



Research article

Some identities of degenerate multi-poly-Changhee polynomials and numbers

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Abstract: Recently, many researchers studied the degenerate multi-special polynomials as degenerate versions of the multi-special polynomials and obtained some identities and properties of the those polynomials. The aim of this paper was to introduce the degenerate multi-poly-Changhee polynomials arising from multiple logarithms and investigate some interesting identities and properties of these polynomials that determine the relationship between multi-poly-Changhee polynomials, the Stirling numbers of the second kind, degenerate Stirling numbers of the first kind and falling factorial sequences. In addition, we investigated the phenomenon of scattering the zeros of these polynomials.

Keywords: degenerate multi-poly-Changhee polynomials; multiple logarithm; the zeros of polynomials

1. Introduction

Carlitz [1] initiated a degenerate exponential function for the generalization of the Stirling numbers and attempted to generalize Bernoulli and Eulerian numbers using these functions in 1979. Since then, studies on degenerate versions of various special functions have been actively conducted by many researchers. For $r \in \mathbb{N}$, Kim-Jang-Kwon studied the generalized degenerate Bernoulli polynomials, which are given by the generating function to be

$$\frac{t^r}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [2]}). \tag{1.1}$$

When $x = 0$, $\mathcal{B}_{n,\lambda}^{(r)} = \mathcal{B}_{n,\lambda}^{(r)}(0)$ are called the generalized Bernoulli numbers. In particular, when $r = 1$, then $\mathcal{B}_{n,\lambda}^{(1)}(x) = B_n(x)$. Note that $\mathcal{B}_{0,\lambda}^{(r)}(x) = \mathcal{B}_{1,\lambda}^{(r)}(x) = \dots = \mathcal{B}_{r-1,\lambda}^{(r)}(x) = 0$.

For $\lambda (\neq 0) \in \mathbb{C}$, the degenerate exponential function is defined by

$$e_{\lambda}^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} \quad (\text{see [2-13]}). \quad (1.2)$$

We note that $\lim_{\lambda \rightarrow 0} e_{\lambda}^x(t) = e^{xt}$, $e_{\lambda}^{(1)}(t) = e_{\lambda}(t)$ and

$$e_{\lambda}^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad (1.3)$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda)$, $(n \geq 1)$.

The *Changhee polynomials* $Ch_n(x)$ and the *Changhee polynomials of order k* $Ch_n^{(k)}(x)$ are respectively given by the generating functions to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!} \quad (\text{see [14]}). \quad (1.4)$$

and

$$\left(\frac{2}{t+2}\right)^k (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{t^n}{n!} \quad (\text{see [14]}). \quad (1.5)$$

The *Stirling numbers of the first kind* are defined as

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0) \quad (\text{see [6, 9]}), \quad (1.6)$$

where $(x)_0 = 1$, $(x)_n = x(x - 1) \cdots (x - n + 1)$, $(n \geq 1)$, and the *Stirling numbers of the second kind* are given by

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l, \quad (n \geq 0) \quad (\text{see [6, 9]}). \quad (1.7)$$

Let $\log_{\lambda}(t)$ be the compositional inverse function of $e_{\lambda}(t)$ such that $e_{\lambda}(\log_{\lambda}(t)) = \log_{\lambda}(e_{\lambda}(t)) = t$. By Newton's binomial expansion, we see that

$$\log_{\lambda}(1+t) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1} (1)_{n,1/\lambda}}{n!} t^n \quad (\text{see [6, 9, 10]}), \quad (1.8)$$

and by (1.8), we see that $\lim_{\lambda \rightarrow 0} \log_{\lambda}(1+t) = \log(1+t)$.

As the degenerate version of the Stirling numbers of the first and the second kind, Kim-Kim considered the *degenerate Stirling numbers of the second kind* $S_{2,\lambda}(n, k)$, $(n, k \geq 0)$, which are given by

$$(x)_{n,\lambda} = \sum_{k=0}^n S_{2,\lambda}(n, k) (x)_k, \quad (n \geq 0) \quad (\text{see [6, 9, 10]}), \quad (1.9)$$

and, as the inversion formula of (1.9), the *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n, k)$ are given by

$$(x)_n = \sum_{k=0}^n S_{1,\lambda}(n, k) (x)_{k,\lambda} \quad (n \geq 0) \quad (\text{see [6, 9, 10]}). \quad (1.10)$$

Since $\lim_{\lambda \rightarrow 0} (x)_{n,\lambda} = x^n$, we see that $\lim_{\lambda \rightarrow 0} S_{1,\lambda}(n, k) = S_1(n, k)$ and $\lim_{\lambda \rightarrow 0} S_{2,\lambda}(n, k) = S_2(n, k)$.

In addition, by (1.9) and (1.10), it is easy to see that

$$\frac{1}{k!} (e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \quad \text{and} \quad \frac{1}{k!} (\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} \quad (\text{see [6, 9, 10]}). \quad (1.11)$$

Special numbers and polynomials are interesting and important things in pure and applied mathematics, combinatorics and engineering (see [15]). In particular, authors gave two new identities involving the famous numbers of Fibonacci, Lucas, Pell and Pell-Lucas numbers in [16]. In [17], authors gave some new connection formulae between two generalized classes of Fibonacci and Lucas polynomials, and in [18], some celebrated orthogonal polynomials and other polynomials were given in terms of Bernoulli polynomials.

As one of the special functions, the Changhee numbers and polynomials were defined by Kim et al. in [14] and generalized these polynomials. In [19], authors defined the twisted Changhee polynomials and found some relationships between the Euler polynomials, the Stirling numbers of the first and the second kind and these polynomials. The (r, s) -analogue of the Changhee polynomials was defined in [20], and in [21] the authors studied the degenerate version of Changhee polynomials. Jang et al. [22] investigated the zeros of twisted Changhee polynomials, and Park gave the degenerate version of twisted q -Changhee polynomials of higher order in [23] and found some identities.

In 1998, Srivastava et al. [24] studied the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$, and Kim et al. [13] introduced the generalized degenerate Bernoulli numbers $\beta_{n,\lambda}^{(\alpha)}$ by means of the Gauss hypergeometric function. Recently, Kim et al. [12] introduced the degenerate multi-poly-Bernoulli polynomials $\beta_{n,\lambda}^{(k_1, k_2, \dots, k_r)}(x)$ by using the multiple polylogarithm. In [25], authors defined degenerate multi-poly-Euler polynomials by using degenerate multi-polyexponential functions and found some interesting identities of these polynomials and numbers. In [26], authors defined multi-poly-Genocchi polynomials with parameters a, b and c by using multiple polylogarithms and found some relationships between logarithm functions, poly-Bernoulli numbers and those polynomials. Kaneko et al. [27] defined the a zeta function that interpolates multi-poly-Bernoulli numbers, and showed that both these zeta functions and the ξ -function defined by Arakawa and Kaneko are related closely to multiple zeta functions. Choi et al. introduced the Legendre-based poly-Euler polynomials and Legendre-based multi-poly-Euler polynomials and found some related identities and formulas in [28].

In this paper, we consider the degenerate multi-poly-Changhee polynomials and numbers and investigate some properties and identities of those polynomials. Furthermore, we find explicit examples of the degenerate multi-poly-Changhee polynomials and investigate the phenomenon of scattering the zeros of these polynomials.

2. The degenerate multi-poly Changhee numbers

For $k_1, k_2, \dots, k_r \in \mathbb{Z}$, the multiple logarithm is given by

$$Li_{k_1, \dots, k_r}(x) = \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{x^{n_r}}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}} \quad (\text{see [12, 25–27]}), \quad (2.1)$$

where the sum is over all integers n_1, n_2, \dots, n_r satisfying $0 < n_1 < n_2 < \dots < n_r$.

From (2.1), Kim et al. [12] obtained the following results:

$$\frac{d}{dx} Li_{k_1, \dots, k_r}(x) = \frac{1}{x} Li_{k_1, \dots, k_{r-1}, k_{r-1}}(x), \quad (2.2)$$

$$Li_{\underbrace{1, \dots, 1}_{r\text{-times}}}(x) = \frac{1}{r!} (-\log(1-x))^r, \quad (2.3)$$

and

$$S_1^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)(n, r) = (-1)^{n-r} S_1(n, r) = \begin{bmatrix} n \\ r \end{bmatrix}, \quad (2.4)$$

where the multi-Stirling number of the first kind is defined by

$$Li_{k_1, \dots, k_r}(t) = \sum_{n=r}^{\infty} S_1^{(k_1, \dots, k_r)}(n, r) \frac{t^n}{n!}. \quad (2.5)$$

Now we consider the degenerate multi-poly-Changhee polynomials, which are given by the generating function to be

$$\frac{2^r Li_{k_1, \dots, k_r}(1 - e^{-t})}{t^r (2+t)^r} e_\lambda^x(\log_\lambda(1+t)) = \sum_{n=0}^{\infty} Ch_n^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!}. \quad (2.6)$$

Theorem 2.1. For $r \in \mathbb{Z}$, $r \geq 1$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$Ch_n^{(k_1, \dots, k_r)}(x) = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} S_{1,\lambda}(l, m) Ch_{n-l}^{(k_1, \dots, k_r)}(x)_{m,\lambda}.$$

In particular, $k_1 = \dots = k_r = 1$, $Ch_n^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x) = \frac{1}{r!} Ch_n^{(r)}(x)$ and $(n \geq 0)$.

Proof. From (2.6), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_n^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} &= \frac{2^r Li_{k_1, \dots, k_r}(1 - e^{-t})}{t^r (2+t)^r} e_\lambda^x(\log_\lambda(1+t)) \\ &= \left(\sum_{n=0}^{\infty} Ch_n^{(k_1, \dots, k_r)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{1}{n!} (\log_\lambda(1+t))^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} S_{1,\lambda}(l, m) (x)_{m,\lambda} Ch_{n-l}^{(k_1, \dots, k_r)} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.7)$$

Therefore, by comparing the coefficients on the both sides of (2.7), we proved the theorem. \square

Theorem 2.2. For $r \in \mathbb{Z}$, $r \geq 1$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} Ch_n^{(k_1, \dots, k_r)}(x) &= \sum_{b=0}^n \sum_{l=0}^b \sum_{n_r=1}^{l+1} \sum_{m=0}^{\infty} \binom{n}{b} \binom{b}{l} \binom{k_r + m - 1}{m} n_r! \\ &\times \frac{(-1)^{l+1+n_r+m} S_2(l+1, n_r)}{l+1} n_r^{-k_r-m} Ch_{b-l}^{(k_1, \dots, k_{r-1-m})} Ch_{n-b}(x). \end{aligned}$$

Proof. From (2.6), we note that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \text{Ch}_n^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} = \frac{2^r \text{Li}_{k_1, \dots, k_r}(1 - e^{-t})}{t^r (2+t)^r} e_\lambda^x(\log_\lambda(1+t)) \\
&= \frac{2^r}{t^r (2+t)^r} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{1}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}}} \sum_{n_r = n_{r-1} + 1}^{\infty} \frac{(1 - e^{-t})^{n_r}}{n_r^{k_r}} e_\lambda^x(\log_\lambda(1+t)) \\
&= \frac{2^r}{t^r (2+t)^r} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{(1 - e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}}} \sum_{n_r = 1}^{\infty} \frac{(1 - e^{-t})^{n_r}}{(n_r + n_{r-1})^{k_r}} e_\lambda^x(\log_\lambda(1+t)) \\
&= \frac{2^r}{t^r (2+t)^r} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{(1 - e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}}} \\
&\times \sum_{n_r = 1}^{\infty} n_r! \frac{1}{n_r!} (1 - e^{-t})^{n_r} \frac{1}{(n_r + n_{r-1})^{k_r}} e_\lambda^x(\log_\lambda(1+t)) \\
&= \frac{2}{(2+t)} e_\lambda^x(\log_\lambda(1+t)) \frac{1}{t} \frac{2^{r-1}}{t^{r-1} (2+t)^{r-1}} \\
&\times \sum_{0 < n_1 < \dots < n_{r-1}} \frac{(1 - e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}}} \frac{(1 - e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1} - m}} \sum_{m=0}^{\infty} (-1)^m \binom{k_r + m - 1}{m} \tag{2.8} \\
&= \frac{2}{(2+t)} e_\lambda^x(\log_\lambda(1+t)) \frac{2^{r-1}}{t^{r-1} (2+t)^{r-1}} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{(1 - e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1} - m}} \\
&\times \sum_{l=0}^{\infty} \left(\sum_{n_r=1}^{l+1} \sum_{m=0}^{\infty} (-1)^m \binom{k_r + m - 1}{m} n_r! \frac{(-1)^{l+1-n_r} S_2(l+1, n_r)}{l+1} n_r^{-k_r-m} \right) \frac{t^l}{l!} \\
&= \left(\sum_{n=0}^{\infty} \text{Ch}_n(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \text{Ch}_n^{(k_1, \dots, k_{r-1}-m)} \frac{t^n}{n!} \right) \\
&\times \sum_{l=0}^{\infty} \left(\sum_{n_r=1}^{l+1} \sum_{m=0}^{\infty} (-1)^m \binom{k_r + m - 1}{m} n_r! \frac{(-1)^{l+1-n_r} S_2(l+1, n_r)}{l+1} n_r^{-k_r-m} \right) \frac{t^l}{l!} \\
&= \left(\sum_{n=0}^{\infty} \text{Ch}_n(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{l=0}^n \sum_{n_r=1}^{l+1} \sum_{m=0}^{\infty} \binom{n}{l} (-1)^m \binom{k_r + m - 1}{m} n_r! \right. \\
&\times \left. \frac{(-1)^{l+1-n_r} S_2(l+1, n_r)}{l+1} n_r^{-k_r-m} \text{Ch}_{n-l}^{(k_1, \dots, k_{r-1}-m)} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{b=0}^n \sum_{l=0}^b \sum_{n_r=1}^{l+1} \sum_{m=0}^{\infty} \binom{n}{b} \binom{b}{l} (-1)^m \binom{k_r + m - 1}{m} n_r! \right. \\
&\times \left. \frac{(-1)^{l+1-n_r} S_2(l+1, n_r)}{l+1} n_r^{-k_r-m} \text{Ch}_{b-l}^{(k_1, \dots, k_{r-1}-m)} \text{Ch}_{n-b}(x) \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, by comparing the coefficients on the both sides of (2.8), our proof is completed. \square

Corollary 2.3. For $r \in \mathbb{Z}$, $r \geq 1$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} & \text{Ch}_n^{(k_1, \dots, -k_r)}(x) \\ &= \sum_{b=0}^n \sum_{l=0}^b \sum_{n_r=1}^{l+1} \sum_{m=0}^{\infty} \binom{n}{b} \binom{b}{l} \binom{k_r}{m} n_r! \frac{(-1)^{l+1+n_r} S_2(l+1, n_r)}{l+1} n_r^{-k_r-m} \text{Ch}_{b-l}^{(k_1, \dots, k_{r-1}-m)} \text{Ch}_{n-b}(x). \end{aligned}$$

Proof. Replacing k_r by $-k_r$, then

$$\binom{-k_r + m - 1}{m} = (-1)^m \binom{k_r}{m}$$

and so our corollary is proved. \square

From now on, we get the recurrence relation of the degenerate multi-poly-Changhee polynomials.

Theorem 2.4. For $r \in \mathbb{Z}$, $r \geq 1$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} & \text{Ch}_n^{(k_1, \dots, k_r)}(x+1) - \text{Ch}_n^{(k_1, \dots, k_r)}(x) \\ &= \sum_{k=1}^n \sum_{a=0}^{n-k} \sum_{n_r=1}^k \sum_{m=0}^{\infty} \binom{n}{k} \binom{n-k}{a} (-1)^{m+k-n_r} \binom{k_r + m - 1}{m} n_r! n_r^{-k_r-m} \\ & \quad \times S_2(k, n_r) \text{Ch}_{n-k, \lambda}^{(k_1, \dots, k_{r-1})}(x) \text{Ch}_{n-k-a}. \quad (n \geq 1). \end{aligned}$$

Proof. Consider that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left(\text{Ch}_n^{(k_1, \dots, k_r)}(x+1) - \text{Ch}_n^{(k_1, \dots, k_r)}(x) \right) \frac{t^n}{n!} = \frac{2^r e_\lambda^x (\log_\lambda(1+t))}{t^{r-1}(2+t)^r} \text{Li}_{k_1, \dots, k_r}(1-e^{-t}) \\
& = \frac{2^r e_\lambda^x (\log_\lambda(1+t))}{t^{r-1}(2+t)^r} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{1}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}}} \sum_{n_r=n_{r-1}+1}^{\infty} \frac{(1-e^{-t})^{n_r}}{n_r^{k_r}} \\
& = \frac{2^r e_\lambda^x (\log_\lambda(1+t))}{t^{r-1}(2+t)^r} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}}} \sum_{n_r=1}^{\infty} \frac{(1-e^{-t})^{n_r} n_r!}{(n_r+n_{r-1})^{k_r} n_r!} \\
& = \frac{2}{2+t} \frac{2^{r-1}}{t^{r-1}(1+t)^{r-1}} e_\lambda^x (\log_\lambda(1+t)) \sum_{0 < n_1 < \dots < n_{r-1}} \frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}-m}} \\
& \times \sum_{n_r=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{k_r+m-1}{m} n_r! n_r^{-k_r-m} \sum_{l=n_r}^{\infty} (-1)^{n_r+l} S_2(l, n_r) \frac{t^l}{l!} \\
& = \left(\sum_{n=0}^{\infty} \text{Ch}_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \text{Ch}_{n,\lambda}^{(k_1, \dots, k_{r-1})}(x) \frac{t^n}{n!} \right) \tag{2.9} \\
& \times \left(\sum_{k=1}^{\infty} \sum_{n_r=1}^k \sum_{m=0}^{\infty} (-1)^{m+k-n_r} \binom{k_r+m-1}{m} n_r! n_r^{-k_r-m} S_2(k, n_r) \frac{t^k}{k!} \right) \\
& = \sum_{m=0}^{\infty} \left(\sum_{a=0}^m \binom{m}{a} \text{Ch}_{m,\lambda}^{(k_1, \dots, k_{r-1})}(x) \text{Ch}_{m-a} \frac{t^m}{m!} \right) \\
& \times \left(\sum_{k=1}^{\infty} \sum_{n_r=1}^k \sum_{m=0}^{\infty} (-1)^{m+k-n_r} \binom{k_r+m-1}{m} n_r! n_r^{-k_r-m} S_2(k, n_r) \frac{t^k}{k!} \right) \\
& = \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \sum_{a=0}^{n-k} \sum_{n_r=1}^k \sum_{m=0}^{\infty} \binom{n}{k} \binom{n-k}{a} (-1)^{m+k-n_r} \binom{k_r+m-1}{m} n_r! n_r^{-k_r-m} \right. \\
& \left. \times S_2(k, n_r) \text{Ch}_{n-k,\lambda}^{(k_1, \dots, k_{r-1})}(x) \text{Ch}_{n-k-a} \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, by comparing the coefficients on the both sides of (2.9), we proved our theorem. \square

We will get another result of the recurrence relation as follows:

Theorem 2.5. For $r \in \mathbb{Z}$, $r \geq 1$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
& \text{Ch}_{n,\lambda}^{(k_1, \dots, k_r)}(x+1) - \text{Ch}_{n,\lambda}^{(k_1, \dots, k_r)}(x) \\
& = 2 \sum_{a=0}^n \sum_{n_r=1}^{a+1} \sum_{m=0}^{\infty} \binom{n}{a} n_r! n_r^{-k_r-m} \frac{(-1)^{a+1-n_r+m} S_2(a+1, n_r)}{a+1}.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left(\text{Ch}_n^{(k_1, \dots, k_r)}(x+1) - \text{Ch}_n^{(k_1, \dots, k_r)}(x) \right) \frac{t^n}{n!} = \frac{2^r e_\lambda^x (\log_\lambda(1+t))}{t^r (2+t)^{r-1}} \text{Li}_{k_1, \dots, k_r}(1-e^{-t}) \\
&= \frac{2}{t} \frac{2^{r-1} e_\lambda^x (\log_\lambda(1+t))}{t^{r-1} (2+t)^{r-1}} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{1}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}}} \sum_{n_r=n_{r-1}+1}^{\infty} \frac{(1-e^{-t})^{n_r}}{n_r^{k_r}} \\
&= 2 \frac{2^{r-1} e_\lambda^x (\log_\lambda(1+t))}{t^{r-1} (2+t)^{r-1}} \sum_{0 < n_1 < \dots < n_{r-1}} \frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1} \dots n_{r-1}^{k_{r-1}-m}} \\
&\times \sum_{a=1}^{\infty} \sum_{n_r=1}^a \sum_{m=0}^{\infty} n_r! (-1)^{a-n_r+m} S_2(a, n_r) \binom{k_r+m-1}{m} n_r^{-k_r-m} \frac{t^a}{a!} \\
&= 2 \left(\sum_{n=0}^{\infty} \text{Ch}_{n,\lambda}^{(k_1, \dots, k_{r-1-m})}(x) \frac{t^n}{n!} \right) \\
&\times \left(\sum_{a=0}^{\infty} \sum_{n_r=1}^{a+1} \sum_{m=0}^{\infty} n_r! \frac{(-1)^{a+1-n_r+m} S_2(a+1, n_r) \binom{k_r+m-1, m}{n}^{-k_r-m}}{a+1} \right) \frac{t^a}{a!} \\
&= 2 \sum_{n=0}^{\infty} \sum_{a=0}^n \sum_{n_r=1}^{a+1} \sum_{m=0}^{\infty} \binom{n}{a} n_r! n_{r-1}^{-k_r-m} \frac{(-1)^{a+1-n_r+m} S_2(a+1, n_r) t^a}{a+1} \frac{t^a}{a!}.
\end{aligned} \tag{2.10}$$

Therefore, by comparing the coefficients on both sides of (2.10), our theorem is proved. \square

Theorem 2.6. For $r \in \mathbb{Z}$, $r \geq 1$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\text{Ch}_{n,\lambda}^{(k_1, \dots, k_r)}(x+y) = \sum_{m=0}^n \text{Ch}_{n-m,\lambda}^{(k_1, \dots, k_r)}(x)(y)_m.$$

Proof. From (2.6), we observe that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \text{Ch}_{n,\lambda}^{(k_1, \dots, k_r)}(x+y) \frac{t^n}{n!} \\
&= \frac{2^r e_\lambda^{x+y} (\log_\lambda(1+t))}{t^r (2+t)^r} \text{Li}_{k_1, \dots, k_r}(1-e^{-t}) \\
&= \frac{2^r}{t^r (2+t)^r} \text{Li}_{k_1, \dots, k_r}(1-e^{-t}) e_\lambda^x (\log_\lambda(1+t)) e_\lambda^y (\log_\lambda(1+t)) \\
&= \left(\sum_{n=0}^{\infty} \text{Ch}_{n,\lambda}^{(k_1, \dots, k_r)}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^n \text{Ch}_{n-m,\lambda}^{(k_1, \dots, k_r)}(x)(y)_m \frac{t^n}{n!}.
\end{aligned} \tag{2.11}$$

Therefore, by comparing the coefficients on both sides of (2.10), we obtain our theorem. \square

3. The zeros of degenerate multi-poly-Changhee polynomials

It is well known that finding the roots of polynomials is a very important task in applied mathematics. There is no formula for the roots of polynomials of degree five or higher, and there are formulas for the roots of polynomials of degree four or less, but their expressions are very complex. Hence, we want to investigate the numerical pattern of the roots of the polynomials $\text{Ch}_n^{(k_1, \dots, k_r)}(x)$. Before investigating the numerical pattern of the roots of the polynomials $\text{Ch}_n^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)$ with $n = 5$ for $r = 2, 3, 4, 5$ and of $\text{Ch}_n^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)$ with $r = 5$ for $n = 2, 3, 4, 5$ in Figure 1.

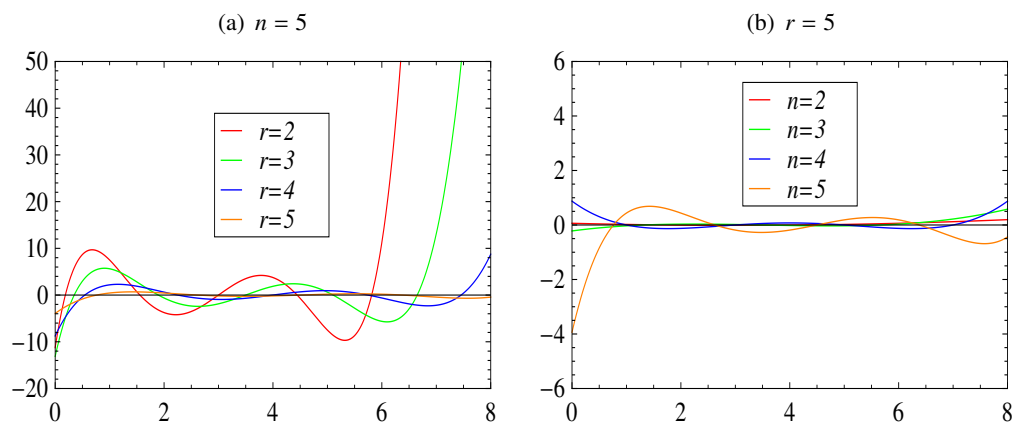


Figure 1. The graph of $\text{Ch}_n^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)$.

The polynomial $\text{Ch}_n^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)$ can be explicitly expressed by using Mathematica. For example,

$$\text{Ch}_1^{(1,1)}(x) = \frac{1}{2}(-1 + x),$$

$$\text{Ch}_2^{(1,1)}(x) = \frac{1}{4}(3 - 6x + 2x^2),$$

$$\text{Ch}_3^{(1,1)}(x) = \frac{1}{4}(-6 + 19x - 12x^2 + 2x^3),$$

$$\text{Ch}_1^{(1,1,1)}(x) = \frac{1}{12}(-3 + 2x),$$

$$\text{Ch}_2^{(1,1,1)}(x) = \frac{1}{6}(3 - 4x + x^2),$$

$$\text{Ch}_3^{(1,1,1)}(x) = \frac{1}{12}(-15 + 31x - 15x^2 + 2x^3).$$

We want to observe the impact of the parameter r on the distribution of the roots of the polynomials. For the aims, we set the degree of the polynomial as $n = 40$. Using the Mathematical tool with 100 working precision, the roots of the polynomial $\text{Ch}_n^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)$ are computed. The absolute numerical

error is bounded as the following:

$$\sum_{i=1}^{40} |\text{Ch}_{40}^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x_i)| < 10^{-62},$$

where x_i denotes the roots of the polynomial. Hence, the numerical roots are reliable. We compute the numerical roots of $\text{Ch}_{40}^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)$ with $r = 2, 3, 10$. The numerical results are plotted in Figure 2.

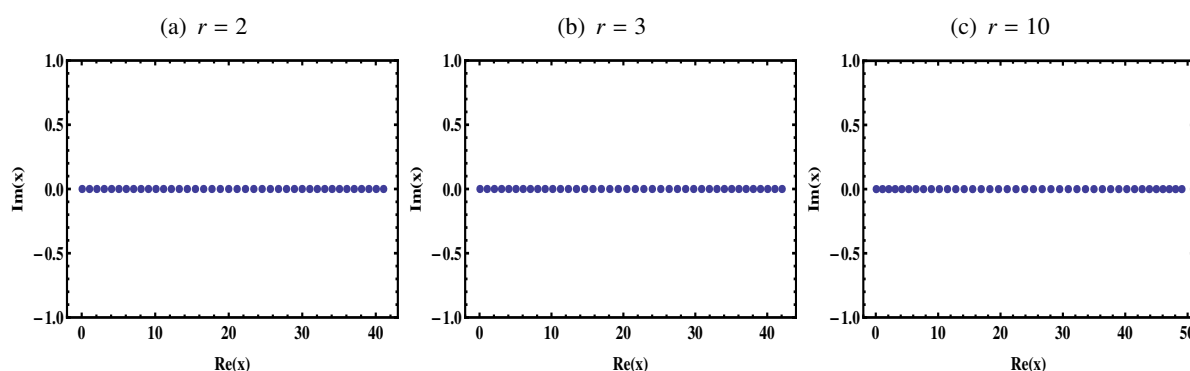


Figure 2. The computed roots of $\text{Ch}_{40}^{\overbrace{(1, \dots, 1)}^{r\text{-times}}}(x)$.

4. Conclusions

Degenerate versions of some special polynomials and numbers were initiated by Carlitz, and many researchers have introduced many degenerate numbers and polynomials using Carlitz's degeneracy theory. Moreover, researchers also studied the identities and properties of these degenerate numbers and polynomials. Here, we would like to mention that the study of degenerate versions is not limited only to polynomials but can be extended also to transcendental functions like gamma functions, degenerate umbral calculus and degenerate probability density function.

In this paper, we considered the degenerate multi-poly-Changhee numbers and polynomials, which are defined by means of the multiple polylogarithms. We investigated some properties for those numbers and polynomials and also presented explicit examples of the polynomials by using the Mathematica tool. In order to better understanding the polynomials, the distribution of roots was presented.

In research following this paper, we plan to expand various special polynomials and numbers using multiple polylogarithm functions or multiple poly-exponential functions and find their properties.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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