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Research article

Two-grid H^1-Galerkin mixed finite elements combined with $L1$ scheme for nonlinear time fractional parabolic equations

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Abstract: In this paper, we propose a two-grid algorithm for nonlinear time fractional parabolic equations by H^1-Galerkin mixed finite element discreitzation. First, we use linear finite elements and Raviart-Thomas mixed finite elements for spatial discretization, and $L1$ scheme on graded mesh for temporal discretization to construct a fully discrete approximation scheme. Second, we derive the stability and error estimates of the discrete scheme. Third, we present a two-grid method to linearize the nonlinear system and discuss its stability and convergence. Finally, we confirm our theoretical results by some numerical examples.

Keywords: two-grid algorithm; H^1-Galerkin mixed finite element; nonlinear time fractional partial differential equations

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1. Introduction

We focus on the following nonlinear time fractional partial differential equation (TFPDE):

$$\begin{cases} \mathcal{D}_t^\alpha y - \operatorname{div}\boldsymbol{Y} = f(y), & \text{in } J \times \Omega, \\ \boldsymbol{Y} = \nabla y, & \text{in } J \times \Omega, \\ y = 0, & \text{on } J \times \partial\Omega, \\ y(0,x) = y_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $J = (0,T]$ and $\Omega \subset \mathbb{R}^2$ is a convex polygonal with boundary $\partial\Omega$, the nonlinear function $f(y)$ fulfills

$$|f'(y)| + |f''(y)| \leq M, \quad \forall y \in \mathbf{R}. \tag{1.2}$$

The Caputo derivative [1] $\mathcal{D}_t^\alpha$ defined as follows:

$$\mathcal{D}_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(\kappa)}{(t-\kappa)^\alpha} d\kappa, \quad 0 < \alpha < 1.$$

Here, $\Gamma(z) = \int_0^{+\infty} \kappa^{z-1} e^{-\kappa} d\kappa$ is the usual Gamma function.

Due to the excellent performance in describing memory and genetic properties, TFPDEs have been widely used to model anomalous diffusion, electromagnetic wave, hydrodynamics, signal processing, material science etc. Unfortunately, the analytical solutions of most TFPDEs are almost impossible to calculate. For the past twenty years, many researchers proposed extensive numerical methods of TFPDEs. Related books can be found in [2–4]. In the current literatures, we can see finite difference [5–8], spectral method [9, 10], finite volume [11], finite element (FE) [12–14], weak FE [15], mixed finite element (MFE) [16–19], virtual element method [20], fast algorithm [21–23], higher-order schemes [24–27] and so on, which mostly focused on the linear fractional differential equation case. Recently, Zheng et al. [28] established the stability and error estimates of fully discrete FE for a hidden-memory variable-order time-fractional optimal control model.

In recent years, scholars pay much attention to nonlinear TFPDEs. In [29], Li et al. considered a numerical approximation of nonlinear TFPDEs. In [30], the authors introduced a Newton linearized Galerkin FE approximation for nonlinear TFPDEs with non-smooth solutions. The two-grid method was first proposed by Xu [31] for solving nonsymmetric and nonlinear partial differential equations (PDEs) under finite element approximation. As an effective numerical method, it has been expanded to solve various nonlinear fractional PDEs [32–35].

For the classic MFE, its discrete space pairs must satisfy the Ladyženskaja-Babuška-Brezzi (LBB) condition. To avoid the LBB constraints, Pehlivanov et al. [36] developed a least-squares MFE, Pani [37] presented an H^1-Galerkin mixed finite element (HMFE) approximation by requiring extra regularity on the solution and Yang [38] proposed a splitting positive definite MFE by separating the pressure equation from flux equation. Recently, Liu and Hou investigated the HMFE approximation of one-dimensional TFPDEs [39, 40] and semilinear parabolic integro-differential equations [41], respectively. Our aim of this article is to provide a two-grid algorithm (TGA) for the nonlinear problem (1.1) by HMFE combined with $L1$ scheme discretization and analyze its stability and convergence.

The layout of the paper is as follows. We give a fully discrete HMFE approximation scheme (FDHAS) of the problem (1.1) in Section 2. The stability and convergence of the discrete scheme is analyzed in Section 3. The TGA of (1.1) and its stability and convergence are discussed in Section 4. In Section 5, we give several examples to verify our theoretical findings.

2. FDHAS of nonlinear TFPEDs

The FDHAS of the nonlinear TFPDE (1.1) will be constructed in this section. Throughout this article, we use standard notations for Sobolev spaces $W^{m,q}(\Omega)$ with a semi-norm $|w|_{m,q}$ and a norm $\|w\|_{m,q}$. For $q = 2$, let $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = \{w \in H^m(\Omega) : w|_{\partial\Omega} = 0\}$, $\|\cdot\| = \|\cdot\|_{0,2}$, $\|\cdot\|_m = \|\cdot\|_{m,2}$. $L^p(J; W^{m,q}(\Omega))$ be all L^p integrable function space from J into $W^{m,q}(\Omega)$ with norm $\|w\|_{L^p(J;W^{m,q}(\Omega))} = \left(\int_0^T \|w\|^p_{W^{m,q}(\Omega)} dt\right)^{1/p}$ for $p \in [1, \infty)$, and the standard modification for $p = \infty$. In addition, $c > 0$ and $C > 0$ are generic constant.

Let us assume that the Eq (1.1) has a solution $y(t, x)$ such that

$$\left|\frac{\partial^l y(t,x)}{\partial t^l}\right| \le c t^{\alpha-l}, \quad 0 \le l \le 3.$$

Set $W = H_0^1(\Omega)$ and $\boldsymbol{V} = H(\text{div}, \Omega)$ with

$$H(\text{div}, \Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2 : \nabla \cdot \boldsymbol{v} \in L^2(\Omega)\}.$$

Like in [42], an H^1-Galerkin mixed variational formulation of (1.1) is: Find $\{y, \boldsymbol{Y}\} : (0, T] \to W \times \boldsymbol{V}$, for any $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$, such that

$$(\mathcal{D}_t^\alpha \boldsymbol{Y}, \boldsymbol{v}) + (\text{div}\boldsymbol{Y}, \text{div}\boldsymbol{v}) = -(f(y), \text{div}\boldsymbol{v}), \tag{2.1}$$

$$(\boldsymbol{Y}, \nabla w) = (\nabla y, \nabla w), \tag{2.2}$$

$$y(0, x) = y_0(x). \tag{2.3}$$

Let $\mathcal{T}_h$ be a uniform rectangular partition of Ω, where $h = \max\limits_{e \in \mathcal{T}_h}\{h_e\}$ denotes the spatial mesh size and h_e deotes the diameter of element e. Associated with the rectangular partition $\mathcal{T}_h$ of Ω, we define subspaces $W_h \times \boldsymbol{V}_h \subset W \times \boldsymbol{V}$ as follows [43]:

$$W_h := \{w_h \in C(\Omega) : w_h|_e \in Q_{1,1}(e), \forall e \in \mathcal{T}_h, w_h|_{\partial\Omega} = 0\},$$
$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \boldsymbol{v}_h|_e \in Q_{1,0} \times Q_{0,1}(e), \forall e \in \mathcal{T}_h\},$$

where $Q_{m,n}(e)$ be the space of polynomials with degree no more than m and n in the x_1 direction and the x_2 direction, respectively.

Then a semidiscrete HMFE approximation of (1.1) is: Find $\{y_h, \boldsymbol{Y}_h\} : (0, T] \to W_h \times \boldsymbol{V}_h$, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, such that

$$(\mathcal{D}_t^\alpha \boldsymbol{Y}_h, \boldsymbol{v}_h) + (\text{div}\boldsymbol{Y}_h, \text{div}\boldsymbol{v}_h) = -(f(y_h), \text{div}\boldsymbol{v}_h), \tag{2.4}$$

$$(\boldsymbol{Y}_h, \nabla w_h) = (\nabla y_h, \nabla w_h), \tag{2.5}$$

$$y_h(0, x) = P_h y_0(x), \tag{2.6}$$

where [41] $P_h : W \to W_h$, which fulfills: For any $w_h \in W_h$ and $\varphi \in W$

$$(\nabla(P_h\varphi - \varphi), \nabla w_h) = 0,$$
$$\|\varphi - P_h\varphi\|_r \le Ch^{2-r}\|\varphi\|_2, \quad \forall \varphi \in H^2(\Omega), r = 0, 1. \tag{2.7}$$

We introduce [43,44] $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which fulfills: For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\boldsymbol{\psi} \in \boldsymbol{V}$

$$(\text{div}(\Pi_h\boldsymbol{\psi} - \boldsymbol{\psi}), \text{div}\boldsymbol{v}_h) = 0,$$

$$\|\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi}\| \le Ch\|\boldsymbol{\psi}\|_1, \quad \forall \boldsymbol{\psi} \in \left(H^1(\Omega)\right)^2, \tag{2.8}$$

$$\|\text{div}(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi})\|_{-r} \le Ch^{1+r}\|\text{div}\boldsymbol{\psi}\|_1, \quad \forall \text{div}\boldsymbol{\psi} \in H^1(\Omega), r = 0, 1, \tag{2.9}$$

$$(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi}, \boldsymbol{v}_h) \le Ch^2\left(\|\boldsymbol{\psi}\|_2\|\boldsymbol{v}_h\| + \|\boldsymbol{\psi}\|_1\|\text{div}\boldsymbol{v}_h\|\right), \quad \forall \boldsymbol{\psi} \in \left(H^2(\Omega)\right)^2. \tag{2.10}$$

Since the solution may be weakly singular at $t = 0$, we use the graded mesh for time discretization. For $n = 0, 1, \cdots, N$ with $N \in \mathbb{Z}^+$, let $t_n = T(n/N)^\gamma$, where $\gamma \geq 1$ will be adapted to the strength of the singularity and chosen by the user and the temporal mesh size $\tau = \max\{\tau_n\}_{n=1}^N$ with $\tau_n = t_n - t_{n-1}$. We set $\varphi^n = \varphi(t_n, x)$ for $n = 0, 1, \cdots, N$. The $L1$ approximation scheme is given by [2]:

$$\begin{aligned}\mathcal{D}_t^\alpha \varphi^n &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{\varphi^{k+1} - \varphi^k}{\tau_{k+1}} \int_{t_k}^{t_{k+1}} \frac{ds}{(t_n - s)^\alpha} + R_\varphi^n \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{\varphi^{k+1} - \varphi^k}{\tau_{k+1}} \left[(t_n - t_k)^{1-\alpha} - (t_n - t_{k+1})^{1-\alpha} \right] + R_\varphi^n \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{n-1} (d_{n,k+1} - d_{n,k}) \varphi^{n-k} + \frac{d_{n,1}}{\Gamma(2-\alpha)} \varphi^n - \frac{d_{n,n}}{\Gamma(2-\alpha)} \varphi^0 + R_\varphi^n \\ &:= \mathcal{D}_N^\alpha \varphi^n + R_\varphi^n,\end{aligned} \tag{2.11}$$

where

$$d_{n,k} = \frac{(t_n - t_{n-k})^{1-\alpha} - (t_n - t_{n-k+1})^{1-\alpha}}{\tau_{n-k+1}}, k = 1, 2, \cdots, n.$$

Lemma 2.1. *([22]) If $|\varphi''(t)| \leq Ct^{\alpha-2}, 0 < t \leq T$, there exists a constant C independent of τ, such that*

$$\left| R_\varphi^n \right| := \left| \mathcal{D}_t^\alpha \varphi^n - \mathcal{D}_N^\alpha \varphi^n \right| \leq Cn^{-r}, \quad n = 1, 2, \cdots, N, \tag{2.12}$$

where $r = \min\{2 - \alpha, \gamma\alpha\}$.

Then the FDHAS of (1.1) is as follows: Find $\left(y_h^n, \boldsymbol{Y}_h^n\right) \in W_h \times \boldsymbol{V}_h, n = 1, 2, \cdots, N$ for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, such that

$$(\mathcal{D}_N^\alpha \boldsymbol{Y}_h^n, \boldsymbol{v}_h) + (\mathrm{div} \boldsymbol{Y}_h^n, \mathrm{div} \boldsymbol{v}_h) = -\left(f(y_h^n), \mathrm{div} \boldsymbol{v}_h\right), \tag{2.13}$$

$$(\boldsymbol{Y}_h^n, \nabla w_h) = (\nabla y_h^n, \nabla w_h), \tag{2.14}$$

$$y_h^0 = P_h y_0(x). \tag{2.15}$$

3. Stability and convergence analysis of the FDHAS

The stability and convergence of the FDHAS (2.13)–(2.15) will be analyzed in this section. The discrete Grönwall lemma will be used in the following analysis.

Lemma 3.1. *([26, 45]) Let $\{\xi^n\}_{n=1}^N$, $\{g^n\}_{n=1}^N$ and $\{\lambda_n\}_{n=0}^{N-1}$ be given nonnegative sequences. If there is a constant Λ independent of τ satisfies*

$$\tau \leq \frac{1}{\sqrt[\alpha]{2\Gamma(2-\alpha)\Lambda}} \text{ and } \sum_{n=0}^{N-1} \lambda_n \leq \Lambda.$$

Then, for any nonnegative sequence $\{V^k\}_{k=0}^N$ and $1 \leq n \leq N$ such that

$$\mathcal{D}_N^\alpha (V^n)^2 \leq \sum_{k=1}^n \lambda_{n-k} \left(V^k\right)^2 + \xi^n V^n + (g^n)^2, \tag{3.1}$$

it holds that

$$V^n \le 2E_\alpha\left(2\Lambda t_n^\alpha\right)\left(V^0 + \Gamma(1-\alpha)\max_{1\le k\le n}\left\{t_k^\alpha \xi^k\right\} + \sqrt{\Gamma(1-\alpha)}\max_{1\le k\le n}\left\{t_k^{\alpha/2} g^k\right\}\right), \tag{3.2}$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$ *is the Mittag-Leffler function.*

3.1. Stability analysis

We first show the stability of the solution to (2.13)–(2.15).

Theorem 3.1. *Let* $(y_h, \boldsymbol{Y}_h)$ *be the solution of (2.13)–(2.15) and all the conditions in Lemma 3.1 are valid. Then*

$$\left\|\boldsymbol{Y}_h^n\right\| \le C\left(\left\|\boldsymbol{Y}_h^0\right\| + \|f(0)\|\right), \tag{3.3}$$

$$\left\|y_h^n\right\| \le C\left(\left\|\boldsymbol{Y}_h^0\right\| + \|f(0)\|\right). \tag{3.4}$$

Proof. Choosing $\boldsymbol{v}_h = 2\boldsymbol{Y}_h^n$ in (2.13) leads to

$$\begin{aligned} 2\left(\mathcal{D}_N^\alpha \boldsymbol{Y}_h^n, \boldsymbol{Y}_h^n\right) + 2\left(\mathrm{div}\boldsymbol{Y}_h^n, \mathrm{div}\boldsymbol{Y}_h^n\right) =& 2\left(f(0) - f(y_h^n) - f(0), \mathrm{div}\boldsymbol{Y}_h^n\right) \\ \le& 2\left(\|f(0) - f(y_h^n)\|\|\mathrm{div}\boldsymbol{Y}_h^n\| + \|f(0)\|\|\mathrm{div}\boldsymbol{Y}_h^n\|\right) \\ \le& 2C(\varepsilon)\left(M\|y_h^n\|^2 + \|f(0)\|^2\right) + 4\varepsilon\|\mathrm{div}\boldsymbol{Y}_h^n\|^2, \end{aligned} \tag{3.5}$$

where we have used the Taylor's formula, ε-Cauchy inequality and the condition (1.2).

From the definition of $\mathcal{D}_N^\alpha \boldsymbol{Y}_h^n$ and Hölder inequality, we can derive

$$\frac{1}{2}\mathcal{D}_N^\alpha \left\|\boldsymbol{Y}_h^n\right\|^2 \le \left(\mathcal{D}_N^\alpha \boldsymbol{Y}_h^n, \boldsymbol{Y}_h^n\right), \quad \text{for } 1 \le n \le N. \tag{3.6}$$

Taking $w_h = y_h^n$ in (2.14) and using Hölder's inequality, we have

$$\|\nabla y_h^n\|^2 = \left(\boldsymbol{Y}_h^n, \nabla y_h^n\right) \le C\|\boldsymbol{Y}_h^n\|\|\nabla y_h^n\|. \tag{3.7}$$

Employing the Poincaré inequality and (3.5)–(3.7), we obtain

$$\mathcal{D}_N^\alpha \left\|\boldsymbol{Y}_h^n\right\|^2 \le 2C(\varepsilon)M\|\boldsymbol{Y}_h^n\|^2 + 2C(\varepsilon)\|f(0)\|^2. \tag{3.8}$$

Then (3.3) follows from Lemma 3.1 and (3.8). From (3.3) and (3.7), we can easily arrive at (3.4). □

3.2. Convergence analysis

From (2.1)–(2.3), (2.11) and (2.13)–(2.15), we obtain error equations

$$\left(\mathcal{D}_N^\alpha\left(\boldsymbol{Y}^n - \boldsymbol{Y}_h^n\right), \boldsymbol{v}_h\right) + \left(\mathrm{div}\left(\boldsymbol{Y}^n - \boldsymbol{Y}_h^n\right), \mathrm{div}\boldsymbol{v}_h\right) = \left(f(y_h^n) - f(y^n), \mathrm{div}\boldsymbol{v}_h\right) - \left(R_{\boldsymbol{Y}}^n, \boldsymbol{v}_h\right), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.9}$$

$$\left(\boldsymbol{Y}^n - \boldsymbol{Y}_h^n, \nabla w_h\right) = \left(\nabla\left(y^n - y_h^n\right), \nabla w_h\right), \quad \forall w_h \in W_h. \tag{3.10}$$

For convenience, we set

$$\begin{aligned} y - y_h &= y - P_h y + P_h y - y_h := \zeta + \eta, \\ \boldsymbol{Y} - \boldsymbol{Y}_h &= \boldsymbol{Y} - \Pi_h \boldsymbol{Y} + \Pi_h \boldsymbol{Y} - \boldsymbol{Y}_h := \boldsymbol{\theta} + \boldsymbol{\vartheta}. \end{aligned}$$

Theorem 3.2. *Let $(y, \boldsymbol{Y})$ and $(y_h, \boldsymbol{Y}_h)$ be the solutions of (2.1)–(2.3) and (2.13)–(2.15), respectively. Suppose that $y \in L^\infty(J; H^2(\Omega)) \cap H^2(J; L^2(\Omega))$, $\boldsymbol{Y} \in (L^\infty(J; H^1(\Omega))^2 \cap (H^2(J; H^1(\Omega)^2$ and all the conditions in Lemma 2.1 and Theorem 3.1 are valid. Then for $n = 1, 2, \cdots, N$, there hold*

$$\left\|\boldsymbol{Y}^n - \boldsymbol{Y}_h^n\right\| \le C\left(h + N^{-r}\right), \tag{3.11}$$
$$\left\|y^n - y_h^n\right\| \le C\left(h^2 + N^{-r}\right). \tag{3.12}$$

Proof. From the definitions of the projection operators P_h and Π_h and error Eqs (3.9)–(3.10). For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, we have

$$(\mathcal{D}_N^\alpha \boldsymbol{\vartheta}^n, \boldsymbol{v}_h) + (\mathrm{div}\boldsymbol{\vartheta}^n, \mathrm{div}\boldsymbol{v}_h) = \left(f(y_h^n) - f(y^n), \mathrm{div}\boldsymbol{v}_h\right) - (R_{\boldsymbol{Y}}^n, \boldsymbol{v}_h) - (\mathcal{D}_N^\alpha \boldsymbol{\theta}^n, \boldsymbol{v}_h), \tag{3.13}$$
$$(\boldsymbol{\vartheta}^n, \nabla w_h) = (\nabla \eta^n, \nabla w_h) - (\boldsymbol{\theta}^n, \nabla w_h). \tag{3.14}$$

Taking $\boldsymbol{v}_h = \boldsymbol{\vartheta}^n$ in (3.13), then there yields

$$(\mathcal{D}_N^\alpha \boldsymbol{\vartheta}^n, \boldsymbol{\vartheta}^n) + (\mathrm{div}\boldsymbol{\vartheta}^n, \mathrm{div}\boldsymbol{\vartheta}^n) = \left(f(y_h^n) - f(y^n), \mathrm{div}\boldsymbol{\vartheta}^n\right) - (R_{\boldsymbol{Y}}^n, \boldsymbol{\vartheta}^n) - (\mathcal{D}_N^\alpha \boldsymbol{\theta}^n, \boldsymbol{\vartheta}^n). \tag{3.15}$$

According to the mean-value theorem, the conditions (1.2) and (2.7), we obtain

$$\begin{aligned}\left\|f(y_h^n) - f(y^n)\right\| &\le M\left\|y^n - y_h^n\right\| \\ &= M\left(\|y^n - P_h y^n\| + \left\|P_h y^n - y_h^n\right\|\right) \\ &\le M\left(Ch^2\|y^n\|_2 + \|\eta^n\|\right).\end{aligned} \tag{3.16}$$

It follows from Hölder's inequality and (2.12) that

$$\begin{aligned}(R_{\boldsymbol{Y}}^n, \boldsymbol{\vartheta}^n) &= (\mathcal{D}_t^\alpha \boldsymbol{Y}^n - \mathcal{D}_N^\alpha \boldsymbol{Y}^n, \boldsymbol{\vartheta}^n) \\ &\le C\left\|\mathcal{D}_t^\alpha \boldsymbol{Y}^n - \mathcal{D}_N^\alpha \boldsymbol{Y}^n\right\|\|\boldsymbol{\vartheta}^n\| \\ &\le Cn^{-r}\|\boldsymbol{\vartheta}^n\|.\end{aligned} \tag{3.17}$$

From (2.10) and Poincaré inequality, we have

$$\begin{aligned}(\mathcal{D}_N^\alpha \boldsymbol{\theta}^n, \boldsymbol{\vartheta}^n) &\le Ch^2\left(\mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_2\|\boldsymbol{\vartheta}^n\| + \mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_1\|\mathrm{div}\boldsymbol{\vartheta}^n\|\right) \\ &\le Ch^2\left(\mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_2 + \mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_1\right)\|\mathrm{div}\boldsymbol{\vartheta}^n\|.\end{aligned} \tag{3.18}$$

Using (3.15)–(3.18), Hölder's inequality and ε-Cauchy inequality, we get

$$\begin{aligned}&(\mathcal{D}_N^\alpha \boldsymbol{\vartheta}^n, \boldsymbol{\vartheta}^n) + \|\mathrm{div}\boldsymbol{\vartheta}^n\|^2 \\ &\le \|f(y_h^n) - f(y^n)\|\|\mathrm{div}\boldsymbol{\vartheta}^n\| + \|R_{\boldsymbol{Y}}^n\|\|\boldsymbol{\vartheta}^n\| + Ch^2\left(\mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_2 + \mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_1\right)\|\mathrm{div}\boldsymbol{\vartheta}^n\| \\ &\le C(\varepsilon)\left(\|f(y_h^n) - f(y^n)\|^2 + h^4\left(\mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_2 + \mathcal{D}_N^\alpha\|\boldsymbol{Y}^n\|_1\right)^2\right) + \|R_{\boldsymbol{Y}}^n\|\|\boldsymbol{\vartheta}^n\| + 2\varepsilon\|\mathrm{div}\boldsymbol{\vartheta}^n\|^2 \\ &\le C(\varepsilon)\left(M\|\eta^n\|^2 + h^4\right) + Cn^{-r}\|\boldsymbol{\vartheta}^n\| + 2\varepsilon\|\mathrm{div}\boldsymbol{\vartheta}^n\|^2.\end{aligned} \tag{3.19}$$

Noting that $\left(\mathcal{D}_N^\alpha \boldsymbol{\vartheta}^n, \boldsymbol{\vartheta}^n\right) \ge \frac{1}{2}\mathcal{D}_N^\alpha\|\boldsymbol{\vartheta}^n\|^2$, from (3.16)–(3.19), we have

$$\frac{1}{2}\mathcal{D}_N^\alpha\|\boldsymbol{\vartheta}^n\|^2 + (1 - 2\epsilon)\|\mathrm{div}\boldsymbol{\vartheta}^n\|^2 \le C(\varepsilon)\left(M\|\eta^n\|^2 + h^4\right) + Cn^{-r}\|\boldsymbol{\vartheta}^n\|. \tag{3.20}$$

Choosing $w_h = \eta^n$ in (3.14), we get

$$(\boldsymbol{\vartheta}^n, \nabla\eta^n) = (\nabla\eta^n, \nabla\eta^n) - (\boldsymbol{\theta}^n, \nabla\eta^n). \tag{3.21}$$

Using (3.21), Cauchy-Schwartz inequality, (2.9) and Poincaré inequality, we derive

$$\begin{aligned} \|\nabla\eta^n\|^2 &= (\boldsymbol{\vartheta}^n, \nabla\eta^n) - (\mathrm{div}\boldsymbol{\theta}^n, \eta^n) \\ &\le C\left(\|\boldsymbol{\vartheta}^n\|\|\nabla\eta^n\| + \|\mathrm{div}\boldsymbol{\theta}^n\|_{-1}\|\eta^n\|_1\right) \\ &\le C\left(\|\boldsymbol{\vartheta}^n\| + Ch^2\|\mathrm{div}\boldsymbol{Y}^n\|_1\right)\|\nabla\eta^n\|. \end{aligned} \tag{3.22}$$

According to (3.22), $\boldsymbol{Y} \in (L^\infty(J; H^2(\Omega))^2$ and Poincaré inequality, we get

$$\|\eta^n\| \le C\|\nabla\eta^n\| \le C\left(\|\boldsymbol{\vartheta}^n\| + h^2\right). \tag{3.23}$$

Combining (3.20) and (3.23) yields

$$\mathcal{D}_N^\alpha\|\boldsymbol{\vartheta}^n\|^2 \le CM\|\boldsymbol{\vartheta}^n\|^2 + C\left(Mh^2 + n^{-r}\right)\|\boldsymbol{\vartheta}^n\| + Ch^4. \tag{3.24}$$

By (3.24) and Lemma 3.1, we obtain

$$\|\boldsymbol{\vartheta}^n\| = \left\|\Pi_h\boldsymbol{Y}^n - \boldsymbol{Y}_h^n\right\| \le C\left(h^2 + N^{-r}\right). \tag{3.25}$$

Noting that $\boldsymbol{Y} \in (L^\infty(J; H^2(\Omega))^2$ and using triangle inequality, (2.8) and (3.25), we derive

$$\left\|\boldsymbol{Y}^n - \boldsymbol{Y}_h^n\right\| \le \|\boldsymbol{Y}^n - \Pi_h\boldsymbol{Y}^n\| + \|\Pi_h\boldsymbol{Y}^n - \boldsymbol{Y}_h^n\| \le C\left(h + N^{-r}\right). \tag{3.26}$$

From (3.23) and (3.25), we have

$$\|\eta^n\| \le C\left(h^2 + N^{-r}\right). \tag{3.27}$$

It follows from (2.7), triangle inequality and (3.27) that

$$\left\|y^n - y_h^n\right\| \le \|y^n - P_hy^n\| + \left\|P_hy^n - y_h^n\right\| \le C\left(h^2 + N^{-r}\right). \tag{3.28}$$

Thus, the proof is completed. □

4. TGA and error estimates

A two-grid HMFE algorithm to solve the nonlinear TFPDE (1.1) will be proposed in this section. Let $\mathcal{T}_H$ and $\mathcal{T}_h$ be two uniform rectangular partitions of Ω with different size H and h ($h \ll H$). The fine mesh $\mathcal{T}_h$ is obtained by uniformly refined the coarse mesh $\mathcal{T}_H$. Associated with $\mathcal{T}_H$ and $\mathcal{T}_h$ are HMFE spaces $W_H \times \boldsymbol{V}_H$ and $W_h \times \boldsymbol{V}_h$, respectively. It is obvious that $W_H \times \boldsymbol{V}_H \subset W_h \times \boldsymbol{V}_h$. We show the TGA as follows:

Two-grid algorithm (TGA).

Step 1. On $\mathcal{T}_H$: Find $\left(y_H^n, \boldsymbol{Y}_H^n\right) \in W_H \times \boldsymbol{V}_H$ for $n = 0, 1, \cdots, N$ and any $\boldsymbol{v}_H \in \boldsymbol{V}_H$, $w_H \in W_H$, such that

$$(\mathcal{D}_N^\alpha\boldsymbol{Y}_H^n, \boldsymbol{v}_H) + (\mathrm{div}\boldsymbol{Y}_H^n, \mathrm{div}\boldsymbol{v}_H) = -\left(f(y_H^n), \mathrm{div}\boldsymbol{v}_H\right), \tag{4.1}$$

$$(\boldsymbol{Y}_H^n, \nabla w_H) = (\nabla y_H^n, \nabla w_H), \tag{4.2}$$
$$y_H^0 = P_H y_0(x). \tag{4.3}$$

Step 2. On $\mathcal{T}_h$: Given y_H^n for $n = 1, 2, \cdots, N$, find $\left(y_h^{*,n}, \boldsymbol{Y}_h^{*,n}\right) \in W_h \times \boldsymbol{V}_h$ for $n = 0, 1, \cdots, N$ and any $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $w_h \in W_h$, such that

$$\left(\mathcal{D}_N^\alpha \boldsymbol{Y}_h^{*,n}, \boldsymbol{v}_h\right) + \left(\mathrm{div}\boldsymbol{Y}_h^{*,n}, \mathrm{div}\boldsymbol{v}_h\right) = -\left(f(y_H^n) + f'(y_H^n)\left(y_h^{*,n} - y_H^n\right), \mathrm{div}\boldsymbol{v}_h\right), \tag{4.4}$$
$$\left(\boldsymbol{Y}_h^{*,n}, \nabla w_h\right) = \left(\nabla y_h^{*,n}, \nabla w_h\right), \tag{4.5}$$
$$y_h^{*,0} = P_h y_0(x). \tag{4.6}$$

4.1. Stability analysis

Now, we analyze the stability of the TGA.

Theorem 4.1. *Let $\left(y_h^*, \boldsymbol{Y}_h^*\right)$ be the solution of the two-grid algorithm (4.1)–(4.6) and all the conditions in Lemma 3.1 are valid. Then we have*

$$\left\|\boldsymbol{Y}_h^{*,n}\right\| \leq C\left(\left\|\boldsymbol{Y}_H^0\right\| + \left\|\boldsymbol{Y}_h^{*,0}\right\| + \|f(0)\|\right), \tag{4.7}$$
$$\left\|y_h^{*,n}\right\| \leq C\left(\left\|\boldsymbol{Y}_H^0\right\| + \left\|\boldsymbol{Y}_h^{*,0}\right\| + \|f(0)\|\right). \tag{4.8}$$

Proof. Setting $\boldsymbol{v}_h = 2\boldsymbol{Y}_h^{*,n}$ in (4.4), we get

$$\begin{aligned}
&2\left(\mathcal{D}_N^\alpha \boldsymbol{Y}_h^{*,n}, \boldsymbol{Y}_h^{*,n}\right) + 2\left(\mathrm{div}\boldsymbol{Y}_h^{*,n}, \mathrm{div}\boldsymbol{Y}_h^{*,n}\right)\\
=&-2\left(f(y_H^n) + f'(y_H^n)\left(y_h^{*,n} - y_H^n\right), \mathrm{div}\boldsymbol{Y}_h^{*,n}\right)\\
=&2\left(f(0) - f(y_H^n) - f(0), \mathrm{div}\boldsymbol{Y}_h^{*,n}\right) + 2\left(f'(y_H^n)\left(y_H^n - y_h^{*,n}\right), \mathrm{div}\boldsymbol{Y}_h^{*,n}\right)\\
\leq&2C(\varepsilon)\left(2M\|y_H^n\|^2 + \|f(0)\|^2 + M\|y_h^{*,n}\|^2\right) + 6\varepsilon\left\|\mathrm{div}\boldsymbol{Y}_h^{*,n}\right\|^2,
\end{aligned} \tag{4.9}$$

where we have used the Taylor's formula, ε-Cauchy inequality and the condition (1.2).

Choosing $w_h = y_h^{*,n}$ in (4.5) and using Hölder's inequality, we obtain

$$\left\|\nabla y_h^{*,n}\right\|^2 = \left(\boldsymbol{Y}_h^{*,n}, \nabla y_h^{*,n}\right) \leq C\|\boldsymbol{Y}_h^{*,n}\|\|\nabla y_h^{*,n}\|. \tag{4.10}$$

Noting $2\left(\mathcal{D}_N^\alpha \boldsymbol{Y}_h^{*,n}, \boldsymbol{Y}_h^{*,n}\right) \geq \mathcal{D}_N^\alpha \left\|\boldsymbol{Y}_h^{*,n}\right\|^2$ for $n = 1, 2, \cdots, N$. Applying the Poincaré inequality, (3.7), (4.9) and (4.10), we derive

$$\mathcal{D}_N^\alpha \left\|\boldsymbol{Y}_h^{*,n}\right\|^2 \leq 2C(\varepsilon)\left(2M\|\boldsymbol{Y}_H^n\|^2 + M\|\boldsymbol{Y}_h^{*,n}\|^2 + \|f(0)\|^2\right). \tag{4.11}$$

Then (4.7) follows from (4.11) and Lemma 3.1. According to (4.7) and (4.10), it is easy to get (4.8). □

4.2. Error estimates

Subtracting (4.4) and (4.5) from (2.1) and (2.2), for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, we can obtain equations

$$\begin{aligned}
&\left(\mathcal{D}_N^\alpha \left(\boldsymbol{Y}^n - \boldsymbol{Y}_h^{*,n}\right), \boldsymbol{v}_h\right) + \left(\mathrm{div}\left(\boldsymbol{Y}^n - \boldsymbol{Y}_h^{*,n}\right), \mathrm{div}\boldsymbol{v}_h\right)\\
=&\left(f(y_H^n) - f(y^n) + f'(y_H^n)\left(y_h^{*,n} - y_H^n\right), \mathrm{div}\boldsymbol{v}_h\right) - \left(R_{\boldsymbol{Y}}^n, \boldsymbol{v}_h\right),
\end{aligned} \tag{4.12}$$

$$\left(\boldsymbol{Y}^n - \boldsymbol{Y}_h^{*,n}, \nabla w_h\right) = \left(\nabla\left(y^n - y_h^{*,n}\right), \nabla w_h\right). \tag{4.13}$$

In order to conveniently derive error estimation results, we set

$$\Pi_h \boldsymbol{Y} - \boldsymbol{Y}_h^* := \boldsymbol{\rho},\ P_h y - y_h^* := \xi.$$

Theorem 4.2. *Let* $(y, \boldsymbol{Y})$ *and* $\left(y_h^*, \boldsymbol{Y}_h^*\right)$ *be the solutions of (2.1)–(2.3) and (4.1)–(4.6), respectively. Suppose that* $y \in L^\infty(J; H^2(\Omega)) \cap H^2(J; L^2(\Omega))$*,* $\boldsymbol{Y} \in (L^\infty(J; H^2(\Omega))^2 \cap (H^2(J; L^2(\Omega)^2$ *and all the conditions in Theorem 3.2 and Theorem 4.1 are valid. Then for* $n = 1, 2, \cdots, N$*, there hold*

$$\left\|\boldsymbol{Y}^n - \boldsymbol{Y}_h^{*,n}\right\| \leq C\left(h + H^2 + N^{-r}\right), \tag{4.14}$$

$$\left\|y^n - y_h^{*,n}\right\| \leq C\left(h^2 + H^2 + N^{-r}\right). \tag{4.15}$$

Proof. Subtracting (4.2) from (2.2) and utilizing the definition of P_H, we have

$$\left(\boldsymbol{Y}^n - \boldsymbol{Y}_H^n, \nabla w_H\right) = \left(\nabla\left(P_H y^n - y_H^n\right), \nabla w_H\right), \quad \forall w_H \in W_H. \tag{4.16}$$

Choosing $w_H = P_H y^n - y_H^n$ in (4.16), we derive

$$\begin{aligned} \left\|\nabla\left(P_H y^n - y_H^n\right)\right\|^2 &= \left(\boldsymbol{Y}^n - \boldsymbol{Y}_H^n, \nabla\left(P_H y^n - y_H^n\right)\right) \\ &\leq C\|\boldsymbol{Y}^n - \boldsymbol{Y}_H^n\|\|\nabla\left(P_H y^n - y_H^n\right)\|. \end{aligned} \tag{4.17}$$

From Poincaré inequality, (4.17) and Theorem 3.2, we obtain

$$\begin{aligned} \left\|P_H y^n - y_H^n\right\|_1 &\leq C\left\|\nabla\left(P_H y^n - y_H^n\right)\right\| \\ &\leq C\|\boldsymbol{Y}^n - \boldsymbol{Y}_H^n\| \\ &\leq C\left(H + N^{-r}\right). \end{aligned} \tag{4.18}$$

According to triangle inequality, (2.7) and (4.18), we arrive at

$$\|y^n - y_H^n\|_1 \leq \|y^n - P_H y^n\|_1 + \|P_H y^n - y_H^n\|_1 \leq C\left(H + N^{-r}\right). \tag{4.19}$$

Using (4.12) and (4.13) and the definitions of P_h and Π_h, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, we get

$$\begin{aligned} &\left(\mathcal{D}_N^\alpha \boldsymbol{\rho}^n, \boldsymbol{v}_h\right) + \left(\mathrm{div}\boldsymbol{\rho}^n, \mathrm{div}\boldsymbol{v}_h\right) \\ &= \left(f(y_H^n) - f(y^n) + f'\left(y_H^n\right)\left(y_h^{*,n} - y_H^n\right), \mathrm{div}\boldsymbol{v}_h\right) - \left(R_{\boldsymbol{Y}}^n, \boldsymbol{v}_h\right) - \left(\mathcal{D}_N^\alpha \boldsymbol{\theta}^n, \boldsymbol{v}_h\right), \end{aligned} \tag{4.20}$$

$$\left(\boldsymbol{\rho}^n, \nabla w_h\right) = \left(\nabla \xi^n, \nabla w_h\right) - \left(\boldsymbol{\theta}^n, \nabla w_h\right). \tag{4.21}$$

From the Taylor expansion formula, we have

$$f(y^n) = f\left(y_H^n\right) + f'\left(y_H^n\right)\left(y^n - y_H^n\right) + \frac{1}{2} f''(\delta)\left(y^n - y_H^n\right)^2, \tag{4.22}$$

where δ between y^n and y_H^n.

Substituting (4.22) into (4.20) and choosing $\boldsymbol{v}_h = \boldsymbol{\rho}^n$ yields

$$\begin{aligned}
&(\mathcal{D}_N^{\alpha}\boldsymbol{\rho}^n,\boldsymbol{\rho}^n) + (\mathrm{div}\boldsymbol{\rho}^n, \mathrm{div}\boldsymbol{\rho}^n) \\
&= \left(f'\left(y_H^n\right)\left(y_h^{*,n} - y^n\right) - \frac{1}{2}f''\left(\delta\right)\left(y^n - y_H^n\right)^2, \mathrm{div}\boldsymbol{\rho}^n\right) - (R_{\boldsymbol{Y}}^n,\boldsymbol{\rho}^n) - (\mathcal{D}_N^{\alpha}\boldsymbol{\theta}^n,\boldsymbol{\rho}^n).
\end{aligned} \tag{4.23}$$

We can estimate the right-hand side of (4.23) as follows:

$$\begin{aligned}
&\left(f'\left(y_H^n\right)\left(y_h^{*,n} - y^n\right) - \frac{1}{2}f''\left(\delta\right)\left(y^n - y_H^n\right)^2, \mathrm{div}\boldsymbol{\rho}^n\right) \\
&= \left(-f'\left(y_H^n\right)\left(y^n - P_h y^n\right) - f'\left(y_H^n\right)\left(P_h y^n - y_h^{*,n}\right) - \frac{1}{2}f''\left(\delta\right)\left(y^n - y_H^n\right)^2, \mathrm{div}\boldsymbol{\rho}^n\right) \\
&\le C(\varepsilon)M\left(\|\xi^n\|^2 + \|\zeta^n\|^2 + \frac{1}{2}\|y^n - y_H^n\|_{L^4(\Omega)}^4\right) + 3\varepsilon\|\mathrm{div}\boldsymbol{\rho}^n\|^2 \\
&\le C(\varepsilon)M\left(\|\xi^n\|^2 + \|\zeta^n\|^2 + \frac{1}{2}\|y^n - y_H^n\|_1^4\right) + 3\varepsilon\|\mathrm{div}\boldsymbol{\rho}^n\|^2 \\
&\le C(\varepsilon)M\|\xi^n\|^2 + C(\varepsilon)M\left(h^4 + (H + n^{-r})^4\right) + 3\varepsilon\|\mathrm{div}\boldsymbol{\rho}^n\|^2
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
(R_{\boldsymbol{Y}}^n,\boldsymbol{\rho}^n) &= (\mathcal{D}_t^{\alpha}\boldsymbol{Y}^n - \mathcal{D}_N^{\alpha}\boldsymbol{Y}^n,\boldsymbol{\rho}^n) \\
&\le C\left\|\mathcal{D}_t^{\alpha}\boldsymbol{Y}^n - \mathcal{D}_N^{\alpha}\boldsymbol{Y}^n\right\|\|\boldsymbol{\rho}^n\| \\
&\le Cn^{-r}\|\boldsymbol{\rho}^n\|.
\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}
(\mathcal{D}_N^{\alpha}\boldsymbol{\theta}^n,\boldsymbol{\rho}^n) &\le Ch^2\left(\mathcal{D}_N^{\alpha}\|\boldsymbol{\theta}^n\|_2\|\boldsymbol{\rho}^n\| + \|\boldsymbol{\theta}^n\|_1\|\mathrm{div}\boldsymbol{\rho}^n\|\right) \\
&\le Ch^2\left(\mathcal{D}_N^{\alpha}\|\boldsymbol{\theta}^n\|_2 + \|\boldsymbol{\theta}^n\|_1\right)\|\mathrm{div}\boldsymbol{\rho}^n\| \\
&\le C(\varepsilon)h^4\left(\mathcal{D}_N^{\alpha}\|\boldsymbol{\theta}^n\|_2 + \|\boldsymbol{\theta}^n\|_1\right)^2 + \varepsilon\|\mathrm{div}\boldsymbol{\rho}^n\|^2.
\end{aligned} \tag{4.26}$$

where we have used (2.7), (2.10), Lemma 2.1, Hölder inequality, embedding theorem, Theorem 3.2 and Poincaré inequality.

Taking $w_h = \xi^n$ in (4.21), we get

$$(\boldsymbol{\rho}^n, \nabla\xi^n) = (\nabla\xi^n, \nabla\xi^n) - (\boldsymbol{\theta}^n, \nabla\xi^n). \tag{4.27}$$

From (2.9), (4.27) and Cauchy-Schwartz inequality, we derive

$$\begin{aligned}
\|\nabla\xi^n\|^2 &= (\boldsymbol{\rho}^n, \nabla\xi^n) - (\mathrm{div}\boldsymbol{\theta}^n, \xi^n) \\
&\le C\left(\|\boldsymbol{\rho}^n\|\|\nabla\xi^n\| + \|\mathrm{div}\boldsymbol{\theta}^n\|_{-1}\|\xi^n\|_1\right) \\
&\le C\left(\|\boldsymbol{\rho}^n\| + Ch^2\|\mathrm{div}\boldsymbol{Y}^n\|_1\right)\|\nabla\xi^n\|.
\end{aligned} \tag{4.28}$$

By (4.28) and Poincaré inequality, we have

$$\|\xi^n\| \le C\|\nabla\xi^n\| \le C\left(\|\boldsymbol{\rho}^n\| + h^2\|\mathrm{div}\boldsymbol{Y}^n\|_1\right). \tag{4.29}$$

Combining (4.23)–(4.29) and using $(a+b)^4 \le 8(a^4+b^4)$ with $a, b > 0$, we yield

$$\mathcal{D}_N^{\alpha}\|\boldsymbol{\rho}^n\|^2 \le C(\varepsilon)M\|\boldsymbol{\rho}^n\|^2 + Cn^{-r}\|\boldsymbol{\rho}^n\| + CM\left(h^4 + H^4 + n^{-4r}\right). \tag{4.30}$$

It follows from Lemma 3.1 and (4.30) that

$$\|\boldsymbol{\rho}^n\| \le C\left(h^2 + H^2 + N^{-r}\right). \tag{4.31}$$

Using triangle inequality, (2.8) and (4.31), we arrive at

$$\left\|\boldsymbol{Y}^n - \boldsymbol{Y}_h^{*,n}\right\| \le \|\boldsymbol{Y}^n - \Pi_h\boldsymbol{Y}^n\| + \left\|\Pi_h\boldsymbol{Y}^n - \boldsymbol{Y}_h^{*,n}\right\| \le C\left(h + H^2 + N^{-r}\right). \tag{4.32}$$

From (4.29) and (4.31), we have

$$\|\xi^n\| \le C\left(h^2 + H^2 + N^{-r}\right). \tag{4.33}$$

It follows from (2.7), triangle inequality and (4.33) that

$$\left\|y^n - y_h^{*,n}\right\| \le \|y^n - P_hy^n\| + \left\|P_hy^n - y_h^{*,n}\right\| \le C\left(h^2 + H^2 + N^{-r}\right). \tag{4.34}$$

We complete the proof of Theorem 4.2. □

5. Numerical experiment

Several examples are provided to demonstrate our theoretical findings in this section. All numerical examples will be solved by the FDHAS described as (2.13)–(2.15) and TGA described as (4.1)–(4.6), where the program codes are based on AFEPack [46]. Let $J = (0, 1]$ and $\Omega = (0, 1)^2$. We numerically solve the following nonlinear time fractional equation:

$$\begin{cases} \mathcal{D}_t^{\alpha} y - \mathrm{div}\boldsymbol{Y} = f(y) + g, \quad \text{in } J \times \Omega, \\ \boldsymbol{Y} = \nabla y, \quad \text{in } J \times \Omega, \\ y = 0, \quad \text{on } J \times \partial\Omega, \\ y(0, x) = y_0(x), \quad \text{in } \Omega. \end{cases}$$

For simplicity, we define $|||\psi||| = \max\limits_{0\le n\le N}\{\|\psi^n\|\}$. The convergence rate are computed by $Rate = \frac{\ln(|||\psi_{k+1}|||)-\ln(|||\psi_k|||)}{\ln(s_{k+1})-\ln(s_k)}$, where $|||\psi_{k+1}|||$ ($|||\psi_k|||$) is the error with the spatial or temporal mesh step s_{k+1} (s_k).

Example 1. The initial condition and the right function $g(t, x)$ are suitably chosen such that $y(t, x) = t^2\sin(\pi x_1)\sin(\pi x_2)$ and the nonlinear term $f(y) = y(1 - y)$.

When the spatial step $h = \frac{1}{100}$ and $H = \sqrt{h} = \frac{1}{10}$ are fixed, in Table 1, we present the errors $|||\boldsymbol{Y}_h - \boldsymbol{Y}|||$ and $|||\boldsymbol{Y}_h^* - \boldsymbol{Y}|||$, the temporal convergence orders and the CPU time for $\alpha = 0.4, 0.5$ and 0.8 by using the general FDHAS (2.13)–(2.15) and TGA (4.1)–(4.6), respectively.

Table 1. Errors, convergence rates and CPU time of TGA and FDHAS with $h = \frac{1}{100}$.

(α, γ)	N	$\vert\vert\vert \boldsymbol{Y}_h^* - \boldsymbol{Y} \vert\vert\vert$	Rate	CPU(s)	$\vert\vert\vert \boldsymbol{Y}_h - \boldsymbol{Y} \vert\vert\vert$	Rate	CPU(s)
(0.4, 4)	10	3.60838×10^{-1}	–	65.824	2.85026×10^{-1}	–	73.826
	20	1.99016×10^{-1}	1.5895	82.139	9.42388×10^{-2}	1.5967	108.85
	40	6.57236×10^{-2}	1.5984	265.74	3.11347×10^{-2}	1.5978	367.74
	80	2.16987×10^{-2}	1.5988	847.65	1.02785×10^{-2}	1.5989	1229.6
(0.5, 3)	10	3.28475×10^{-1}	–	51.654	2.51374×10^{-1}	–	71.685
	20	1.17152×10^{-1}	1.4874	71.294	8.90098×10^{-2}	1.4978	113.89
	40	4.15431×10^{-2}	1.4957	257.65	3.15046×10^{-2}	1.4984	370.96
	80	1.46897×10^{-2}	1.4998	737.5	1.11471×10^{-2}	1.4989	1285.7
(0.8, 1.5)	10	3.48107×10^{-1}	–	68.32	3.25583×10^{-1}	–	76.508
	20	1.52006×10^{-1}	1.1954	84.74	1.42092×10^{-2}	1.1962	120.54
	40	6.62287×10^{-2}	1.1986	286.53	6.19435×10^{-3}	1.1978	388.45
	80	2.88337×10^{-2}	1.1997	852.15	2.69830×10^{-3}	1.1989	1324.6

Table 2. Errors and convergence rates of TGA and FDHAS with $N = 1000$.

(α, γ)	h	$\vert\vert\vert \boldsymbol{Y}_h^* - \boldsymbol{Y} \vert\vert\vert$	Rate	$\vert\vert\vert y_h - y \vert\vert\vert$	Rate	$\vert\vert\vert \boldsymbol{Y}_h - \boldsymbol{Y} \vert\vert\vert$	Rate
(0.4, 4)	1/16	4.28805×10^{-1}	–	2.87246×10^{-2}	–	4.21063×10^{-1}	–
	1/36	1.95527×10^{-1}	0.9684	7.29352×10^{-3}	1.9776	1.90430×10^{-1}	0.9785
	1/64	1.11558×10^{-1}	0.9753	1.84538×10^{-3}	1.9827	1.08269×10^{-1}	0.9814
	1/100	7.18895×10^{-2}	0.9846	7.58503×10^{-4}	1.9922	6.95959×10^{-2}	0.9902
(0.5, 3)	1/16	4.19853×10^{-1}	–	2.60964×10^{-2}	–	4.18365×10^{-1}	–
	1/36	1.90392×10^{-1}	0.9752	6.58544×10^{-3}	1.9865	1.87850×10^{-1}	0.9874
	1/64	1.08429×10^{-1}	0.9785	1.65908×10^{-3}	1.9889	1.06538×10^{-1}	0.9857
	1/100	6.98731×10^{-2}	0.9848	6.81898×10^{-4}	1.9923	6.84832×10^{-2}	0.9901
(0.8, 1.5)	1/16	4.30674×10^{-1}	–	3.31085×10^{-2}	–	4.22057×10^{-1}	–
	1/36	1.97049×10^{-1}	0.9642	8.45808×10^{-3}	1.9688	1.92919×10^{-1}	0.9654
	1/64	1.12239×10^{-1}	0.9782	2.13322×10^{-3}	1.9873	1.09357×10^{-1}	0.9785
	1/100	7.23284×10^{-2}	0.9846	8.76696×10^{-4}	1.9925	7.02953×10^{-3}	0.9902

The numerical results in Table 1 show that the TGA (4.1)–(4.6) can save significant computational costs compared with FDHAS (2.13)–(2.15) without losing accuracy. Fixed $N = 1000$, $\gamma = \frac{2-\alpha}{\alpha}$ and $h = H^2$, the results in Table 2 reflect $|||\boldsymbol{Y}_h - \boldsymbol{Y}||| = O(h)$, $|||y_h - y||| = O\left(h^2\right)$ and $|||\boldsymbol{Y}_h^* - \boldsymbol{Y}||| = O(h)$. The convergence rate results in time and space direction are consistent with our theoretical results.

Example 2. The initial condition and the right function $g(t, x)$ are suitably chosen such that $y(t, x) = \left(E_\alpha(-t^\alpha) + t^3\right) x_1(1 - x_1)x_2(1 - x_2)$ and the nonlinear term $f(y) = y^3$.

In Table 3, we also give the numerical results with the fixed spatial step $h = \frac{1}{100}$ and $H = \sqrt{h} = \frac{1}{10}$ for $\alpha = 0.4, 0.5$ and 0.8 by utilizing the general FDHAS (2.13)–(2.15) and TGA (4.1)–(4.6), respectively. For fixed $N = 1000$, $\gamma = \frac{2-\alpha}{\alpha}$ and $h = H^2$, we show the numerical results in Table 4. The numerical results demonstrate that the TGA is more efficient than the FDHAS. It is agreement with our theoretical analysis.

Table 3. Errors, convergence rates and CPU time of TGA and FDHAS with $h = \frac{1}{100}$.

(α, γ)	N	$\|\|\|\boldsymbol{Y}_h^* - \boldsymbol{Y}\|\|\|$	Rate	CPU(s)	$\|\|\|\boldsymbol{Y}_h - \boldsymbol{Y}\|\|\|$	Rate	CPU(s)
(0.4, 4)	10	3.85284×10^{-1}	–	68.245	3.24606×10^{-1}	–	83.537
	20	1.29005×10^{-1}	1.5785	88.326	1.08064×10^{-1}	1.5868	111.56
	40	4.27094×10^{-2}	1.5948	295.87	3.57072×10^{-2}	1.5976	372.84
	80	1.40967×10^{-2}	1.5992	864.75	1.17880×10^{-3}	1.5989	1228.7
(0.5, 3)	10	4.28354×10^{-1}	–	59.582	3.86095×10^{-1}	–	81.605
	20	1.54007×10^{-1}	1.4758	81.936	1.38047×10^{-1}	1.4838	108.75
	40	5.50875×10^{-2}	1.4832	265.83	4.88950×10^{-2}	1.4974	350.68
	80	1.96091×10^{-2}	1.4902	753.62	1.73002×10^{-2}	1.4989	1184.5
(0.8, 1.5)	10	4.52136×10^{-1}	–	70.352	4.10875×10^{-1}	–	86.452
	20	1.97829×10^{-1}	1.1925	92.875	1.79278×10^{-2}	1.1965	116.48
	40	8.61937×10^{-2}	1.1986	286.53	7.81544×10^{-3}	1.1978	384.56
	80	3.75310×10^{-2}	1.1995	876.14	3.40234×10^{-3}	1.1998	1245.6

Table 4. Errors and convergence rates of TGA and FDHAS with $N = 1000$.

(α, γ)	h	$\|\|\|\boldsymbol{Y}_h^* - \boldsymbol{Y}\|\|\|$	Rate	$\|\|\|y_h - y\|\|\|$	Rate	$\|\|\|\boldsymbol{Y}_h - \boldsymbol{Y}\|\|\|$	Rate
(0.4, 4)	1/16	6.20865×10^{-1}	–	3.25801×10^{-2}	–	6.08254×10^{-1}	–
	1/36	2.83309×10^{-1}	0.9675	8.32367×10^{-3}	1.9687	2.75559×10^{-1}	0.9764
	1/64	1.61707×10^{-1}	0.9746	2.11465×10^{-3}	1.9768	1.56571×10^{-1}	0.9825
	1/100	1.04150×10^{-1}	0.9858	8.71667×10^{-4}	1.9858	1.00465×10^{-1}	0.9942
(0.5, 3)	1/16	6.10598×10^{-1}	–	3.10528×10^{-2}	–	5.81075×10^{-1}	–
	1/36	2.76589×10^{-1}	0.9765	7.90547×10^{-3}	1.9738	2.61353×10^{-1}	0.9853
	1/64	1.57505×10^{-1}	0.9787	2.00479×10^{-3}	1.9794	1.48047×10^{-1}	0.9878
	1/100	1.01444×10^{-1}	0.9857	8.26346×10^{-4}	1.9859	9.49957×10^{-2}	0.9943
(0.8, 1.5)	1/16	6.43177×10^{-1}	–	3.38926×10^{-2}	–	6.21307×10^{-1}	–
	1/36	2.94014×10^{-1}	0.9653	8.69628×10^{-3}	1.9625	2.83258×10^{-1}	0.9686
	1/64	1.67441×10^{-1}	0.9785	2.20671×10^{-3}	1.9785	1.60270×10^{-1}	0.9898
	1/100	1.07844×10^{-1}	0.9859	9.09493×10^{-4}	1.9861	1.02839×10^{-1}	0.9942

6. Conclusions

In this paper, we proposed the TGA for the nonlinear TFPDEs (1.1) discretized by H^1-Galerkin mixed finite element on spatial rectangular mesh combined with $L1$ scheme on temporal graded mesh. The stability and optimal convergence of the TGA are rigorously proved. Our theoretical results seem to be new in the literature. Numerical experimental results show that the TGA (4.1)–(4.6) can save a lot of computing cost compared with FDHAS (2.13)–(2.15) without losing accuracy. Although our TGA in this paper focuses on a two-dimensional case, it can be directly applied to three-dimensional problems. Future work includes the developments of two-grid finite element methods combined with some higher-order schemes or fast algorithm for nonlinear TFPDEs.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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