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Research note

The Frobenius problem for special progressions

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Abstract: Let S be a given finite set of positive and relatively prime integers. Denote $L(S)$ to be the set of integers obtained by taking all nonnegative integer linear combinations of integers in S. It is well known that there are finitely many positive integers that are not in $L(S)$. Let $g(S)$ and $n(S)$ represent the greatest integer that does not belong to $L(S)$ and the number of nonnegative integers that do not belong to $L(S)$, respectively. The Frobenius problem is to determine $g(S)$ and $n(S)$. In 2016, Tripathi obtained results on $g(S)$ and $n(S)$ when $S = \{a, ha + d, ha + db, ha + db^2, \ldots, ha + db^k\}$. In this paper, for $S_c := \{a, ha+d, ha+c+db, ha+2c+db^2, \ldots, ha+kc+db^k\}$ with h, c being nonnegative integers, a, b, d being positive integers and $\gcd(a, d) = 1$, we focused the investigation on formulas for $g(S_c)$ and $n(S_c)$. Actually, we gave formulas for $g(S_c)$ and $n(S_c)$ for all sufficiently large values of d when c is any multiple of d or certain multiples of a. This generalized the results of Tripathi in 2016.

Keywords: Frobenius problem; Frobenius number; special progressions; linear form; euclidean division

1. Introduction

Let n be a positive integer greater than 1. For a given set $S = \{a_1, a_2, \ldots, a_n\}$ with each a_i being a positive integer and $\gcd(S) := \gcd(a_1, a_2, \ldots, a_n) = 1$, define $L(S)$ to be the set of integers represented as nonnegative integer linear forms of integers in S; that is,

$$L(S) = \{x_1a_1 + x_2a_2 + \cdots + x_na_n | x_1, \ldots, x_n \in \mathbb{N}\},$$

where $\mathbb{N}$ represents the natural number set containing zero. It is well known that (see, for example, Theorem 1.16 in [1]) for any positive integer z if

$$z \geq (a_n - 1) \sum_{i=1}^{n-1} a_i,$$

then z belongs to $L(S)$. It follows that there are finitely many nonnegative integers that are not in $L(S)$. So, if $L^c(S) := \mathbb{N} \setminus L(S)$, then $L^c(S)$ is a finite set. Define

$$g(S) := \max L^c(S), \text{ and } n(S) := |L^c(S)|,$$

where $|\cdot|$ denotes the size of a set. Perhaps Sylvester was the first person who asked to determine $g(S)$ and $n(S)$. He also proved that

$$g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1 \text{ and } n(a_1, a_2) = \frac{1}{2}(a_1 - 1)(a_2 - 1).$$

During the early part of the twentieth century, Frobenius raised the same problem in his lectures (according to [2]). Frobenius was largely instrumental in giving this problem the early recognition, and it was after him that the problem was named. This problem of determining $g(S)$ is called the *Frobenius problem* and $g(S)$ is the *Frobenius number*. The Frobenius problem has a rich and long history, with several applications, extensions and connections to several areas of research. A comprehensive survey covering all aspects of the problem was given by Ramíres-Alfonsín [3].

Exact determination of the $g(S)$ and $n(S)$ is a difficult problem in general. There are only a few cases where the $g(S)$ and $n(S)$ have been exactly determined for any n variables. Table 1 shows some of the results in the case of S being special progressions, which were obtained by researchers over the past few decades.

Table 1. Progress on the Frobenius problem for special progressions

S	Condition	$g(S)$	$n(S)$	Year	Reference
$a, a+1, a+2, \ldots, a+k$		✓		1942	Brauer [2]
$a, a+d, a+2d, \ldots, a+kd$		✓		1956	Roberts [4]
$a, a+d, a+bd, \ldots, a+b^kd$	For sufficiently large d	✓		1966	Hofmeister [5]
$a, a+d, a+2d, \ldots, a+kd$			✓	1973	Grant [6]
$a, ha+d, ha+2d, \ldots, ha+kd$		✓		1977	Selmer [7]
$a, a+1, a+2, a+2^2, \ldots, a+2^k$	$a > (k-3)2^k + 1$	✓	✓	1977	Selmer [7]
$a^k, a^{k-1}b, a^{k-2}b^2, \ldots, b^k$		✓	✓	2008	Tripathi [8]
$a, ha+d, ha+2d, \ldots, ha+kd$			✓	2013	Tripathi [9]
$a, ha+d, ha+bd, \ldots, ha+b^kd$	$d \geq h(k(b-1) - \lfloor \frac{a}{b^k} \rfloor)$	✓	✓	2016	Tripathi [10]

Here, ✓ means that the responding result was obtained. The results in the table show that scholars were gradually attacking the Frobenius problem for more general sets. For some very recent development aspects of this field, one can refer to [11–13].

For any nonnegative integer c, let

$$S_c := \{a, ha+d, ha+c+bd, ha+2c+b^2d, \ldots, ha+kc+b^kd\},$$

where k, a, b, d and h are positive integers with $\gcd(a, d) = 1$. It is natural to consider the Frobenius problem for S_c. In the paper, we present formulas for $g(S_c)$ and $n(S_c)$ for all sufficiently large values of d in the condition that c is a multiple of a or a multiple of d.

In order to state our theorem well, let us first define some notations. Let b, k and u be nonnegative integers with $b \geq 2$ and $k \geq 1$. For any positive integer x, define two sequences of nonnegative integers, denoted by $\{q_i(x)\}_{i=0}^{k}$ and $\{r_i(x)\}_{i=0}^{k}$, as follows:

$$\begin{aligned}
x &= q_k(x)(b^k + uk) + r_k(x),\\
r_k(x) &= q_{k-1}(x)(b^{k-1} + u(k-1)) + r_{k-1}(x),\\
r_{k-1}(x) &= q_{k-2}(x)(b^{k-2} + u(k-2)) + r_{k-2}(x),\\
&\vdots\\
r_2(x) &= q_1(x)(b^1 + u \cdot 1) + r_1(x),\\
r_1(x) &= q_0(x)(b^0 + u \cdot 0) + r_0(x),
\end{aligned}$$

where $0 \leq r_i(x) \leq b^i + ui - 1$ for each $1 \leq i \leq k$ and $r_0(x) = 0$. From Euclidean division, we know that $\{q_i(x)\}_{i=0}^{k}$ and $\{r_i(x)\}_{i=0}^{k}$ are uniquely determined by x, so they are well defined. It is immediate that

$$q_k(x) = \left\lfloor \frac{x}{b^k + uk} \right\rfloor,\ q_0(x) = r_1(x) \leq b + u - 1,$$

and

$$q_i(x) = \frac{r_{i+1}(x) - r_i(x)}{b^i + ui} \leq \frac{r_{i+1}(x)}{b^i + ui} < \frac{b^{i+1} + u(i+1)}{b^i + ui} = \frac{b(b^i + ui) + u(i + 1 - bi)}{b^i + ui} \leq b,$$

i.e., $0 \leq q_i(x) \leq b$ for any $1 \leq i \leq k-1$. In particular, when $u = 0$, we use $\{\overline{q}_i(x)\}_{i=0}^{k}$ and $\{\overline{r}_i(x)\}_{i=0}^{k}$ to replace $\{q_i(x)\}_{i=0}^{k}$ and $\{r_i(x)\}_{i=0}^{k}$, respectively. Thus,

$$\overline{q}_k(x) = \left\lfloor \frac{x}{b^k} \right\rfloor,\ \overline{q}_j(x) = \left\lfloor \frac{x - \sum_{i=j+1}^{k} \overline{q}_i(x) b^i}{b^j} \right\rfloor, 0 \leq j \leq k-1, \tag{1.1}$$

and

$$\overline{r}_m(x) = x - \sum_{i=m}^{k} \overline{q}_i(x) b^i = \sum_{i=0}^{m-1} \overline{q}_i(x) b^i, 0 \leq m \leq k \tag{1.2}$$

is the b-adic representation of $\overline{r}_m(x)$. Let

$$\mathtt{W}_{b,u}(x) := \sum_{i=0}^{k-1} q_i(x) \text{ and } \mathtt{V}_{b,v}(x) := \sum_{i=0}^{k-1} (1 + iv) \overline{q}_i(x), \tag{1.3}$$

where v is a nonnegative integer.

Now, we can report the main theorem as follows.

Theorem 1.1. *Let k, a, b, d and h be positive integers with* $\gcd(a, d) = 1$ *and $b \geq 2$. For any nonnegative integer c, let $S_c := \{a, ha + d, ha + c + bd, ha + 2c + b^2 d, \ldots, ha + kc + b^k d\}$. The following statements are true.*

(a) If $c = ud$ for a nonnegative integer u and $d \geq ah(k(b-1)+u)$, then

$$g(S_c) = ah\left(\left\lfloor \frac{a-1}{b^k + uk} \right\rfloor + \mathtt{W}_{b,u}(a-1)\right) + d(a-1)$$

and

$$n(S_c) = h\sum_{x=1}^{a-1} \mathtt{W}_{b,u}(x) + \frac{1}{2}hq(q-1)(b^k + ku) + hq(r+1) + \frac{1}{2}(a-1)(d-1).$$

(b) If $c = va$ for a nonnegative integer v with $v \leq b-1$ and $d \geq \frac{1}{2}akh(b-1)(2+(k-1)v)$, then

$$g(S_c) = ah\left((1+kv)\left\lfloor \frac{a-1}{b^k} \right\rfloor + \mathtt{V}_{b,v}(a-1)\right) + d(a-1)$$

and

$$n(S_c) = \frac{1}{2}(a-1)(ah+d-1) + h\sum_{i=1}^{k}(v(b+i-ib)+1-b)M_i,$$

where $M_i = \frac{1}{2}b^i\left\lfloor \frac{a-1}{b^i}\right\rfloor\left(\left\lfloor \frac{a-1}{b^i}\right\rfloor - 1\right) + \left\lfloor \frac{a-1}{b^i}\right\rfloor\left(a - b^i\left\lfloor \frac{a-1}{b^i}\right\rfloor\right)$.

Here, $\mathtt{W}_{b,u}$ and $\mathtt{V}_{b,v}$ are defined as (1.3).

Clearly, Theorem 1.1 extends a result of Tripathi [10]. The paper is organized as follows. First, in Section 2, two key lemmas are given. Second, in Section 3, we present the proof of Theorem 1.1. Finally, in Section 4, another possible direction of the generalizations of S for the Frobenius problem is introduced, which can be served as a future research for the interested.

2. Lemmas

Let S be any given finite set of positive integers with $\gcd(S) = 1$. Let a be a positive integer of S. For any integer x with $1 \leq x \leq a-1$, denote $\langle x \rangle$ to be the residue class of integers congruent to x modulo a. Let $\boldsymbol{m}(S, x)$ represent the least positive integer in $L(S) \cap \langle x \rangle$. Brauer and Shockley [14] obtained the following results, which give $g(S)$ and $n(S)$ from $\boldsymbol{m}(S, x)$.

Lemma 2.1. *Let S be any finite set of positive integers with* $\gcd(S) = 1$. *For any $a \in S$,*

$$g(S) = \max_{1 \leq x \leq a-1} \boldsymbol{m}(S, x) - a \text{ and } n(S) = \frac{1}{a}\sum_{x=1}^{a-1} \boldsymbol{m}(S, x) - \frac{1}{2}(a-1).$$

The second lemma presents results of $\min(L(S) \cap \langle x \rangle)$.

Lemma 2.2. *Let c be a fixed nonnegative integer, k, a, b, d and h be positive integers with* $\gcd(a, d) = 1$ *and $b \geq 2$. Let $S_c = \{a, ha+d, ha+c+bd, ha+2c+b^2d, \ldots, ha+kc+b^kd\}$. For any integer x with $1 \leq x \leq a-1$, let $\boldsymbol{m}(x) := \min(L(S_c) \cap \langle dx \rangle)$, where $\langle dx \rangle$ denotes the residue class of integers congruent to dx modulo a. The following results are derived.*

(a) *if $c = ud$ for a nonnegative integer u, then for any positive integer x with $x \le a-1$ we have*

$$\boldsymbol{m}(x) = \min_{t\in\mathbb{N}}\left(ah\left(\left\lfloor \frac{at+x}{b^k+uk}\right\rfloor + \mathtt{W}_{b,u}(at+x)\right) + d(at+x)\right).$$

In particular, if $d \ge h\left(k(b-1)+u-\left\lfloor \frac{a}{b^k+uk}\right\rfloor\right)$ with $\gcd(a,d)=1$, then

$$\boldsymbol{m}(x) = ah\left(\left\lfloor \frac{x}{b^k+uk}\right\rfloor + \mathtt{W}_{b,u}(x)\right) + dx.$$

(b) *if $c = va$ for a nonnegative integer v with $v \le b-1$, then for any positive integer x with $x \le a-1$ we have*

$$\boldsymbol{m}(x) = \min_{s\in\mathbb{N}}\left(ah(1+kv)\left\lfloor \frac{as+x}{b^k}\right\rfloor + ah\mathtt{V}_{b,v}(as+x) + d(as+x)\right).$$

Moreover, if $d \ge \frac{1}{2}kh(b-1)(2+(k-1)v) - h(1+kv)\left\lfloor \frac{a}{b^k}\right\rfloor$, then

$$\boldsymbol{m}(x) = ah(1+kv)\left\lfloor \frac{x}{b^k}\right\rfloor + ah\mathtt{V}_{b,v}(x) + dx.$$

Here, $\mathtt{W}_{b,u}$ and $\mathtt{V}_{b,v}$ are defined as (1.3).

Proof. Let x be a positive integer with $x \le a-1$. For any $N \in L(S_c)\cap\langle dx\rangle$, one writes

$$\begin{aligned} N &= x_{-1}a + x_0(ha+d) + x_1(ha+c+bd) + \cdots + x_k(ha+kc+b^kd) \\ &= a\left(x_{-1} + h\sum_{i=0}^{k} x_i\right) + d\sum_{i=0}^{k} b^i x_i + c\sum_{i=0}^{k} ix_i, \end{aligned}$$

with each x_i being a nonnegative integer.

First, let $c = ud$ for a nonnegative integer u. Rewrite

$$N = a\left(x_{-1} + h\sum_{i=0}^{k} x_i\right) + d\sum_{i=0}^{k}(b^i+ui)x_i. \tag{2.1}$$

Since $N \in \langle dx\rangle$, then (2.1) implies that

$$\sum_{i=0}^{k}(b^i+ui)x_i \equiv x \pmod{a}.$$

Let $\sum_{i=0}^{k}(b^i+ui)x_i = at+x$ for some nonnegative integer t. On the one hand, we can regard as N a function of the variables $t, x_{-1}, x_0, x_1, \ldots, x_k$ and write $N = N(t; x_{-1}, x_0, x_1, \ldots, x_k)$. It then follows that

$$\min\left(L(S_c)\cap\langle dx\rangle\right) = \min_{t\in\mathbb{N}}\left(\min_{x_{-1},\ldots,x_k\in\mathbb{N}} N(t; x_{-1}, x_0, x_1, \ldots, x_k)\right).$$

On the other hand, for any fixed t, one readily finds that

$$\min_{x_{-1},\ldots,x_k\in\mathbb{N}} N(t; x_{-1}, x_0, x_1, \ldots, x_k)$$

$$
\begin{aligned}
&= N(t; 0, q_0(at+x), q_1(at+x), \ldots, q_k(at+x)) \\
&= ah \sum_{i=0}^{k} q_i(at+x) + d(at+x) \\
&= ah\left(\left\lfloor \frac{at+x}{b^k+uk} \right\rfloor + \mathtt{W}_{b,u}(at+x)\right) + d(at+x),
\end{aligned}
$$

where $\mathtt{W}_{b,u}$ is defined in (1.3). Therefore,

$$
\boldsymbol{m}(x) = \min_{t\in\mathbb{N}} \left(ah\left(\left\lfloor \frac{at+x}{b^k+uk} \right\rfloor + \mathtt{W}_{b,u}(at+x)\right) + d(at+x) \right).
$$

Let $F(t) := ah\left(\left\lfloor \frac{at+x}{b^k+uk} \right\rfloor + \mathtt{W}_{b,u}(at+x)\right) + d(at+x)$, and consider $F(t+1) - F(t)$, which equals

$$
ah\left(\left\lfloor \frac{at+a+x}{b^k+uk} \right\rfloor - \left\lfloor \frac{at+x}{b^k+uk} \right\rfloor + \mathtt{W}_{b,u}(at+a+x) - \mathtt{W}_{b,u}(at+x)\right) + da. \tag{2.2}
$$

Note that

$$
\left\lfloor \frac{at+a+x}{b^k+uk} \right\rfloor - \left\lfloor \frac{at+x}{b^k+uk} \right\rfloor = \left\lfloor \frac{a}{b^k+uk} \right\rfloor \text{ or } \left\lceil \frac{a}{b^k+uk} \right\rceil
$$

and

$$
-k(b-1) - u \le \mathtt{W}_{b,u}(at+a+x) - \mathtt{W}_{b,u}(at+x) \le k(b-1) + u.
$$

By (2.2), one then has that $F(t+1) \ge F(t)$ for any nonnegative integer t if $d \ge h\left(k(b-1) + u - \left\lfloor \frac{a}{b^k+uk} \right\rfloor\right)$. Thus

$$
\boldsymbol{m}(x) = \min_{t\in\mathbb{N}} F(t) = F(0) = ah\left(\left\lfloor \frac{x}{b^k+uk} \right\rfloor + \mathtt{W}_{b,u}(x)\right) + dx,
$$

whenever $d \ge h\left(k(b-1) + u - \left\lfloor \frac{a}{b^k+uk} \right\rfloor\right)$, as desired. Item (a) is proved.

Next, let $c = va$ for an integer v with $0 \le v \le b-1$, then

$$
N = a\left(x_{-1} + h\sum_{i=0}^{k}(1+vi)x_i\right) + d\sum_{i=0}^{k} b^i x_i. \tag{2.3}
$$

Note that $N \equiv dx \pmod a$, then by (2.3) one may let $\sum_{i=0}^{k} b^i x_i = as + x$ for some nonnegative integer s. For any fixed nonnegative integer s, let

$$
f(x_0, \ldots, x_k) := \sum_{i=0}^{k}(1+vi)x_i \tag{2.4}
$$

be a function of the variables $x_0, \ldots, x_k$ subject to $\sum_{i=0}^{k} b^i x_i = as + x$, then one claims that

$$
\min_{\substack{x_0,\ldots,x_k\in\mathbb{N} \\ \sum_{i=0}^{k} b^i x_i = as+x}} f(x_0, \ldots, x_k) = f(\overline{q}_0(as+x), \overline{q}_1(as+x), \ldots, \overline{q}_k(as+x)), \tag{2.5}
$$

where each $\overline{q}_i$ is defined as (1.1). Now, let us prove the claim by mathematical induction on k as follows.

• Let $k = 1$. Let $as + x = x_0 + bx_1 = g_1$. Write $g_1 = \overline{q}_1(g_1)b + \overline{r}_1(g_1)$ with $0 \le \overline{r}_1(g_1) \le b-1$ and let $\overline{q}_0(g_1) = \overline{r}_1(g_1)$. Note that $x_1 \le \lfloor \frac{g_1}{b} \rfloor = \overline{q}_1(g_1)$, then we have

$$
f(x_0, x_1) = x_0 + (v+1)x_1
$$

$$
\begin{aligned}
&= g_1 + (v-(b-1))x_1 \\
&\geq g_1 + (v-(b-1))\overline{q}_1(g_1) \\
&= \overline{q}_0(g_1) + (v+1)\overline{q}_1(g_1) = f(\overline{q}_0(g_1), \overline{q}_1(g_1)),
\end{aligned}
$$

where the inequality in the third line holds because $v \leq b-1$. It follows that

$$
\min_{\substack{x_0, x_1 \in \mathbb{N} \\ x_0 + bx_1 = as+x}} f(x_0, x_1) = f(\overline{q}_0(as+x), \overline{q}_1(as+x)).
$$

This is to say that the claim of (2.5) is true for $k = 1$.

• Assume that the claim of (2.5) holds for $k-1$ with $k \geq 2$. Let $g_k := \sum_{i=0}^{k} b^i x_i = as + x$ and $\widetilde{f}(x_0, x_1, \ldots, x_{k-1}) := \sum_{i=0}^{k-1}(1+vi)x_i$. By (2.4), one then deduces that

$$
\begin{aligned}
f(x_0, \ldots, x_k) &= (1+vk)x_k + \widetilde{f}(x_0, x_1, \ldots, x_{k-1}) \\
&\geq (1+vk)x_k + \min_{\substack{x_0, \ldots, x_k \in \mathbb{N} \\ \sum_{i=0}^{k-1} b^i x_i = g_k - b^k x_k}} \widetilde{f}(x_0, x_1, \ldots, x_{k-1}).
\end{aligned} \tag{2.6}
$$

By the inductive hypothesis we have

$$
\min_{\substack{x_0, \ldots, x_k \in \mathbb{N} \\ \sum_{i=0}^{k-1} b^i x_i = g_k - b^k x_k}} \widetilde{f}(x_0, x_1, \ldots, x_{k-1}) = \widetilde{f}(\overline{q'}_0(g_k - b^k x_k), \ldots, \overline{q'}_{k-1}(g_k - b^k x_k)),
$$

where

$$
\overline{q'}_{k-1}(g_k - b^k x_k) = \left\lfloor \frac{g_k - b^k x_k}{b^{k-1}} \right\rfloor
$$

and

$$
\overline{q'}_j(g_k - b^k x_k) = \left\lfloor \frac{g_k - b^k x_k - \sum_{i=j+1}^{k-1} b^i \overline{q'}_j(g_k - b^k x_k)}{b^j} \right\rfloor
$$

for $j = k-2, k-3, \ldots, 0$. For $g_k = as + x$, let $\{\overline{q}_i(g_k)\}_{i=0}^{k}$ be the sequence defined as (1.1). It is observed that $g_k - x_k b^k \pmod{b^{k-1}}$ is independent on x_k. One then finds that

$$
\overline{q}_j(g_k) = \overline{q'}_j(g_k - x^k b^k)
$$

for any $0 \leq j \leq k-2$. This together with (2.6) implies that

$$
\begin{aligned}
f(x_0, \ldots, x_k) &\geq (1+vk)x_k + (1+v(k-1)) \left\lfloor \frac{g_k - b^k x_k}{b^{k-1}} \right\rfloor + \sum_{i=0}^{k-2}(1+iv)\overline{q}_i(g_k) \\
&= (1+vk)x_k + (1+v(k-1))(b\overline{q}_k(g_k) + \overline{q}_{k-1}(g_k) - bx_k) + \sum_{i=0}^{k-2}(1+iv)\overline{q}_i(g_k) \\
&= x_k(1+kv-b-bv(k-1)) + \overline{q}_k(g_k)b + \overline{q}_k(g_k)bv(k-1) + \sum_{i=0}^{k-1}(1+iv)\overline{q}_i(g_k) \\
&\geq \overline{q}_k(g_k)(1+kv-b-bv(k-1)) + \overline{q}_k(g_k)b + \overline{q}_k(g_k)bv(k-1) + \sum_{i=0}^{k-1}(1+iv)\overline{q}_i(g_k);
\end{aligned}
$$

that is,

$$f(x_0,\dots,x_k) \geq \sum_{i=0}^{k}(1+iv)\overline{q}_i(g_k) = f(\overline{q}_0(g_k),\overline{q}_1(g_k),\dots,\overline{q}_k(g_k)).$$

Hence, we arrive at

$$\min_{\substack{x_0,\dots,x_k\in\mathbb{N}\\ \sum_{i=0}^{k} b^i x_i = g_k}} f(x_0,\dots,x_k) = f(\overline{q}_0(g_k),\overline{q}_1(g_k),\dots,\overline{q}_k(g_k)),$$

which means that (2.5) is true for k, so the claim of (2.5) is proved. It then follows from (2.3)–(2.5) that

$$\boldsymbol{m}(x) = \min_{s\in\mathbb{N}}\left(ah\sum_{i=0}^{k}(1+vi)\overline{q}_i(as+x) + d(as+x)\right).$$

Let $G(s) := ah\sum_{i=0}^{k}(1+vi)\overline{q}_i(as+x)+d(as+x)$. It is direct to check that $G(s+1)-G(s) = ah(1+kv)\left(\left\lfloor\frac{as+x+a}{b^k}\right\rfloor - \left\lfloor\frac{as+x}{b^k}\right\rfloor\right) + ah\sum_{i=0}^{k-1}(1+iv)(\overline{q}_i(as+x+a)-\overline{q}_i(as+x))$. Note that

$$\left\lfloor\frac{as+x+a}{b^k}\right\rfloor - \left\lfloor\frac{as+x}{b^k}\right\rfloor \geq \left\lfloor\frac{a}{b^k}\right\rfloor$$

and

$$1-b \leq \overline{q}_i(as+x+a)-\overline{q}_i(as+x) \leq b-1$$

for any $0 \leq i \leq k-1$. It infers that

$$G(s+1)-G(s) \geq ah(1+kv)\left\lfloor\frac{a}{b^k}\right\rfloor - \frac{1}{2}kah(b-1)(2+(k-1)v) + da,$$

so $G(s+1)-G(s) \geq 0$ for any nonnegative integer s whenever

$$d \geq \frac{1}{2}kh(b-1)(2+(k-1)v) - h(1+kv)\left\lfloor\frac{a}{b^k}\right\rfloor.$$

Therefore,

$$\boldsymbol{m}(x) = \min_{s\in\mathbb{N}} G(s) = G(0) = ah\sum_{i=0}^{k}(1+vi)\overline{q}_i(x) + dx$$

for these values of $d \geq \frac{1}{2}kh(b-1)(2+(k-1)v) - h(1+kv)\left\lfloor\frac{a}{b^k}\right\rfloor$. The proof of Item (b) is finished.

This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.1

In this section, we use Lemmas 2.1 and 2.2 to prove Theorem 1.1.

Proof of Theorem 1.1. First, let $c = ud$ for a nonnegative integer and $d \geq ah(k(b-1)+u)$. Note that $\gcd(a,d) = 1$, then by Lemmas 2.1 and 2.2 we have that

$$g(S_c) = \max_{\langle dx\rangle\neq\langle 0\rangle} \boldsymbol{m}(x) - a$$

$$
\begin{aligned}
&= \max_{1\le x\le a-1} \boldsymbol{m}(x) - a\\
&= \max_{1\le x\le a-1} \left(ah\left(\left\lfloor \frac{x}{b^k+uk}\right\rfloor + \mathtt{W}_{b,u}(x)\right) + dx\right) - a. \qquad (3.1)
\end{aligned}
$$

Let $H(x) := ah\left(\left\lfloor \frac{x}{b^k+uk}\right\rfloor + \mathtt{W}_{b,u}(x)\right) + dx$, then

$$
\begin{aligned}
H(x+1) - H(x) &= ah\left(\left\lfloor \frac{x+1}{b^k+uk}\right\rfloor - \left\lfloor \frac{x}{b^k+uk}\right\rfloor + \mathtt{W}_{b,u}(x+1) - \mathtt{W}_{b,u}(x)\right) + d\\
&\ge d - ah(k(b-1)+u) \ge 0
\end{aligned}
$$

for any positive integer x. It follows from (3.1) that

$$
g(S_c) = H(a-1) = ah\left(\left\lfloor \frac{a-1}{b^k+uk}\right\rfloor + \mathtt{W}_{b,u}(a-1)\right) + da - d - a.
$$

Next, applying Lemmas 2.1 and 2.2 to computing $n(S_c)$, one has that

$$
\begin{aligned}
n(S_c) &= \frac{1}{a}\sum_{\langle dx\rangle\ne\langle 0\rangle} \boldsymbol{m}(x) - \frac{1}{2}(a-1)\\
&= \frac{1}{a}\sum_{x=1}^{a-1} \boldsymbol{m}(x) - \frac{1}{2}(a-1)\\
&= h\sum_{x=1}^{a-1}\left(\left\lfloor \frac{x}{b^k+uk}\right\rfloor + \mathtt{W}_{b,u}(x)\right) + \frac{1}{2}(a-1)(d-1). \qquad (3.2)
\end{aligned}
$$

Write $a-1 = q(b^k+ku)+r$ with $0\le r\le b^k-1$, then

$$
\begin{aligned}
\sum_{x=1}^{a-1}\left\lfloor \frac{x}{b^k+uk}\right\rfloor &= \sum_{i=0}^{q-1}\sum_{x=i(b^k+uk)}^{(i+1)(b^k+ku)-1}\left\lfloor \frac{x}{b^k+uk}\right\rfloor + \sum_{x=q(b^k+ku)}^{q(b^k+ku)+r}\left\lfloor \frac{x}{b^k+uk}\right\rfloor\\
&= \sum_{i=0}^{q-1}\sum_{x=i(b^k+uk)}^{(i+1)(b^k+ku)-1} i + \sum_{x=q(b^k+ku)}^{q(b^k+ku)+r} q\\
&= \frac{1}{2}q(q-1)(b^k+ku) + (r+1)q. \qquad (3.3)
\end{aligned}
$$

Putting (3.3) into (3.2), we derive that

$$
n(S_c) = h\sum_{x=1}^{a-1}\mathtt{W}_{b,u}(x) + \frac{1}{2}hq(q-1)(b^k+ku) + hq(r+1) + \frac{1}{2}(a-1)(d-1),
$$

as desired. Item (a) is proved.

Now, we let $c = va$ for a nonnegative integer v with $v\le b-1$ and $d\ge \frac{1}{2}akh(b-1)(2+(k-1)v)$. To be similar as the proof of Item (a), we also can obtain that

$$
g(S_c) = ah\left((1+kv)\left\lfloor \frac{a-1}{b^k}\right\rfloor + \mathtt{V}_{b,v}(a-1)\right) + d(a-1).
$$

Next, let us compute $n(S_c)$ in the following. First, by employing Lemmas 2.1 and 2.2 we have that

$$\begin{aligned} n(S_c) &= \frac{1}{a}\sum_{\langle dx\rangle\neq\langle 0\rangle} \boldsymbol{m}(x) - \frac{1}{2}(a-1) \\ &= \frac{1}{a}\sum_{x=1}^{a-1} \boldsymbol{m}(x) - \frac{1}{2}(a-1) \\ &= h\sum_{x=1}^{a-1}\left((1+kv)\left\lfloor\frac{x}{b^k}\right\rfloor + \mathtt{V}_{b,v}(x)\right) + \frac{1}{2}(a-1)(d-1). \end{aligned} \tag{3.4}$$

Second, we compute $\sum_{x=1}^{a-1}\left\lfloor\frac{x}{b^k}\right\rfloor$ and $\sum_{x=1}^{a-1}\mathtt{V}_{b,v}(x)$. For the purpose, define two sequences $\{q_i\}_{i=1}^k$ and $\{r_i\}_{i=1}^k$ by

$$q_i := \left\lfloor\frac{a-1}{b^i}\right\rfloor,\ r_i := a-1-q_ib^i. \tag{3.5}$$

Then, for any $0 \le i \le k$, one deduces that

$$\begin{aligned} \sum_{x=1}^{a-1}\left\lfloor\frac{x}{b^i}\right\rfloor &= \sum_{j=0}^{q_i-1}\sum_{x=jb^i}^{(j+1)b^i-1}\left\lfloor\frac{x}{b^i}\right\rfloor + \sum_{x=q_ib^i}^{q_ib^i+r_i}\left\lfloor\frac{x}{b^i}\right\rfloor \\ &= \sum_{j=0}^{q_i-1}\sum_{x=jb^i}^{(j+1)b^i-1} j + \sum_{x=q_ib^i}^{q_ib^i+r_i} q_i \\ &= \frac{1}{2}b^iq_i(q_i-1) + q_i(r_i+1), \end{aligned} \tag{3.6}$$

which is denoted by M_i for brevity. Recall that $\overline{q}_i(x)$ is defined as (1.1), and it is checked that

$$\overline{q}_i(x) = \left\lfloor\frac{x}{b^i}\right\rfloor - b\left\lfloor\frac{x}{b^{i+1}}\right\rfloor$$

for any $0 \le i \le k-1$. It then follows that

$$\begin{aligned} \sum_{x=1}^{a-1}\mathtt{V}_{b,v}(x) &= \sum_{x=1}^{a-1}\sum_{i=0}^{k-1}(1+iv)\overline{q}_i(x) \\ &= \sum_{i=0}^{k-1}(1+iv)\sum_{x=1}^{a-1}\overline{q}_i(x) \\ &= \sum_{i=0}^{k-1}(1+iv)\sum_{x=1}^{a-1}\left(\left\lfloor\frac{x}{b^i}\right\rfloor - b\left\lfloor\frac{x}{b^{i+1}}\right\rfloor\right) \\ &= \sum_{i=0}^{k-1}(1+iv)(M_i - bM_{i+1}). \end{aligned} \tag{3.7}$$

Finally, putting (3.6) and (3.7) into (3.4) we arrive at

$$n(S_c) = \frac{1}{2}(a-1)(ah+d-1) + h\sum_{i=1}^{k}(v(b+i-ib)+1-b)M_i,$$

where $M_i = \frac{1}{2}b^i q_i(q_i - 1) + q_i(r_i + 1)$, q_i and r_i are defined as (3.5). Thus, the proof of Item (b) is done.

The proof of Theorem 1.1 is completed. □

4. Conclusions

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of positive integers with $\gcd(a_1, a_2, \ldots, a_n) = 1$. The celebrated Frobenius problem is to find $g(S)$ and $n(S)$, which is the largest natural number that is not representable as a nonnegative integer combination of $a_1, a_2, \ldots, a_n$ and the number of natural numbers that are not nonnegative integer combinations of $a_1, a_2, \ldots, a_n$, respectively. In this paper, we determined $g(S_c)$ and $n(S_c)$ for $S_c = \{a, ha + d, ha + c + db, ha + 2c + db^2, \ldots, ha + kc + db^k\}$ in the case of that c is divided by one of a and d. In fact, we presented a formula for $\boldsymbol{m}(S_c, x)$ by determining the minimum value of certain functions with multi variables, where $\boldsymbol{m}(S_c, x)$ is the least positive integer in $L(S) \cap \langle x \rangle$ (see Lemma 2.2). By employing Lemmas 2.1 and 2.2 and with some technical calculations on some complex sums, we finally derived the explicit expressions of $g(S_c)$ and $n(S_c)$, so the paper extended a result of Tripathi in 2016. However, in this paper we do not say anything about this problem when $a \nmid c$ and $d \nmid c$. Maybe it needs more new ideas to settle the problem for that case. In addition, the following generalization direction for the Frobenius problem is attractive as well.

Problem 4.1. *Let a, k, h, d and s be positive integers with $\gcd(a, d) = 1$. Let $b_1, \ldots, b_s \geq 2$ be distinct positive integers. For*

$$S = \{a, ha + d, ha + d(b_1 + \cdots + b_s), ha + d(b_1^2 + \cdots + b_s^2), \ldots, ha + d(b_1^k + \cdots + b_s^k)\},$$

find $g(S)$ and $n(S)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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