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Research article

A nonlinear delay integral equation related to infectious diseases

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Abstract: A class of nonlinear integral equations with delay, related to infectious diseases, is studied. Making use of some tools from operators theory, we deal with the well-posedness in an adequate functional space, approximation of solution, estimates of lower/upper solutions and the data dependence of solutions.

Keywords: nonlinear delay integral equation; existence/uniqueness; approximation; lower/upper solutions; data dependence

1. Introduction

We are concerned with the study of the integral equation

$$\upsilon(s) = \left(\int_{s-\mu_1}^{s} \iota_1(t, \upsilon(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, \upsilon(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, \upsilon(t))\, dt\right), \quad s \in \mathbb{R}, \tag{1.1}$$

where m is a positive integer and $\mu_i > 0$, $i = 1, 2, \cdots, m$, are constants. Namely, we study the well-posedness of (1.1), the upper and lower solutions and the data dependence of solutions w.r.t. small perturbations of the functions ι_j, $j = 1, 2, \cdots, m$.

In the particular case $m = 1$, (1.1) reduces to

$$\upsilon(s) = \int_{s-\mu_1}^{s} \iota_1(t, \upsilon(t))\, dt, \quad \in \mathbb{R}. \tag{1.2}$$

The above equation has been used as a model of the propagation of some infectious diseases with a rate of contact that depends on seasons, see e.g., [1, 2]. Due to the importance of (1.2) in Biosciences, several mathematical studies of this equation have been done. Namely, different numerical methods for solving integral equations of the form (1.2) have been developed (see e.g., [3–6] and the references therein). Moreover, various results concerning the qualitative behavior of solutions have been obtained. For instance, in [1], Eq (1.2) has been studied, where ι_1 is continuous and periodic in t, and satisfies $\iota_1(t, 0) = 0$. Namely, a threshold theorem has been established in the following sense:

(i) If $\mu_1 > 0$ is small enough, then every nonnegative solution to (1.2) tends to 0 as $s \to +\infty$;
(ii) If μ_1 is sufficiently large, then (1.2) admits a positive periodic solution with the same period as ι_1.

In [7], sufficient conditions for the existence of a positive periodic solution to (1.2) have been obtained using the theory of fixed point index. In [8], the existence of positive almost periodic solutions to (1.2) has been studied when $\mu_1 = \mu_1(t)$. In [9], the same question has been studied by means of Hilbert's projective metric. Using the theory of Picard operators developed by Rus and his collaborators (see [10–15]), Dobriţoiu et al. [16] provided a detailed study of (1.2) concerning the well-posedness, lower/upper solutions and the data dependence.

The goal of this work is to extend the study made in [16] to the integral equation (1.1). Namely, in Section 2, the existence/uniqueness of solutions is established making use of Prešić fixed point theorem [17]. We also provide and iterative process for approximating the solution. Next, some estimates involving lower/upper solutions to (1.1) are obtained in Section 3. In Section 4, we study the data dependence of solutions w.r.t. small perturbations of the functions ι_j, $j = 1, 2, \cdots, m$.

As we mentioned above, the proof of our well-posedness result makes use of Prešić fixed point theorem. We recall below this result.

Lemma 1.1 (see [17]). *Let (V, δ) be a metric space. Let $\Gamma : V^m \to V$, where m is a positive integer. Assume that*

- *(V, δ) is complete;*
- *There exists a finite sequence $\{\zeta_i\}_{i=1}^m \subset [0, \infty)$ with $0 < \sum_{i=1}^m \zeta_i < 1$ such that*

$$\delta(\Gamma(v_0, v_1, \cdots, v_{m-1}), \Gamma(v_1, v_2, \cdots, v_m)) \leq \zeta_1 \delta(v_0, v_1) + \zeta_2 \delta(v_1, v_2) + \cdots + \zeta_m \delta(v_{m-1}, v_m)$$

for every $\{v_j\}_{j=0}^m \subset V$.

Then, the equation

$$v = \Gamma(v, v, \cdots, v), \quad v \in V$$

has a unique solution $v^* \in V$. Moreover, for any $\{v_j\}_{j=0}^{m-1} \subset V$, the sequence

$$v_{j+1} = \Gamma(v_{j-m+1}, \cdots, v_j), \quad j \geq m - 1$$

converges to v^*.

2. Well-posedness

Assume that the conditions below hold:

(i) $\iota_j \in C(\mathbb{R} \times \mathbb{I}, \mathbb{J})$, $j = 1, 2, \cdots, m$, $m \geq 2$, where $\mathbb{I}$ and $\mathbb{J}$ are two closed and bounded intervals of $\mathbb{R}$;

(ii) for all $j = 1, 2, \cdots, m$, there exists $L_{\iota_j} > 0$ such that

$$|\iota_j(t, w) - \iota_j(t, z)| \leq L_{\iota_j}|w - z|$$

for all $t \in \mathbb{R}$ and $w, z \in \mathbb{I}$;

(iii) $0 < \sum_{j=1}^{m} L_{\iota_j} \prod_{k=1, k\neq j}^{m} \|\iota_k\| < \left(\prod_{i=1}^{m} \mu_i\right)^{-1}$, where $\|\iota_k\| = \sup_{(t,z)\in\mathbb{R}\times\mathbb{I}} |\iota_k(t, z)|$;

(iv) there exists a closed subset X of $C(\mathbb{R}, \mathbb{I})$ such that $\Gamma(X^m) \subset X$, where

$$\Gamma(v_1, v_2, \cdots, v_m)(s) = \left(\int_{s-\mu_1}^{s} \iota_1(t, v_1(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, v_2(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, v_m(t))\, dt\right), \quad s \in \mathbb{R} \tag{2.1}$$

for all $v_1, v_2, \cdots, v_m \in X$ and $C(\mathbb{R}, \mathbb{I})$ is equipped with the norm

$$\|v\|_\infty = \sup_{s\in\mathbb{R}} |v(s)|, \quad v \in C(\mathbb{R}, \mathbb{I}).$$

We have the following result.

Theorem 2.1. *Assume that* (i)–(iv) *hold. Then,*

(I) (1.1) *admis a unique solution* $\upsilon^* \in X$;

(II) for all $v_0, v_1, \cdots, v_{m-1} \in X$, *the sequence* $\{v_j\} \subset X$ *defined by*

$$v_{j+1}(s) = \left(\int_{s-\mu_1}^{s} \iota_1(t, v_{j-m+1}(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, v_{j-m+2}(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, v_j(t))\, dt\right) \tag{2.2}$$

for all $j \geq m - 1$, *converges uniformly to* υ^*, *that is,*

$$\lim_{j\to+\infty} \|v_j - \upsilon^*\|_\infty = 0.$$

Proof. From (iv), the mapping

$$\Gamma : X^m \to X$$

is well-defined. Furthermore, for all $v_0, v_1, v_2, \cdots, v_m \in X$ and $s \in \mathbb{R}$, we have

$$
\begin{aligned}
&\Gamma(v_1, v_2, v_3, \cdots, v_m)(s) - \Gamma(v_0, v_1, v_2, \cdots, v_{m-1})(s) \\
&= \left(\int_{s-\mu_1}^{s} \iota_1(t, v_1(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, v_2(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \iota_3(t, v_3(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, v_m(t))\, dt\right) \\
&\quad - \left(\int_{s-\mu_1}^{s} \iota_1(t, v_0(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, v_1(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \iota_3(t, v_2(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, v_{m-1}(t))\, dt\right) \\
&= \left(\int_{s-\mu_1}^{s} (\iota_1(t, v_1(t)) - \iota_1(t, v_0(t))\ dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, v_2(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \iota_3(t, v_3(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, v_m(t))\, dt\right) \\
&\quad + \left(\int_{s-\mu_2}^{s} (\iota_2(t, v_2(t)) - \iota_2(t, v_1(t))\ dt\right)\left(\int_{s-\mu_1}^{s} \iota_1(t, v_0(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \iota_3(t, v_3(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, v_m(t))\, dt\right) \\
&\quad + \left(\int_{s-\mu_3}^{s} (\iota_3(t, v_3(t)) - \iota_3(t, v_2(t))\ dt\right)\left(\int_{s-\mu_1}^{s} \iota_1(t, v_0(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, v_1(t))\, dt\right) \\
&\qquad \left(\int_{s-\mu_4}^{s} \iota_4(t, v_4(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, v_m(t))\, dt\right) \\
&\quad + \cdots \\
&\quad + \left(\int_{s-\mu_m}^{s} (\iota_m(t, v_m(t)) - \iota_m(t, v_{m-1}(t))\ dt\right)\left(\int_{s-\mu_1}^{s} \iota_1(t, v_0(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, v_1(t))\, dt\right) \\
&\qquad \cdots\left(\int_{s-\mu_{m-1}}^{s} \iota_{m-1}(t, v_{m-2}(t))\, dt\right),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&|\Gamma(v_1, v_2, v_3, \cdots, v_m)(s) - \Gamma(v_0, v_1, v_2, \cdots, v_{m-1})(s)| \\
&\leq L_{\iota_1}\mu_1\mu_2\cdots\mu_m\|\iota_2\|\|\iota_3\|\cdots\|\iota_m\|\|v_1 - v_0\|_\infty \ + L_{\iota_2}\mu_1\mu_2\cdots\mu_m\|\iota_1\|\|\iota_3\|\cdots\|\iota_m\|\|v_2 - v_1\|_\infty \\
&\quad + \cdots + L_{\iota_m}\mu_1\mu_2\cdots\mu_m\|\iota_1\|\|\iota_2\|\cdots\|\iota_{m-1}\|\|v_m - v_{m-1}\|_\infty.
\end{aligned}
$$

Hence, it holds that

$$
\|\Gamma(v_0, v_1, v_2, \cdots, v_{m-1}) - \Gamma(v_1, v_2, v_3, \cdots, v_m)\|_\infty \leq \zeta_1\|v_0 - v_1\|_\infty + \zeta_2\|v_1 - v_2\|_\infty + \cdots + \zeta_m\|v_m - v_{m-1}\|_\infty, \quad (2.3)
$$

where

$$
\zeta_j = \prod_{i=1}^{m} \mu_i\, L_{\iota_j} \prod_{k=1, k\neq j}^{m} \|\iota_k\|, \quad j = 1, 2, \cdots, m.
$$

On the other hand, by (iii), we have

$$
0 < \sum_{j=1}^{m} \zeta_j < 1.
$$

Hence, by Lemma 1.1, there exists a unique $\upsilon^* \in X$ such that

$$
\upsilon^* = \Gamma(\upsilon^*, \upsilon^*, \cdots, \upsilon^*),
$$

that is, υ^* is the unique solution to (1.1) in X, which proves (I). Finally, (II) follows from the convergence result in Lemma 1.1.

We now take $\mathbb{I} = [a, b]$ and $\mathbb{J} = [c, d]$, where $a < b$ and $0 < c < d$.

Corollary 2.2. Assume that (i)–(iii) hold, and

$$a \leq c^m \prod_{i=1}^{m} \mu_i, \quad b \geq d^m \prod_{i=1}^{m} \mu_i. \tag{2.4}$$

Then the statements (I) and (II) of Theorem 2.1 hold true, where $X = C(\mathbb{R}, \mathbb{I})$.

Proof. We have just to show that condition (iv) is satisfied. Then, from Theorem 2.1, (I) and (II) follow. Let $v_1, v_2, \cdots, v_m \in C(\mathbb{R}, \mathbb{I})$. For all $j = 1, 2, \cdots, m$, one has

$$0 < c \leq \iota_j(t, v_j(t)) \leq d,$$

which implies that

$$0 < c\mu_j \leq \int_{s-\mu_j}^{s} \iota_j(t, v_j(t))\, dt \leq d\mu_j.$$

Then, for all $s \in \mathbb{R}$, it holds that

$$c^m \prod_{i=1}^{m} \mu_i \leq \Gamma(v_1, v_2, \cdots, v_m)(s) \leq d^m \prod_{i=1}^{m} \mu_i.$$

Taking into consideration (2.4), we obtain

$$a \leq \Gamma(v_1, v_2, \cdots, v_m)(s) \leq b,$$

which shows that $\Gamma(v_1, v_2, \cdots, v_m) \in C(\mathbb{R}, \mathbb{I})$. Consequently, we have $\Gamma(X^m) \subset X$, where $X = C(\mathbb{R}, \mathbb{I})$.

We provide below an example to illustrate Theorem 2.1.

Example 2.3. Consider the integral equation

$$\upsilon(s) = \left(\int_{s-\mu_1}^{s} \frac{1}{t^2+1} \ln(1 + \upsilon(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \frac{1}{(t^2+1)^2} \ln(1 + \upsilon^2(t))\, dt\right), \quad s \in \mathbb{R}, \tag{2.5}$$

where $\mu_1, \mu_2 > 0$ are constants and

$$\mu_1\mu_2 < \frac{1}{5 \ln 2}. \tag{2.6}$$

Let $X = C(\mathbb{R}, [0, 1])$. We claim that

(I) (2.5) admis a unique solution $\upsilon^* \in X$;
(II) for all $v_0, v_1 \in X$, the sequence $\{v_j\} \subset X$ defined by

$$v_{j+1}(s) = \left(\int_{s-\mu_1}^{s} \frac{1}{t^2+1} \ln(1 + v_{j-1}(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \frac{1}{(t^2+1)^2} \ln(1 + v_j^2(t))\, dt\right) \tag{2.7}$$

for all $j \geq 1$, converges uniformly to υ^*, that is,

$$\lim_{j \to +\infty} \|v_j - \upsilon^*\|_\infty = 0.$$

Indeed, for all $k = 1, 2$, let

$$\iota_k : \mathbb{R} \times [0, 1] \to [0, \ln 2]$$

be the functions defined by

$$\iota_k(t, z) = \frac{1}{(t^2 + 1)^k} \ln(1 + z^k), \quad t \in \mathbb{R},\ z \in [0, 1].$$

Then (2.5) is a special case of (1.1) with $m = 2$.

Let $\mathbb{I} = [0, 1]$ and $\mathbb{J} = [0, \ln 2]$. For all $t \in \mathbb{R}$ and $w, z \in \mathbb{I}$, by the mean value theorem, we have

$$|\iota_1(t, w) - \iota_1(t, z)| = \frac{1}{t^2 + 1} |\ln(1 + w) - \ln(1 + z)| \leq L_{\iota_1} |w - z|,$$

where $L_{\iota_1} = 1$. Similarly,

$$|\iota_2(t, w) - \iota_2(t, z)| = \frac{1}{(t^2 + 1)^2} \left|\ln(1 + w^2) - \ln(1 + z^2)\right| \leq 2|w - z||w + z| \leq L_{\iota_2} |w - z|,$$

where $L_{\iota_2} = 4$. Furthermore, by (2.6), we have

$$\sum_{j=1}^{2} L_{\iota_j} \prod_{k=1, k \neq j}^{2} \|\iota_k\| = L_{\iota_1} \|\iota_2\| + L_{\iota_2} \|\iota_1\| \leq 5 \ln 2 < (\mu_1 \mu_2)^{-1}.$$

On the other hand, for all $v_1, v_2 \in X$, we have

$$\Gamma(v_1, v_2)(s) = \left(\int_{s-\mu_1}^{s} \iota_1(t, v_1(t))\, dt \right) \left(\int_{s-\mu_2}^{s} \iota_2(t, v_2(t))\, dt \right) \leq \|\iota_1\| \|\iota_2\| \mu_1 \mu_2 \leq \frac{(\ln 2)^2}{5 \ln 2} = \frac{\ln 2}{5} < 1,$$

which shows that $\Gamma(X \times X) \subset X$. Hence, all the assumptions (i)–(iv) of Theorem 2.1 are satisfied. Then the claims (I) and (II) follow from Theorem 2.1.

3. Lower/upper solutions

We say that υ is a lower solution to (1.1), if

$$\upsilon(s) \leq \left(\int_{s-\mu_1}^{s} \iota_1(t, \upsilon(t))\, dt \right) \left(\int_{s-\mu_2}^{s} \iota_2(t, \upsilon(t))\, dt \right) \cdots \left(\int_{s-\mu_m}^{s} \iota_m(t, \upsilon(t))\, dt \right), \ s \in \mathbb{R}.$$

If υ verifies

$$\upsilon(s) \geq \left(\int_{s-\mu_1}^{s} \iota_1(t, \upsilon(t))\, dt \right) \left(\int_{s-\mu_2}^{s} \iota_2(t, \upsilon(t))\, dt \right) \cdots \left(\int_{s-\mu_m}^{s} \iota_m(t, \upsilon(t))\, dt \right), \ s \in \mathbb{R},$$

then υ is called an upper solution to (1.1).

Let us consider (1.1) under conditions (i)–(iv). Then, by Theorem 2.1, (1.1) admits a unique solution $\upsilon^* \in X$.

We have the following result.

Theorem 3.1. *Assume that* (i)–(iv) *hold. Suppose also that* $\mathbb{J} \subset [0, +\infty)$ *and for all* $t \in \mathbb{R}$ *and* $j = 1, 2, \cdots, m$*, we have*

$$w, z \in \mathbb{I},\ w \leq z \implies \iota_j(t, w) \leq \iota_j(t, z). \tag{3.1}$$

If $\upsilon \in X$ *is a lower solution to* (1.1)*, then*

$$\upsilon(s) \leq \upsilon^*(s),\ \ s \in \mathbb{R}. \tag{3.2}$$

Proof. Let $\Gamma : X^m \to X$ be the mapping defined by (2.1). By (3.1) and since $\mathbb{J} \subset [0, +\infty)$, if $w \in X$ and $(v_1, \cdots, v_{j-1}, v_j, v_{j+1}, \cdots v_m) \in X^m$ are such that $v_j(s) \leq w(s)$ for all $s \in \mathbb{R}$, then

$$\Gamma(v_1, v_2, \cdots, v_m)(s) \leq \Gamma(v_1, \cdots, v_{j-1}, w, v_{j+1}, \cdots v_m)(s),\ \ s \in \mathbb{R}.$$

Let $\upsilon \in X$ be a lower solution to (1.1). Then, for all $s \in \mathbb{R}$,

$$\upsilon(s) \leq \Gamma(\upsilon, \upsilon, \cdots, \upsilon)(s). \tag{3.3}$$

Let $v_0 = v_1 = \cdots = v_{m-1} = \upsilon$ and

$$v_{j+1} = \Gamma(v_{j-m+1}, \cdots, v_j),\ \ j \geq m - 1.$$

We claim that

$$\upsilon(s) \leq v_{j+1}(s),\ \ j \geq m - 1. \tag{3.4}$$

By (3.1) and (3.3), we have

$$\begin{aligned} \upsilon(s) &\leq \Gamma(v_0, v_1, \cdots, v_{m-1})(s) \\ &= v_m(s), \end{aligned} \tag{3.5}$$

which shows that (3.4) holds true for $j = m - 1$. On the other hand, by (3.1) and (3.5), we have

$$\begin{aligned} v_m(s) &= \Gamma(v_0, v_1, \cdots, v_{m-2}, \upsilon)(s) \\ &\leq \Gamma(v_0, v_1, \cdots, v_{m-2}, v_m)(s) \\ &= \Gamma(v_1, v_2, \cdots, v_{m-1}, v_m)(s) \\ &= v_{m+1}(s). \end{aligned} \tag{3.6}$$

Then, (3.5) and (3.6) yield

$$\upsilon(s) \leq v_{m+1}(s),$$

which shows that (3.4) holds true for $j = m$. Repeating the same argument, by the induction principle, we obtain (3.4). Now, taking the limit as $j \to +\infty$ in (3.4) and using the convergence result provided by part (II) of Theorem 2.1, we obtain (3.2).

Proceeding as in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. *Assume that* (i)–(iv) *hold. Suppose also that* $\mathbb{J} \subset [0, +\infty)$ *and for all* $t \in \mathbb{R}$ *and* $j = 1, 2, \cdots, m$*,* (3.1) *holds. If* $\upsilon \in X$ *is an upper solution to* (1.1)*, then*

$$\upsilon(s) \geq \upsilon^*(s),\ \ s \in \mathbb{R}.$$

We now take $\mathbb{I} = [a, b]$ and $\mathbb{J} = [c, d]$, where $a < b$ and $0 < c < d$. For all $j = 1, 2, \cdots, m$, let $\iota_{j1}, \iota_{j2}, \iota_{j3} \in C(\mathbb{R} \times \mathbb{I}, \mathbb{J})$. Assume that conditions below hold:

(c_1) for all all $j = 1, 2, \cdots, m$ and $i = 1, 2, 3$, there exists $L_{\iota_{ji}} > 0$ such that

$$|\iota_{ji}(t, w) - \iota_{ji}(t, z)| \leq L_{\iota_{ji}}|w - z|$$

for all $t \in \mathbb{R}$ and $w, z \in I$;

(c_2) for all $i = 1, 2, 3$, we have

$$0 < \sum_{j=1}^{m} L_{\iota_{ji}} \prod_{k=1,k\neq j}^{m} \|\iota_{ki}\| < \left(\prod_{i=1}^{m} \mu_i\right)^{-1};$$

(c_3) $a \leq c^m \prod_{i=1}^{m} \mu_i, \ \ b \geq d^m \prod_{i=1}^{m} \mu_i;$

(c_4) for all $j = 1, 2, \cdots, m$, we have

$$\iota_{j1}(t, w) \leq \iota_{j2}(t, w) \leq \iota_{j3}(t, w), \ \ (j, w) \in \mathbb{R} \times \mathbb{I};$$

(c_5) for all $j = 1, 2, \cdots, m$ and $t \in \mathbb{R}$, we have

$$w, z \in \mathbb{I}, \ w \leq z \implies \iota_{j2}(t, w) \leq \iota_{j2}(t, z).$$

Observe that under the above conditions, thanks to Theorem 2.1 and Corollary 2.2, for all $i = 1, 2, 3$, the problem

$$\upsilon(s) = \left(\int_{s-\mu_1}^{s} \iota_{1i}(t, \upsilon(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_{2i}(t, \upsilon(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_{mi}(t, \upsilon(t))\, dt\right), \ \ s \in \mathbb{R} \tag{3.7}$$

admits a unique solution $\upsilon_i^* \in C(\mathbb{R}, \mathbb{I})$.

The following result holds.

Corollary 3.3. Under conditions (c_1)–(c_5), it holds that

$$\upsilon_1^*(s) \leq \upsilon_2^*(s) \leq \upsilon_3^*(s), \ \ s \in \mathbb{R}.$$

Proof. We have

$$\upsilon_1^*(s) = \left(\int_{s-\mu_1}^{s} \iota_{11}(t, \upsilon_1^*(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_{21}(t, \upsilon_1^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_{m1}(t, \upsilon_1^*(t))\, dt\right)$$

for all $s \in \mathbb{R}$. Due to (c_4), we obtain

$$\upsilon_1^*(s) \leq \left(\int_{s-\mu_1}^{s} \iota_{12}(t, \upsilon_1^*(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_{22}(t, \upsilon_1^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_{m2}(t, \upsilon_1^*(t))\, dt\right),$$

which shows that υ_1^* is a lower solution to (3.7) with $i = 2$. Hence, by Theorem 3.1, we get $\upsilon_1^*(s) \leq \upsilon_2^*(s), \ \ s \in \mathbb{R}$. Similarly, we have

$$\upsilon_3^*(s) = \left(\int_{s-\mu_1}^{s} \iota_{13}(t, \upsilon_3^*(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_{23}(t, \upsilon_3^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_{m3}(t, \upsilon_3^*(t))\, dt\right)$$

for all $s \in \mathbb{R}$. Due to (c$_4$), we obtain

$$\upsilon_3^*(s) \geq \left(\int_{s-\mu_1}^{s} \iota_{12}(t, \upsilon_3^*(t))\,dt\right)\left(\int_{s-\mu_2}^{s} \iota_{22}(t, \upsilon_3^*(t))\,dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_{m2}(t, \upsilon_3^*(t))\,dt\right),$$

which shows that υ_3^* is an upper solution to (3.7) with $i = 2$. Then, by Theorem 3.2, we obtain $\upsilon_3^*(s) \geq \upsilon_2^*(s),\ s \in \mathbb{R}$.

4. Data dependence of solutions

We now consider the perturbed problem

$$\widetilde{\upsilon}(s) = \left(\int_{s-\mu_1}^{s} \widetilde{\iota}_1(t, \widetilde{\upsilon}(t))\,dt\right)\left(\int_{s-\mu_2}^{s} \widetilde{\iota}_2(t, \widetilde{\upsilon}(t))\,dt\right)\cdots\left(\int_{s-\mu_m}^{s} \widetilde{\iota}_m(t, \widetilde{\upsilon}(t))\,dt\right),\ s \in \mathbb{R}, \tag{4.1}$$

where $m \geq 2$ is an integer, $\mu_i > 0$, $i = 1, 2, \cdots, m$, $\widetilde{\iota}_j \in C(\mathbb{R} \times \mathbb{I}, \mathbb{J})$, $j = 1, 2, \cdots, m$, $\mathbb{I}$ and $\mathbb{J}$ are two closed and bounded intervals of $\mathbb{R}$.

We have the following dependence data result.

Theorem 4.1. *Assume that* (i)–(iv) *hold, and let* $\upsilon^* \in X$ *be the unique solution to* (1.1). *Suppose that for all* $j = 1, 2, \cdots, m$, *there exits* $\eta_j > 0$ *such that*

$$|\iota_j(t, w) - \widetilde{\iota}_j(t, w)| \leq \eta_j \tag{4.2}$$

for all $(t, w) \in \mathbb{R} \times \mathbb{I}$. *If* $\widetilde{\upsilon}^* \in X$ *is a solution to* (4.1), *then*

$$\|\upsilon^* - \widetilde{\upsilon}^*\|_\infty \leq \mathcal{E}, \tag{4.3}$$

where

$$\mathcal{E} = \frac{\prod_{i=1}^{m} \mu_i \left(\sum_{j=2}^{m-1} \eta_j \prod_{k=j+1}^{m} \|\iota_k\| \prod_{k=1}^{j-1} \|\widetilde{\iota}_k\| + \eta_1 \prod_{k=2}^{m} \|\iota_k\| + \eta_m \prod_{k=1}^{m-1} \|\widetilde{\iota}_k\|\right)}{1 - \prod_{i=1}^{m} \mu_i \sum_{j=1}^{m} L_{\iota_j} \prod_{k=1, k\neq j}^{m} \|\iota_k\|}.$$

Proof. Let

$$\widetilde{\Gamma}(\widetilde{\upsilon}, \widetilde{\upsilon}, \cdots, \widetilde{\upsilon})(s) = \left(\int_{s-\mu_1}^{s} \widetilde{\iota}_1(t, \widetilde{\upsilon}(t))\,dt\right)\left(\int_{s-\mu_2}^{s} \widetilde{\iota}_2(t, \widetilde{\upsilon}(t))\,dt\right)\cdots\left(\int_{s-\mu_m}^{s} \widetilde{\iota}_m(t, \widetilde{\upsilon}(t))\,dt\right),\ s \in \mathbb{R}.$$

For all $s \in \mathbb{R}$, we have

$$\begin{aligned} |\upsilon^*(s) - \widetilde{\upsilon}^*(s)| &= |\Gamma(\upsilon^*, \upsilon^*, \cdots, \upsilon^*)(s) - \widetilde{\Gamma}(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s)| \\ &\leq |\Gamma(\upsilon^*, \upsilon^*, \cdots, \upsilon^*)(s) - \Gamma(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s)| \\ &\quad + |\Gamma(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s) - \widetilde{\Gamma}(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s)|. \end{aligned} \tag{4.4}$$

On the other hand, by (2.3), we have

$$|\Gamma(\upsilon^*, \upsilon^*, \cdots, \upsilon^*)(s) - \Gamma(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s)| \leq \zeta \|\upsilon^* - \widetilde{\upsilon}^*\|_\infty, \tag{4.5}$$

where

$$0 < \zeta = \prod_{i=1}^{m} \mu_i \sum_{j=1}^{m} L_{\iota_j} \prod_{k=1, k\neq j}^{m} \|\iota_k\| < 1.$$

Furthermore, we have

$$\begin{aligned}
&\Gamma(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s) - \widetilde{\Gamma}(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s) \\
&= \left(\int_{s-\mu_1}^{s} \iota_1(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \iota_3(t, \widetilde{\upsilon}^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, \widetilde{\upsilon}^*(t))\, dt\right) \\
&\quad - \left(\int_{s-\mu_1}^{s} \widetilde{\iota}_1(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \widetilde{\iota}_2(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \widetilde{\iota}_3(t, \widetilde{\upsilon}^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \widetilde{\iota}_m(t, \widetilde{\upsilon}^*(t))\, dt\right) \\
&= \left(\int_{s-\mu_1}^{s} (\iota_1(t, \widetilde{\upsilon}^*(t)) - \widetilde{\iota}_1(t, \widetilde{\upsilon}^*(t))\ dt\right)\left(\int_{s-\mu_2}^{s} \iota_2(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \iota_3(t, \widetilde{\upsilon}^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, \widetilde{\upsilon}^*(t))\, dt\right) \\
&\quad + \left(\int_{s-\mu_2}^{s} (\iota_2(t, \widetilde{\upsilon}^*(t)) - \widetilde{\iota}_2(t, \widetilde{\upsilon}^*(t))\ dt\right)\left(\int_{s-\mu_1}^{s} \widetilde{\iota}_1(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_3}^{s} \iota_3(t, \widetilde{\upsilon}^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, \widetilde{\upsilon}^*(t))\, dt\right) \\
&\quad + \left(\int_{s-\mu_3}^{s} (\iota_3(t, \widetilde{\upsilon}^*(t)) - \widetilde{\iota}_3(t, \widetilde{\upsilon}^*(t))\ dt\right)\left(\int_{s-\mu_1}^{s} \widetilde{\iota}_1(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \widetilde{\iota}_2(t, \widetilde{\upsilon}^*(t))\, dt\right) \\
&\qquad \left(\int_{s-\mu_4}^{s} \iota_4(t, \widetilde{\upsilon}^*(t))\, dt\right)\cdots\left(\int_{s-\mu_m}^{s} \iota_m(t, \widetilde{\upsilon}^*(t))\, dt\right) \\
&\quad + \cdots \\
&\quad + \left(\int_{s-\mu_m}^{s} (\iota_m(t, \widetilde{\upsilon}^*(t)) - \widetilde{\iota}_m(t, \widetilde{\upsilon}^*(t))\ dt\right) \\
&\qquad \left(\int_{s-\mu_1}^{s} \widetilde{\iota}_1(t, \widetilde{\upsilon}^*(t))\, dt\right)\left(\int_{s-\mu_2}^{s} \widetilde{\iota}_2(t, \widetilde{\upsilon}^*(t))\, dt\right)\cdots\left(\int_{s-\mu_{m-1}}^{s} \widetilde{\iota}_{m-1}(t, \widetilde{\upsilon}^*(t))\, dt\right),
\end{aligned}$$

which implies by (4.2) that

$$\begin{aligned}
&|\Gamma(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s) - \widetilde{\Gamma}(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s)| \\
&\leq \eta_1\mu_1\mu_2\cdots\mu_m\|\iota_2\|\|\iota_3\|\cdots\|\iota_m\| + \eta_2\mu_1\mu_2\cdots\mu_m\|\widetilde{\iota}_1\|\|\iota_3\|\cdots\|\iota_m\| \\
&\quad + \eta_3\mu_1\mu_2\cdots\mu_m\|\widetilde{\iota}_1\|\|\widetilde{\iota}_2\|\|\iota_4\|\cdots\|\iota_m\| + \cdots + \eta_m\mu_1\mu_2\cdots\mu_m\|\widetilde{\iota}_1\|\|\widetilde{\iota}_2\|\cdots\|\widetilde{\iota}_{m-1}\|,
\end{aligned}$$

that is,

$$\begin{aligned}
&|\Gamma(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s) - \widetilde{\Gamma}(\widetilde{\upsilon}^*, \widetilde{\upsilon}^*, \cdots, \widetilde{\upsilon}^*)(s)| \\
&\leq \prod_{i=1}^{m} \mu_i \left(\sum_{j=2}^{m-1} \eta_j \prod_{k=j+1}^{m} \|\iota_k\| \prod_{k=1}^{j-1} \|\widetilde{\iota}_k\| + \eta_1 \prod_{k=2}^{m} \|\iota_k\| + \eta_m \prod_{k=1}^{m-1} \|\widetilde{\iota}_k\|\right).
\end{aligned} \tag{4.6}$$

Therefore, from (4.4)–(4.6), it follows that

$$\|\upsilon^* - \widetilde{\upsilon}^*\|_\infty \leq \zeta\|\upsilon^* - \widetilde{\upsilon}^*\|_\infty + \prod_{i=1}^{m} \mu_i \left(\sum_{j=2}^{m-1} \eta_j \prod_{k=j+1}^{m} \|\iota_k\| \prod_{k=1}^{j-1} \|\widetilde{\iota}_k\| + \eta_1 \prod_{k=2}^{m} \|\iota_k\| + \eta_m \prod_{k=1}^{m-1} \|\widetilde{\iota}_k\|\right),$$

which yields (4.3).

5. Conclusions

The nonlinear integral equation (1.1) is investigated. We first studied the well-posedness of the problem in $C(\mathbb{R}, \mathbb{I})$, where $\mathbb{I}$ is a closed and bounded subset of $\mathbb{R}$ (see Theorem 2.1). Namely, by means of Prešić fixed point theorem (see Lemma 1.1), we proved that under conditions (i)–(iv), (1.1) admits a unique solution that can be approximated by the iterative sequence (2.7). We next established some estimates involving upper and lower solutions to (1.1) (see Theorems 3.1 and 3.2). Finally, we studied the dependence of the solution to (1.1) with respect to perturbations of the functions ι_j, $j = 1, 2, \cdots, m$ (see Theorem 4.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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