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Research article

Gradient estimates for the double phase problems in the whole space

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Abstract: This paper presents Calderón-Zygmund estimates for the weak solutions of a class of nonuniformly elliptic equations in $\mathbb{R}^n$, which are obtained through the use of the iteration-covering method. More precisely, a global Calderón-Zygmund type result

$$|f|^{p_1} + a(x)|f|^{p_2} \in L^s(\mathbb{R}^n) \Rightarrow |Du|^{p_1} + a(x)|Du|^{p_2} \in L^s(\mathbb{R}^n) \quad \text{for any } s > 1$$

is established for the weak solutions of

$$-\mathrm{div}A(x, Du) = -\mathrm{div}F(x, f) \quad \text{in } \mathbb{R}^n,$$

which are modeled on

$$-\mathrm{div}(|Du|^{p_1-2}Du + a(x)|Du|^{p_2-2}Du) = -\mathrm{div}(|f|^{p_1-2}f + a(x)|f|^{p_2-2}f),$$

where $0 \leq a(\cdot) \in C^{0,\alpha}(\mathbb{R}^n)$, $\alpha \in (0, 1]$ and $1 < p_1 < p_2 < p_1 + \frac{\alpha p_1}{n}$.

Keywords: regularity; gradient; double phase; Calderón-Zygmund estimate; non-uniform ellipticity

1. Introduction and main result

The main goal of this article is to derive the Calderón-Zygmund estimates for solutions to non-uniformly elliptic equations

$$-\,\mathrm{div}A(x, Du) = -\mathrm{div}F(x, f) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is given. Let $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field that is $C^1(\mathbb{R}^n \setminus \{0\})$-regular in $h \in \mathbb{R}^n$ and satisfies that

$$\begin{cases} |A(x,h)| + |D_h A(x,h)|\,|h| \leq L(|h|^{p_1-1} + a(x)\,|h|^{p_2-1}), \\ l\left(|h|^{p_1-2} + a(x)|h|^{p_2-2}\right)|\xi|^2 \leq \langle D_h A(x,h)\xi, \xi\rangle\,, \\ |A(x_1,h) - A(x_2,h)| \leq L\,|a(x_1) - a(x_2)|\,|h|^{p_2-1}, \end{cases} \tag{1.2}$$

for any $x, x_1, x_2 \in \mathbb{R}^n, h \in \mathbb{R}^n\backslash\{0\}$ and $\xi \in \mathbb{R}^n$. Here, l and L denote fixed constants with $0 < l \leq 1 \leq L$. The symbol D_h stands for the partial differentiation in h and $\langle\cdot,\cdot\rangle$ stands for the standard inner product. The function $a(\cdot) : \mathbb{R}^n \to [0,\infty)$ meets the following condition:

$$0 \leq a(\cdot) \in C^{0,\alpha}(\mathbb{R}^n),\ \alpha \in (0,1]. \tag{1.3}$$

The numbers p_1, p_2 satisfy

$$1 < p_1 < p_2, \tag{1.4}$$

along with

$$\frac{p_2}{p_1} < 1 + \frac{\alpha}{n}. \tag{1.5}$$

To the right of Eq (1.1), the vector field $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous in h and measurable in x. In addition, F satisfies

$$|F(x,h)| \leq L\left(|h|^{p_1-1} + a(x)|h|^{p_2-1}\right). \tag{1.6}$$

Equation (1.1) is modeled on the following Euler-Lagrange equation

$$-\mathrm{div}(p_1|Du|^{p_1-2}Du + a(x)p_2|Du|^{p_2-2}Du) = -\mathrm{div}(|f|^{p_1-2}f + a(x)|f|^{p_2-2}f),$$

for the functional

$$u \mapsto \int_{\mathbb{R}^n} (|Du|^{p_1} + a(x)|Du|^{p_2})\,dx - \int_{\mathbb{R}^n} \left\langle |f|^{p_1-2}f + a(x)|f|^{p_2-2}f, Du\right\rangle dx.$$

Let

$$\mathcal{P}(u,U) := \int_U (|Du|^{p_1} + a(x)|Du|^{p_2})\,dx$$

be the double phase functional, whenever $u \in W^{1,1}(U)$, $U \subset \mathbb{R}^n$ is open and $n \geq 2$. Zhikov [1] was the first to propose and investigate the functional $\mathcal{P}$. When studying the characterization of materials exhibiting strong anisotropy, Zhikov discovered that their hardening properties drastically change by the point; for example, refer to [2]. According to Marcellini's terminology in [3], the functional $\mathcal{P}$ is one of the functionals, which are defined by integrals with nonstandard growth conditions. In the functional $\mathcal{P}$, the coefficient function $a(\cdot)$ serves as an auxiliary tool to control the mixing between two distinct materials, which exhibits power hardening behaviors with rates p_1 and p_2. If $a(x) > 0$, the composite includes the p_2-material as one of its constituents, while the p_1-material is the sole constituent when $a(x) = 0$. The functional $\mathcal{P}$ also provides a new instance of the Lavrentiev phenomenon in action, see [4]. To find out more properties of the functional $\mathcal{P}$ and its current research status, readers can review [5–7] and the references contained within.

An interesting topic is the Calderón-Zygmund type estimates (L^p estimates) of the double phase equations in the whole space $\mathbb{R}^n$. The primary objective of L^p estimates is to derive the L^p bounds of various operators and solutions to equations in Sobolev spaces, which have been demonstrated to be a tool in numerous areas of partial differential equations and harmonic analysis.

Iwaniec [8] is credited with initiating the study of the nonlinear Calderón-Zygmund theory. One of his groundbreaking contributions is the proof of the inequality

$$||\nabla u||_p \leq c||f||_p,$$

for $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with $1 < p < \infty$. This inequality gives estimates of solutions to the equation

$$\text{div}(\nabla u) = \text{div} f.$$

In addition to this, Iwaniec also proved a local regularity result for weak solutions of

$$-\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = -\text{div}\left(|f|^{p-2} f\right),$$

in the subdomain $\Omega^{'}$ of $\mathbb{R}^n$. Specifically, he showed that for every $s > 1$ there is

$$|f|^p \in L^s_{loc} \Rightarrow |\nabla u|^p \in L^s_{loc}.$$

DiBenedetto and Manfredi [9] generalized Iwaniec's results to the case of vector-valued functions in the context of the p-laplace system. They have made important contributions to the development of this theory. DiBenedetto and Manfredi [9] also proved the following global L^p estimate

$$\int_{\mathbb{R}^n} |\nabla u|^{pq} dx \leq C \int_{\mathbb{R}^n} |f|^{pq} dx, \tag{1.7}$$

for weak solutions of

$$\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \text{div}\left(|f|^{p-2} f\right) \ \ in \ \mathbb{R}^n, \tag{1.8}$$

where $1 < p \leq q$.

Yao [10] extended it to the subsequent quasilinear equations

$$\text{div}\left(g(|\nabla u|)\nabla u\right) = \text{div}\left(g(|f|) f\right) \ \ in \ \mathbb{R}^n, \tag{1.9}$$

where the function $g : (0, \infty) \to (0, \infty) \in C^1(0, \infty)$ satisfies

$$0 \leq \inf_{t>0} \frac{tg'(t)}{g(t)} \leq \sup_{t>0} \frac{tg'(t)}{g(t)} < \infty, \tag{1.10}$$

and the following conclusion is given by defining $B(t) = \int_0^t \tau g(\tau) d\tau$,

$$\int_{\mathbb{R}^n} [B(|\nabla u|)]^q dx \leq C \int_{\mathbb{R}^n} [B(|f|)]^q dx. \tag{1.11}$$

In particular, if $g(t) = t^{p-2}$ for $p \geq 2$, then Eq (1.9) degenerates into Eq (1.8), where $B(t) = t^p$. In this case, the corresponding conclusion (1.11) also becomes (1.7). In addition, regularity studies of solutions to other related equations in $\mathbb{R}^n$ can also be seen in [11–13].

As yet, there have been no studies of Calderón-Zygmund estimates for the double phase problems in $\mathbb{R}^n$. But, there are many results about the regularity of related double phase operators and equations in bounded domains in $\mathbb{R}^n$, where $n \geq 2$. Throughout the rest of this article, $\Omega \subset \mathbb{R}^n$ is used to denote the bounded open set. It is worth noting that Colombo and Mingione [5] proved that $(|f|^{p_1} + a(x)|f|^{p_2}) \in L^s_{loc}(\Omega)$, which implies $(|Du|^{p_1} + a(x)|Du|^{p_2}) \in L^s_{loc}(\Omega)$ for weak solutions of (1.1) in Ω. This has greatly contributed to the development of regularity estimates for the double phase problems. Additionally, there exists a plethora of relevant literatures about the regularity theory of the double phase problems, for example [14–19].

Inspired by the above papers, the Calderón-Zygmund estimates of double phase equations in $\mathbb{R}^n$ are established in this paper by using a method similar to that in [10]. But in this article, the introduction of the weight function $a(x)$ in the double phase problems prevents the direct application of Lemma 2.4 in [10] when estimating the function. In this case, the frozen function equation needs to be introduced, and the inverse Hölder inequality needs to be used in the comparison estimates. However, the prerequisite for the inverse Hölder inequality to be used is that it is discussed in the sphere B_R, $R \leq 1$, which leads to the fact that in this paper we need to discuss the region of integration during the final integration.

It should be indicated that the optimal bound (1.5) is inevitable for the regularity under consideration here, see [20]. The fulfillment of condition (1.5) is critical in ensuring the stability of all the constants involved and ultimately maintaining the smallness of certain essential positive quantities.

The following notation is used in this article to simplify the description,

$$G(x, h) := |h|^{p_1} + a(x)|h|^{p_2}, \tag{1.12}$$

where $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$.

Denote

$$D \equiv D\Big(n, p_1, p_2, l, L, \alpha, [a]_{0,\alpha}, \|G(\cdot, Du)\|_{L^1(\mathbb{R}^n)}, \|G(\cdot, f)\|_{L^1(\mathbb{R}^n)}\Big),$$

to shorten the notation. Then the primary result of this paper is as follows.

Theorem 1.1. *If u is the weak solution to problem (1.1) subject to the assumptions (1.2)–(1.6), and provided that $G(x, Du)$ and $G(x, f)$ belong to $L^1(\mathbb{R}^n)$, then for every $s > 1$, the following result holds:*

$$G(x, f) \in L^s(\mathbb{R}^n) \Rightarrow G(x, Du) \in L^s(\mathbb{R}^n). \tag{1.13}$$

Moreover, for every $s > 1$, there exists $C \equiv C(D, s)$ such that

$$\int_{\mathbb{R}^n} [G(x, Du)]^s dx \leq C \int_{\mathbb{R}^n} [G(x, f)]^s dx + C. \tag{1.14}$$

Remark 1.2. *The difference between the conclusion presented in (1.14) of Theorem 1.1 and the conclusion presented in (1.11) from [10] lies in the presence of an additional constant C on the right side of (1.14). It is also interesting to obtain a result without the constant term. So, our subsequent research will be devoted to obtaining a result of the same form as (1.11). Specifically, our goal is to verify whether the following inequality holds:*

$$\int_{\mathbb{R}^n} [G(x, Du)]^s dx \leq C \int_{\mathbb{R}^n} [G(x, f)]^s dx.$$

Theorem 1.1 can be proved by the technique developed in [21, 22], which involves a new iteration-covering method introduced by Acerbi and Mingione. This approach utilizes an exit time argument and Vitali's covering lemma, instead of the Calderón-Zygmund decomposition and maximal functions, and has now become widely adopted in L^p-type regularity theory. Additionally, the applications of Calderón-Zygmund theory can be found in [23, 24].

The structure of this paper is arranged as follows. The subsequent section presents preliminary definitions and lemmas that are necessary for the discussion that follows. In the final section, several significant lemmas are presented, and the main conclusions are proved.

2. Preliminaries

In this paper, the notation c stands for a general constant and $c \geq 1$, which differs depending on the line. Similar notations will be used to denote special occurrences as $\overline{c}, \tilde{c}, c_1$ and C. Furthermore, parentheses will be employed to emphasize the relevant dependence on parameters. For example, $c \equiv c(D)$ signifies that c depends on D. The set $B_r(x_0)$ is defined as $\{x \in \mathbb{R}^n : |x - x_0| < r\}$. Additionally, it is stated that $B_1 = B_1(0)$ unless otherwise specified. Moreover, for functions g_1 and g_2 on $\mathbb{R}^n$, when $g_1 \sim g_2$ appears in this paper, it implies that there exist constants $m, m_0 > 0$ making $mg_1 \leq g_2 \leq m_0 g_1$ hold.

Given a function $b : \mathbb{R}^n \to \mathbb{R}$ and a subset $\mathcal{B} \subset \mathbb{R}^n$, where $\alpha \in (0, 1]$ is a given number, the notation is defined as follows:

$$[b]_{0,\alpha;\mathcal{B}} := \sup_{x,y\in\mathcal{B}, x\neq y} \frac{|b(x) - b(y)|}{|x - y|^\alpha}, \ [b]_{0,\alpha} \equiv [b]_{0,\alpha;\mathbb{R}^n}.$$

Furthermore, for a measurable set $Q \subset \mathbb{R}^n$ with $0 < |Q| < \infty$, and a locally integrable map $d : Q \to \mathbb{R}^k$ where $k \geq 1$, the integral average is represented as:

$$(d)_Q \equiv \fint_Q d(x)dx := \tfrac{1}{|Q|} \int_Q d(x)dx.$$

The following presents the definition of a weak solution and several lemmas that are required for subsequent use in this paper.

Definition 2.1. *A function $u \in W^{1,1}(\mathbb{R}^n)$ is defined as a weak solution of problem (1.1) with $G(x, Du), G(x, f) \in L^1(\mathbb{R}^n)$, if the following identity*

$$\int_{\mathbb{R}^n} A(x, Du) \cdot D\varphi dx = \int_{\mathbb{R}^n} F(x, f) \cdot D\varphi dx$$

holds for all $\varphi \in W_0^{1,1}(\mathbb{R}^n)$.

A result similar to Theorem 3.1 in [5] is given below, which also holds when the region under study changes from a bounded domain to $\mathbb{R}^n$.

Lemma 2.2. *If conditions (1.2)–(1.5) hold, the function $z_0 \in W^{1,1}(\tilde{B})$ satisfies $G(x, Dz_0) \in L^1(\tilde{B})$, where $\tilde{B} \subset \mathbb{R}^n$ is open, then there exists a unique solution $z \in z_0 + W_0^{1,p_1}(B)$ for the following Dirichlet problem*

$$\begin{cases} -\text{div}A(x, Dz) = 0 \ \textit{in}\ B, \\ z \in z_0 + W_0^{1,p_1}(B), \end{cases}$$

such that $G(x, Dz) \in L^1(B)$, where $B \Subset \tilde{B} \subset \mathbb{R}^n$. Moreover, it follows that

$$Dz \in L_{loc}^{\frac{np_1}{n-2\alpha_1}}(B) \cap W_{loc}^{\min\{\frac{2\alpha_1}{p_1},\alpha_1\},p_1}(B), \quad \text{for every } \alpha_1 < \alpha,$$

and

$$\int_B G(x, Dz)dx \leq c \int_B G(x, Dz_0)dx,$$

where $c \equiv c(n, p_1, p_2, l, L)$. In particular, it can be inferred that

$$Dz \in L_{loc}^{2p_2-p_1}(B) \subset L_{loc}^{p_2}(B).$$

Proof. The main proof process can be found in the proof of Theorem 3.1 in [5]. It should be noted that the first step in its proof still holds when the region changes from a bounded domain Ω to $\mathbb{R}^n$. This is because the first conclusion of Theorem 4.1 in [6] is still valid when the region is $\mathbb{R}^n$.

The specific reason is that remark 4 of [6] still allows us to take $z_0 \in W^{1,p_1}(\mathbb{R}^n)$ with the property that $G(x, Dz_0) \in L^1(\tilde{B})$ and find a sequence $\{\tilde{z}_k\} \subset C^\infty(B)$ which satisfies the property that $\tilde{z}_k \to z_0$ strongly in $W^{1,p_1}(B)$ and

$$\int_B G(x, D\tilde{z}_k)dx \to \int_B G(x, Dz_0)dx.$$

Since the rest of the discussion is in B, the conclusion is valid in $\mathbb{R}^n$.

Once $Dz \in L_{loc}^{2p_2-p_1}(B)$ is established, the conditional reverse Hölder type inequality can be derived.

Lemma 2.3. ([5],Theorem 4.1) *Consider $z \in W^{1,p_1}(B)$, which is a solution to*

$$-\mathrm{div}A(x, Dz) = 0 \quad in\ B,$$

under the assumptions (1.2)–(1.5) and $G(x, Dz) \in L^1(B)$. In addition, assume

$$\max_{x \in \overline{B}_r} a(x) \leq M_1 [a]_{0,\alpha} r^\alpha,$$

where $B_r \subset B \subset \mathbb{R}^n$, $r \leq 1$, and $M_1 \geq 1$. Then, for all q' less than $np_1/(n - 2\alpha)$, if $\alpha = 1$ and $n = 2$, then $q' = \infty$, and there exists a constant $c \equiv c(D, M_1, q')$ such that the following inequality holds:

$$\left(\fint_{B_{r/2}} |Du|^{q'} dx\right)^{1/q'} \leq c \left(\fint_{B_r} |Du|^{p_1} dx\right)^{1/p_1},$$

and the constant c increases monotonically with respect to $\|Dh_1\|_{L^{p_1}(B_r)}$.

3. The proof of Theorem 1.1

Since Lemma 2.3 needs to be used in the proof process, $R \leq 1$ is selected first and will be determined later. Next, set

$$\lambda_0 := \frac{20^n}{|B_1| R^n} \int_{\mathbb{R}^n} \left[G(x, Du) + \frac{1}{\delta} G(x, f)\right] dx. \tag{3.1}$$

Take into account the sets

$$E(Du, \lambda) := \{x \in \mathbb{R}^n : G(x, Du(x)) > \lambda\}, \quad \lambda > 0,$$

and define a function Φ for any fixed point $x_0 \in E(Du, \lambda)$ such that

$$\Phi(B_\rho(x_0)) := \fint_{B_\rho(x_0)} \left[G(x, Du) + \tfrac{1}{\delta}G(x, f)\right] dx, \tag{3.2}$$

where $B_\rho(x_0) \subset \mathbb{R}^n$ and $0 < \delta < 1$ are to be determined later. It should be noted that Φ is monotonically decreasing with respect to ρ.

For the subsequent proof, the following iteration-covering lemma will be given, which is a pure PDE method and draws significant inspiration from [21].

Lemma 3.1. *For any $\lambda > \lambda_0$, there is a collection of disjoint balls $\{B_{\rho_i}(x_i)\}_{i\geqslant 1}$ satisfying $x_i \in E(Du, \lambda)$ and $0 < \rho_i = \rho(x_i, \lambda) \leq \frac{R}{20}$ such that*

$$E(Du, \lambda) \subset \bigcup_{i\geq 1} B_{5\rho_i}(x_i) \cup \text{negligible set}, \tag{3.3}$$

and

$$\Phi[B_{\rho_i}(x_i)] = \lambda,\ \Phi[B_\rho(x_i)] < \lambda \quad \text{for every } \rho \in (\rho_i, R]. \tag{3.4}$$

Moreover,

$$\left|B_{\rho_i}(x_i)\right| \leq \frac{2}{\lambda} \int_{\left\{x\in B_{\rho_i}(x_i):G(x,Du(x))>\frac{\lambda}{4}\right\}} G(x, Du)dx + \frac{2}{\delta\lambda} \int_{\left\{x\in B_{\rho_i}(x_i):G(x,f(x))>\frac{\delta\lambda}{4}\right\}} G(x, f)dx. \tag{3.5}$$

Proof. For almost every $x_0 \in E(Du, \lambda)$, the Lebesgue differentiation theorem implies that

$$\lim_{\rho\to 0} \Phi(B_\rho(x_0)) > \lambda. \tag{3.6}$$

However , for any $x_0 \in \mathbb{R}^n$ and $\rho \in \left[\frac{R}{20}, R\right]$, it can be shown that

$$\Phi(B_\rho(x_0)) \leq \frac{20^n}{|B_1|R^n} \int_{\mathbb{R}^n} \left[G(x, Du) + \tfrac{1}{\delta}(G(x, f)\right] dx = \lambda_0 < \lambda. \tag{3.7}$$

Since Φ is monotonic, it follows from (3.6) and (3.7) that for almost every $x_0 \in E(Du, \lambda)$, there is a radius $\rho_0 \in \left(0, \frac{R}{20}\right)$ such that

$$\Phi\left(B_{\rho_0}(x_0)\right) = \lambda \quad \text{and} \quad \Phi(B_\rho(x_0)) < \lambda \quad \text{for all } \rho \in (\rho_0, R].$$

Then, the family $\left\{B_{\varrho_0}(x_0)\right\}$ covers $E(Du, \lambda)$ up to a negligible set. By Vitali's covering lemma, there is a countable collection of mutually disjoint balls $\left\{B_{\rho_i}(x_i)\right\}_{i=1}^{\infty}$, where $x_i \in E(Du, \lambda)$ and $\rho_i \in \left(0, \frac{R}{20}\right)$ such that

$$E(Du, \lambda) \subset \bigcup_{i\geq 1} B_{5\rho_i}(x_i) \cup \text{negligible set},$$

and

$$\Phi\left(B_{\rho_i}(x_i)\right) = \lambda \text{ and } \Phi(B_\rho(x_i)) < \lambda \quad \text{, for all } \rho \in (\rho_i, R].$$

That is, (3.3) and (3.4) have been confirmed.

Next, it follows from (3.2) and $(3.4)_1$ that

$$\fint_{B_{\rho_i}(x_i)} \left[G(x, Du) + \tfrac{1}{\delta}G(x, f)\right] dx = \lambda. \tag{3.8}$$

Obviously, by decomposing the integral region in (3.8), it is clear that the following equality holds:

$$\begin{aligned}\lambda\left|B_{\rho_i}(x_i)\right| &\leq \int_{\left\{x\in B_{\rho_i}(x_i):G(x,Du(x))>\frac{\lambda}{4}\right\}} G(x,Du)dx+\frac{\lambda}{4}\left|B_{\rho_i}(x_i)\right| \\ &\quad +\frac{1}{\delta}\int_{\left\{x\in B_{\rho_i}(x_i):G(x,f(x))>\frac{\delta\lambda}{4}\right\}} G(x,f)dx+\frac{\lambda}{4}\left|B_{\rho_i}(x_i)\right|.\end{aligned}$$

Clearly, (3.5) holds.

The comparison estimates, which are similar to those obtained in [5], will be discussed in the family of countable balls obtained in Lemma 3.1. The proof in [5] will also be modified in this paper to obtain suitable results.

Before proceeding, two related problems need to be introduced. For every ball $B_{5\rho_i}$ that is considered in (3.3), Lemma 2.2 states that $v_1 \in u + W_0^{1,p_1}(B_{20\rho_i}(x_i))$ can be established as the solution to

$$\begin{cases} -\mathrm{div}A(x,Dv_1)=0 \ in\ B_{20\rho_i}(x_i), \\ v_1 \in u+W_0^{1,p_1}(B_{20\rho_i}(x_i)). \end{cases} \tag{3.9}$$

It follows that

$$v_1 \in W_{loc}^{1,2p_2-p_1}(B_{20\rho_i}(x_i)) \text{ and } Dv_1 \in L_{loc}^{np_1/(n-2\alpha_1)}(B_{20\rho_i}(x_i)), \text{ for every } \alpha_1<\alpha, \tag{3.10}$$

and

$$⨍_{B_{20\rho_i}(x_i)} G(x,Dv_1)\,dx \leq c⨍_{B_{20\rho_i}(x_i)} G(x,Du)dx, \tag{3.11}$$

where $c \equiv c(n,p_1,p_2,l,L)$. Furthermore, a comparison estimate can be obtained as presented below.

Lemma 3.2. *For any $\lambda > \lambda_0$, if u is the weak solution of (1.1) in $\mathbb{R}^n$, then the inequality*

$$⨍_{B_{20\rho_i}(x_i)} G(x,Du-Dv_1)\,dx \leq \epsilon_1\lambda \tag{3.12}$$

holds for every $\epsilon_1 \in (0,1)$.

Proof. For the convenience of subsequent proof, here we show that for arbitrary $\eta_1 \in (0,1)$, $\rho \in (\rho_i, R]$ and $h_1, h_2 \in \mathbb{R}^n$, there is

$$\begin{aligned}&⨍_{B_\rho(x_i)} G(x,h_1-h_2)\,dx \\ &\leq 2^{p_2-1}\eta_1⨍_{B_\rho(x_i)}(G(x,h_1)+G(x,h_2))\,dx \\ &\quad +\frac{1}{\eta_1}⨍_{B_\rho(x_i)}\left[(|h_1|+|h_2|)^{p_1-2}+a(x)(|h_1|+|h_2|)^{p_2-2}\right]|h_1-h_2|^2dx.\end{aligned} \tag{3.13}$$

Specifically distinguished into three cases to discuss.

Case 1: $2 \leq p_1 < p_2$. The definition of $G(x,h)$ in (1.12) and the triangle inequality leads directly to the following estimate:

$$⨍_{B_\rho(x_i)} G(x,h_1-h_2)\,dx \leq ⨍_{B_\rho(x_i)}\left[(|h_1|+|h_2|)^{p_1-2}+a(x)(|h_1|+|h_2|)^{p_2-2}\right]|h_1-h_2|^2dx. \tag{3.14}$$

Since $\eta_1 \in (0,1)$, (3.13) holds.

Case 2: $1 < p_1 < p_2 \le 2$. For $1 < k \le 2$ and any $b_1 \in L^\infty(B_\rho(x_i))$, using Young's inequality with $\eta_1 \in (0,1)$ gives that

$$\begin{aligned}
&\fint_{B_\rho(x_i)} b_1(x)|h_1-h_2|^k dx \\
&= \fint_{B_\rho(x_i)} b_1(x)\left(|h_1|+|h_2|\right)^{\frac{k(2-k)}{2}}\left[\left(|h_1|+|h_2|\right)^{\frac{k(k-2)}{2}}|h_1-h_2|^k\right]dx \\
&\le \eta_1 \fint_{B_\rho(x_i)} b_1(x)\left(|h_1|+|h_2|\right)^k dx \\
&\quad + \frac{k}{2}\left(\frac{2}{2-k}\right)^{\frac{k-2}{k}}\eta_1^{\frac{k-2}{k}}\fint_{B_\rho(x_i)} b_1(x)\left(|h_1|+|h_2|\right)^{k-2}|h_1-h_2|^2 dx.
\end{aligned}$$

And since for $1 \le t < \infty$, the inequality

$$(a'+b')^t \le \frac{1}{2}(2a')^t + \frac{1}{2}(2b')^t \le 2^{t-1}(a')^t + 2^{t-1}(b')^t \tag{3.15}$$

holds for arbitrary $a', b' > 0$, it can be further deduced that

$$\fint_{B_\rho(x_i)} b_1(x)\left(|h_1|+|h_2|\right)^k dx \le 2^{k-1}\fint_{B_\rho(x_i)} b_1(x)\left(|h_1|^k+|h_2|^k\right)dx. \tag{3.16}$$

Combining (3.16) with the fact that $1 < k \le 2$, then there is

$$\fint_{B_\rho(x_i)} b_1(x)|h_1-h_2|^k dx \le 2^{k-1}\eta_1\fint_{B_\rho(x_i)} b_1(x)\left(|h_1|^k+|h_2|^k\right)dx + \frac{1}{\eta_1}\fint_{B_\rho(x_i)} b_1(x)\left(|h_1|+|h_2|\right)^{k-2}|h_1-h_2|^2 dx.$$

Thus, it can be seen that

$$\begin{aligned}
&\fint_{B_\rho(x_i)} G(x, h_1-h_2)\, dx \\
&\le 2^{p_1-1}\eta_1\fint_{B_\rho(x_i)} \left(|h_1|^{p_1}+|h_2|^{p_1}\right)dx + \frac{1}{\eta_1}\fint_{B_\rho(x_i)} \left(|h_1|+|h_2|\right)^{p_1-2}|h_1-h_2|^2 dx \\
&\quad + 2^{p_2-1}\eta_1\fint_{B_\rho(x_i)} a(x)\left(|h_1|^{p_2}+|h_2|^{p_2}\right)dx + \frac{1}{\eta_1}\fint_{B_\rho(x_i)} a(x)\left(|h_1|+|h_2|\right)^{p_2-2}|h_1-h_2|^2 dx \\
&\le 2^{p_2-1}\eta_1\fint_{B_\rho(x_i)} \left(G(x,h_1)+G(x,h_2)\right)dx + \frac{1}{\eta_1}\fint_{B_\rho(x_i)}\left[\left(|h_1|+|h_2|\right)^{p_1-2} + a(x)\left(|h_1|+|h_2|\right)^{p_2-2}\right]|h_1-h_2|^2 dx.
\end{aligned}$$

Case 3: $1 < p_1 < 2 < p_2$. Merging the two aforementioned situations leads to the conclusion that

$$\begin{aligned}
&\fint_{B_\rho(x_i)} G(x, h_1-h_2)\, dx \\
&\le 2^{p_1-1}\eta_1\fint_{B_\rho(x_i)} \left(|h_1|^{p_1}+|h_2|^{p_1}\right)dx + \frac{1}{\eta_1}\fint_{B_\rho(x_i)} \left(|h_1|+|h_2|\right)^{p_1-2}|h_1-h_2|^2 dx \\
&\quad + \fint_{B_\rho(x_i)} a(x)\left(|h_1|+|h_2|\right)^{p_2-2}|h_1-h_2|^2 dx \\
&\le 2^{p_2-1}\eta_1\fint_{B_\rho(x_i)} \left(G(x,h_1)+G(x,h_2)\right)dx \\
&\quad + \frac{1}{\eta_1}\fint_{B_\rho(x_i)}\left[\left(|h_1|+|h_2|\right)^{p_1-2} + a(x)\left(|h_1|+|h_2|\right)^{p_2-2}\right]|h_1-h_2|^2 dx.
\end{aligned}$$

Thus, (3.13) is proved.

Here, make $\rho = 20\rho_i$, $h_1 = Du$ and $h_2 = Dv_1$ in (3.13), and combined with (3.11), it is concluded that

$$\begin{aligned}
&\fint_{B_{20\rho_i}(x_i)} G(x, Du-Dv_1)\, dx \\
&\le c\eta_1\fint_{B_{20\rho_i}(x_i)} G(x, Du)dx \\
&\quad + \frac{1}{\eta_1}\fint_{B_{20\rho_i}(x_i)}\left[\left(|Du|+|Dv_1|\right)^{p_1-2} + a(x)\left(|Du|+|Dv_1|\right)^{p_2-2}\right]|Du-Dv_1|^2 dx,
\end{aligned} \tag{3.17}$$

where $c \equiv c(n, p_1, p_2, l, L)$. For the first term on the right-hand side of (3.17), it is clear from $(3.4)_2$ (with $\rho = 20\rho_i$) and (3.2) that

$$\fint_{B_{20\rho_i}(x_i)} G(x, Du)dx \le \lambda. \tag{3.18}$$

For the second term on the right-hand side of (3.17), it is known from $(1.2)_2$ that for all $h_3, h_4 \in \mathbb{R}^n$ and a.e., $x \in \mathbb{R}^n$, there is

$$\begin{aligned}&\fint_{B_{20\rho_i}(x_i)} \left[(|h_3| + |h_4|)^{p_1-2} + a(x)\,(|h_3| + |h_4|)^{p_2-2}\right] |h_3 - h_4|^2 dx\\ &\leq c\fint_{B_{20\rho_i}(x_i)} \langle D_{h'}A(x,h')\,(h_3 - h_4)\,, (h_3 - h_4)\rangle\, dx,\end{aligned} \tag{3.19}$$

where $c \equiv c(l)$, $|h'| = |h_3| + |h_4|$. Then, using Lemma 19 in [25], we get that

$$D_{h'}A(x,h') \sim \int_0^1 D_{(\theta h_3 + (1-\theta)h_4)} A(x, \theta h_3 + (1-\theta)\,h_4) d\theta, \tag{3.20}$$

where $c \equiv c(p_1, p_2)$. Multiplying both sides of inequality (3.20) by $(h_3 - h_4)$ simultaneously gives that

$$D_{h'}A(x,h')\,(h_3 - h_4) \sim A(x,h_3) - A(x,h_4), \tag{3.21}$$

where $c \equiv c(p_1, p_2)$. Combining (3.19) and (3.21), and substituting $h_3 = Du$ and $h_4 = Dv_1$, we obtain that

$$\begin{aligned}&\fint_{B_{20\rho_i}(x_i)} \left[(|Du| + |Dv_1|)^{p_1-2} + a(x)\,(|Du| + |Dv_1|)^{p_2-2}\right] |Du - Dv_1|^2 dx\\ &\leq c\fint_{B_{20\rho_i}(x_i)} \langle A(x,Du) - A(x,Dv_1), (Du - Dv_1)\rangle\, dx,\end{aligned} \tag{3.22}$$

where $c \equiv c(p_1, p_2, l)$. Next we estimate the term to the right of (3.22). First, since v_1 is a solution to problem (3.9), using the test function $\varphi = u - v_1$ we have that

$$\fint_{B_{20\rho_i}(x_i)} \langle A(x,Dv_1), (Du - Dv_1)\rangle\, dx = 0.$$

Naturally, we have

$$\fint_{B_{20\rho_i}(x_i)} \langle A(x,Du) - A(x,Dv_1), (Du - Dv_1)\rangle\, dx = \fint_{B_{20\rho_i}(x_i)} \langle F(x,f), (Du - Dv_1)\rangle\, dx. \tag{3.23}$$

Then, combining (3.22) with (3.23) and (1.6), and utilizing Young's inequality with η_2 taken from the interval $(0,1)$, we get that

$$\begin{aligned}&\fint_{B_{20\rho_i}(x_i)} \left[(|Du| + |Dv_1|)^{p_1-2} + a(x)\,(|Du| + |Dv_1|)^{p_2-2}\right] |Du - Dv_1|^2 dx\\ &\leq c\fint_{B_{20\rho_i}(x_i)} \left[|f|^{p_1-1} + a(x)|f|^{p_2-1}\right] [|Du| + |Dv_1|]\, dx\\ &\leq c\left[\eta_2 \fint_{B_{20\rho_i}(x_i)} (G(x,Du) + G(x,Dv_1))\, dx + \eta_2^{-\frac{1}{p_1-1}} \fint_{B_{20\rho_i}(x_i)} G(x,f) dx\right].\end{aligned}$$

Recalling (3.11), $(3.4)_2$ (with $\rho = 20\rho_i$) and (3.2) yields that

$$\fint_{B_{20\rho_i}(x_i)} \left[(|Du| + |Dv_1|)^{p_1-2} + a(x)\,(|Du| + |Dv_1|)^{p_2-2}\right] |Du - Dv_1|^2 dx \leq c(\eta_2 + \eta_2^{-\frac{1}{p_1-1}}\delta)\lambda.$$

It can be inferred that

$$\fint_{B_{20\rho_i}(x_i)} \left[(|Du| + |Dv_1|)^{p_1-2} + a(x)\,(|Du| + |Dv_1|)^{p_2-2}\right] |Du - Dv_1|^2 dx \leq c\delta^{\kappa}\lambda, \tag{3.24}$$

by taking $\eta_2 = \delta^{\frac{p_1-1}{2}} \in (0,1)$, $\kappa = \min\{\frac{p_1-1}{2}, \frac{1}{2}\}$. Finally, from (3.24), (3.17) and (3.18), it shows that

$$\fint_{B_{20\rho_i}(x_i)} G\,(x, Du - Dv_1)\, dx \leq c\left(\eta_1 + \tfrac{1}{\eta_1}\delta^{\kappa}\right)\lambda \leq 2c\delta^{\frac{\kappa}{2}}\lambda \leq \epsilon_1\lambda,$$

by selecting $\eta_1 = \delta^{\frac{\kappa}{2}}$, $\delta = \left(\frac{\epsilon_1}{2c}\right)^{\frac{2}{\kappa}}$.

In the context of the above problem, considering a point $\tilde{x} \in \overline{B}_{10\rho_i}(x_i)$ such that

$$a(\tilde{x}) = \max_{x \in \overline{B}_{10\rho_i}(x_i)} a(x).$$

It is known from Lemma 2.2 that $v_2 \in v_1 + W_0^{1,p_1}(B_{10\rho_i}(x_i))$ can be established as the solution to

$$\begin{cases} -\mathrm{div} A(\tilde{x}, Dv_2) = 0 \ in \ B_{10\rho_i}(x_i), \\ v_2 \in v_1 + W_0^{1,p_1}(B_{10\rho_i}(x_i)), \end{cases}$$

and

$$\fint_{B_{10\rho_i}(x_i)} G(\tilde{x}, Dv_2)\, dx \le c \fint_{B_{10\rho_i}(x_i)} G(\tilde{x}, Dv_1) dx,$$

where $c \equiv c(n, p_1, p_2, l, L)$. Then, comparison estimates are made in two cases, as shown below:

$$\min_{x \in \overline{B}_{10\rho_i}(x_i)} a(x) > M[a]_{0,\alpha} \rho_i^{\alpha}, \tag{3.25}$$

$$\min_{x \in \overline{B}_{10\rho_i}(x_i)} a(x) \le M[a]_{0,\alpha} \rho_i^{\alpha}, \tag{3.26}$$

for a constant $M \ge 10$ that will be determined based on D. (3.25) and (3.26) are respectively called (p_1, p_2)-phase and p_1-phase. In the case of (3.26), the calculation of the comparison estimate requires the use of Lemma 2.3, which is established because of (3.10). Finally, the following two inequalities are obtained through calculation,

$$\begin{aligned} &\fint_{B_{10\rho_i}(x_i)} \left[(|Dv_1| + |Dv_2|)^{p_1 - 2} + a(\tilde{x}) (|Dv_1| + |Dv_2|)^{p_2 - 2} \right] |Dv_1 - Dv_2|^2 dx \\ &\le \left(\frac{\bar{c}}{M} + \tilde{c} R^{\sigma} \right) \fint_{B_{20\rho_i}(x_i)} G(x, Du) dx, \end{aligned} \tag{3.27}$$

where $\bar{c} \equiv \bar{c}(n, p_1, p_2, l, L)$, $\tilde{c} \equiv \tilde{c}(D, M)$ and $\sigma = \alpha - n(\frac{p_2}{p_1} - 1) > 0$, and

$$\max_{x \in \overline{B}_{5\rho_i}(x_i)} G(x, Dv_2) \le \max_{x \in \overline{B}_{5\rho_i}(x_i)} G(\tilde{x}, Dv_2) \le c \fint_{B_{10\rho_i}(x_i)} G(\tilde{x}, Dv_2) dx \le c \fint_{B_{20\rho_i}(x_i)} G(x, Du) dx, \tag{3.28}$$

where $c \equiv c(n, p_1, p_2, l, L, \alpha, [a]_{0,\alpha}, \|G(\cdot, Du)\|_{L^1(\mathbb{R}^n)})$. For specific calculation of comparison estimates, please refer to steps 5–9 in the proof of Theorem 1.1 in [5].

Next we give a few important lemmas to be used in this paper.

Lemma 3.3. *For any $\lambda > \lambda_0$, the following inequality*

$$\fint_{B_{10\rho_i}(x_i)} G(x, Dv_1 - Dv_2)\, dx \le \epsilon_2 \lambda \tag{3.29}$$

holds for every $\epsilon_2 \in (0, 1)$.

Proof. It can be known from (3.27) that the inequality

$$\fint_{B_{10\rho_i}(x_i)} \left[(|Dv_1| + |Dv_2|)^{p_1 - 2} + a(\tilde{x}) (|Dv_1| + |Dv_2|)^{p_2 - 2} \right] |Dv_1 - Dv_2|^2 dx \le \eta_3 \fint_{B_{20\rho_i}(x_i)} G(x, Du) dx \tag{3.30}$$

holds for any $\eta_3 \in (0,1)$ by taking $M := \frac{2\bar{c}}{\eta_3}$, $R := (\frac{\eta_3}{2\bar{c}})^{\frac{1}{\sigma}}$. Combining (3.30) with (3.13) and setting $x = \tilde{x}$, $\rho = 10\rho_i$, $h_1 = Dv_1$, $h_2 = Dv_2$, $\eta_1 = \eta_1'$, then, using the definition of $a(\tilde{x})$, we have

$$\begin{aligned}
&\fint_{B_{10\rho_i}(x_i)} G(x, Dv_1 - Dv_2)\,dx \\
&\leq 2^{p_2-1}\eta_1' \fint_{B_{10\rho_i}(x_i)} [G(x, Dv_1) + G(x, Dv_2)]\,dx \\
&\quad + \frac{1}{\eta_1'} \fint_{B_{10\rho_i}(x_i)} \left[(|Dv_1| + |Dv_2|)^{p_1-2} + a(x)\,(|Dv_1| + |Dv_2|)^{p_2-2}\right] |Dv_1 - Dv_2|^2 dx \\
&\leq 2^{p_2-1}\eta_1' \fint_{B_{10\rho_i}(x_i)} [G(x, Dv_1) + G(\tilde{x}, Dv_2)]\,dx \\
&\quad + \frac{1}{\eta_1'} \fint_{B_{10\rho_i}(x_i)} \left[(|Dv_1| + |Dv_2|)^{p_1-2} + a(\tilde{x})\,(|Dv_1| + |Dv_2|)^{p_2-2}\right] |Dv_1 - Dv_2|^2 dx.
\end{aligned}$$

Recalling again (3.11), (3.28), (3.18) and (3.27), we end up with that

$$\begin{aligned}
&\fint_{B_{10\rho_i}(x_i)} G(x, Dv_1 - Dv_2)\,dx \\
&\leq c\eta_1' \fint_{B_{20\rho_i}(x_i)} G(x, Du) dx \\
&\quad + \frac{1}{\eta_1'} \fint_{B_{10\rho_i}(x_i)} \left[(|Dh| + |Dv|)^{p_1-2} + a(x)\,(|Dh| + |Dv|)^{q-2}\right] |Dh - Dv|^2 dx \\
&\leq c\left(\eta_1' + \frac{\eta_3}{\eta_1'}\right)\lambda.
\end{aligned}$$

By selecting $\eta_1' = \eta_3^{\frac{1}{2}}$ and $\eta_3 = \left(\frac{\epsilon_2}{2c}\right)^2$, it can be shown that (3.29) holds true.

Combining Lemma 3.2 with Lemma 3.3, the following final comparison estimate can be obtained.

Lemma 3.4. *If u is the weak solution of (1.1) in $\mathbb{R}^n$, then for each $\epsilon \in (0,1)$ there is*

$$\fint_{B_{10\rho_i}(x_i)} G(x, Du - Dv_2)\,dx \leq \epsilon\lambda. \tag{3.31}$$

Proof. It can be seen from (3.15) that for any $h_3, h_4 \in \mathbb{R}^n$, the following inequality is valid:

$$\begin{aligned}
G(x, h_3 + h_4) &\leq (|h_3| + |h_4|)^{p_1} + a(x)\,(|h_3| + |h_4|)^{p_2} \\
&\leq 2^{p_1-1}|h_3|^{p_1} + 2^{p_1-1}|h_4|^{p_1} + 2^{p_2-1}a(x)|h_3|^{p_2} + 2^{p_2-1}a(x)|h_4|^{p_2} \\
&\leq c_1 G(x, h_3) + c_1 G(x, h_4),
\end{aligned} \tag{3.32}$$

where $c_1 \equiv c_1(p_1, p_2)$. From (3.12) and (3.29), it can be seen that

$$\begin{aligned}
\fint_{B_{10\rho_i}(x_i)} G(x, Du - Dv_2)\,dx &\leq c_1 \fint_{B_{10\rho_i}(x_i)} G(x, Du - Dv_1)\,dx + c_1 \fint_{B_{10\rho_i}(x_i)} G(x, Dv_1 - Dv_2)\,dx \\
&\leq c_1'\,(\epsilon_1 + \epsilon_2)\,\lambda,
\end{aligned}$$

where $c_1' \equiv c_1'(n, p_1, p_2)$. The final result (3.31) is obtained by selecting $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2c_1'}$.

Lemma 3.5. *For any $\lambda > \lambda_0$, there exists $M_0(n, p_1, p_2, l, L, \alpha, [a]_{0,\alpha}, \|G(\cdot, Du)\|_{L^1(\mathbb{R}^n)}) \geq 1$ such that*

$$\max_{x \in \overline{B}_{5\rho_i}(x_i)} G(x, Dv_2) dx \leq M_0\lambda. \tag{3.33}$$

Proof. It is straightforward to use (3.28) in combination with (3.18) to get (3.33).

Based on the above estimated results and (3.32), for any $\lambda > \lambda_0$, it can be deduced as follows:

$$\begin{aligned}
&\left|\left\{x \in B_{5\rho_i}(x_i) : G(x, Du) > 2c_1M_0\lambda\right\}\right| \\
&\leq \left|\left\{x \in B_{5\rho_i}(x_i) : G(x, Du - Dv_2) > M_0\lambda\right\}\right| + \left|\left\{x \in B_{5\rho i}(x_i) : G(x, Dv_2) > M_0\lambda\right\}\right| \\
&= \left|\left\{x \in B_{5\rho i}(x_i) : G(x, Du - Dv_2) > M_0\lambda\right\}\right| \\
&\leq \frac{1}{M_0\lambda} \int_{B_{10\rho i}(x_i)} G(x, Du - Dv_2)dx \\
&\leq \epsilon|B_{10\rho i}(x_i)| \\
&= 10^n\epsilon|B_{\rho i}(x_i)|.
\end{aligned}$$

Hence, it can be inferred from (3.5) that

$$\begin{aligned}
&\left|\left\{x \in B_{5\rho_i}(x_i) : G(x, Du) > 2c_1M_0\lambda)\right\}\right| \\
&\leq \frac{2\cdot 10^n\epsilon}{\lambda}\left(\int_{\{x\in B_{\rho_i(x_i)}:G(x,Du)>\frac{\lambda}{4}\}} G(x, Du)dx + \frac{1}{\delta}\int_{\{x\in B_{\rho_i(x_i)}:G(x,f)>\frac{\delta\lambda}{4}\}} G(x, f)dx\right).
\end{aligned} \tag{3.34}$$

Referring to (3.3) again, we know that the balls in $\{B_{\rho_i}(x_i)\}_{i\in N}$ are disjoint and

$$E(Du, 2c_1M_0\lambda) = \{x \in \mathbb{R}^n : G(x, Du) > 2c_1M_0\lambda\} \subset \bigcup_{i\in N} B_{5\rho_i}(x_i) \cup \text{negligible set},$$

for any $\lambda > \lambda_0$. Then, by summing up (3.34) over $i \in N$, we have

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : G(x, Du) > 2c_1M_0\lambda)\}| \\
&\leq \sum_i \left|\left\{x \in B_{5\rho_i}(x_i) : G(x, Du) > 2c_1M_0\lambda)\right\}\right| \\
&\leq \frac{2\cdot 10^n\epsilon}{\lambda}\left(\int_{E(Du,\frac{\lambda}{4})} G(x, Du)dx + \frac{1}{\delta}\int_{\{x\in\mathbb{R}^n:G(x,f)>\frac{\delta\lambda}{4}\}} G(x, f)dx\right).
\end{aligned} \tag{3.35}$$

The proof of Theorem 1.1 is given below.

Proof of Theorem 1.1. We first give two important equations to be used subsequently.

Elementary measure theory yields the following equations,

$$\int_{\mathbb{R}^n} |F|^\gamma dx = \gamma \int_{\lambda>0} \lambda^{\gamma-1} |\{x \in \mathbb{R}^n : |F| > \lambda\}| \, d\lambda, \tag{3.36}$$

which is Theorem 1.9 in [26], and

$$\int_{\mathbb{R}^n} |F|^\gamma dx = (\gamma - 1) \int_{\lambda>0} \lambda^{\gamma-2} \int_{\{x\in\mathbb{R}^n:|F|>\lambda\}} |F| dx d\lambda, \tag{3.37}$$

which can be seen in [10]. By (3.36), the following calculation can be performed:

$$\begin{aligned}
\int_{\mathbb{R}^n} [G(x, Du)]^s dx &= s(2c_1M_0)^s \int_0^\infty \lambda^{s-1} |\{x \in \mathbb{R}^n : G(x, Du) > 2c_1M_0\lambda\}| \, d\lambda \\
&= s(2c_1M_0)^s \int_0^{\lambda_0} \lambda^{s-1} |\{x \in \mathbb{R}^n : G(x, Du) > 2c_1M_0\lambda\}| \, d\lambda \\
&\quad + s(2c_1M_0)^s \int_{\lambda_0}^\infty \lambda^{s-1} |\{x \in \mathbb{R}^n : G(x, Du) > 2c_1M_0\lambda\}| \, d\lambda \\
&=: I_1 + I_2.
\end{aligned}$$

Then, we can obtain that

$$
\begin{aligned}
I_1 &\le s(2c_1M_0)^{s-1}\int_0^{\lambda_0}\lambda^{s-2}\int_{\{x\in\mathbb{R}^n:G(x,Du)>2c_1M_0\lambda\}}G(x,Du)dxd\lambda\\
&\le s(2c_1M_0)^{s-1}\int_0^{\lambda_0}\lambda^{s-2}\int_{\mathbb{R}^n}G(x,Du)dxd\lambda\\
&\le \tfrac{s(2c_1M_0)^{s-1}\lambda_0^{s-1}}{s-1}\int_{\mathbb{R}^n}G(x,Du)dx\\
&\le \tfrac{s(2c_1M_0)^{s-1}}{s-1}\tfrac{|B_1|R^n}{20^n}\lambda_0^s\\
&\le c_2,
\end{aligned}
\tag{3.38}
$$

where c_2 depends on D, s. The definition of λ_0 in (3.1) is used in the calculation, from which we can obtain that

$$
\lambda_0 \le \tfrac{20^n}{|B_1|R^n}\left[\|G(x,Du)\|_{L^1(\mathbb{R}^n)}+\tfrac{1}{\delta}\|G(x,f)\|_{L^1(\mathbb{R}^n)}\right],
$$

where R is selected earlier in Lemma 3.3.

For the estimate of I_2, obviously through (3.35) and (3.37), there is

$$
\begin{aligned}
I_2 &\le c\epsilon\left\{\int_{\lambda_0}^{\infty}\lambda^{s-2}\int_{E(Du,\frac{\lambda}{4})}G(x,Du)dxd\lambda+\tfrac{1}{\delta}\int_{\lambda_0}^{\infty}\lambda^{s-2}\int_{\{x\in\mathbb{R}^n:G(x,f)>\frac{\delta\lambda}{4}\}}G(x,f)dxd\lambda\right\}\\
&\le c\epsilon\left\{\int_0^{\infty}\lambda^{s-2}\int_{E(Du,\frac{\lambda}{4})}G(x,Du)dxd\lambda+\tfrac{1}{\delta}\int_0^{\infty}\lambda^{s-2}\int_{\{x\in\mathbb{R}^n:G(x,f)>\frac{\delta\lambda}{4}\}}G(x,f)dxd\lambda\right\}\\
&\le c_3\epsilon\int_{\mathbb{R}^n}[G(x,Du)]^sdx+c_4\int_{\mathbb{R}^n}[G(x,f)]^sdx,
\end{aligned}
\tag{3.39}
$$

where $c_3 \equiv c_3(n, p_1, p_2, l, L, s)$ and $c_4 \equiv c_4(n, p_1, p_2, l, L, s, \epsilon)$. Combining (3.38) with (3.39), we get that

$$
\int_{\mathbb{R}^n}[G(x,Du)]^sdx \le c_3\epsilon\int_{\mathbb{R}^n}[G(x,Du)]^sdx+c_4\int_{\mathbb{R}^n}[G(x,f)]^s+c_2.
$$

Eventually, selecting suitable ϵ such that $c_3\epsilon = 1/2$, this yields

$$
\int_{\mathbb{R}^n}[G(x,Du)]^sdx \le C\int_{\mathbb{R}^n}[G(x,f)]^sdx+C,
$$

where C depends on D, s, and then (1.13) holds. In summary, Theorem 1.1 is substantiated.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.*, **29** (1987), 33–66. https://doi.org/10.1070/IM1987v029n01ABEH000958
2. V. V. Jikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer Berlin, Heidelberg, 1994. https://doi.org/10.1007/978-3-642-84659-5
3. P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, *Arch. Ration. Mech. Anal.*, **105** (1989), 267–284. https://doi.org/10.1007/BF00251503
4. M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.*, **218** (2015), 219–273. https://doi.org/10.1007/s00205-015-0859-9
5. M. Colombo, G. Mingione, Calderón-Zygmund estimates and nonuniformly elliptic operators, *J. Funct. Anal.*, **270** (2016), 1416–1478. https://doi.org/10.1016/j.jfa.2015.06.022
6. M. Colombo, G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.*, **215** (2015), 443–496. https://doi.org/10.1007/s00205-014-0785-2
7. P. Baroni, M. Colombo, G. Mingione, Non-autonomous functionals, borderline cases and related function classes, *St. Petersburg Math. J.*, **27** (2016), 347–379. https://doi.org/10.1090/SPMJ/1392
8. T. Iwaniec, Projections onto gradient fields and L^p-estimates for degenerate elliptic equations, *Stud. Math.*, **75** (1983), 293–312. https://doi.org/10.4064/sm-75-3-293-312
9. E. DiBenedetto, J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems, *Am. J. Math.*, **115** (1993), 1107–1134. https://doi.org/10.2307/2375066
10. F. Yao, C. Zhang, S. Zhou, Global regularity estimates for a class of quasilinear elliptic equations in the whole space, *Nonlinear Anal.*, **194** (2020), 111307. https://doi.org/10.1016/j.na.2018.07.004
11. L. Diening, S. Schwarzacher, Global gradient estimates for the $p(x)$-Laplacian , *Nonlinear Anal. Theory Methods Appl.*, **106** (2014), 70–85. https://doi.org/10.1016/J.NA.2014.04.006
12. C. Zhang, S. Zhou, B. Ge, Gradient estimates for the $p(x)$-Laplacian equation in $\mathbb{R}^N$, *Ann. Pol. Math.*, **114** (2015), 45–65. https://doi.org/10.4064/ap114-1-4
13. F. Yao, S. Zhou, Global estimates in Orlicz spaces for p-Laplacian systems in $\mathbb{R}^N$, *J. Partial Differ. Equations*, **25** (2012), 103–114. https://doi.org/10.4208/jpde.v25.n2.1
14. S. S. Byun, S. Ryu, P. Shin, Calderón–Zygmund estimates for ω-minimizers of double phase variational problems, *Appl. Math. Lett.*, **86** (2018), 256–263. https://doi.org/10.1016/j.aml.2018.07.009
15. S. Baasandorj, S. S. Byun, J. Oh, Calderón–Zygmund estimates for generalized double phase problems, *J. Funct. Anal.*, **279** (2020), 108670. https://doi.org/10.1016/j.jfa.2020.108670
16. S. S. Byun, H. S. Lee, Calderón–Zygmund estimates for elliptic double phase problems with variable exponents, *J. Math. Anal. Appl.*, **501** (2021), 124015. https://doi.org/10.1016/j.jmaa.2020.124015
17. P. Shin, Calderón–Zygmund estimates for general elliptic operators with double phase, *Nonlinear Anal.*, **194** (2020), 111409. https://doi.org/10.1016/j.na.2018.12.020

18. P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differ. Equations*, **57** (2018), 62. https://doi.org/10.1007/s00526-018-1332-z
19. S. Liang, S. Z. Zheng, On $W^{1,\gamma(\cdot)}$-regularity for nonlinear non-uniformly elliptic equations, *Manuscr. Math.*, **159** (2019), 247–268. https://doi.org/10.1007/s00229-018-1053-9
20. L. Esposito, F. Leonetti, G. Mingione, Sharp regularity for functionals with (p, q) growth, *J. Differ. Equations*, **204** (2004), 5–55. https://doi.org/10.1016/j.jde.2003.11.007
21. E. Acerbi, G. Mingione, Gradient estimates for a class of parabolic systems, *Duke Math. J.*, **136** (2007), 285–320. https://doi.org/10.1215/S0012-7094-07-13623-8
22. G. Mingione, The Calderón-Zygmund theory for elliptic problems with measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **6** (2007), 195–261. https://doi.org/10.2422/2036-2145.2007.2.01
23. K. Taira, Singular integrals and Feller semigroups with jump phenomena, *Rend. Circ. Mat. Palermo Ser. II*, 2023. https://doi.org/10.1007/s12215-023-00907-2
24. N. S. Papageorgiou, C. Vetro, F. Vetro, Robin problems with general potential and double resonance, *Appl. Math. Lett.*, **68** (2017), 122–128. https://doi.org/10.1016/j.aml.2017.01.003
25. L. Diening, F. Ettwein, Fractional estimates for non-differentiable elliptic systems with general growth, *Forum Math.*, **20** (2008), 523–556. https://doi.org/10.1515/FORUM.2008.027
26. J. Malý, W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, American Mathematical Society, 1997.