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Research article

Global existence, blow-up and mass concentration for the inhomogeneous nonlinear Schrödinger equation with inverse-square potential

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Abstract: In the current paper, the Cauchy problem for the inhomogeneous nonlinear Schrödinger equation including inverse-square potential is considered. First, some criteria of global existence and finite-time blow-up in the mass-critical and mass-supercritical settings with $0 < c \leq c^*$ are obtained. Then, by utilizing the potential well method and the sharp Sobolev constant, the sharp condition of blow-up is derived in the energy-critical case with $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$. Finally, we establish the mass concentration property of explosive solutions, as well as the dynamic behaviors of the minimal-mass blow-up solutions in the L^2-critical setting for $0 < c < c^*$, by means of the variational characterization of the ground-state solution to the elliptic equation, scaling techniques and a suitable refined compactness lemma. Our results generalize and supplement the ones of some previous works.

Keywords: inhomogeneous nonlinear Schrödinger equation; inverse-square potential; blow-up; global existence; mass concentration

1. Introduction

This article discusses the Cauchy problem for the following inhomogeneous nonlinear Schrödinger equation (NLS) with inverse-square potential:

$$\begin{cases} i\varphi_t = (-\Delta - c|x|^{-2})\varphi - K(x)|\varphi|^{p-2}\varphi, \ t > 0, \ x \in \mathbb{R}^N, \\ \varphi(0, x) = \varphi_0 \in H^1(\mathbb{R}^N), \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $0 < T \leq \infty$, $\varphi : [0, T) \times \mathbb{R}^N \to \mathbb{C}$, $K(x) \in C^1(\mathbb{R}^N, \mathbb{R})$, $2 < p \leq p^* = \frac{2N}{N-2}$ and $0 < c \leq c^*$, where $c^* = \frac{(N-2)^2}{4}$ represents the sharp constant of the following Hardy's inequality:

$$c^* \int_{\mathbb{R}^N} |x|^{-2}|\varphi|^2 dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2 dx, \ \varphi \in H^1(\mathbb{R}^N).$$

The model of inhomogeneous NLS (1.1) with inverse-square potential can be applied to a variety of physical environments, such as black hole solutions of Einstein's equations or quantum field equations (see e.g., [1–3]) and quantum gas theory (see e.g., [4–6]).

In the present work, our interest focuses on the optimal criteria of global existence and finite-time blow-up, as well as the L^2-concentration property and the dynamics of blow-up solutions for Eq (1.1), which are pursued in both mathematics and physics.

Before going further, we recall some existing results. When $c = 0$, Eq (1.1) becomes the following common inhomogeneous NLS:

$$i\varphi = -\Delta\varphi - K(x)|\varphi|^{p-2}\varphi, \tag{1.2}$$

which is widely used in fields such as quantum mechanics, nonlinear optics and Bose-Einstein condensation. In the past several decades, this kind of NLS has garnered a great deal of interest. Under the condition of $K(x) = 1$ in Eq (1.2), Weinstein [7] not only applied the ground-state solution to the scalar equation

$$\Delta\varphi - \varphi + |\varphi|^{\frac{4}{N}} = 0 \tag{1.3}$$

to establish the sharp criterion of global and blow-up solutions, he also showed the existence of the unstable standing wave solutions in the L^2-critical case ($p = 2 + \frac{4}{N}$). Merle, in [8], proved the mass concentration of explosive solutions and classified the minimal-mass blow-up solutions on non-radial data in the L^2-critical setting by utilizing the concentration lemma and pseudo-conformal transformation, together with the variational characterization of the ground-state solution to Eq (1.3). In [9], Hmidi and Keraani proposed a refined compactness lemma by applying the profile decomposition to bounded sequences in $H^1(\mathbb{R}^N)$. In view of the obtained compactness result, they also gave the simpler proofs to the concentration property of solutions blowing up in finite time, the limiting profile and determination of the minimal blow-up solutions to the homogeneous L^2-critical NLS.

Under the condition that $K(x) \neq constant$ and satisfies some appropriate assumptions, Merle [10] explored, in detail, the existence or nonexistence for the minimal-mass blow-up solutions, as well as the blow-up dynamical properties of solutions to Eq (1.2). Shu and Zhang [11] derived a sharp condition for the existence of a global solution to Eq (1.2) with the mass critical exponent $p = 2 + \frac{4}{N}$ by constructing some cross-invariant manifolds and variational problems. It is also worth noting that, for $p = 2 + \frac{4}{N-2}$ and $K(x) \neq constant$ satisfying $(A_1) - (A_2)$ (see Section 2), Liu [12] established a new sharp criterion for the explosive solutions to the H^1-critical inhomogeneous Eq (1.2) with $p = 2 + \frac{4}{N-2}$ for the non-radial case by using the potential well method. These studies are strongly dependent on the hypotheses imposed on the inhomogeneous coefficient $K(x)$. Then, a natural problem arises: Are these results valid for the inhomogeneous NLS (1.1) with inverse-square potential ($c \neq 0$) and variable coefficient $K(x) \neq constant$? It is one of our starting points to study problem (1.1) in the current article.

For the case that $c \neq 0$ and $K(x) = 1$, Eq (1.1) corresponds to the homogeneous NLS with inverse-square potential, which has been extensively discussed due to the singular property of inverse-square potential $c|x|^{-2}$. In [13, 14], the authors researched the global scattering and blow-up problems for the focusing and defocusing NLS in the intercritical and energy-critical settings, respectively. By making full use of the ground state to the elliptic equation

$$-\Delta\varphi - c|x|^{-2}\varphi - |\varphi|^{p-2}\varphi + \varphi = 0,\ \varphi \in H^1(\mathbb{R}^N), \tag{1.4}$$

Dinh [15] showed the finite-time blow-up and global existence of radially symmetric and non-radial solutions of Eq (1.1) in the mass-critical and mass-supercritical settings with $c < 0$ or $0 < c < c^*$, as well as the criterion of blow-up solutions for the energy-critical case with $c \neq 0$ and $c < \frac{N^2+4N}{(N+2)^2}c^*$, respectively. By replacing the power-type nonlinearity by $-(I_\alpha * |\varphi|^p)|\varphi|^{p-2}\varphi$ in Eq (1.1), in light of Dinh [15], Li [16] investigated the criteria of global existence versus blow-up for non-radial and radial solutions to the L^2-critical and L^2-supercritical Choquard equation in the cases of $0 < c < c^*$ and $c < 0$, respectively. Under the condition that $N \geq 3$, $p = 2 + \frac{4}{N}$ and $0 < c < c^*$ in Eq (1.1), Csobo and Genoud [17] showed that the ground-state solution to the mass critical Eq (1.4) exists, and they obtained an optimal condition for global existence. In addition, the ground-state solution of Eq (1.4) and pseudo-conformal transformation were applied to classify the explosive solutions with minimal-mass for Eq (1.1) in [17], and a detailed characterization of minimal blow-up solutions was given via the variational characteristic of the ground-state solution to Eq (1.4) and the concentration compactness principle related to the Hardy functional. For the critical case of $c = c^*$ in Eq (1.1), Mukherjee et al. [18] was the first to verify the existence and uniqueness of the ground-state solution to Eq (1.4) for $2 < p < p^*$, which was applied to establish the criteria of blow-up versus global existence dichotomy for the homogeneous Eq (1.1). Moreover, by taking advantage of the variational characterization to the ground-state solution of Eq (1.4) and the corresponding concentration compactness principle, they studied the mass concentration phenomenon of explosive solutions and gave a complete characterization of the minimal-mass blow-up solutions. Regarding the case that $p = 2 + \frac{4}{N}$ and $0 < c < c^*$, Bensouilah [19] proposed a refined compactness lemma related to the Hardy functional by using the profile decomposition technique in $H^1(\mathbb{R}^N)$, and they applied it to investigate the L^2-concentration property of solutions blowing up at time $0 < T < \infty$. Based on [17] and [19], Pan and Zhang [20] demonstrated the accurate L^2-concentration property for the explosive solutions with a minimal mass by using the variational characterization to the ground state of Eq (1.4) and the compactness lemma proposed by [19]. For $0 < c < c^*$ or $c < 0$ in Eq (1.1), the authors of [21, 22] studied the stability and strong instability of standing waves for different values of p. For the homogeneous NLS including combined nonlinearities and inverse-square potential, Xia [23], Cao [24] and Li and Zou [25] researched the global existence and scattering, the existence of stable standing waves or the existence and properties of normalized solutions, respectively. Regarding the standing waves, Goubet and Manoubi [26] researched the existence and orbital stability of standing waves for a class of NLS involving a discontinuous dispersion. Zuo et al. [27] applied a variational approach based on the scaling function method to investigate the existence of normalized solutions for a kind of fractional NLS with bounded parametric potential; the solutions have attracted widespread attention due to their important applications in many settings, such as physics.

For $K(x) = \lambda|x|^{-b}$ with $\lambda = \pm 1$, $0 < b < 2$ and $0 < c < c^*$, Campos and Guzmán [28] and An et al. [29] considered the blow-up and global solutions to Eq (1.1). Pan and Zhang [30] studied the existence of a ground-state solution to the corresponding L^2-critical inhomogeneous elliptic equation with $K(x) = |x|^{-b}$, and they proved the uniqueness of the minimal-mass blow-up solutions by using the concentration compactness principle, with respect to the Hardy functional and inhomogeneous nonlinearity, as well as variational characterization to the ground state. To the best of our knowledge, there is no literature concerning the inhomogeneous Eq (1.1) that includes inverse-square potential and the general C^1 variable coefficient $K(x)$, which has significant differences with the cases of [15, 17–20]. Therefore, it is particularly meaningful for us to study Eq (1.1).

Motivated by the works mentioned above, the aims of this paper are to gain the criteria for the existence of global or finite-time blow-up solutions to Eq (1.1) in the L^2-critical, L^2-supercritical cases and the sharp criterion of blow-up in the energy-critical case, as well as the blow-up dynamics of solutions in finite time. The main difficulty stems from the presence of inverse-square potential $c|x|^{-2}$ and variable coefficient $K(x)$, leading to the loss of pseudo-conformal symmetry and scaling invariance, which play a vital role in the research on the blow-up dynamics; see [9, 18] for example. To overcome the difficulty, we utilize the unique ground-state solution of Eq (3.1) to establish the global existence and blow-up results to the mass-critical and mass-supercritical inhomogeneous NLS and apply the ground state to characterize the blow-up dynamic behavior of solutions to Eq (1.1). To be more precise, enlightened by [15, 18], we first obtain some sharp thresholds for global existence and finite-time blow-up in the L^2-critical and L^2-supercritical cases for $0 < c \leq c^*$ in terms of the ground state for Eq (3.1). It is worth mentioning that, in this work, for $0 < c < c^*$ and $p = 2 + \frac{4}{N}$, the argument regarding the existence of explosive solutions is established through scaling techniques, which differs from the methods of [15, 18]. Then, under the assumptions $(A_1) - (A_2)$ on the inhomogeneous coefficient $K(x)$ (see Section 2), following the ideas of [12] and [15], we derive the sharp criterion of blow-up in the energy-critical case with $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$ by utilizing the potential well method and the sharp Sobolev constant. Finally, in light of [10, 19, 20], we establish the mass concentration property of explosive solutions, as well as the dynamic behaviors of the minimal-mass blow-up solutions in the L^2-critical setting for $0 < c < c^*$. The main ingredients we use in the proofs of the dynamics are the variational characterization of the ground state for Eq (3.1), scaling techniques and a refined compactness lemma proposed by Bensouilah [19]. Our results generalize and supplement the results of [12, 15, 18–20].

The remaining parts of the present article is structured as follows. Section 2 gives some notations and important hypotheses, as well as some useful lemmas. Section 3 considers the criteria for the global existence and finite-time blow-up of Eq (1.1) in the L^2-critical and L^2-supercritical cases, as well as the sharp blow-up criterion in the energy-critical case, respectively. The last section focuses on the blow-up dynamics of solutions in the L^2-critical setting with $0 < c < c^*$.

2. Notations and preliminaries

To simplify the symbols, we use the abbreviation $\int \cdot \, dx$ to represent $\int_{\mathbb{R}^N} \cdot \, dx$, and we denote $\|\cdot\|_{L^p}$ $(1 \leq p < \infty)$ by $\|\cdot\|_p$. C represents a positive constant, which may vary from line to line.

Hereafter, we assume that the inhomogeneous coefficient $K(x)$ satisfies some of the following hypotheses: there exist $K_2 \geq K_1 > 0$ such that

(A_1) $\forall x \in \mathbb{R}^N,\ K_1 = inf_{x\in\mathbb{R}^N} K(x) \leq K(x) \leq sup_{x\in\mathbb{R}^N} K(x) = K_2 < \infty;$

(A_2) $\forall x \in \mathbb{R}^N,\ x \cdot \nabla K(x) \leq 0$ *and* $|x \cdot \nabla K(x)| \leq C;$

(A_3) *there is* $x_0 \in \mathbb{R}^N$ *satisfying that* $K(x_0) = K_2$.

In accordance with Dinh [15] and Okazawa et al. [31], we have the following argument regarding the local well-posedness of solutions to Eq (1.1).

Proposition 2.1. *Let* $\varphi_0 \in H^1(\mathbb{R}^N)$, $2 < p < p^*$ *and* $0 < c \leq c^*$ *or* $p = p^*$ *and* $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$, *and assume that* (A_1) *holds ; then, for* $T \in (0, \infty]$ *(maximal existence time), the unique solution* $\varphi(t, x) \in C([0, T), H^1(\mathbb{R}^N))$ *for Eq (1.1) exists. Meanwhile, one has the alternative* $T = \infty$ *(global existence),* *or else* $T < \infty$ *and* $\lim_{t\to T}\|\varphi(t,x)\|_{H^1(\mathbb{R}^N)} = \infty$ *(blow-up). Furthermore, for all* $t \in [0, T)$*, the solution* $\varphi(t, x)$ *possesses the conserved quantities of mass and energy as shown below:*

$$\int |\varphi(t,x)|^2 dx = \int |\varphi_0|^2 dx, \tag{2.1}$$

$$E(\varphi_0) = E(\varphi(t,x)) = \frac{1}{2}\int |\nabla\varphi|^2 dx - \frac{c}{2}\int |x|^{-2}|\varphi|^2 dx - \frac{1}{p}\int K(x)|\varphi|^p dx. \tag{2.2}$$

Define the Hardy functional as below:

$$H(\varphi) = \int |\nabla\varphi|^2 dx - c\int |x|^{-2}|\varphi|^2 dx,$$

which is of great importance in analyzing the dynamical properties for blow-up solutions. Taking account of the hypothesis on c, the semi-norm defined by $H(\varphi)$ on $H^1(\mathbb{R}^N)$ is equivalent to $\|\nabla\varphi\|_2^2$. Thus, the solution $\varphi(t, x)$ to Eq (1.1) blows up at $T > 0$ if and only if $\lim_{t\to T} H(\varphi) = \infty$.

To go further, we review some useful lemmas.

Lemma 2.2. *(Hardy-Gagliardo-Nirenberg inequality ([18, 32])) Let* $N \geq 3$, $0 < c \leq c^*$ *and* $2 < p < p^*$*. Then, we obtain*

$$\|\varphi\|_p \leq \frac{1}{C_{HGN}} H(\varphi)^{\frac{\theta}{2}} \|\varphi\|_2^{1-\theta},\ \theta = \frac{N}{2} - \frac{N}{p}, \tag{2.3}$$

for all $\varphi \in H^1(\mathbb{R}^N)$*, with the sharp constant*

$$C_{HGN} = \|Q(x)\|_2^{\frac{p-2}{p}} (1-\theta)^{\frac{1}{p}} (\frac{\theta}{1-\theta})^{\frac{N(p-2)}{4p}},$$

where $Q(x)$ *is the unique radial positive solution to the elliptic equation (1.4).*

Lemma 2.3. *([7]) Assume that* $\varphi \in H^1(\mathbb{R}^N)$*; then, we get*

$$\int |\varphi|^2 dx \leq \frac{2}{N}\Big(\int |\nabla\varphi|^2 dx\Big)^{\frac{1}{2}}\Big(\int |x|^2|\varphi|^2 dx\Big)^{\frac{1}{2}}.$$

Lemma 2.4. *(Sharp Sobolev embedding ([33])) Let* $N \geq 3$ *and* $0 < c < c^*$*; then, we have*

$$\|f\|_{p^*} \leq C_{SE}(c) H(f)^{\frac{1}{2}},$$

where the sharp Sobolev constant $C_{SE}(c)$ *is as follows:*

$$C_{SE}(c) = \sup\{\|f\|_{p^*} \div H(f)^{\frac{1}{2}} : f \in H^1(\mathbb{R}^N)\}. \tag{2.4}$$

3. Global existence and blow-up of the solutions to Eq (1.1)

In the current section, we are devoted to researching the criteria for the global existence and finite-time blow-up to Eq (1.1) in the L^2-critical and L^2-supercritical settings, as well as the sharp threshold of blow-up in the energy-critical case, respectively. As we know, the ground state has a crucial role in the criteria for blow-up versus global existence, and it is the unique positive radially symmetric solution of the following elliptic equation with power nonlinearity and inverse-square potential:

$$-\Delta\varphi - c|x|^{-2}\varphi - K_2|\varphi|^{p-2}\varphi + \varphi = 0,\ \varphi \in H^1(\mathbb{R}^N), \tag{3.1}$$

where K_2 is same as the hypotheses $(A_1) - (A_3)$. Furthermore, we will apply the ground-state solution of Eq (3.1) to characterize the dynamical properties of explosive solutions in the next section.

Assume that $Q(x)$ is the unique radial positive solution to Eq (1.4); by the scaling transformation $Q_{K_2}(x) = K_2^{-\frac{1}{p-2}}Q(x)$, it is easy to obtain that $Q_{K_2}(x)$ is the positive ground state of Eq (3.1). Combining Eqs (1.4) and (3.1), we immediately get the scaling identity and Pohožaev identities as follows:

$$\|Q_{K_2}\|_2^2 = K_2^{-\frac{2}{p-2}}\|Q\|_2^2, \tag{3.2}$$

$$\|Q_{K_2}\|_2^2 = \frac{4-(N-2)(p-2)}{N(p-2)}H(Q_{K_2}),\ \|Q_{K_2}\|_p^p = \frac{2p}{NK_2(p-2)}H(Q_{K_2}). \tag{3.3}$$

Define the following functionals:

$$E_{K_2}(\varphi) = \frac{1}{2}H(\varphi) - \frac{K_2}{p}\|\varphi\|_p^p,$$

$$L(c) = E_{K_2}(Q_{K_2})\|Q_{K_2}\|_2^{2\sigma},\ \ G(c) = H(Q_{K_2})^{\frac{1}{2}}\|Q_{K_2}\|_2^{\sigma}, \tag{3.4}$$

where $\sigma = \frac{2N+2p-Np}{Np-2N-4}$ when $2+\frac{4}{N} < p < 2+\frac{4}{N-2}$. From Eqs (2.3), (3.2) and (3.3), one has

$$\begin{aligned} C_{HGN}^p &= \frac{2p-Np+2N}{2p}K_2\|Q_{K_2}\|_2^{p-2}\Big(\frac{Np-2N}{2p-Np+2N}\Big)^{\frac{N(p-2)}{4}} \\ &= \frac{N(p-2)}{2p}K_2H(Q_{K_2})^{\frac{p-2}{2}}\Big[\frac{4-(N-2)(p-2)}{N(p-2)}\Big]^{\frac{2p+2N-Np}{4}} \\ &= K_2^{\frac{p}{2}}\|Q_{K_2}\|_p^{\frac{p(p-2)}{2}}\frac{[4-(N-2)(p-2)]^{\frac{2p+2N-Np}{4}}[N(p-2)]^{\frac{Np-2N}{4}}}{(2p)^{\frac{p}{2}}} \end{aligned} \tag{3.5}$$

and

$$E_{K_2}(Q_{K_2}) = \frac{Np-2N-4}{2[4-(N-2)(p-2)]}\|Q_{K_2}\|_2^2 = \frac{Np-2N-4}{2N(p-2)}H(Q_{K_2}). \tag{3.6}$$

It follows from Eqs (3.2)–(3.6) that

$$L(c) = \frac{Np-2N-4}{2(Np-2N)}\Big[\frac{2pC_{HGN}^p}{(Np-2N)K_2}\Big]^{\frac{4}{Np-2N-4}}$$

and

$$G(c) = \Big[\frac{2pC_{HGN}^p}{(Np-2N)K_2}\Big]^{\frac{2}{Np-2N-4}}. \tag{3.7}$$

In particular, we have

$$L(c) = \frac{Np-2N-4}{2(Np-2N)}G^2(c). \tag{3.8}$$

Now, we consider the virial-type identities, which play a key role in the research of the existence of explosive solutions to Eq (1.1). Let

$$\Sigma = \{\varphi \in H^1(\mathbb{R}^N) : x\varphi \in L^2(\mathbb{R}^N)\},$$

and for $\varphi(t, x) \in \Sigma$, we introduce the variance functional

$$V(t) = \int |x|^2|\varphi(t, x)|^2 dx.$$

Then, we are able to derive the following conclusion.

Proposition 3.1. *Assume that* $2 < p < p^*$ *and* $0 < c \leq c^*$ *or* $p = p^*$ *and* $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$, *and let* $\varphi(t, x)$ *be a solution of problem (1.1) on* $t \in [0, T)$. *If* $\varphi_0 \in H^1(\mathbb{R}^N)$ *and* $|x|\varphi_0 \in L^2(\mathbb{R}^N)$, *then* $\varphi(t, x) \in \Sigma$ *for any* $t \in [0, T)$ *and* $V(t)$ *satisfies the following identities:*

$$V'(t) = 4Im \int x\nabla\varphi\bar{\varphi}dx$$

and

$$\begin{aligned} V''(t) &= 8H(\varphi) + \frac{8N - 4Np}{p}\int K(x)|\varphi|^p dx + \frac{8}{p}\int |\varphi|^p \cdot x \cdot \nabla K(x)dx \\ &= 16E(\varphi_0) + \frac{4(4 + 2N - Np)}{p}\int K(x)|\varphi|^p dx + \frac{8}{p}\int |\varphi|^p \cdot x \cdot \nabla K(x)dx. \end{aligned} \tag{3.9}$$

Proof. Based on the work of Csobo and Genoud [17] (see also Cazenave [34]), by a formal computation, it is easy to obtain that

$$\begin{aligned} V'(t) &= 2Re\int |x|^2\bar{\varphi}\varphi_t dx \\ &= 2Re(-i)\int |x|^2\bar{\varphi}(-\Delta\varphi - c|x|^{-2}\varphi - K(x)|\varphi|^{p-2}\varphi)dx \\ &= 4Im\int x\nabla\varphi\bar{\varphi}dx \end{aligned}$$

and

$$\begin{aligned} V''(t) &= 4Im\frac{d}{dt}\int x\nabla\varphi\bar{\varphi}dx \\ &= 4(-Im\int N\varphi_t\bar{\varphi}dx + 2Im\int x\nabla\varphi\bar{\varphi}_t dx) = 4(I_1 + I_2), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} I_1 &= -Im\int N\bar{\varphi}\varphi_t dx = NRe\int \bar{\varphi}(-\Delta\varphi - c|x|^{-2}\varphi - K(x)|\varphi|^{p-2}\varphi)dx \\ &= N\int (|\nabla\varphi|^2 - c|x|^{-2}|\varphi|^2 - K(x)|\varphi|^p)dx, \end{aligned} \tag{3.11}$$

$$\begin{aligned} I_2 &= 2Im\int x\nabla\varphi\bar{\varphi}_t dx = -2Im\int x\nabla\bar{\varphi}\varphi_t dx \\ &= 2Re\int x\nabla\bar{\varphi}(-\Delta\varphi - c|x|^{-2}\varphi - K(x)|\varphi|^{p-2}\varphi)dx \\ &= -(N-2)\int|\nabla\varphi|^2dx - (2-N)\int c|x|^{-2}|\varphi|^2dx \\ &\quad + \frac{2N}{p}\int K(x)|\varphi|^p dx + \frac{2}{p}\int|\varphi|^p\cdot x\cdot\nabla K(x)dx. \end{aligned} \tag{3.12}$$

From Eqs (3.10)–(3.12), we claim that Eq (3.9) holds.

From Proposition 3.1 and Lemma 2.3, we get the following sufficient conditions for blow-up.

Corollary 3.2. *Assume that $2+\frac{4}{N}\le p\le p^*$ and (A_1) and (A_2) hold; if $\varphi_0\in\Sigma$ and φ_0 meets one of the three conditions below:*
Case 1): $E(\varphi_0)<0$;
Case 2): $E(\varphi_0)=0$ and $Im\int x\nabla\varphi_0\bar{\varphi}_0dx<0$;
Case 3): $E(\varphi_0)>0$ and $Im\int x\nabla\varphi_0\bar{\varphi}_0dx+(2V(0)E(\varphi_0))^{\frac{1}{2}}\le 0$,
then blow-up of the solution $\varphi(t,x)$ to Eq (1.1) occurs in finite time.

Proof. Since $K(x)$ satisfies (A_1) and (A_2), using Proposition 3.1, we have

$$V''(t)\le 16E(\varphi_0).$$

Thus

$$\begin{aligned} 0\le V(t) &= V(0)+V'(0)t+\int_0^t(t-s)V''(s)ds \\ &\le V(0)+V'(0)t+8E(\varphi_0)t^2. \end{aligned}$$

Then, for each of the cases 1, 2 or 3, one can deduce that $0<T<+\infty$ must exist and satisfy

$$\lim_{t\to T^-}V(t)=\lim_{t\to T^-}\int|x|^2|\varphi(t)|^2dx=0.$$

This, together with Lemma 2.3, implies that

$$\lim_{t\to T^-}\|\nabla\varphi(t)\|_2=+\infty.$$

Therefore, the explosion of the solution $\varphi(t,x)$ to Eq (1.1) happens in the time period of $0<T<+\infty$.

3.1. *L^2-critical case*

The first argument of our work is about the global existence and blow-up of Eq (1.1) in the L^2-critical setting (i.e., $p=2+\frac{4}{N}$).

Theorem 3.3. *Assume that $p = 2 + \frac{4}{N}$ and $\varphi_0 \in H^1(\mathbb{R}^N)$. Let $Q_{K_2}(x)$ be the positive ground state of Eq (3.1).*
(i) Global existence: Assume that (A_1) holds true. If $0 < c \le c^*$ and $\|\varphi_0\|_2 < \|Q_{K_2}\|_2$, then the solution $\varphi(t,x)$ of Eq (1.1) exists globally in $t \in [0,+\infty)$.
(ii) Blow-up: Assume that (A_1) and (A_2) hold.
(a) Then, for $0 < c < c^*$, any $\lambda > 0$ and any real constant C_1 satisfying that $|C_1| \ge (\frac{K_2}{K_1})^{\frac{N}{4}} \ge 1$, there exists $\varphi_0 = C_1\lambda^{\frac{N}{2}}Q_{K_2}(\lambda x) \in \Sigma$ such that

$$\|\varphi_0\|_2^2 \ge \|Q_{K_2}\|_2^2,$$

and blow-up of the corresponding solution $\varphi(t,x)$ to problem (1.1) occurs in $0 < T < +\infty$.
(b) For $c = c^*$, if $E(\varphi_0) < 0$ and $|x|\varphi_0 \in L^2(\mathbb{R}^N)$, then the blow-up of the solution $\varphi(t,x)$ for Eq (1.1) happens in the time period of $0 < T < +\infty$.

Proof. (i) Since $p = 2 + \frac{4}{N}$, we have that $p\theta = 2$ and $pC_{HGN}^p = 2\|Q\|_2^{\frac{4}{N}}$. Thus, from Eq (2.2), (A_1), Eqs (2.3) and (2.1), we have the following estimate:

$$\begin{aligned} E(\varphi_0) = E(\varphi) &\ge \frac{1}{2}H(\varphi) - \frac{K_2}{2\|Q\|_2^{\frac{4}{N}}}H(\varphi)^{\frac{p\theta}{2}}\|\varphi_0\|_2^{p(1-\theta)} \\ &= \frac{1}{2}H(\varphi) - \frac{K_2}{2}H(\varphi)(\frac{\|\varphi_0\|_2}{\|Q\|_2})^{\frac{4}{N}} \\ &= \frac{1}{2}H(\varphi) - \frac{1}{2}H(\varphi)(\frac{\|\varphi_0\|_2}{\|Q_{K_2}\|_2})^{\frac{4}{N}} \\ &= \frac{1}{2}(1 - \frac{\|\varphi_0\|_2}{\|Q_{K_2}\|_2})^{\frac{4}{N}}H(\varphi). \end{aligned}$$

Since $\|\varphi_0\|_2 < \|Q_{K_2}\|_2$, $H(\varphi)$ is bounded uniformly for $t \in [0,+\infty)$. From Proposition 2.1, we claim that $\varphi(t,x)$ must be a global solution.

Next, we prove part (ii) of Theorem 3.3. For $0 < c < c^*$, let

$$\varphi_0 = C_1\lambda^{\frac{N}{2}}Q_{K_2}(\lambda x)$$

for any $\lambda > 0$, and $C_1 \in \mathbb{R}$ will be determined later. Based on the scaling arguments, one has that

$$\int |\varphi_0|^2 dx = |C_1|^2 \int Q_{K_2}^2 dx; \tag{3.13}$$

$$\int |\nabla\varphi_0|^2 dx = |C_1|^2\lambda^2 \int |\nabla Q_{K_2}|^2 dx; \tag{3.14}$$

$$\int |\varphi_0|^p dx = |C_1|^p\lambda^{\frac{N(p-2)}{2}} \int Q_{K_2}^p dx; \tag{3.15}$$

$$\int |x|^{-2}|\varphi_0|^2 dx = |C_1|^2\lambda^2 \int |x|^{-2}Q_{K_2}^2 dx. \tag{3.16}$$

Take

$$|C_1| \ge \left(\frac{K_2}{K_1}\right)^{\frac{N}{4}} \ge 1;$$

then, we have that $\varphi_0 \in H^1(\mathbb{R}^N)$ and $|x|\varphi_0 \in L^2(\mathbb{R}^N)$. Indeed, according to Bensouilah-Dinh-Zhu's conclusion in [21], one has that $Q(x) \in H^1(\mathbb{R}^N)$ and $|x|Q(x) \in L^2(\mathbb{R}^N)$. Thus, we obtain that $Q_{K_2}(x) \in H^1(\mathbb{R}^N)$ and $|x|Q_{K_2}(x) \in L^2(\mathbb{R}^N)$, which yields that $\varphi_0 = C_1\lambda^{\frac{N}{2}}Q_{K_2}(\lambda x) \in \Sigma$. Moreover, it follows from Eq (3.13) that

$$\|\varphi_0\|_2^2 = |C_1|^2\|Q_{K_2}\|_2^2 \geq \left(\frac{K_2}{K_1}\right)^{\frac{N}{2}}\|Q_{K_2}\|_2^2 \geq \|Q_{K_2}\|_2^2.$$

From Eq (2.2), (A_1), Eqs (3.14)–(3.16) and the Pohožaev identities given by Eq (3.3), we have

$$\begin{aligned} E(\varphi_0) &= \frac{1}{2}H(\varphi_0) - \frac{1}{p}\int K(x)|\varphi_0|^p dx \\ &\leq \frac{1}{2}H(\varphi_0) - \frac{K_1}{p}\int |\varphi_0|^p dx \\ &= \frac{1}{2}|C_1|^2\lambda^2 H(Q_{K_2}) - \frac{K_1}{p}C_1^p\lambda^{\frac{N(p-2)}{2}}\int Q_{K_2}^p dx \\ &= \frac{1}{2}|C_1|^2\lambda^2\Big(1 - \frac{K_1}{K_2}C_1^{p-2}\lambda^{\frac{N(p-2)}{2}-2}\Big)H(Q_{K_2}) \\ &< 0, \end{aligned}$$

where the last inequality is based on the fact that $p = 2 + \frac{4}{N}$ and $|C_1| \geq (\frac{K_2}{K_1})^{\frac{N}{4}} \geq 1$. Thus, we infer from Corollary 3.2 that blow-up of the corresponding solution $\varphi(t,x)$ to problem (1.1) occurs in $0 < T < +\infty$.

For $c = c^*$, if $E(\varphi_0) < 0$ and $|x|\varphi_0 \in L^2(\mathbb{R}^N)$, then, by Corollary 3.2, it is easy to derive that blow-up of the solution $\varphi(t,x)$ for Eq (1.1) happens in the time period of $0 < T < +\infty$.

3.2. *L^2-supercritical case*

Then, we analyze the L^2-supercritical case (i.e., $2 + \frac{4}{N} < p < p^* = 2 + \frac{4}{N-2}$). For this case, we obtain the threshold for global existence as shown below.

Theorem 3.4. *Suppose that $N \geq 3$, $0 < c \leq c^*$ and $2 + \frac{4}{N} < p < p*$. Let $\varphi_0 \in H^1(\mathbb{R}^N)$ and $\varphi(t,x)$ be the corresponding solution of Eq (1.1). Suppose that*

$$E(\varphi_0)\|\varphi_0\|_2^{2\sigma} < L(c). \tag{3.17}$$

(i) Global existence: Assume that (A_1) holds. If

$$H(\varphi_0)^{\frac{1}{2}}\|\varphi_0\|_2^{\sigma} < G(c), \tag{3.18}$$

then the global solution $\varphi(t,x)$ to Eq (1.1) exists. In addition,

$$H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma} < G(c) \ for\ any\ t > 0.$$

(ii) Blow-up: Assume that $(A_1) - (A_2)$ hold. If

$$H(\varphi_0)^{\frac{1}{2}}\|\varphi_0\|_2^{\sigma} > G(c) \tag{3.19}$$

and $|x|\varphi_0 \in L^2(\mathbb{R}^N)$, then the finite-time blow-up solution $\varphi(t,x)$ of Eq (1.1) exists and

$$H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma} > G(c) \tag{3.20}$$

for any $t \in [0,T)$. Furthermore, the finite-time blow-up result still holds true if we assume that $E(\varphi_0) < 0$, in place of Eqs (3.17) and (3.19).

Proof. (i) From Eq (2.2), (A_1) and Eq (2.3), one has

$$\begin{aligned}
E(\varphi_0)\|\varphi_0\|_2^{2\sigma} &= E(\varphi)\|\varphi\|_2^{2\sigma} \\
&\geq \frac{1}{2}(H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma})^2 - \frac{K_2}{p}\|\varphi\|_p^p\|\varphi\|_2^{2\sigma} \\
&\geq \frac{1}{2}(H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma})^2 - \frac{K_2}{pC_{HGN}^p}H(\varphi)^{\frac{Np-2N}{4}}\|\varphi\|_2^{\frac{2p+2N-Np}{2}}\|\varphi\|_2^{2\sigma} \\
&= \frac{1}{2}(H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma})^2 - \frac{K_2}{pC_{HGN}^p}(H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma})^{\frac{Np-2N}{2}} \\
&= f(H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma}),
\end{aligned}$$

where

$$f(x) = \frac{1}{2}x^2 - \frac{K_2}{pC_{HGN}^p}x^{\frac{Np-2N}{2}}. \tag{3.21}$$

Enlightened by the idea of [35] (the function f in [35] differs from ours; however, this change does not make a significant difference), we will utilize an important fact that f strictly increases in $[0, G(c)]$ and strictly decreases in $[G(c), \infty)$. Moreover, from Eqs (3.21), (3.7) and (3.8), we get

$$f(G(c)) = \frac{1}{2}G^2(c) - \frac{K_2}{pC_{HGN}^p}G^{\frac{Np-2N}{2}}(c) = (\frac{1}{2} - \frac{2}{Np-2N})G^2(c) = L(c). \tag{3.22}$$

Thus, combining this with Eqs (3.17), (3.21) and (3.22), we have

$$f(H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma}) \leq E(\varphi_0)\|\varphi_0\|_2^{2\sigma} < L(c) = f(G(c)). \tag{3.23}$$

From the above inequality, Eq (3.18) and the continuity argument, we deduce that

$$H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma} < G(c) \; for\ any\ t > 0.$$

From Eq (2.1), we obtain the boundedness of $H(\varphi)^{\frac{1}{2}}$, which implies that the solution $\varphi(t, x)$ of Eq (1.1) exists globally.

We next treat part (ii) of Theorem 3.4. For the case that $E(\varphi_0) \geq 0$, we claim that Eq (3.20) holds. Indeed, from Eqs (3.23) and (3.19) and the continuity argument, one has

$$H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma} > G(c) \; for\ any\ t < T,$$

which means that Eq (3.20) holds true.

On the other hand, from Eq (3.17) and the continuity argument, we can take $\delta > 0$ small enough such that

$$E(\varphi_0)\|\varphi_0\|_2^{2\sigma} \leq (1-\delta)L(c), \tag{3.24}$$

which yields that

$$f(H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma}) \leq (1-\delta)L(c). \tag{3.25}$$

Applying Eqs (3.21), (3.7) and (3.8) to Eq (3.25), one has that

$$\frac{Np-2N}{Np-2N-4}\Big(\frac{H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma}}{G(c)}\Big)^2 - \frac{4}{Np-2N-4}\Big(\frac{H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma}}{G(c)}\Big)^{\frac{Np-2N}{2}} \leq 1-\delta.$$

By making use of the continuity argument again, we deduce from Eq (3.19) that there exists $\delta' > 0$ that relies upon δ satisfying

$$H(\varphi)^{\frac{1}{2}}\|\varphi\|_2^{\sigma} \geq (1+\delta')G(c). \tag{3.26}$$

Moreover, taking

$$LHS = 8H(\varphi) + \frac{8N-4Np}{p}\int K(x)|\varphi|^p dx + \varepsilon,$$

then, for any $t \in [0, T)$, we claim that

$$LHS < -C < 0$$

for $\varepsilon > 0$ small enough. In fact, multiplying LHS by $\|\varphi\|_2^{2\sigma}$, we obtain

$$LHS \times \|\varphi\|_2^{2\sigma} = (4pN - 8N)E(\varphi)\|\varphi\|_2^{2\sigma} + (8 + 4N - 2Np + \varepsilon)H(\varphi)\|\varphi\|_2^{2\sigma}.$$

Utilizing Eqs (2.1), (2.2), (3.24), (3.26) and (3.8), one has

$$\begin{aligned}
LHS \times \|\varphi_0\|_2^{2\sigma} &\leq (4pN - 8N)(1-\delta)L(c) + (8 + 4N - 2Np + \varepsilon)(1+\delta')^2 G^2(c)\\
&= [2(Np - 2N - 4)(1-\delta) + (8 + 4N - 2Np + \varepsilon)(1+\delta')^2]G^2(c)\\
&= [2(Np - 2N - 4)(1 - \delta - (1+\delta')^2) + \varepsilon(1+\delta')^2]G^2(c).
\end{aligned}$$

We readily obtain $LHS \leq -C < 0$ by taking $\varepsilon > 0$ small enough. Then, it follows from the virial identity (3.9) and (A_2) that

$$V''(t) = 8H(\varphi) + \frac{8N-4Np}{p}\int K(x)|\varphi|^p dx + \frac{8}{p}\int |\varphi|^p \cdot x\cdot\nabla K(x)dx < -C < 0.$$

This yields that blow-up of the solution $\varphi(t, x)$ must occur in the time period of $0 < T < +\infty$.

The case that $E(\varphi_0) < 0$ is easy. By Corollary 3.2, we conclude that the finite-time blow-up solution $\varphi(t, x)$ of Eq (1.1) exists.

3.3. Energy-critical setting

Finally, the energy-critical setting (i.e., $p = p^* = 2 + \frac{4}{N-2}$) for problem (1.1) is considered in this subsection. According to Dinh [15], taking account the sharp constant (see also Eq (2.4))

$$S = \frac{1}{C_{SE}^2(c)} = \inf_{\varphi\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{H(\varphi)}{\left(\int |\varphi|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{N}}},$$

then we are able to get the optimal blow-up criterion.

Theorem 3.5. *Assume that $p = p^*$, $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$, $\varphi_0 \in H^1(\mathbb{R}^N)$ and $E(\varphi_0) < \frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}}$. Let $\varphi(t, x)$ be the corresponding solution of Eq (1.1), defined on $[0, T)\times\mathbb{R}^N$, $0 < T \leq \infty$.*
(i) Uniform boundedness: Assume that $K(x)$ satisfies (A_1). If $H(\varphi_0) < \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$, then the solution $\varphi(t, x)$ of Eq (1.1) is bounded in $H^1(\mathbb{R}^N)$ for $t \in [0, T)$ and $H(\varphi(t)) < \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$.
(ii) Blow-up: Assume that $K(x)$ satisfies (A_1) – (A_2). If $H(\varphi_0) > \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$ and $|x|\varphi_0 \in L^2(\mathbb{R}^N)$, then there exists $0 < T < \infty$ such that the solution $\varphi(t, x)$ of Eq (1.1) blows up at T.

Proof. (i) Assume that $\varphi_0 \in H^1(\mathbb{R}^N)$ satisfies that $E(\varphi_0) < \frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}}$ and $H(\varphi_0) < \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$. We claim that

$$H(\varphi) < \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}} \; for \; any \; t \in [0, T). \tag{3.27}$$

We demonstrate Eq (3.27) by contradiction. Suppose that there exists $t_1 \in (0, T)$ such that $H(\varphi(t_1)) = \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$ by using the continuity of the solution $\varphi(t, x)$ in $H^1(\mathbb{R}^N)$ at time t. By Eq (2.2), (A_1) and Lemma 2.4, we found that

$$\begin{aligned}
\frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}} &> E(\varphi_0) = E(\varphi(t_1)) \\
&\geq \frac{1}{2}H(\varphi(t_1)) - \frac{1}{p*}\int K_2|\varphi(t_1)|^{p*}dx \\
&\geq \frac{1}{2}H(\varphi(t_1)) - \frac{K_2}{p*}S^{-\frac{N}{N-2}}H(\varphi(t_1))^{\frac{N}{N-2}} \\
&= \frac{1}{2}\frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}} - \frac{K_2}{p*}S^{-\frac{N}{N-2}}\Big(\frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}\Big)^{\frac{N}{N-2}} = \frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}},
\end{aligned}$$

which contradicts the fact that $H(\varphi(t_1)) = \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$. Therefore, Eq (3.27) holds. This means that the solution $\varphi(t, x)$ is bounded in $H^1(\mathbb{R}^N)$ for $t \in [0, T)$.

Now, we prove part (ii) of Theorem 3.5. Since $\varphi_0 \in H^1(\mathbb{R}^N)$ satisfies that $E(\varphi_0) < \frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}}$ and $H(\varphi_0) > \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$. It remains to be proven that

$$H(\varphi) > \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}} \; for \; any \; t \in [0, T). \tag{3.28}$$

If otherwise, since $\varphi(t, x)$ is continuous with respect to time t in $H^1(\mathbb{R}^N)$, we deduce that there exists $t_2 \in (0, T)$ satisfying that $H(\varphi(t_2)) = \frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}$. Combining Eq (2.2) with (A_1), and by Lemma 2.4, one has the following estimate:

$$\begin{aligned}
\frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}} &> E(\varphi_0) = E(\varphi(t_2)) \\
&\geq \frac{1}{2}H(\varphi(t_2)) - \frac{1}{p*}\int K_2|\varphi(t_2)|^{p*}dx \\
&\geq \frac{1}{2}H(\varphi(t_2)) - \frac{K_2}{p*}S^{-\frac{N}{N-2}}H(\varphi(t_2))^{\frac{N}{N-2}} \\
&= \frac{1}{2}\frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}} - \frac{K_2}{p*}S^{-\frac{N}{N-2}}\Big(\frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}\Big)^{\frac{N}{N-2}} = \frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}},
\end{aligned}$$

which gives a contradiction. Thus, Eq (3.28) holds. Keeping in mind that $p^* = 2+\frac{4}{N-2}$ and $\varphi_0 \in H^1(\mathbb{R}^N)$ with $|x|\varphi_0 \in L^2(\mathbb{R}^N)$, we have

$$V''(t) = 16E(\varphi_0) - \frac{16}{N}\int K(x)|\varphi|^{\frac{2N}{N-2}}dx + \frac{4(N-2)}{N}\int x\cdot\nabla K(x)|\varphi|^{\frac{2N}{N-2}}dx.$$

Then, we infer from Eqs (3.28) and (2.2) that

$$\begin{aligned} -\int K(x)|\varphi_0|^{\frac{2N}{N-2}}dx &= \frac{2N}{N-2}(E(\varphi_0) - \frac{1}{2}H(\varphi_0)) \\ &< \frac{2N}{N-2}\Big(\frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}} - \frac{S^{\frac{N}{2}}}{2K_2^{\frac{N-2}{2}}}\Big) = -\frac{S^{\frac{N}{2}}}{K_2^{\frac{N-2}{2}}}. \end{aligned} \tag{3.29}$$

Inserting Eq (3.29) into $V''(t)$, and then from (A_2), one has the estimate

$$V''(t) \leq 16E(\varphi_0) - \frac{16S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}} = 16\Big(E(\varphi_0) - \frac{S^{\frac{N}{2}}}{NK_2^{\frac{N-2}{2}}}\Big) < 0,$$

from which we know that explosion of the solution $\varphi(t,x)$ to Eq (1.1) must happen within the time period of $0 < T < \infty$.

Remark 3.6. *(i) Under the conditions that $K(x) = 1$, $c = c^*$ and $2 < p < p^*$ in Eq (1.1), Mukherjee et al. [18] studied the criterion of blow-up versus global existence in the L^2-critical and L^2-supercritical cases (see [18], Theorem 3); Dinh [15] obtained the blow-up and global existence results for Eq (1.1) for non-radial data in the mass-critical and mass-supercritical cases with $0 < c < c^*$, as well as the sharp blow-up criterion in the energy-critical case with $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$ (see [15], Theorems 1.3, 1.6 and 1.12). Our results (Theorems 3.3–3.5) generalize the results of [15, 18] to the case of the inhomogeneous NLS involving inverse-square potential and a bounded positive variable coefficient.*
(ii) Given $c = 0$, Liu [12] verified the sharp existence of non-radial finite-time blow-up solutions to the inhomogeneous Eq (1.2) in the energy-critical case $p = \frac{2N}{N-2}$ (see [12], Theorem 1.2). We extend this conclusion to the case of the inhomogeneous NLS with inverse-square potential for $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$ (see Theorem 3.5).
(iii) For $0 < c < \frac{N^2+4N}{(N+2)^2}c^*$ and $p = p^*$, we can derive that the global solution of Eq (1.1) exists for some initial value small enough.

4. Dynamics of blow-up solutions in the L^2-critical setting

In the present part, we investigate the dynamics of blow-up solutions in the L^2-critical setting ($p = 2 + \frac{4}{N}$) with $0 < c < c^*$, including the mass concentration phenomenon of blow-up solutions and the dynamical properties of blow-up solutions with a minimal mass for Eq (1.1). To achieve these goals, we first recall a key compactness lemma established by Bensouilah [19].

Lemma 4.1. *Assume that $\{v_n\}_{n=1}^{\infty}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying*

$$\limsup_{n\to\infty} H(v_n) \leq M, \quad \limsup_{n\to\infty} \|v_n\|_p \geq m.$$

Then, there exists $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ *such that, up to a subsequence,*

$$v_n(x + x_n) \rightharpoonup U \ \ weakly\ in\ \ H^1(\mathbb{R}^N),$$

with $\|U\|_2 \geq (\frac{N}{N+2})^{\frac{N}{4}} \frac{m^{\frac{N}{2}+1}}{M^{\frac{N}{4}}} \|Q(x)\|_2$, *where* $Q(x)$ *is the ground-state solution to Eq (1.4).*

With Lemma 4.1 in hand, we are able to obtain the following concentration property of the explosive solutions to Eq (1.1).

Theorem 4.2. *(L^2-concentration) Assume that* $K(x)$ *satisfies* (A_1) *and* (A_2). *Suppose that* $\varphi(t, x)$ *is a solution to Eq (1.1) blowing up at finite time* T, *and that* $s(t)$ *is a nonnegative real-valued function on* $[0, T)$ *such that* $s(t)\|\nabla\varphi(t)\|_2 \to +\infty$ *as* $t \to T$. *Then, there exists a function* $x(t) \in \mathbb{R}^N$ *for* $t < T$ *satisfying*

$$\liminf_{t\to T} \int_{|x-x(t)|\leq s(t)} |\varphi(t, x)|^2 dx \geq \int Q_{K_2}^2 dx, \tag{4.1}$$

where $Q_{K_2}(x)$ *denotes the ground state for Eq (3.1).*

Proof. Take

$$\rho(t) = \left[\frac{H(Q_{K_2})}{H(\varphi)}\right]^{\frac{1}{2}} \ \ and\ \ v(t, x) = \rho(t)^{\frac{N}{2}} \varphi(t, \rho(t)x). \tag{4.2}$$

Suppose that $\{t_n\}_{n=1}^{\infty}$ is an arbitrary time sequence satisfying that $t_n \to T$ as $n \to \infty$, and denote $\rho_n = \rho(t_n)$ and $v_n(x) = v(t_n, x)$. It follows from Eq (2.1) and the definition of v_n that

$$\|v_n\|_2 = \|\varphi(t_n)\|_2 = \|\varphi_0\|_2,\ \ H(v_n) = \rho_n^2 H(\varphi) = H(Q_{K_2}). \tag{4.3}$$

For $v(x) \in H^1(\mathbb{R}^N)$, we define the functional

$$F(v) = H(v) - \frac{2K_2}{p}\|v\|_p^p.$$

From (A_1) and Eqs (4.3), (2.2) and (4.2), we find that

$$\begin{aligned} \frac{1}{2}F(v_n) &= \frac{1}{2}H(v_n) - \frac{K_2}{p}\|v_n\|_p^p \\ &\leq \frac{1}{2}H(v_n) - \frac{1}{p}\int K(x)|v_n|^p dx \\ &= \rho_n^2(\frac{1}{2}H(\varphi) - \frac{1}{p}\int K(x)|\varphi|^p dx) \\ &= \rho_n^2 E(\varphi_0) \to 0 \ \ since\ \ \rho_n \to 0\ \ as\ n \to \infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \int |v_n|^p dx \geq \frac{p}{2K_2} H(Q_{K_2})\ \ as\ n \to \infty. \tag{4.4}$$

Take $M = H(Q_{K_2})$ and $m^{2+\frac{4}{N}} = \frac{p}{2K_2}H(Q_{K_2})$. Thanks to Lemma 4.1, it is sufficient to demonstrate that there exist $U(x) \in H^1(\mathbb{R}^N)$ and $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(\cdot + x_n) = \rho_n^{\frac{N}{2}} \varphi(t_n, \rho_n \cdot + x_n) \rightharpoonup U \ \ weakly\ \ in\ H^1(\mathbb{R}^N), \tag{4.5}$$

with

$$\|U\|_2 \geq (\frac{N}{N+2})^{\frac{N}{4}} \frac{m^{\frac{N}{2}+1}}{M^{\frac{N}{4}}} \|Q\|_2 = (\frac{N}{N+2})^{\frac{N}{4}} \frac{\left[(\frac{N+2}{NK_2}) H(Q_{K_2})\right]^{\frac{N}{4}}}{H(Q_{K_2})^{\frac{N}{4}}} \|Q\|_2 = \|Q_{K_2}\|_2, \tag{4.6}$$

which leads to

$$v_n(\cdot + x_n) \rightharpoonup U \ \ weakly \ \ in \ L^2(\mathbb{R}^N).$$

This, together with the lower semi-continuity of the L^2-norm, yields

$$\liminf_{n\to\infty} \int_{|x|\leq A} \rho_n^N |\varphi(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x|\leq A} |U|^2 dx \ \ for\ any\ A > 0. \tag{4.7}$$

Since

$$\lim_{n\to\infty} \frac{s(t_n)}{\rho_n} = \lim_{n\to\infty} \frac{s(t_n) H(\varphi)^{\frac{1}{2}}}{H(Q_{K_2})^{\frac{1}{2}}} = \infty,$$

there exists $n_0 > 0$ such that for any $n > n_0$, we obtain that $A\rho_n < s(t_n)$. Combining this result and Eq (4.7), one has

$$\begin{aligned} \liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq s(t_n)} |\varphi(t_n, x)|^2 dx &\geq \liminf_{n\to\infty} \int_{|x-x_n|\leq A\rho_n} |\varphi(t_n, x)|^2 dx \\ &= \liminf_{n\to\infty} \int_{|x|\leq A} \rho_n^N |\varphi(t_n, \rho_n x + x_n)|^2 dx \\ &\geq \int_{|x|\leq A} |U|^2 dx, \ \ for\ any\ A > 0, \end{aligned}$$

which means that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq s(t_n)} |\varphi(t_n, x)|^2 dx \geq \int |U|^2 dx = \|U\|_2^2.$$

According to the arbitrariness of the sequence $\{t_n\}_{n=1}^{\infty}$, using the fact that $\|U\|_2 \geq \|Q_{K_2}\|_2$, we obtain

$$\liminf_{t\to T} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq s(t)} |\varphi(t, x)|^2 dx \geq \|Q_{K_2}\|_2^2. \tag{4.8}$$

For any fixed $t \in [0, T)$, it is simple to infer that the function $g(y) := \int_{|x-y|\leq s(t)} |\varphi(t, x)|^2 dx$ is continuous on $y \in \mathbb{R}^N$ and $\lim_{|y|\to\infty} g(y) = 0$. Thus, for any $0 \leq t < T$, there exists a function $x(t) \in \mathbb{R}^N$ that satisfies

$$\sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq s(t)} |\varphi(t, x)|^2 dx = \int_{|x-x(t)|\leq s(t)} |\varphi(t, x)|^2 dx. \tag{4.9}$$

Therefore, from Eqs (4.8) and (4.9) we infer that Eq (4.1) holds true.

Corollary 4.3. *Suppose that $\varphi(t, x)$ is a solution of Eq (1.1) blowing up within the time period of $0 < T < \infty$. Then, for any $l > 0$, there exists $x(t) \in \mathbb{R}^N$ for $0 < t < T$ satisfying that*

$$\liminf_{t\to T}\int_{B(x(t),l)}|\varphi(t,x)|^2dx \geq \int Q_{K_2}^2dx,$$

where $Q_{K_2}(x)$ denotes the ground state for Eq (3.1) and $B(x(t),l) = \{x \in \mathbb{R}^N \| x - x(t)| \leq l\}$.

Now, in terms of Theorem 4.2 and Corollary 4.3, we are concerned with the dynamics of blow-up solutions with the mass $\|\varphi_0\|_2 = \|Q_{K_2}\|_2$.

Theorem 4.4. *(Limiting profile) Let $K(x)$ satisfy (A_1) – (A_2) and assume that $\|\varphi_0\|_2 = \|Q_{K_2}\|_2$. Assume that $\varphi(t,x)$ is a corresponding solution to problem (1.1) blowing up within the time period of $0 < T < \infty$; then, there exist two functions $\theta(t) \in [0,2\pi)$ and $x(t) \in \mathbb{R}^N$ that satisfy*

$$\rho(t)^{\frac{N}{2}}e^{i\theta(t)}\varphi(t,\rho(t)x + x(t)) \to Q_{K_2} \ \ \textit{strongly in } H^1(\mathbb{R}^N) \textit{ when } t \to T,$$

where $Q_{K_2}(x)$ denotes the ground state for Eq (3.1) and $\rho(t) = \left[\frac{H(Q_{K_2})}{H(\varphi)}\right]^{\frac{1}{2}}$.

Proof. From Theorem 4.2, we have that $\|U\|_2 \geq \|Q_{K_2}\|_2$ (see Eq (4.6)). Then, by using the assumption that $\|\varphi_0\|_2 = \|Q_{K_2}\|_2$ and Eq (2.1), we obtain

$$\|Q_{K_2}\|_2^2 \leq \|U\|_2^2 \leq \liminf_{n\to\infty}\|v_n\|_2^2 = \liminf_{n\to\infty}\|\varphi(t_n)\|_2^2 = \|\varphi_0\|_2^2 = \|Q_{K_2}\|_2^2,$$

which implies that

$$\lim_{n\to\infty}\|v_n\|_2^2 = \|U\|_2^2 = \|Q_{K_2}\|_2^2. \tag{4.10}$$

It follows from Eqs (4.1) and (4.10) that

$$v_n(\cdot + x_n) = \rho_n^{\frac{N}{2}}\varphi(t_n,\rho_n\cdot + x_n) \to U \ \ \textit{strongly in } L^2(\mathbb{R}^N) \ \textit{as} \ n\to\infty. \tag{4.11}$$

From Eq (2.3), one has

$$\begin{aligned}
\|v_n(x+x_n) - U\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} &\leq \frac{1}{C_{HGN}^p}H(v_n(x+x_n) - U)^{\frac{\theta p}{2}}\|v_n(x+x_n) - U\|_2^{(1-\theta)p} \\
&= \frac{p}{2\|Q\|_2^{\frac{4}{N}}}H(v_n(x+x_n) - U)\|v_n(x+x_n) - U\|_2^{\frac{4}{N}} \\
&\leq C\|v_n(x+x_n) - U\|_2^{\frac{4}{N}}(\|\nabla v_n(x+x_n)\|_2^2 + \|\nabla U\|_2^2) \\
&\leq C\|v_n(x+x_n) - U\|_2^{\frac{4}{N}},
\end{aligned} \tag{4.12}$$

where the last inequality is based on the boundedness of $v_n(x + x_n)$ in $H^1(\mathbb{R}^N)$. Thus, Eqs (4.11) and (4.12) give us that

$$v_n(\cdot + x_n) \to U \ \ \textit{in } L^{2+\frac{4}{N}}(\mathbb{R}^N) \ \textit{as} \ n\to\infty. \tag{4.13}$$

Now, we claim that

$$v_n(\cdot + x_n) \to U \ \ \textit{strongly in } H^1(\mathbb{R}^N) \ \textit{when} \ n\to\infty. \tag{4.14}$$

Indeed, we deduce from Eqs (4.3) and (4.4) that

$$\lim_{n\to\infty}\int |v_n|^p dx \geq \frac{p}{2K_2}H(Q_{K_2}) = \frac{p}{2K_2}\lim_{n\to\infty}H(v_n). \tag{4.15}$$

Then, from Eqs (4.5), (4.15), (4.13), (2.3) and (4.10), one has

$$\begin{aligned} H(U) &\leq \liminf_{n\to\infty} H(v_n) = H(Q_{K_2}) \\ &\leq \frac{2K_2}{p}\lim_{n\to\infty}\int |v_n|^p dx = \frac{2K_2}{p}\int |U|^p dx \\ &\leq \frac{K_2}{\|Q\|_2^{\frac{4}{N}}} H(U)\|U\|_2^{\frac{4}{N}} \\ &= \frac{1}{\|Q_{K_2}\|_2^{\frac{4}{N}}} H(U)\|U\|_2^{\frac{4}{N}} = H(U), \end{aligned}$$

which implies that

$$\liminf_{n\to\infty} H(v_n(x+x_n)) = H(U) = H(Q_{K_2}) = \frac{2K_2}{p}\int |U|^p dx.$$

From this and Eq (4.5), we get Eq (4.14) and

$$F(U) = H(U) - \frac{2K_2}{p}\int |U|^p dx = 0.$$

In summary, we identify the properties of the profile U as below:

$$\|U\|_2^2 = \|Q_{K_2}\|_2^2, \quad H(U) = H(Q_{K_2}), \quad F(U) = 0,$$

from which we deduce that there exist $\theta \in [0, 2\pi)$ and $x_0 \in \mathbb{R}^N$ such that

$$U(x) = e^{i\theta} Q_{K_2}(x + x_0)$$

and

$$\rho_n^{\frac{N}{2}} \varphi(t_n, \rho_n \cdot + x_0) \to e^{i\theta} Q_{K_2}(\cdot + x_0) \ \ \textit{strongly in } H^1(\mathbb{R}^N) \ \ \textit{as} \ \ n \to \infty,$$

where we utilize the variational characterization to the ground state Q_{K_2}. Since the sequence $\{t_n\}_{n=1}^{\infty}$ is arbitrary and tends to T as $n \to \infty$, one can infer that there exist two functions $\theta(t) \in [0, 2\pi)$ and $x(t) \in \mathbb{R}^N$ such that

$$\lambda(t)^{\frac{N}{2}} e^{i\theta(t)} \varphi(t, \lambda(t)x + x(t)) \to Q_{K_2} \ \ \textit{strongly in } H^1(\mathbb{R}^N) \ \ \textit{when} \ \ t \to T,$$

with $\rho(t) = \left[\frac{H(Q_{K_2})}{H(\varphi)}\right]^{\frac{1}{2}} \to 0$ as $t \to T$.

Theorem 4.5. *Assume that $K(x)$ satisfies $(A_1) - (A_3)$ and that $\|\varphi_0\|_2 = \|Q_{K_2}\|_2$. Denote $W = \{x \in \mathbb{R}^N | K(x) = K_2\}$. Let $\varphi(t, x)$ be the corresponding solution to Eq (1.1) blowing up within the time period of $0 < T < \infty$; then the following holds true:*
(i) (Location of L^2-concentration point) There exists $x_0 \in W$ such that

$$\lim_{t\to T} x(t) = x_0 \textit{ and } |\varphi(t, x)|^2 \to \|Q_{K_2}\|_2^2 \delta_{x=x_0} \textit{ in the distribution sense as } t \to T, \tag{4.16}$$

where Q_{K_2} is the ground state for Eq (3.1).
(ii) (Blow-up rate) There is a positive constant $C > 0$ that satisfies

$$\|\nabla\varphi(t)\|_2 \geq \frac{C}{T-t} \textit{ for all } t \in [0, T). \tag{4.17}$$

Proof. (i) From Eq (2.1) and $\|\varphi_0\|_2 = \|Q_{K_2}\|_2$, we deduce that

$$\|\varphi\|_2 = \|\varphi_0\|_2 = \|Q_{K_2}\|_2 \ for\ t < T. \tag{4.18}$$

On the other side, according to Theorem 4.2 and Corollary 4.3, for any $l > 0$, one has that

$$\|Q_{K_2}\|_2^2 \le \liminf_{t\to T} \int_{|x-x(t)|\le l} |\varphi(t,x)|^2 dx \le \liminf_{t\to T} \int |\varphi(t,x)|^2 dx \le \|\varphi_0\|_2^2. \tag{4.19}$$

From Eqs (4.18) and (4.19), we have

$$\liminf_{t\to T} \int_{|x-x(t)|<l} |\varphi(t,x)|^2 dx = \|Q_{K_2}\|_2^2,$$

which shows that

$$|\varphi(t, x + x(t))|^2 \to \|Q_{K_2}\|_2^2 \delta_{x=0} \ in\ the\ distribution\ sense\ as\ t \to T. \tag{4.20}$$

We shall demonstrate in what follows that there exists $x_0 \in W$ satisfying that

$$|\varphi(t,x)|^2 \to \|Q_{K_2}\|_2^2 \delta_{x=x_0} \ in\ the\ sense\ of\ distribution\ as\ t \to T,$$

which implies that

$$\lim_{t\to T} \int w(x)|\varphi(t,x)|^2 dx = w(x_0)\|Q_{K_2}\|_2^2,\ for\ any\ w(x) \in C_0^\infty(\mathbb{R}^N).$$

As a matter of fact, for any real-valued function $\theta(x) \in \mathbb{R}^N$ and any $\beta \in \mathbb{R}$, from (A_1) and Eq (2.3), one has the following estimate

$$\begin{aligned}
E(e^{i\beta\theta(x)}\varphi) &= \frac{1}{2}H(e^{i\beta\theta(x)}\varphi) - \frac{1}{p}\int K(x)|e^{i\beta\theta(x)}\varphi|^p dx \\
&\ge \frac{1}{2}H(e^{i\beta\theta(x)}\varphi) - \frac{K_2}{p}\|e^{i\beta\theta(x)}\varphi\|_p^p \\
&\ge \frac{1}{2}H(e^{i\beta\theta(x)}\varphi) - \frac{K_2}{p}\frac{1}{C_{HGN}^p} H(e^{i\beta\theta(x)}\varphi)^{\frac{\theta p}{2}} \|e^{i\beta\theta(x)}\varphi\|_2^{(1-\theta)p} \\
&= \frac{1}{2}H(e^{i\beta\theta(x)}\varphi) - \frac{K_2}{2}\left(\frac{\|\varphi_0\|_2}{\|Q\|_2}\right)^{\frac{4}{N}} H(e^{i\beta\theta(x)}\varphi) \\
&= \frac{1}{2}H(e^{i\beta\theta(x)}\varphi) - \frac{K_2}{2}\left(\frac{\|Q_{K_2}\|_2}{\|Q\|_2}\right)^{\frac{4}{N}} H(e^{i\beta\theta(x)}\varphi) \\
&= 0,
\end{aligned}$$

where θ is defined in Lemma 2.2. Here in the last equality, we have used the fact that $\|Q_{K_2}\|_2 = K_2^{-\frac{N}{4}}\|Q\|_2$. Therefore, from Eqs (2.1) and (2.2), for any $\beta \in \mathbb{R}$, we obtain

$$0 \le E(e^{i\beta\theta(x)}\varphi) = \frac{1}{2}H(e^{i\beta\theta(x)}\varphi) - \frac{1}{p}\int K(x)|e^{i\beta\theta(x)}\varphi|^p dx$$

$$\begin{aligned} &= \frac{1}{2}\Big[\int \beta^2|\nabla\theta(x)\cdot\varphi|^2dx + 2\beta Im\int \nabla\theta(x)\nabla\varphi\overline{\varphi}dx + \int|\nabla\varphi|^2dx\Big] \\ &\quad -\frac{1}{2}\int c|x|^{-2}|e^{i\beta\theta(x)}\varphi|^2dx - \frac{1}{p}\int K(x)|e^{i\beta\theta(x)}\varphi|^pdx \\ &= \frac{1}{2}\beta^2\int|\nabla\theta(x)|^2|\varphi|^2dx + \beta Im\int\nabla\theta(x)\cdot\nabla\varphi\cdot\overline{\varphi}dx + E(\varphi_0), \end{aligned}$$

which implies that

$$\left|Im\int\nabla\theta(x)\cdot\nabla\varphi\cdot\overline{\varphi}dx\right| \le \Big[2E(\varphi_0)\int|\nabla\theta(x)|^2|\varphi|^2dx\Big]^{\frac{1}{2}}. \tag{4.21}$$

Then, choosing $\theta_j(x) = x_j$ for $j = 1, 2, \cdots, N$ in Eq (4.21), it follows from Eqs (1.1), (2.1) and (2.2) that

$$\begin{aligned} \left|\frac{d}{dt}\int|\varphi(t,x)|^2x_jdx\right| &= \left|2Im\int i\varphi_t\cdot\overline{\varphi}\cdot x_jdx\right| \\ &= \left|2Im\int[(-\Delta-|x|^{-2})\varphi - K(x)|\varphi|^{p-2}\varphi]\overline{\varphi}\cdot x_jdx\right| \\ &= \left|2Im\int\nabla\varphi\cdot\overline{\varphi}\cdot\nabla x_jdx\right| \\ &\le 2\Big(2E(\varphi_0)\int|\varphi_0|^2dx\Big)^{\frac{1}{2}} = C. \end{aligned} \tag{4.22}$$

Take any two sequences $\{t_n\}_{n=1}^{\infty}$, $\{t_m\}_{m=1}^{\infty} \subset [0, T)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{m\to\infty} t_m = T$. Thus, for all $j = 1, 2, \cdots, N$, from Eq (4.22), we have

$$\begin{aligned} \left|\int|\varphi(t_n,x)|^2x_jdx - \int|\varphi(t_m,x)|^2x_jdx\right| &\le \int_{t_m}^{t_n}\left|\frac{d}{dt}\int|\varphi(t,x)|^2x_jdx\right|dt \\ &\le C|t_n - t_m| \to 0 \;\; as\;\; n,\; m\to\infty, \end{aligned}$$

which means that

$$\lim_{t\to T}\int|\varphi(t,x)|^2x_jdx \;\; exists\; for\; any\; j = 1, 2, \cdots, N.$$

Thus,

$$\lim_{t\to T}\int|\varphi(t,x)|^2xdx \;\; exists.$$

Let $x_0 = \frac{\lim_{t\to T}\int|\varphi(t,x)|^2xdx}{\|Q_{K_2}\|_2^2}$; then, one has that $x_0 \in \mathbb{R}^N$ and

$$\lim_{t\to T}\int|\varphi(t,x)|^2xdx = x_0\|Q_{K_2}\|_2^2. \tag{4.23}$$

On the other hand, notice that $p = 2 + \frac{4}{N}$; then, from Proposition 3.1 and (A_2), we have

$$V''(t) = 16E(\varphi_0) + \frac{4(4+2N-Np)}{p}\int K(x)|\varphi|^pdx + \frac{8}{p}\int|\varphi|^p\cdot x\cdot\nabla K(x)dx \le 16E(\varphi_0).$$

Therefore, there is a positive constant $c_0 > 0$ satisfying that

$$V(t) \le c_0\; for\; each\; t\in[0, T).$$

Thus, one has the following estimate:

$$\begin{aligned}\int |x|^2|\varphi(t,x+x(t))|^2dx &\leq 2\int |x+x(t)|^2|\varphi(t,x+x(t))|^2dx \\ &\quad + 2\int |x(t)|^2|\varphi(t,x+x(t))|^2dx \\ &\leq 2c_0 + 2\|\varphi_0\|_2^2|x(t)|^2 \\ &= 2c_0 + 2\|Q_{K_2}\|_2^2|x(t)|^2. \end{aligned} \tag{4.24}$$

From Eq (4.20), one has

$$\begin{aligned}\limsup_{t\to T}|x(t)|^2\|Q_{K_2}\|_2^2 &= \limsup_{t\to T}\int_{|x|<1}|x+x(t)|^2|\varphi(t,x+x(t))|^2dx \\ &\leq \int |x|^2|\varphi(t,x)|^2dx \leq c_0. \end{aligned} \tag{4.25}$$

We derive from Eq (4.25) that

$$\limsup_{t\to T}|x(t)| \leq \frac{\sqrt{c_0}}{\|Q_{K_2}\|_2}. \tag{4.26}$$

Combining Eq (4.24) with Eq (4.26), one has

$$\limsup_{t\to T}\int |x|^2|\varphi(t,x+x(t))|^2dx \leq C.$$

Thus, we have the following for any $l_0 > 0$:

$$\limsup_{t\to T}\int_{|x|\geq l_0} l_0|x||\varphi(t,x+x(t))|^2dx \leq \limsup_{t\to T}\int_{|x|\geq l_0}|x|^2|\varphi(t,x+x(t))|^2dx \leq C.$$

Therefore, for any $\varepsilon > 0$, there exists a large enough $l_0 = l_0(\varepsilon) > 0$ satisfying that

$$\limsup_{t\to T}\left|\int_{|x|\geq l_0} x|\varphi(t,x+x(t))|^2dx\right| \leq \frac{C}{l_0} < \varepsilon. \tag{4.27}$$

Then, using Eqs (4.27) and (4.20), we infer that, for any $\varepsilon > 0$,

$$\begin{aligned}\limsup_{t\to T}\left|\int x|\varphi(t,x)|^2dx - x(t)\|Q_{K_2}\|_2^2\right| &= \limsup_{t\to T}\left|\int x|\varphi(t,x)|^2dx - x(t)\int|\varphi(t,x)|^2dx\right| \\ &= \limsup_{t\to T}\left|\int |\varphi(t,x)|^2(x-x(t))dx\right| \\ &\leq \limsup_{t\to T}\left|\int_{|x|\leq l_0}|\varphi(t,x+x(t))|^2xdx\right| + \varepsilon \\ &= \varepsilon. \end{aligned} \tag{4.28}$$

Combining Eqs (4.23) and (4.28), we immediately get

$$\lim_{t\to T}x(t) = x_0 \ \ and \ \ \limsup_{t\to T}\int x|\varphi(t,x)|^2dx = x_0\|Q_{K_2}\|_2^2. \tag{4.29}$$

Thus, there exists $x_0 \in \mathbb{R}^N$ (see Eq (4.23)) such that

$$|\varphi(t,x)|^2 \to \|Q_{K_2}\|_2^2 \delta_{x=x_0} \ \ in\ the\ sense\ of\ distribution\ when\ t \to T.$$

Now, we claim that $x_0 \in W = \{x \in \mathbb{R}^N | K(x) = K_2\}$. Namely, x_0 is a global maximal point of $K(x)$. Indeed, since $K(x) \in C^1$, we deduce from Corollary 4.3 and Eq (4.29) that

$$\begin{aligned} 1 \leq \liminf_{t\to T} \frac{\|\varphi(t,x)\|^2_{L^2(B(x(t),l))}}{\|Q_{K(x(t))}\|_2^2} &\leq \liminf_{t\to T} \frac{K_2^{-\frac{N}{2}}\|\varphi_0\|_2^2}{[K(x(t))]^{-\frac{N}{2}}\|Q_{K_2}\|_2^2} \\ &= \liminf_{t\to T} \left[\frac{K(x(t))}{K_2}\right]^{\frac{N}{2}} \\ &= \left[\frac{K(x_0)}{K_2}\right]^{\frac{N}{2}} \leq 1, \end{aligned}$$

where $Q_{K(x(t))} = K(x(t))^{-\frac{N}{4}} Q(x(t))$. This implies that $K(x_0) = K_2$. Thus,

$$x_0 \in W = \{x \in \mathbb{R}^N | K(x) = K_2\}.$$

Therefore, it is clear that Eq (4.16) holds true.

(ii) Choosing $\theta(x) = |x - x_0|^2$ in Eq (4.21), we get

$$\begin{aligned} \left|\frac{d}{dt}\int |\varphi(t,x)|^2 |x-x_0|^2 dx\right| &= \left|2Im\int -\Delta\varphi \cdot \overline{\varphi} \cdot |x-x_0|^2 dx\right| \\ &\leq 4\Big(2E(\varphi_0)\int |\varphi(t,x)|^2|x-x_0|^2dx\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int |\varphi(t,x)|^2|x-x_0|^2dx\Big)^{\frac{1}{2}}, \end{aligned}$$

which indicates that

$$\left|\frac{d}{dt}\Big(\int |\varphi(t,x)|^2|x-x_0|^2dx\Big)^{\frac{1}{2}}\right| \leq C.$$

Therefore, for any $t \in [0,T)$, by integrating on both sides of this inequality from t to T, we derive

$$\Big(\int |\varphi(t,x)|^2|x-x_0|^2dx\Big)^{\frac{1}{2}} \leq C(T-t).$$

Based on the uncertainty principle and Hölder's inequality, we obtain

$$\begin{aligned} \|\varphi_0\|_2^2 = \int |\varphi(t,x)|^2 dx &= -\frac{2}{N} Re \int \nabla\varphi \cdot \overline{\varphi} \cdot (x-x_0) dx \\ &\leq C\Big(\int |\varphi(t,x)|^2|x-x_0|^2dx\Big)^{\frac{1}{2}}\Big(\int |\nabla\varphi|^2dx\Big)^{\frac{1}{2}} \\ &\leq C(T-t)\|\nabla\varphi(t)\|_2, \end{aligned}$$

which means that

$$\|\nabla\varphi(t)\|_2 \geq \frac{C}{T-t} \ \ for\ \ \forall\, t \in [0,T).$$

Therefore, the conclusion Eq (4.17) holds.

Remark 4.6. *(i) Given* $K(x) = 1$ *and* $0 < c < c^*$*, Bensouilah [19] verified the* L^2*-concentration of explosive solutions to Eq (1.1) in the* L^2*-critical case (see [19], Theorem 1). Furthermore, Pan and Zhang [20] obtained the precise concentration behavior for blow-up solutions with a critical mass (see [20], Theorem 1.1). Our conclusions generalize and supplement the corresponding results of [19, 20] for the inhomogeneous NLS with inverse-square potential and the coefficient* $0 < K(x) \neq$ *constant (see Theorems 4.2, 4.4 and 4.5).*
(ii) For the case that $c = 0$*,* $K(x) \neq$ *constant and* $(A_1) - (A_3)$ *and other proper conditions are satisfied, the mass concentration property of blow-up solutions and the dynamical behaviors of the* L^2*-minimal blow-up solutions have been derived (see [10], Theorems 1 and 2). Our results extend the results of [10] to the inhomogeneous NLS involving inverse-square potential for* $0 < c < c^*$ *(see Theorems 4.2 and 4.5).*

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors make the declaration that there are no competing interests existing.

References

1. H. E. Camblong, L. N. Epele, H. Fanchiotti, C. A. G. Canal, Quantum anomaly in molecular physics, *Phys. Rev. Lett.*, **87** (2001), 220402. https://doi.org/10.1103/PhysRevLett.87.220402
2. K. M. Case, Singular potentials, *Phys. Rev.*, **80** (1950), 797–806. https://doi.org/10.1103/PhysRev.80.797
3. H. Kalf, U. W. Schmincke, J. Walter, R. Wüst, On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials, in *Spectral Theory and Differential Equations*, Springer press, **448** (1975), 182–226. https://doi.org/10.1007/BFB0067087
4. G. E. Astrakharchik, B. A. Malomed, Quantum versus mean-field collapse in a many-body system, *Phys. Rev. A*, **92** (2015), 043632. https://doi.org/10.1103/PhysRevA.92.043632
5. H. Sakaguchi, B. A. Malomed, Suppression of the quantum-mechanical collapse by repulsive interactions in a quantum gas, *Phys. Rev. A*, **83** (2011), 013607. https://doi.org/10.1103/PhysRevA.83.013607
6. H. Sakaguchi, B. A. Malomed, Suppression of the quantum collapse in binary bosonic gases, *Phys. Rev. A*, **88** (2013), 043638. https://doi.org/10.1103/PhysRevA.88.043638

7. M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Commun. Math. Phys.*, **87** (1983), 567–576. https://doi.org/10.1007/BF01208265

8. F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power, *Duke Math. J.*, **69** (1993), 427–454. https://doi.org/10.1215/S0012-7094-93-06919-0

9. T. Hmidi, S. Keraani, Blowup theory for the critical nonlinear Schrödinger equations revisited, *Int. Math. Res. Not.*, **2005** (2005), 2815–2818. https://doi.org/10.1155/IMRN.2005.2815

10. F. Merle, Nonexistence of minimal blow-up solutions of equations $iu_t = -\Delta u - k(x)|u|^{\frac{4}{N}}u$ in $\mathbb{R}^N$, *Ann. Inst. Henri Poincare*, **64** (1996), 33–85.

11. J. Shu, J. Zhang, Sharp criterion of global existence for a class of nonlinear Schrödinger equation with critical exponent, *Appl. Math. Comput.*, **182** (2006), 1482–1487. https://doi.org/10.1016/j.amc.2006.05.036

12. Z. Liu, On a class of inhomogeneous, energy-critical, focusing, nonlinear Schrödinger equations, *Acta Math. Sci.*, **33** (2013), 1522–1530. https://doi.org/10.1016/S0252-9602(13)60101-0

13. J. Lu, C. X. Miao, J. Murphy, Scattering in H^1 for the intercritical NLS with an inverse-square potential, *J. Differ. Equations*, **264** (2018), 3174–3211. https://doi.org/10.1016/J.JDE.2017.11.015

14. K. Yang, Scattering of the energy-critical NLS with inverse square potential, *J. Math. Anal. Appl.*, **487** (2020), 124006. https://doi.org/10.1016/j.jmaa.2020.124006

15. V. D. Dinh, Global existence and blowup for a class of the focusing nonlinear Schrödinger equation with inverse-square potential, *J. Math. Anal. Appl.*, **468** (2018), 270–303. https://doi.org/10.1016/j.jmaa.2018.08.006

16. X. F. Li, Global existence and blowup for Choquard equations with an inverse-square potential, *J. Differ. Equations*, **268** (2020), 4276–4319. https://doi.org/10.1016/j.jde.2019.10.028

17. E. Csobo, F. Genoud, Minimal mass blow-up solutions for the L^2 critical NLS with inverse-square potential, *Nonlinear Anal.*, **168** (2018), 110–129. https://doi.org/10.1016/j.na.2017.11.008

18. D. Mukherjee, P. T. Nam, P. T. Nguyen, Uniqueness of ground state and minimal-mass blow-up solutions for focusing NLS with Hardy potential, *J. Funct. Anal.*, **281** (2021), 109092. https://doi.org/10.1016/j.jfa.2021.109092

19. A. Bensouilah, L^2 concentration of blow-up solutions for the mass-critical NLS with inverse-square potential, *Bull. Belg. Math. Soc. Simon Stevin*, **26** (2019), 759–771. https://doi.org/10.36045/bbms/1579402821

20. J. J. Pan, J. Zhang, On the minimal mass blow-up solutions for the nonlinear Schrödinger equation with Hardy potential, *Nonlinear Anal.*, **197** (2020), 111829. https://doi.org/10.1016/j.na.2020.111829

21. A. Bensouilah, V. D. Dinh, S. H. Zhu, On stability and instability of standing waves for the nonlinear Schrödinger equation with an inverse-square potential, *J. Math. Phys.*, **59** (2018), 101505. https://doi.org/10.1063/1.5038041

22. V. D. Dinh, On the instability of standing waves for the nonlinear Schrödinger equation with inverse-square potential, *Complex Var. Elliptic Equations*, **66** (2021), 1699–1716. https://doi.org/10.1080/17476933.2020.1779235

23. S. X. Xia, Energy-critical nonlinear Schrödinger equation with inverse square potential and subcritical perturbation, *J. Math. Anal. Appl.*, **487** (2020), 123955. https://doi.org/10.1016/j.jmaa.2020.123955

24. L. J. Cao, Existence of stable standing waves for the nonlinear Schrödinger equation with the Hardy potential, *Discrete Contin. Dyn. Syst. - Ser. B*, **28** (2023), 1342–1366. https://doi.org/10.3934/dcdsb.2022125

25. H. W. Li, W. M. Zou, Normalized ground state for the Sobolev critical Schrödinger equation involving Hardy term with combined nonlinearities, *Math. Nachr.*, **296** (2023), 2440–2466. https://doi.org/10.1002/mana.202000481

26. O. Goubet, I. Manoubi, Standing waves for semilinear Schrödinger equations with discontinuous dispersion, *Rend. Circ. Mat. Palermo Ser. 2*, **71** (2022), 1159–1171. https://doi.org/10.1007/s12215-022-00782-3

27. J. Zuo, C. Liu, C. Vetro, Normalized solutions to the fractional Schrödinger equation with potential, *Mediterr. J. Math.*, **20** (2023), 216. https://doi.org/10.1007/s00009-023-02422-1

28. L. Campos, C. M. Guzmán, On the inhomogeneous NLS with inverse-square potential, *Z. Angew. Math. Phys.*, **72** (2021), 143. https://doi.org/10.1007/s00033-021-01560-4

29. J. An, R. Jang, J. Kim, Global existence and blow-up for the focusing inhomogeneous nonlinear Schrödinger equation with inverse-square potential, *Discrete Contin. Dyn. Syst. - Ser. B*, **28** (2023), 1046–1067. https://doi.org/10.3934/dcdsb.2022111

30. J. J. Pan, J. Zhang, Blow-up solutions with minimal mass for the nonlinear Schrödinger equation with variable poteantial, *Adv. Nonlinear Anal.*, **11** (2022), 58–71. https://doi.org/10.1515/anona-2020-0185

31. N. Okazawa, T. Suzuki, T. Yokota, Energy methods for abstract nonlinear Schrödinger equations, *Evol. Equations Control Theory*, **1** (2012), 337–354. https://doi.org/10.3934/eect.2012.1.337

32. R. Killip, J. Murphy, M. Visan, J. Q. Zheng, The focusing cubic NLS with inverse-square potential in three space dimensions, *Differ. Integr. Equations*, **30** (2017), 161–206. https://doi.org/10.57262/die/1487386822

33. R. Killip, C. X. Miao, M. Visan, J. Y. Zhang, J. Q. Zheng, The energy-critical NLS with inverse-square potential, *Discrete Contin. Dyn. Syst.*, **37** (2017), 3831–3866. https://doi.org/10.3934/dcds.2017162

34. T. Cazenave, *Semilinear Schrödinger Equations*, *in Courant Lecture Notes in Mathematics*, American Mathematical Society Press, USA, 2003. https://doi.org/10.1090/cln/010

35. J. Holmer, S. Roudenko, On blow-up solutions to the 3D cubic nonlinear Schrödinger equation, *Appl. Math. Res. eXpress*, **2007** (2007), abm004. https://doi.org/10.1093/amrx/abm004

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