1}}_{L^{\frac{p}{p-\gamma_1}}(\Omega)}-\frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|^{\frac{q}{q-\gamma_2}}_{L^{\frac{q}{q-\gamma_2}}(\Omega)}\\ &= \frac{\alpha+\beta-\max\{p,q\}}{\alpha+\beta}M^{\frac{\alpha+\beta}{\alpha+\beta-\max\{p,q\}}}_{(\lambda_1,\lambda_2)}\left(\frac{\max\{p,q\}}{(\alpha+\beta)M_2}\right)^{\frac{\max\{p,q\}}{\alpha+\beta-\max\{p,q\}}} \end{aligned}$$

$$-\frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|^{\frac{p}{p-\gamma_1}}_{L^{\frac{p}{p-\gamma_1}}(\Omega)}-\frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|^{\frac{q}{q-\gamma_2}}_{L^{\frac{q}{q-\gamma_2}}(\Omega)}$$
$$> 0.$$

Let $r_{(\lambda_1,\lambda_2)} = t^{\star}_{(\lambda_1,\lambda_2)}$. Hence, we have come to the conclusion.

Lemma 3.2. *For each (λ_1,λ_2) satisfying (1.11), there exists $(u_{(\lambda_1,\lambda_2)}, v_{(\lambda_1,\lambda_2)}) \in W$ with $\|(u_{(\lambda_1,\lambda_2)}, v_{(\lambda_1,\lambda_2)})\|_W > r_{(\lambda_1,\lambda_2)}$ such that $\psi(u_{(\lambda_1,\lambda_2)}, v_{(\lambda_1,\lambda_2)}) < 0$.*

Proof. For any given $(u,v)\in W$ with $\int_\Omega c(x)|u|^\alpha|v|^\beta d\mu \neq 0$ and any $z\in\mathbb{R}^+$, we have

$$\begin{aligned}
\psi(zu,zv) &= \frac{1}{p}z^p\|u\|^p_{W_0^{m_1,p}(\Omega)} - \frac{\lambda_1}{\gamma_1}z^{\gamma_1}\int_\Omega h_1(x)|u|^{\gamma_1}d\mu \\
&\quad + \frac{1}{q}z^q\|v\|^q_{W_0^{m_2,q}(\Omega)} - \frac{\lambda_2}{\gamma_2}z^{\gamma_2}\int_\Omega h_2(x)|v|^{\gamma_2}d\mu - \frac{1}{\alpha+\beta}z^{\alpha+\beta}\int_\Omega c(x)|u|^\alpha|v|^\beta d\mu \\
&\leq \frac{1}{p}z^p\|u\|^p_{W_0^{m_1,p}(\Omega)} - \frac{\lambda_1 h_1^\star}{\gamma_1}z^{\gamma_1}\int_\Omega |u|^{\gamma_1}d\mu \\
&\quad + \frac{1}{q}z^q\|v\|^q_{W_0^{m_2,q}(\Omega)} - \frac{\lambda_2 h_2^\star}{\gamma_2}z^{\gamma_2}\int_\Omega |v|^{\gamma_2}d\mu - \frac{1}{\alpha+\beta}z^{\alpha+\beta}\int_\Omega c(x)|u|^\alpha|v|^\beta d\mu. \qquad (3.7)
\end{aligned}$$

Note that $\alpha+\beta > \max\{p,q\}$. So, there exists $z_{(\lambda_1,\lambda_2)}$ large enough such that $\|(z_{(\lambda_1,\lambda_2)}u, z_{(\lambda_1,\lambda_2)}v)\|_W > r_{(\lambda_1,\lambda_2)}$ and $\psi(z_{(\lambda_1,\lambda_2)}u, z_{(\lambda_1,\lambda_2)}v) < 0$. Let $u_{(\lambda_1,\lambda_2)} = z_{(\lambda_1,\lambda_2)}u$ and $v_{(\lambda_1,\lambda_2)} = z_{(\lambda_1,\lambda_2)}v$. Then, the proof is finished.

Lemma 3.3. *For each (λ_1,λ_2) satisfying (1.11), ψ satisfies the Palais-Smale condition.*

Proof. For any Palais-Smale sequence $(u_k,v_k)\subseteq W$, there exists a constant c with $c>0$ such that

$$|\psi(u_k,v_k)|\leq c \;\text{ and }\; \psi'(u_k,v_k)\to 0 \;\text{ as } k\to\infty, \quad \text{for all } k\in\mathbb{N}.$$

Hence, the following holds:

$$\begin{aligned}
& c + \|u_k\|_{W_0^{m_1,p}(\Omega)} + \|v_k\|_{W_0^{m_2,q}(\Omega)} \\
\geq\ & \psi(u_k,v_k) - \frac{1}{\alpha+\beta}\langle\psi'(u_k,v_k),(u_k,v_k)\rangle \\
=\ & \left(\frac{1}{p}-\frac{1}{\alpha+\beta}\right)\|u_k\|^p_{W_0^{m_1,p}(\Omega)} + \left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\|v_k\|^q_{W_0^{m_2,q}(\Omega)} \\
& -\lambda_1\left(\frac{1}{\gamma_1}-\frac{1}{\alpha+\beta}\right)\int_\Omega h_1(x)|u_k|^{\gamma_1}d\mu - \lambda_2\left(\frac{1}{\gamma_2}-\frac{1}{\alpha+\beta}\right)\int_\Omega h_2(x)|v_k|^{\gamma_2}d\mu \\
\geq\ & \left(\frac{1}{p}-\frac{1}{\alpha+\beta}\right)\|u_k\|^p_{W_0^{m_1,p}(\Omega)} + \left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\|v_k\|^q_{W_0^{m_2,q}(\Omega)} \\
& -\lambda_1\left(\frac{1}{\gamma_1}-\frac{1}{\alpha+\beta}\right)H_1C^{\gamma_1}_{m_1,p}(\Omega)\|u_k\|^{\gamma_1}_{W_0^{m_1,p}(\Omega)} \\
& -\lambda_2\left(\frac{1}{\gamma_2}-\frac{1}{\alpha+\beta}\right)H_2C^{\gamma_2}_{m_2,q}(\Omega)\|v_k\|^{\gamma_2}_{W_0^{m_2,q}(\Omega)}. \qquad (3.8)
\end{aligned}$$

We claim that $\|(u_k, v_k)\|_W$ is bounded. In fact, if

$$\|u_k\|_{W_0^{m_1,p}(\Omega)} \to \infty \text{ and } \|v_k\|_{W_0^{m_2,q}(\Omega)} \to \infty \text{ as } k \to \infty, \tag{3.9}$$

then it follows from (3.8) that

$$\begin{aligned}
& c + \|(u_k, v_k)\|_W \\
\geq\ & \min\left\{\frac{1}{p} - \frac{1}{\alpha+\beta}, \frac{1}{q} - \frac{1}{\alpha+\beta}\right\}\left(\|u_k\|^p_{W_0^{m_1,p}(\Omega)} + \|v_k\|^q_{W_0^{m_2,q}(\Omega)}\right) \\
& - \max\left\{\lambda_1\left(\frac{1}{\gamma_1} - \frac{1}{\alpha+\beta}\right)H_1 C^{\gamma_1}_{m_1,p}(\Omega), \lambda_2\left(\frac{1}{\gamma_2} - \frac{1}{\alpha+\beta}\right)H_2 C^{\gamma_2}_{m_2,q}(\Omega)\right\}\left(\|u_k\|^{\gamma_1}_{W_0^{m_1,p}(\Omega)} + \|v_k\|^{\gamma_2}_{W_0^{m_2,q}(\Omega)}\right) \\
\geq\ & 2^{1-\min\{p,q\}} \min\left\{\frac{1}{p} - \frac{1}{\alpha+\beta}, \frac{1}{q} - \frac{1}{\alpha+\beta}\right\}\|(u_k, v_k)\|_W^{\min\{p,q\}} \\
& - \max\left\{\lambda_1\left(\frac{1}{\gamma_1} - \frac{1}{\alpha+\beta}\right)H_1 C^{\gamma_1}_{m_1,p}(\Omega), \lambda_2\left(\frac{1}{\gamma_2} - \frac{1}{\alpha+\beta}\right)H_2 C^{\gamma_2}_{m_2,q}(\Omega)\right\}\|(u_k, v_k)\|_W^{\max\{\gamma_1,\gamma_2\}},
\end{aligned}$$

which contradicts (3.9). If

$$\|u_k\|_{W_0^{m_1,p}(\Omega)} \to \infty \text{ as } k \to \infty, \tag{3.10}$$

and $\|v_k\|_{W_0^{m_2,q}(\Omega)}$ is bounded, then (3.8) implies that there exists a constant c_1 with $c_1 > 0$ such that

$$c_1 + \|u_k\|_{W_0^{m_1,p}(\Omega)} \geq \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)\|u_k\|^p_{W_0^{m_1,p}(\Omega)} - \lambda_1\left(\frac{1}{\gamma_1} - \frac{1}{\alpha+\beta}\right)H_1 C^{\gamma_1}_{m_1,p}(\Omega)\|u_k\|^{\gamma_1}_{W_0^{m_1,p}(\Omega)},$$

which contradicts (3.10). Similarly, if

$$\|v_k\|_{W_0^{m_2,q}(\Omega)} \to \infty \text{ as } k \to \infty,$$

and $\|u_k\|_{W_0^{m_1,p}(\Omega)}$ is bounded, we can also obtain a contradiction. Hence, $\|u_k\|_{W_0^{m_1,p}(\Omega)}$ and $\|v_k\|_{W_0^{m_2,q}(\Omega)}$ are bounded. Then, there exist subsequences $\{u_{k_n}\} \subset \{u_k\}$ and $\{v_{k_n}\} \subset \{v_k\}$ such that $u_{k_n} \rightharpoonup u_0$ and $v_{k_n} \rightharpoonup v_0$ for some $u_0 \in W_0^{m_1,p}(\Omega)$ and $v_0 \in W_0^{m_2,q}(\Omega)$ as $n \to \infty$. Moreover, Lemma 2.1 implies that

$$u_{k_n} \to u_0 \text{ in } W_0^{m_1,p}(\Omega) \text{ and } v_{k_n} \to v_0 \text{ in } W_0^{m_2,q}(\Omega) \text{ as } n \to \infty.$$

The proof is complete.

Proof of Theorem 1.1. Applying Lemmas 3.1–3.3 and Lemma 2.3, we obtain that for each (λ_1, λ_2) satisfying (1.11), system (1.9) admits one nontrivial solution (u_0, v_0) which has positive energy. Next, similar to the proof of Theorem 3.3 in [20] and Theorem 1.3 in [21], we prove that system (1.9) admits one nontrivial solution of negative energy. Note that $\gamma_1 < p$ and $\gamma_2 < q$. Then, it follows from (3.7) that there exists z small enough such that

$$\psi(zu, zv) < 0.$$

So,

$$-\infty < \inf\{\psi(u, v) : (u, v) \in \bar{B}_{r(\lambda_1,\lambda_2)}\} < 0,$$

where $r_{(\lambda_1,\lambda_2)}$ is given in Lemma 3.1 and $\bar{B}_{r_{(\lambda_1,\lambda_2)}} = \{(u,v) \in W \big| \|(u,v)\|_W \le r_{(\lambda_1,\lambda_2)}\}$. Moreover, by Lemma 3.1, for each (λ_1, λ_2) satisfying (1.11), the following holds:

$$-\infty < \inf_{\bar{B}_{r_{(\lambda_1,\lambda_2)}}} \psi(u,v) < 0 < \inf_{\partial B_{r_{(\lambda_1,\lambda_2)}}} \psi(u,v).$$

Set

$$\frac{1}{n} \in \left(0, \inf_{\partial B_{r_{(\lambda_1,\lambda_2)}}} \psi(u,v) - \inf_{\bar{B}_{r_{(\lambda_1,\lambda_2)}}} \psi(u,v)\right),\ n \in \mathbb{N}. \tag{3.11}$$

By the definition of an infimum, we obtain that there is a point $(u_n, v_n) \in \bar{B}_{r_{(\lambda_1,\lambda_2)}}$ satisfying

$$\psi(u_n, v_n) \le \inf_{\bar{B}_{r_{(\lambda_1,\lambda_2)}}} \psi(u,v) + \frac{1}{n}. \tag{3.12}$$

Since $\psi(u,v) \in C^1(W, \mathbb{R})$, $\psi(u,v)$ is lower semicontinuous. Hence, using Lemma 2.4, we get

$$\psi(u_n, v_n) \le \psi(u,v) + \frac{1}{n}\|(u,v) - (u_n, v_n)\|_W,\ \forall (u,v) \in \bar{B}_{r_{(\lambda_1,\lambda_2)}}.$$

Moreover, (3.11) and (3.12) imply that

$$\psi(u_n, v_n) \le \inf_{\bar{B}_{r_{(\lambda_1,\lambda_2)}}} \psi(u,v) + \frac{1}{n} < \inf_{\partial B_{r_{(\lambda_1,\lambda_2)}}} \psi(u,v);$$

thus, $(u_n, v_n) \in B_{r_{(\lambda_1,\lambda_2)}}$. Set $M_n : W \to \mathbb{R}$ as

$$M_n(u,v) = \psi(u,v) + \frac{1}{n}\|(u,v) - (u_n, v_n)\|_W.$$

Then, $(u_n, v_n) \in B_{r_{(\lambda_1,\lambda_2)}}$ is the minimum point of M_n on $\bar{B}_{r_{(\lambda_1,\lambda_2)}}$. Hence, for some $(u,v) \in W$ satisfying that $\|(u,v)\|_W = 1$, assume that $t > 0$ is small enough such that $(u_n + tu, v_n + tv) \in \bar{B}_{r_{(\lambda_1,\lambda_2)}}$. Then

$$\frac{M_n(u_n + tu, v_n + tv) - M_n(u_n, v_n)}{t} \ge 0. \tag{3.13}$$

By (3.13) and the definition of M_n, the following holds:

$$\langle \psi'(u_n, v_n), (u,v) \rangle \ge -\frac{1}{n}.$$

Similarly, if $t < 0$ and $|t|$ is small enough, then

$$\langle \psi'(u_n, v_n), (u,v) \rangle \le \frac{1}{n}.$$

Therefore,

$$\|\psi'(u_n, v_n)\| = \sup_{\|(u,v)\|_W = 1} |\langle \psi'(u_n, v_n), (u,v) \rangle| \le \frac{1}{n}. \tag{3.14}$$

Thus, (3.12) and (3.14) imply that

$$\psi(u_n, v_n) \to \inf_{\bar{B}_{r(\lambda_1,\lambda_2)}} \psi(u, v) \quad \text{and} \quad \|\psi'(u_n, v_n)\| \to 0 \quad \text{as } n \to \infty.$$

Hence, by using Lemma 3.3, we know that there exists a subsequence $\{(u_{n_k}, v_{n_k})\} \subset \{(u_n, v_n)\}$ satisfying that $(u_{n_k}, v_{n_k}) \to (u_0^\star, v_0^\star) \in \bar{B}_{r(\lambda_1,\lambda_2)}$ as $k \to \infty$, and that

$$\psi(u_0^\star, v_0^\star) = \inf_{\bar{B}_{r(\lambda_1,\lambda_2)}} \psi(u, v) < 0 \quad \text{and} \quad \psi'(u_0^\star, v_0^\star) = 0.$$

Hence, system (1.9) admits a nontrivial solution $(u_0^\star, v_0^\star)$ which has negative energy.

Proof of Theorem 1.2. For each $\lambda_1 > 0$, assume that $(u, 0)$ is a semi-trivial solution of system (1.9). Then, we have

$$\int_{\Omega\cup\partial\Omega} |\nabla^{m_1} u|^p d\mu = \lambda_1 \int_\Omega h_1(x)|u|^{\gamma_1} d\mu \le \lambda_1 H_1 \int_\Omega |u|^{\gamma_1} d\mu \le \lambda_1 H_1 C_{m_1,p}^{\gamma_1}(\Omega)\|u\|_{W_0^{m_1,p}(\Omega)}^{\gamma_1}.$$

Hence,

$$\|u\|_{W_0^{m_1,p}(\Omega)} \le \left(\lambda_1 H_1 C_{m_1,p}^{\gamma_1}(\Omega)\right)^{\frac{1}{p-\gamma_1}}.$$

Similarly, for each $\lambda_2 > 0$, if $(0, v)$ is a semi-trivial solution of system (1.9), we can also obtain

$$\|v\|_{W_0^{m_2,q}(\Omega)} \le \left(\lambda_2 H_2 C_{m_2,q}^{\gamma_2}(\Omega)\right)^{\frac{1}{q-\gamma_2}}.$$

4. Proofs for Theorem 1.3

In this section, we discuss the existence of a ground-state solution for (1.12) by using Lemma 2.5 and Theorem 3.3 in [4]. In [4], Brown and Wu researched the following operator equation basd on the fibering maps and the Nehari manifold:

$$A(u) - B(u) - C(u) = 0, \quad u \in S, \tag{4.1}$$

where S is a reflexive Banach space, $A, B, C : S \to S^*$ are homogeneous operators of degree $p-1, \alpha-1$ and $\gamma - 1$ with $1 < \gamma < p < \alpha$. The energy functional of (4.1) is

$$J(u) = \frac{1}{p}\langle A(u), u\rangle - \frac{1}{\alpha}\langle B(u), u\rangle - \frac{1}{\gamma}\langle C(u), u\rangle, \tag{4.2}$$

the fibering map is

$$G_u(t) = \frac{1}{p}\langle A(u), u\rangle t^p - \frac{1}{\alpha}\langle B(u), u\rangle t^\alpha - \frac{1}{\gamma}\langle C(u), u\rangle t^\gamma, \tag{4.3}$$

for all $t > 0$, and the Nehari manifold is

$$\mathcal{N} = \{u \in S \setminus \{0\} \big| \langle J'(u), u\rangle = 0\}.$$

Define

$$\phi(u) = \langle J'(u), u\rangle.$$

Then, $\mathcal{N}$ can be divided into the following three parts:

$$\mathcal{N}^+ = \{u \in \mathcal{N} \big| \langle \phi'(u), u\rangle > 0\},$$
$$\mathcal{N}^0 = \{u \in \mathcal{N} \big| \langle \phi'(u), u\rangle = 0\},$$
$$\mathcal{N}^- = \{u \in \mathcal{N} \big| \langle \phi'(u), u\rangle < 0\}.$$

In [4], Brown and Wu obtained the following results.

Lemma 4.1. *([4], Lemma 2.5) For any $u \in S$, when $\langle B(u), u\rangle > 0$ and $\langle C(u), u\rangle > 0$, there exist t_u^+ and t_u^- with $0 < t_u^+ < t_u^-$ such that $G_u(t)$ is increasing on the interval (t_u^+, t_u^-) and decreasing on the interval $(0, t_u^+)$ and interval $(t_u^-, +\infty)$.*

Lemma 4.2. *([4], Theorem 3.3) If the following conditions hold:*
(H_1) $u \to \langle A(u), u\rangle$ is a weakly lower semicontinuous function on S and there is a continuous function $\kappa : [0, +\infty) \to [0, +\infty)$ with $\kappa(s) > 0$ on $(0, +\infty)$ and $\lim_{s\to\infty} \kappa(s) = \infty$ such that for all $u \in S$, $\langle A(u), u\rangle \geq \kappa(\|u\|)\|u\|$;
(H_2) there exist $u_i \in S$, $i = 1, 2$ such that

$$\langle B(u_1), u_1\rangle > 0, \langle C(u_2), u_2\rangle > 0;$$

(H_3) B, C are strongly continuous;
(H_4) there exist two positive constants d_1, d_2 with

$$d_1^{\alpha-p} d_2^{p-\gamma} \leq (p-\gamma)^{p-\gamma}(\alpha-p)^{\alpha-p}(\alpha-\gamma)^{\gamma-\alpha},$$

such that

$$\langle B(u), u\rangle \leq d_1[\langle A(u), u\rangle]^{\frac{\gamma}{p}}, \tag{4.4}$$
$$\langle C(u_), u\rangle \leq d_2[\langle A(u), u\rangle]^{\frac{\alpha}{p}}, \tag{4.5}$$

then (4.1) admits at least two nontrivial solutions u_0^+ and u_0^-, where

$$u_0^+ \in \mathcal{N}^+, \ \ J(u_0^+) = \inf_{u\in\mathcal{N}^+} J(u),$$
$$u_0^- \in \mathcal{N}^-, \ \ J(u_0^-) = \inf_{u\in\mathcal{N}^-} J(u),$$

and $\mathcal{N}^0 = \emptyset$.

In the locally finite graph $G = (V, E)$ setting, let $S = W_0^{m,p}(\Omega)$ and

$$\langle A(u), u\rangle = \|u\|^p_{W_0^{m,p}(\Omega)}, \tag{4.6}$$
$$\langle B(u), u\rangle = \int_\Omega c(x)|u|^\alpha d\mu, \tag{4.7}$$
$$\langle C(u), u\rangle = \lambda \int_\Omega h(x)|u|^\gamma d\mu. \tag{4.8}$$

Similar to the arguments in [4], (H_1)–(H_3) hold with A, B and C respectively defined by (4.6)–(4.8). Note that

$$C_0 = \max_{x\in\Omega} c(x), \quad H_0 = \max_{x\in\Omega} h(x).$$

Lemma 2.1 implies that the following holds:

$$\int_\Omega c(x)|u|^\alpha d\mu \le C_0 C_{m,p}^\alpha(\Omega)\|u\|_{W_0^{m,p}(\Omega)}^\alpha \tag{4.9}$$

and

$$\lambda\int_\Omega h(x)|u|^\gamma d\mu \le \lambda H_0 C_{m,p}^\gamma(\Omega)\|u\|_{W_0^{m,p}(\Omega)}^\gamma. \tag{4.10}$$

Let

$$d_1 = C_0 C_{m,p}^\alpha(\Omega), \quad d_2 = \lambda H_0 C_{m,p}^\gamma(\Omega).$$

Then (4.4) and (4.5) hold. Moreover, note that

$$\lambda_0 = \frac{p-\gamma}{H_0} C_{m,p}^{-\gamma}(\Omega)\left(\left(C_0 C_{m,p}^\alpha(\Omega)\right)^{p-\alpha}(\alpha-p)^{\alpha-p}(\alpha-\gamma)^{\gamma-\alpha}\right)^{\frac{1}{p-\gamma}}.$$

Then, it is easy to see that (H_4) holds if $\lambda \in (0, \lambda_0)$. Thus, by Lemma 4.2, (1.12) admits at least two nontrivial solutions $u_0^+ \in \mathcal{N}^+$ and $u_0^- \in \mathcal{N}^-$, and one of them must be a ground-state solution. Next, we discuss which is the ground-state solution. Note that the energy functional of (1.12) is

$$J(u) = \frac{1}{p}\|u\|_{W_0^{m,p}(\Omega)}^p - \frac{\lambda}{\gamma}\int_\Omega h(x)|u|^\gamma d\mu - \frac{1}{\alpha}\int_\Omega c(x)|u|^\alpha d\mu, \quad \forall u \in W_0^{m,p}(\Omega), \tag{4.11}$$

and for each $u \in W_0^{m,p}(\Omega)\backslash\{0\}$, the corresponding fibering map is

$$G_u(t) = \frac{t^p}{p}\|u\|_{W_0^{m,p}(\Omega)}^p - \frac{\lambda}{\gamma}t^\gamma\int_\Omega h(x)|u|^\gamma d\mu - \frac{t^\alpha}{\alpha}\int_\Omega c(x)|u|^\alpha d\mu, \quad \forall t \in (0, +\infty). \tag{4.12}$$

We can obtain that $G_u(t)$ has positive values if $\lambda \in (0, \lambda_\star)$ where $\lambda_\star$ is defined by (1.13). In fact, we define

$$F_u(t) = \frac{t^p}{p}\|u\|_{W_0^{m,p}(\Omega)}^p - \frac{t^\alpha}{\alpha}\int_\Omega c(x)|u|^\alpha d\mu.$$

By (4.9), we have

$$\begin{aligned} \max_{t>0} F_u(t) &= F_u(t_{0u}) \qquad (4.13)\\ &= \frac{1}{p}\|u\|_{W_0^{m,p}(\Omega)}^p\left(\frac{\|u\|_{W_0^{m,p}(\Omega)}^p}{\int_\Omega c(x)|u|^\alpha d\mu}\right)^{\frac{p}{\alpha-p}} - \frac{1}{\alpha}\int_\Omega c(x)|u|^\alpha d\mu\left(\frac{\|u\|_{W_0^{m,p}(\Omega)}^p}{\int_\Omega c(x)|u|^\alpha d\mu}\right)^{\frac{\alpha}{\alpha-p}}\\ &= \left(\frac{1}{p}-\frac{1}{\alpha}\right)\left(\frac{\|u\|_{W_0^{m,p}(\Omega)}^\alpha}{\int_\Omega c(x)|u|^\alpha d\mu}\right)^{\frac{p}{\alpha-p}} \end{aligned}$$

$$\geq \quad \frac{\alpha - p}{p\alpha} \left(C_0 C_{m,p}^{\alpha}(\Omega) \right)^{\frac{p}{p-\alpha}} \tag{4.14}$$

with

$$t_{0u} = \left(\frac{\|u\|^p_{W_0^{m,p}(\Omega)}}{\int_\Omega c(x)|u|^\alpha d\mu} \right)^{\frac{1}{\alpha - p}}.$$

Furthermore,

$$\begin{aligned}
& \frac{\lambda}{\gamma} t_{0u}^{\gamma} \int_\Omega h(x)|u|^\gamma d\mu \\
\leq \quad & \frac{\lambda H_0}{\gamma} C_{m,p}^{\gamma}(\Omega) \|u\|^{\gamma}_{W_0^{m,p}(\Omega)} t_{0u}^{\gamma} \\
= \quad & \frac{\lambda H_0}{\gamma} C_{m,p}^{\gamma}(\Omega) \|u\|^{\gamma}_{W_0^{m,p}(\Omega)} \left(\frac{\|u\|^p_{W_0^{m,p}(\Omega)}}{\int_\Omega c(x)|u|^\alpha d\mu} \right)^{\frac{\gamma}{\alpha - p}} \\
= \quad & \frac{\lambda H_0}{\gamma} C_{m,p}^{\gamma}(\Omega) \left(\frac{\|u\|^\alpha_{W_0^{m,p}(\Omega)}}{\int_\Omega c(x)|u|^\alpha d\mu} \right)^{\frac{\gamma}{\alpha - p}} \\
= \quad & \frac{\lambda H_0}{\gamma} C_{m,p}^{\gamma}(\Omega) \left(\frac{p\alpha}{\alpha - p} F_u(t_{0u}) \right)^{\frac{\gamma}{p}}.
\end{aligned} \tag{4.15}$$

It follows that

$$\begin{aligned}
G_u(t_{0u}) \quad &= \quad F_u(t_{0u}) - \frac{\lambda}{\gamma} t_{0u}^{\gamma} \int_\Omega h(x)|u|^\gamma d\mu \\
&\geq \quad F_u^{\frac{\gamma}{p}}(t_{0u}) \left(F_u^{\frac{p-\gamma}{p}}(t_{0u}) - \lambda \frac{H_0}{\gamma} C_{m,p}^{\gamma}(\Omega) \left(\frac{p\alpha}{\alpha - p} \right)^{\frac{\gamma}{p}} \right).
\end{aligned} \tag{4.16}$$

Note that

$$\lambda_\star = \frac{\gamma(\alpha - p)}{p\alpha H_0 C_{m,p}^{\gamma}(\Omega)} \left(C_0 C_{m,p}^{\alpha}(\Omega) \right)^{\frac{p-\gamma}{p-\alpha}}.$$

Then, for all $u \in W_0^{m,p}(\Omega)\backslash\{0\}$, if $\lambda \in (0, \lambda_\star)$, it follows that

$$G_u(t_{0u}) > 0. \tag{4.17}$$

Moreover, for all $u \in \mathcal{N}^-$, the following holds:

$$G_u'(1) = \|u\|^p_{W_0^{m,p}(\Omega)} - \int_\Omega h(x)|u|^\gamma d\mu - \int_\Omega c(x)|u|^\alpha d\mu = \langle J'(u), u \rangle = 0,$$

and

$$\begin{aligned}
G_u''(1) \quad &= \quad (p-1)\|u\|^p_{W_0^{m,p}(\Omega)} - (\gamma - 1) \int_\Omega h(x)|u|^\gamma d\mu - (\alpha - 1) \int_\Omega c(x)|u|^\alpha d\mu \\
&= \quad p\|u\|^p_{W_0^{m,p}(\Omega)} - \gamma \int_\Omega h(x)|u|^\gamma d\mu - \alpha \int_\Omega c(x)|u|^\alpha d\mu - \left(\|u\|^p_{W_0^{m,p}(\Omega)} - \int_\Omega h(x)|u|^\gamma d\mu - \int_\Omega c(x)|u|^\alpha d\mu \right)
\end{aligned}$$

$$
\begin{aligned}
&= \langle \phi'(u), u\rangle - \langle J'(u), u\rangle \\
&< 0.
\end{aligned}
$$

Then $G_u(1)$ is a local maximum value of $G_u(t)$ on $(0, +\infty)$. It is easy to see that for any $u \in W_0^{m,p}(\Omega)\backslash\{0\}$, we have

$$
\begin{aligned}
\langle B(u), u\rangle &= \int_\Omega c(x)|u|^\alpha d\mu > 0, \\
\langle C(u), u\rangle &= \lambda \int_\Omega h(x)|u|^\gamma d\mu > 0.
\end{aligned}
$$

Then, by Lemma 4.1, for all $u \in \mathcal{N}^-$, there exist t_u^+ and t_u^- with $0 < t_u^+ < t_u^-$ such that $G_u(t)$ is increasing on the interval (t_u^+, t_u^-) and decreasing on the interval $(0, t_u^+)$ and interval $(t_u^-, +\infty)$, together with $G_u(0) = 0$ and (4.17), which implies that both the local maximum point $t_u = 1$ and t_{0u} belong to the interval $(t_u^+, +\infty)$. Thus, for each $u \in \mathcal{N}^-$, we have

$$
J(u) = G_u(1) \geq G_u(t_{0u}) > 0.
$$

Similarly, for each $u \in \mathcal{N}^+$, we know that $G_u(1)$ is a local minimum value of $G_u(t)$ on $(0, +\infty)$, which is located at $(0, t_u^-)$ by Lemma 4.1. Hence, for each $u \in \mathcal{N}^+$, we have

$$
J(u) = G_u(1) < G_u(0) = 0.
$$

Therefore, we conclude that (1.12) admits a nontrivial ground-state solution $u_0^+ \in \mathcal{N}^+$ if $\lambda \in (0, \lambda_{\star\star})$ where $\lambda_{\star\star} = \min\{\lambda_0, \lambda_\star\}$.

5. Some results on the finite graph

Assume that $G = (V, E)$ is a finite graph. Using similar arguments as for Theorems 1.1 and 1.2, we can obtain similar results for the following poly-Laplacian system on finite graph G:

$$
\begin{cases}
\pounds_{m_1,p}u + a(x)|u|^{p-2}u = \lambda_1 h_1(x)|u|^{\gamma_1-2}u + \frac{\alpha}{\alpha+\beta}c(x)|u|^{\alpha-2}u|v|^\beta, & x \in V, \\
\pounds_{m_2,q}v + b(x)|v|^{q-2}v = \lambda_2 h_2(x)|v|^{\gamma_2-2}v + \frac{\beta}{\alpha+\beta}c(x)|u|^\alpha|v|^{\beta-2}v, & x \in V,
\end{cases}
\tag{5.1}
$$

where $m_i, i = 1, 2$ are positive integers, $p, q, \gamma_1, \gamma_2 > 1$, $\lambda_1, \lambda_2, \alpha, \beta > 0$, $\max\{\gamma_1, \gamma_2\} < \min\{p, q\} \leq \max\{p, q\} < \alpha + \beta$, $a, b, h_1, h_2, c : V \to \mathbb{R}^+$. Moreover, similar to the arguments in Theorem 1.3, we can also obtain a similar result for the following equation:

$$
\pounds_{m,p}u + a(x)|u|^{p-2}u = \lambda h(x)|u|^{\gamma-2}u + c(x)|u|^{\alpha-2}u, \qquad x \in V, \tag{5.2}
$$

where m is a positive integer, $p, \gamma > 1$, $\lambda, \alpha > 0$, $\gamma < p < \alpha$, $a, h, c : V \to \mathbb{R}^+$. For any given m and s with $m \in \mathbb{N}^+$ and $s > 1$, the definition of $W^{m,s}(V)$ is similar to that of $W^{m,s}(\Omega)$, which changed the region from Ω to V; the norm is defined as follows:

$$
\|\psi\|_{W^{m,s}(V)} = \left(\int_V (|\nabla^m \psi(x)|^s + h(x)|\psi(x)|^s) d\mu\right)^{\frac{1}{s}}.
$$

Similarly, for any given $1 \le r < +\infty$, the definition of $L^r(V)$ is also similar to that of $L^r(\Omega)$, and the norm is defined as follows:

$$\|u\|_{L^r(V)} = \left(\int_V |u(x)|^r d\mu\right)^{\frac{1}{r}}.$$

For system (5.1), we work in the space of $W(V) = W^{m_1,p}(V) \times W^{m_2,q}(V)$, and for (5.2), we work in the space of $W^{m,p}(V)$. Both $W(V)$ and $W^{m,p}(V)$ are of finite dimension. See [15] for more details.

Denote

$$\begin{aligned} M_{(\lambda_1,\lambda_2)}(V) &= 2^{1-\max\{p,q\}} \min\left\{\frac{1-\lambda_1 C_p^p(V)}{p}, \frac{1-\lambda_2 C_q^q(V)}{q}\right\}, \\ M_2(V) &= \frac{C_0(V)}{(\alpha+\beta)^2}\left(\alpha C_p^{\alpha+\beta}(V) + \beta C_q^{\alpha+\beta}(V)\right), \end{aligned}$$

where $C_0(V) = \max_{x\in V} c(x)$ and $C_p(V)$ and $C_q(V)$ are embedding constants from $W^{m_1,p}(V)$ and $W^{m_2,q}(V)$ into $L^p(V)$ and $L^q(V)$, respectively, which have been obtained in [22] with

$$C_p(V) = \frac{\left(\sum_{x\in V}\mu(x)\right)^{\frac{1}{p}}}{\mu_{\min}^{\frac{1}{p}} h_{\min}^{\frac{1}{p}}} \text{ and } C_q(V) = \frac{\left(\sum_{x\in V}\mu(x)\right)^{\frac{1}{q}}}{\mu_{\min}^{\frac{1}{q}} h_{\min}^{\frac{1}{q}}}.$$

Next, we state the results similar to Theorems 1.1–1.3. Suppose that λ_1 and λ_2 satisfy the following inequalities:

$$\begin{cases} 0 < \lambda_1 < C_p^{-p}(V), \\ 0 < \lambda_2 < C_q^{-q}(V), \\ M_{(\lambda_1,\lambda_2)}(V) \le \frac{\alpha+\beta}{\max\{p,q\}} M_2(V), \\ \frac{\lambda_1(p-\gamma_1)}{p\gamma_1}\|h_1\|^{\frac{p}{p-\gamma_1}}_{L^{\frac{p}{p-\gamma_1}}(V)} + \frac{\lambda_2(q-\gamma_2)}{q\gamma_2}\|h_2\|^{\frac{q}{q-\gamma_2}}_{L^{\frac{q}{q-\gamma_2}}(V)} < \frac{\alpha+\beta-\max\{p,q\}}{\alpha+\beta} M_1^{\frac{\alpha+\beta}{\alpha+\beta-\max\{p,q\}}} \left(\frac{\max\{p,q\}}{(\alpha+\beta)M_2}\right)^{\frac{\max\{p,q\}}{\alpha+\beta-\max\{p,q\}}}. \end{cases} \tag{5.3}$$

Theorem 5.1. *Assume that $G = (V, E)$ is a finite graph. If (λ_1, λ_2) satisfies (5.3), then system (5.1) admits at least one nontrivial solution of positive energy and one nontrivial solution of negative energy.*

Theorem 5.2. *Assume that $G = (V, E)$ is a finite graph. For each $\lambda_1 > 0$, suppose that $(u, 0)$ is a semi-trivial solution of system (5.1). Then*

$$\|u\|_{W^{m_1,p}(V)} \le \left(\lambda_1 H_1(V) C_p^{\gamma_1}(V)\right)^{\frac{1}{p-\gamma_1}},$$

where $H_1(V) = \max_{x\in V} h_1(x)$. Similarly, for each $\lambda_2 > 0$, suppose that $(0, v)$ is a semi-trivial solution of system (5.1). It follows that

$$\|v\|_{W^{m_2,q}(V)} \le \left(\lambda_2 H_2(V) C_q^{\gamma_2}(V)\right)^{\frac{1}{q-\gamma_2}},$$

where $H_2(V) = \max_{x\in V} h_2(x)$.

Denote

$$\lambda_0(V) = \frac{p-\gamma}{H_0(V)} C_p^{-\gamma}(V)\left(\left(C_0(V)C_p^{\alpha}(V)\right)^{p-\alpha}(\alpha-p)^{\alpha-p}(\alpha-\gamma)^{\gamma-\alpha}\right)^{\frac{1}{p-\gamma}},$$

$$\lambda_\star(V) = \frac{\gamma(\alpha - p)}{p\alpha H_0 C_p^\gamma(V)} \left(C_0(V) C_p^\alpha(V)\right)^{\frac{p-\gamma}{p-\alpha}}, \quad \lambda_{\star\star}(V) = \min\{\lambda_0(V), \lambda_\star(V)\},$$

where $H_0(V) = \max_{x\in V} h(x)$ and $C_0(V) = \max_{x\in V} c(x)$.

Theorem 5.3. *Assume that $G = (V, E)$ is a finite graph. If $\lambda \in (0, \lambda_0(V))$, then (5.2) admits at least one nontrivial solution of positive energy and one nontrivial solution of negative energy. Furthermore, if $\lambda \in (0, \lambda_{\star\star}(V))$, the negative energy solution is the ground-state solution of (5.2).*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state no conflict of interest.

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