Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(12): 7182–7194.
DOI: 10.3934/era.2023363
Received: 31 July 2023
Revised: 05 November 2023
Accepted: 06 November 2023
Published: 10 November 2023

Research article

Quasi-periodic solutions for the incompressible Navier-Stokes equations with nonlocal diffusion

Shuguan Ji* and Yanshuo Li

School of Mathematics and Statistics and Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, China

* **Correspondence:** Email: jishuguan@hotmail.com.

Abstract: This paper studied the incompressible Navier-Stokes (NS) equations with nonlocal diffusion on $\mathbb{T}^d(d \geq 2)$. Driven by a time quasi-periodic force, the existence of time quasi-periodic solutions in the Sobolev space was established. The proof was based on the decomposition of the unknowns into the spatial average part and spatial oscillating one. The former were sought under the Diophantine non-resonance assumption, and the latter by the contraction mapping principle. Moreover, by constructing suitable time weighted function space and using the Banach fixed point theorem, the asymptotic stability of quasi-periodic solutions and the exponential decay of perturbation were proved.

Keywords: Navier-Stokes (NS) equations; nonlocal diffusion; quasi-periodic solutions; asymptotic stability; exponential decay

1. Introduction

Consider the following incompressible Navier-Stokes (NS) system with nonlocal diffusion

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u + \nabla p = -u \cdot \nabla u + \varepsilon f(\omega t, x), \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where the variables $x \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d (d \geq 2), t \in \mathbb{R}$. The field $u(x,t)$ and scalar $p(x,t)$ represent the velocity and pressure of fluid at point (x,t), respectively. The nonlocal operator $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is defined by the Fourier transform on the torus

$$(-\Delta)^\alpha u = \sum_{k \in \mathbb{Z}^d} |k|^{2\alpha} \hat{u}(k) e^{ik\cdot x}.$$

The parameter ε is a small positive constant, and the time-dependent external force $f(\mathbb{T}^\nu, \mathbb{T}^d)$ is quasi-periodic with frequency $\omega = (\omega_1, \cdots, \omega_\nu) \in \mathbb{R}^\nu$.

The incompressible NS system with nonlocal diffusion describes the fluid motion with internal friction interaction [1], and corresponds a stochastic representation given in terms of stochastic differential equations driven by Lévy processes [2,3]. It has received increasing attention over the past few decades, and many interesting results have been established in several works [2, 4–10]. For instance, see [4, 5] for the global well-posedness, [6] for the regularity theory of Caffarelli-Kohn-Nirenberg and [8, 9] for the ill-posedness of weak solutions.

Concerning the time periodic solutions of system (1.1) with $\alpha = 1$, we refer the readers to [11–15] for the bounded domain cases, and to [16–22] for the unbounded domain cases. Serrin [11] proved the existence of time periodic solutions based on the solvability and stability of an initial value problem. Kozono and Nakao [16] converted the original time periodic problem into a mild formulation, and then established the existence and uniqueness of solutions by the L^p-L^q estimates of an associated semigroup and Kato iteration. Based on the decomposition of unknowns into a steady part and a purely periodic part, Kyed [17, 18] derived the maximal regularity of the linearized problem and then established the unique existence of solutions for the nonlinear problem. Galdi, jointly with his collaborators [19–21], handled the scenario of flow around a rigid body, which moved or rotated in a prescribed periodic manner.

Recently, Montalto [23] obtained the existence and stability of quasi-periodic solutions for incompressible NS equations on $\mathbb{T}^d (d \geq 2)$. Some authors have applied the normal form method and celebrated the Kolmogorov-Arnold-Moser (KAM) theory to the NS and Euler equations ($\alpha = 0$), and obtained several intriguing results [24–26]. Baldi and Montalto [24] proved the existence of quasi-periodic solutions for Euler equations on $\mathbb{T}^3$. Montalto [25] considered the inviscid limit problem for the quasi-periodic solutions to the NS equations on $\mathbb{T}^2$. Berti, Hassainia and Masmoudi [26] proved the existence of vortex patches close to the Kirchhoff ellipses. In addition, Crouseilles and Faou [27] constructed an explicit quasi-periodic solution with compact support on $\mathbb{T}^2$, and a generalized version of the higher dimension was established in [28].

However, regarding the time periodic problem for the general case of $\alpha \in (0, 1)$, to the best of our knowledge there are few results available in the literature. Motivated by the works above, we would like to extend the results in [23] to system (1.1) and obtain the existence of quasi-periodic solutions near zero for $\alpha \in [1/2, 1)$. It is worth mentioning that the remained case of solutions near zero for $\alpha \in (0, 1/2)$ is hard to handle, which is due to the lack of the sufficient smoothing effect to control the nonlinear convection type term $u \cdot \nabla u$. Nevertheless, the issues considered in the present paper do not involve the *small divisor* problem due to the presence of dissipation.

On one hand, to seek the quasi-periodic solutions $u_\omega(t) := U(\omega t, x)$, $p_\omega(t) := P(\omega t, x)$ of system (1.1), it suffices to solve the following equations:

$$\begin{cases} \omega \cdot \partial_\varphi U + (-\Delta)^\alpha U + U \cdot \nabla U + \nabla P = \varepsilon f(\varphi, x), \\ \nabla \cdot U = 0, \end{cases} \tag{1.2}$$

then one can decompose system (1.2) as a spatial averaged part and an oscillating one,

$$\omega \cdot \partial_\varphi U_0(\varphi) = \varepsilon f_0(\varphi) \tag{1.3}$$

and

$$\omega \cdot \partial_\varphi U_p + (-\Delta)^\alpha U_p = \mathbb{P}(-(U_p \cdot \nabla U_p) + \varepsilon f_p). \tag{1.4}$$

Here, the terms independent of x appear in the nonlinear term vanishes due to $\nabla \cdot U = 0$. The first equation (1.3) could be solved under the assumption that the frequency ω is Diophantine, i.e., for some $\gamma \in (0,1)$,

$$|\omega \cdot \ell| \geq \frac{\gamma}{|\ell|^{\nu}}, \quad \forall\, \ell \in \mathbb{Z}^{\nu} \backslash \{0\}. \tag{1.5}$$

The existence for the second equation (1.4) is obtained by means of the contraction mapping principle, provided that ε is suitably small. The main results are stated as follows.

Theorem 1.1 (Existence)**.** *Let $\nu \geq 1$, $d \geq 2$ and $\alpha \in [1/2, 1)$, and let $\sigma > \nu/2$, $s > d/2 + 1$ and $N \geq \max\{\sigma + \nu, \sigma + s - 2\alpha\}$ be real numbers. Assume that $f(\omega t, x) \in C^N(\mathbb{T}^{\nu} \times \mathbb{T}^d; \mathbb{R}^d)$ is a time-dependent quasi-periodic function with Diophantine frequency ω and satisfies*

$$\int_{\mathbb{T}^{\nu}\times\mathbb{T}^d} f(\varphi, x)dxd\varphi = 0, \tag{1.6}$$

then there exists a positive constant ε_0 such that if $\varepsilon \leq \varepsilon_0$, system (1.1) *admit time quasi-periodic solutions*

$$u_{\omega}(t, x) = U(\omega t, x) \in C(\mathbb{T}^{\nu}; H^s) \cap C^1(\mathbb{T}^{\nu}; H^{s-2\alpha}),$$
$$p_{\omega}(t, x) = P(\omega t, x) \in C(\mathbb{T}^{\nu}; H^s).$$

Moreover,

$$\int_{\mathbb{T}^{\nu}\times\mathbb{T}^d} U(\varphi, x)dxd\varphi = 0 \tag{1.7}$$

and

$$\|U\|_{C_{\varphi}H^s_x \cap C^1_{\varphi}H^{s-2\alpha}_x} \leq C\varepsilon, \quad \|P\|_{C_{\varphi}H^s_x} \leq C\varepsilon. \tag{1.8}$$

In particular, if f satisfies

$$\int_{\mathbb{T}^d} f(\varphi, x)dx = 0, \quad \forall \varphi \in \mathbb{T}^{\nu}, \tag{1.9}$$

then the above conclusion holds for $\omega \in \mathbb{R}^{\nu}$, and

$$\int_{\mathbb{T}^d} U(\varphi, x)dx = 0, \quad \forall \varphi \in \mathbb{T}^{\nu}.$$

Remark 1.2. *If we look for the quasi-periodic solutions near a constant vector $\zeta \in \mathbb{R}^d$ satisfying a suitable assumption, a similar statement may be obtained for $\alpha \in (0, 1)$ by the normal form method and KAM method. Precisely, solutions are written as*

$$u(t, x) = \zeta + w(t, x),$$

where $\nabla \cdot w = 0$.

Remark 1.3. *It is easy to see that if $U_0(\varphi)$ is the solution of Eq* (1.3)*, then $U_0 + \mathbf{C}(\mathbf{C} \in \mathbb{R}^d)$ is also a solution of Eq* (1.3)*. In this paper, we choose the zero order term as zero.*

On the other hand, let $u(t,x) = u_\omega(t,x) + v(t,x)$ and $p(t,x) = p_\omega(t,x) + q(t,x)$ be the perturbed quasi-periodic solutions. The asymptotic stability of quasi-periodic solutions $u_\omega(t,x)$, $p_\omega(t,x)$ can be investigated by studying the global existence of solutions for the following initial value problem

$$\begin{cases} \partial_t v + (-\Delta)^\alpha v + \nabla q = -u_\omega \cdot \nabla v - v \cdot \nabla u_\omega - v \cdot \nabla v, \\ \nabla \cdot v = 0, \end{cases} \tag{1.10}$$

with initial datum

$$v_0(x) = v(0,x). \tag{1.11}$$

By constructing time-weighted function space, we prove the global existence of small solutions by the Banach fixed point theorem. As a by product, we present the exponential decay estimates of the perturbation v.

We state the asymptotic stability of quasi-periodic solutions as follows.

Theorem 1.4 (Asymptotic stability)**.** *Let $\alpha \in (1/2, 1)$, $\beta \in (0,1)$ and the assumptions of Theorem 1.1 hold. Suppose that $u_0 = u(0,x) \in H^s$ and*

$$\int_{\mathbb{T}^d} (u_0 - u_\omega(0)) dx = 0. \tag{1.12}$$

Put

$$E_1 = \|u_0 - u_\omega(0)\|_{H^s},$$

then there exists a positive constant δ_1 such that if $E_1 \le \delta_1$, the problems (1.10) *and* (1.11) *have a unique global solution $(v,q) \in C([0,\infty), H^s)$. Moreover,*

$$\int_{\mathbb{T}^d} (u - u_\omega) dx = 0, \quad \forall\, t \ge 0, \tag{1.13}$$

and

$$\|(u - u_\omega)(t)\|_{H^s} \le C E_1 e^{-\beta t}, \quad \|(p - p_\omega)(t)\|_{H^s} \le C E_1 e^{-\beta t}, \quad \forall\, t \ge 0. \tag{1.14}$$

The paper is organized as follows. In Section 2, we give some function spaces and useful lemmas. Then, in Section 3, we prove the existence of time quasi-periodic solutions which have the same oscillation frequency with the $f(\omega t, x)$. Finally, the asymptotic stability of time quasi-periodic solutions obtained in Section 3 discussed in Section 4.

2. Preliminaries

This section collects some notations, function space and useful lemmas.

2.1. Notations and function spaces

We introduce some notations and function spaces used in this paper. For $m \in \mathbb{N}$, let $\mathbb{Z}_0^m := \mathbb{Z}^m \backslash \{0\}$. Let $\mathbb{P}$ denote the Leray projection operator on solenoidal vector fields. We denote a generic positive constant by C. For any vectors $u = (u_1, u_2, \cdots, u_d)$ and $v = (v_1, v_2, \cdots, v_d)$, $u \otimes v = (u_i v_j)_{1 \le i,j \le d}$ represents the matrix-valued tensor product of u and v.

For any $u(x) \in L^2(\mathbb{T}^d)$, one can expand it by the Fourier series [29]

$$u(x) = \sum_{k \in \mathbb{Z}^d} \hat{u}(k) e^{ix\cdot k},$$

where the Fourier coefficients are given by

$$\hat{u}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) e^{-ix\cdot k} dx.$$

Let π_0 be the orthogonal projection operator defined by

$$\pi_0 u(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) dx = \hat{u}(0).$$

For any $s \geq 0, \sigma \geq 0$, $H^s = H^s(\mathbb{T}^d)$ is the standard Sobolev space with the norm

$$\|u\|_{H^s(\mathbb{T}^d)} = \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{\frac{s}{2}} |\hat{u}(k)|^2 \right)^{\frac{1}{2}}.$$

And let $\dot{H}^s = \dot{H}^s(\mathbb{T}^d)$ be the corresponding Sobolev space with the norm

$$\|u\|_{\dot{H}^s(\mathbb{T}^d)} = \left(\sum_{k \in \mathbb{Z}^d} |k|^s |\hat{u}(k)|^2 \right)^{\frac{1}{2}}.$$

For any $u(\varphi, x) \in L^2(\mathbb{T}^\nu; L^2(\mathbb{T}^d))$, one can expand it as

$$u(\varphi, x) = \sum_{\ell \in \mathbb{Z}^\nu} \sum_{k \in \mathbb{Z}^d} \hat{u}(\ell, k) e^{i\ell\cdot\varphi} e^{ix\cdot k},$$

where the Fourier coefficients are

$$\hat{u}(\ell, k) = \frac{1}{(2\pi)^{\nu+d}} \int_{\mathbb{T}^d} \int_{\mathbb{T}^\nu} u(\varphi, x) e^{-i\ell\cdot\varphi} e^{-ix\cdot k} d\varphi dx.$$

The function space $H^{\sigma,s} = H^\sigma(\mathbb{T}^\nu; H^s)$ is defined by

$$H^{\sigma,s} = \{u \in L^2(\mathbb{T}^\nu; L^2(\mathbb{T}^d)) \mid \|u\|_{H^\sigma_\varphi H^s_x} < +\infty\},$$

where

$$\|u\|_{H^\sigma_\varphi H^s_x} = \left(\sum_{\ell \in \mathbb{Z}^\nu} \sum_{k \in \mathbb{Z}^d} (1 + |\ell|^2)^{\frac{\sigma}{2}} (1 + |k|^2)^{\frac{s}{2}} |\hat{u}(\ell, k)|^2 \right)^{\frac{1}{2}}.$$

For an interval I and a Banach space X, $C^m(I; X)$ denotes the space of m-times continuously differentiable functions on I with values in X. One can also define $C^m(\mathbb{T}^\nu, X)$ in a similar way.

2.2. Two lemmas

Lemma 2.1. *Let $\sigma > \nu/2$, $\iota \geq 0$, then $H^{\sigma+\iota}(\mathbb{T}^\nu)$ is compactly imbedded in $C^\iota(\mathbb{T}^\nu)$ and*

$$\|u\|_{C^\iota} \leq C\|u\|_{H^{\sigma+\iota}}.$$

Note that $H^s(\mathbb{T}^d)$ is a Banach algebra whenever $s > d/2$. The result below readily follows.

Lemma 2.2. *Let $\sigma > \nu/2$, $s > d/2$, then one can get*

$$\|uv\|_{H^s_x} \leq \|u\|_{H^s_x}\|v\|_{H^s_x} \tag{2.1}$$

for $u(x), v(x) \in H^s(\mathbb{T}^d)$, and

$$\|uv\|_{H^\sigma_\varphi H^s_x} \leq \|u\|_{H^\sigma_\varphi H^s_x}\|v\|_{H^\sigma_\varphi H^s_x} \tag{2.2}$$

for $u(\varphi, x), v(\varphi, x) \in H^\sigma(\mathbb{T}^\nu; H^s(\mathbb{T}^d))$.

3. Existence of quasi-periodic solutions

This section is devoted to establishing the existence of quasi-periodic solutions that have the same oscillation frequency as f for system (1.1).

3.1. Proof of Theorem 1.1

Proof. First, for Eq (1.3), by (1.6) we have

$$U_0(\varphi) = \sum_{\ell\in\mathbb{Z}^\nu_0} \frac{\varepsilon\hat{f}(\ell, 0)}{i\omega\cdot\ell} e^{i\ell\cdot\varphi}.$$

Clearly, $\nabla \cdot U_0 = 0$ and

$$\int_{\mathbb{T}^\nu\times\mathbb{T}^d} U_0(\varphi)dxd\varphi = \int_{\mathbb{T}^d\times\mathbb{T}^\nu} U_0(\varphi)d\varphi dx = 0.$$

From the Diophantine condition (1.5) and the Hölder inequality, it follows that

$$\|U_0(\varphi)\|_{H^\sigma_\varphi} \leq \varepsilon\gamma^{-1}\|f_0\|_{H^{\sigma+\nu}_\varphi} \leq C\varepsilon\gamma^{-1}\|f\|_{H^{\sigma+\nu}_\varphi H^0_x}. \tag{3.1}$$

Next, for Eq (1.4), we would justify the solvability by the contraction mapping principle in the setting $H^\sigma(\mathbb{T}^\nu; H^s(\mathbb{T}^d))$ with $\sigma > \nu/2$, $s > d/2 + 1$. To this end, define the following function space

$$X^{\sigma,s} = \{u \in H^\sigma(\mathbb{T}^\nu; H^s(\mathbb{T}^d)) \ : \ \nabla \cdot u = 0,\ \pi_0 u = 0,\ \|u\|_X < \infty\},$$

where

$$\|u\|_X := \|u\|_{H^\sigma_\varphi H^s_x}.$$

Let

$$\mathcal{L}_\omega := \omega \cdot \partial_\varphi + (-\Delta)^\alpha : X^{\sigma+1,s+2\alpha} \to X^{\sigma,s}.$$

Owing to the operator $\mathcal{L}_\omega$ as invertible on $X^{\sigma,s}$, define the mapping

$$\mathcal{T}(u) := \mathcal{L}_\omega^{-1}\mathbb{P}(\varepsilon f_p - u\cdot\nabla u), \tag{3.2}$$

where

$$\mathcal{L}_\omega^{-1}g(\varphi,x) = \sum_{\ell\in\mathbb{Z}^\nu}\sum_{k\in\mathbb{Z}_0^d}\frac{\hat{g}(\ell,k)}{i\omega\cdot\ell+|k|^{2\alpha}}e^{i\ell\cdot\varphi}e^{ik\cdot x}.$$

Now, it suffices to verify that the mapping $\mathcal{T}$ is a strict contraction on some closed ball in $X^{\sigma,s}$. Define the function space

$$X_R = X_R^{\sigma,s} := \{u\in X^{\sigma,s}:\ \|u\|_X\le R\}.$$

where R is a positive constant that will be determined later. For any $u\in X^{\sigma,s}$, a direct calculation shows that

$$\begin{aligned}\|\mathcal{L}_\omega^{-1}u(\varphi,x)\|^2_{H^\sigma_\varphi H^{s+2\alpha}_x} &\le C\sum_{\ell\in\mathbb{Z}^\nu}\sum_{k\in\mathbb{Z}_0^d}(1+|\ell|^2)^\sigma|k|^{2(s+2\alpha)}\frac{|\hat{u}(\ell,k)|^2}{|i\omega\cdot\ell+|k|^{2\alpha}|^2}\\ &\le C\sum_{\ell\in\mathbb{Z}^\nu}\sum_{k\in\mathbb{Z}_0^d}(1+|\ell|^2)^\sigma|k|^{2s}|\hat{u}(\ell,k)|^2\\ &\le C\|u\|^2_{H^\sigma_\varphi H^s_x},\end{aligned} \tag{3.3}$$

where we have used the equivalence of $\|u\|_{H^\sigma_\varphi H^s_x}$ and $\|u\|_{H^\sigma_\varphi \dot{H}^s_x}$, then for any $u\in X_R$, it is easy to check that

$$\int_{\mathbb{T}^\nu}\int_{\mathbb{T}^d}\mathcal{T}(u)dxd\varphi = 0 \quad\text{and}\quad \nabla\cdot\mathcal{T}(u)=0.$$

From (3.3), Lemma 2.2 and the Sobolev embedding theorem, it follows that

$$\begin{aligned}\|\mathcal{T}(u)\|_{H^\sigma_\varphi H^s_x} &= \|\mathcal{L}_\omega^{-1}\mathbb{P}(\varepsilon f_p - u\cdot\nabla u)\|_{H^\sigma_\varphi H^s_x}\\ &\le C\|(\varepsilon f_p - u\cdot\nabla u)\|_{H^\sigma_\varphi H^{s-2\alpha}_x}\\ &\le C\varepsilon\|f_p\|_{H^\sigma_\varphi H^{s-2\alpha}_x} + C\|u\otimes u\|_{H^\sigma_\varphi H^{s+1-2\alpha}_x}\\ &\le C\varepsilon\|f\|_{C^N} + C\|u\|^2_{H^\sigma_\varphi H^{s+1-2\alpha}_x}\\ &\le C_1\varepsilon + C_2\|u\|^2_{H^\sigma_\varphi H^s_x}.\end{aligned} \tag{3.4}$$

Taking $R=4C_1\varepsilon$ yields

$$\|\mathcal{T}(u)\|_X \le 2C_1\varepsilon \le R, \tag{3.5}$$

whenever $\varepsilon\le 1/(16C_1C_2)$. Moreover, for any $u_1,u_2\in X_R$, from (3.2), (3.4) and Lemma 2.2, we have

$$\begin{aligned}\|\mathcal{T}(u_1)-\mathcal{T}(u_2)\|_{H^\sigma_\varphi H^s_x} &\le C_2\|u_1\otimes(u_1-u_2)+u_2\otimes(u_1-u_2)\|_{H^\sigma_\varphi H^{s+1-2\alpha}_x}\\ &\le C_2(\|u_1\|_{H^\sigma_\varphi H^s_x}+\|u_2\|_{H^\sigma_\varphi H^s_x})\|u_1-u_2\|_{H^\sigma_\varphi H^s_x}\\ &\le 2C_2R\|u_1-u_2\|_{H^\sigma_\varphi H^s_x}.\end{aligned}$$

Noting that $R = 4C_1\varepsilon$ and $\varepsilon \leq 1/(16C_1C_2)$, it gives us

$$\|\mathcal{T}(u_1) - \mathcal{T}(u_2)(t)\|_X \leq \frac{1}{2}\|u_1 - u_2\|_X. \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at $\mathcal{T}$ as a strict contraction from X_R to X_R. Thus, there exists a unique fixed point in X_R of the mapping $\mathcal{T}$, which is a unique time quasi-periodic solution U_p of (1.4).

Finally, we end the proof with the regularity of solution $U = U_0 + U_p$. By Lemma 2.1, we have

$$\|U\|_{C_\varphi H_x^s} \leq R \leq C\varepsilon.$$

For the C^1 regularity with respect to φ, from (1.3) and (1.4), it follows that

$$\|\partial_\varphi U_0\|_{H_\varphi^\sigma} \leq C\varepsilon\|f_0\|_{H_\varphi^\sigma} \leq C\varepsilon$$

and

$$\begin{aligned}\|\partial_\varphi U_p\|_{H_\varphi^\sigma H^{s-2\alpha}} &\leq C\| - (-\Delta)^\alpha U_p + \mathbb{P}(-(U_p \cdot \nabla U_p) + \varepsilon f_p)\|_{H_\varphi^\sigma H^{s-2\alpha}} \\ &\leq C\|U_p\|_{H_\varphi^\sigma H^s} + C\|U_p\|^2_{H_\varphi^\sigma H^s} + C\varepsilon\|f_p\|_{H_\varphi^\sigma H^{s-2\alpha}} \\ &\leq C\varepsilon,\end{aligned}$$

hence,

$$\|U\|_{C_\varphi^1 H_x^{s-2\alpha}} \leq C\varepsilon.$$

For the pressure P,

$$\begin{aligned}\|P\|_{C_\varphi H_x^s} \leq C\|P\|_{H_\varphi^\sigma H_x^s} &\leq C\|\Delta^{-1}\nabla \cdot (\nabla \cdot (U \otimes U) + \varepsilon f)\|_{H_\varphi^\sigma H_x^s} \\ &\leq C\|U\|^2_{H_\varphi^\sigma H_x^s} + \varepsilon\|f\|_{H_\varphi^\sigma H_x^{s-1}} \\ &\leq C\varepsilon.\end{aligned}$$

On the whole, we conclude that $u(\omega t, x) = U(\varphi, x)$ and $p(\omega t, x) = P(\varphi, x)$ are solutions of system (1.1) and satisfy the desired properties (1.7) and estimates (1.8). For the particular case of f satisfying (1.9), Eq (1.3) only admits a trivial solution $U_0 = \mathbf{C} \in \mathbb{R}^d$ and the verification of Eq (1.4) is similar with the above one. We omit here for simplification. The whole proof of Theorem 1.1 is completed.

4. Stability

In this section, we prove the asymptotic stability of quasi-periodic solutions.

Lemma 4.1. *Let $\alpha \in [1/2, 1)$, $s \geq 0$. Assume $u_0 \in \dot{H}^s$ satisfies $\pi_0 u_0 = 0$, then it holds that*

$$\|e^{-t(-\Delta)^\alpha} u_0\|_{\dot{H}^s} \leq e^{-t}\|u_0\|_{\dot{H}^s}, \quad \forall\, t > 0. \tag{4.1}$$

Let $1 \leq s_0 \leq s + 1$ and $\beta \in (0, 1)$, then for $g \in \dot{H}^{s+1-s_0}$ satisfying $\pi_0 g = 0$, it holds that

$$\|e^{-t(-\Delta)^\alpha}\nabla \cdot g\|_{\dot{H}^s} \leq Ct^{-\frac{s_0}{2\alpha}}(1 - \beta)^{-\frac{s_0}{2\alpha}} e^{-\beta t}\|g\|_{\dot{H}^{s+1-s_0}}, \quad \forall\, t > 0. \tag{4.2}$$

Proof. We only prove (4.2). From Parseval's equality and the basic property of rapidly decaying functions, one can get

$$\begin{aligned}
\|e^{-t(-\Delta)^\alpha}\nabla\cdot g\|_{\dot H^s}^2 &= \sum_{k\in\mathbb{Z}_0^d}|k|^{2(s+1)}e^{-2t|k|^{2\alpha}}|\hat g(k)|^2\\
&= \sum_{k\in\mathbb{Z}_0^d}e^{-2t\beta|k|^{2\alpha}}|k|^{2(s+1-s_0)}|k|^{2s_0}e^{-2t(1-\beta)|k|^{2\alpha}}|\hat g(k)|^2\\
&\le e^{-2\beta t}(t(1-\beta))^{-\frac{s_0}{\alpha}}\sum_{k\in\mathbb{Z}_0^d}|k|^{2(s+1-s_0)}|\hat g(k)|^2\\
&\quad\times(\sqrt{t(1-\beta)}|k|^\alpha)^{\frac{2s_0}{\alpha}}e^{-2t(1-\beta)|k|^{2\alpha}}\\
&\le Ce^{-2\beta t}(t(1-\beta))^{-\frac{s_0}{\alpha}}\sum_{k\in\mathbb{Z}_0^d}|k|^{2(s+1-s_0)}|\hat g(k)|^2\\
&= Ce^{-2\beta t}(t(1-\beta))^{-\frac{s_0}{\alpha}}\|g\|_{\dot H^{s+1-s_0}}^2.
\end{aligned}$$

The lemma is proved.

Denote

$$B(u,v) := -\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u\otimes v)(\tau)d\tau. \tag{4.3}$$

Lemma 4.2. *Let $\alpha\in(1/2,1)$, $\beta\in(0,1)$ and $s>d/2$. There holds that*

$$\sup_{t\ge0}e^{\beta t}\|B(u,v)\|_{H^s}\le C_3\sup_{t\ge0}e^{\beta t}\|v\|_{H^s}\sup_{t\ge0}\|u\|_{H^s}, \tag{4.4}$$

$$\sup_{t\ge0}e^{\beta t}\|B(u,v)\|_{H^s}\le C_3\sup_{t\ge0}\|v\|_{H^s}\sup_{t\ge0}e^{\beta t}\|u\|_{H^s} \tag{4.5}$$

and

$$\sup_{t\ge0}e^{\beta t}\|B(v,v)\|_{H^s}\le C_3(\sup_{t\ge0}e^{\beta t}\|v\|_{H^s})^2, \tag{4.6}$$

for $u,v\in C([0,\infty);H^s)$ and $\sup_{t\ge0}e^{\beta t}\|v\|_{H^s}<\infty$.

Proof. Since the proof of (4.4) and (4.5) are similar, we only prove the former. If $0<t\le1$, by (4.2) with $s_0=1$ and Lemma 2.1, we have

$$\begin{aligned}
\|B(u,v)\|_{H^s} &\le C\int_0^t\|e^{-(t-\tau)(-\Delta)^\alpha}(u\otimes v)(\tau)\|_{H^{s+1}}d\tau\\
&\le C\int_0^t e^{-\beta(t-\tau)}(t-\tau)^{-\frac{1}{2\alpha}}(1-\beta)^{-\frac{1}{2\alpha}}\|(u\otimes v)(\tau)\|_{H^s}d\tau\\
&\le Ce^{-\beta t}\sup_{t\ge0}e^{\beta t}\|v\|_{H^s}\sup_{t\ge0}\|u\|_{H^s}\int_0^1\tau^{-\frac{1}{2\alpha}}d\tau\\
&\le Ce^{-\beta t}\sup_{t\ge0}e^{\beta t}\|v\|_{H^s}\sup_{t\ge0}\|u\|_{H^s}.
\end{aligned}$$

If $t > 1$, we decompose the integral as

$$\begin{aligned}
&\int_0^t \|e^{-(t-\tau)(-\Delta)^\alpha}(u\otimes v)(\tau)\|_{H^{s+1}}d\tau \\
=&\int_0^{t-1}(\cdots)d\tau + \int_{t-1}^t(\cdots)d\tau \\
=:&J_1 + J_2.
\end{aligned}$$

By changing variables, one can treat J_2 similarly with the case $0 < t \le 1$. For the term J_1, from the Sobolev imbedding inequality and (4.2) with $s_0 = s + 1$, it follows that

$$\begin{aligned}
J_1 &\le \int_0^{t-1}(t-\tau)^{-\frac{s+1}{2\alpha}}(1-\beta)^{-\frac{s+1}{2\alpha}}e^{-\beta(t-\tau)}\|(u\otimes v)(\tau)\|_{L^2}d\tau \\
&\le Ce^{-\beta t}\sup_{t\ge 0}e^{\beta t}\|v\|_{H^s}\sup_{t\ge 0}\|u\|_{H^s}\int_1^t \tau^{-\frac{s+1}{2\alpha}}d\tau \\
&\le Ce^{-\beta t}\sup_{t\ge 0}e^{\beta t}\|v\|_{H^s}\sup_{t\ge 0}\|u\|_{H^s}.
\end{aligned}$$

Since $e^{-\beta t} \le 1$ for any $t \ge 0$, the estimates (4.6) can be proved in a similar way. The lemma is proved.

Now we turn back to the proof of Theorem 1.4.

Proof. Note that

$$q = (-\Delta)^{-1}\left(\nabla\cdot(u_\omega\cdot\nabla v + v\cdot\nabla u_\omega + v\cdot\nabla v)\right)$$

and, therefore, we mainly focus on the unknown $v(t,x)$ in the following. Due to Duhamel's principle, one can deduce that

$$v = e^{-t(-\Delta)^\alpha}v_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}(v\cdot\nabla u_\omega + u_\omega\cdot\nabla v + v\cdot\nabla v)(\tau)d\tau.$$

Therefore, define the following mapping

$$\begin{aligned}
\mathcal{M}v &= e^{-t(-\Delta)^\alpha}v_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u_\omega\otimes v + v\otimes u_\omega + v\otimes v)(\tau)d\tau \\
&:= V_0 + B(u_\omega, v) + B(v, u_\omega) + B(v,v),
\end{aligned} \tag{4.7}$$

where $B(\cdot,\cdot)$ is defined in (4.3). For $0 < \beta < 1$ and $s > d/2$, define the following functional space

$$Y_R := \{u \in C([0,\infty); H^s(\mathbb{T}^d)) \ : \ \|u\|_Y \le R\},$$

where

$$\|u\|_Y := \sup_{t\ge 0}e^{\beta t}\|u\|_{H^s}.$$

For the linear term, from (4.1) we get

$$\sup_{t\geq 0} e^{\beta t}\|V_0\|_{H^s} \leq \sup_{t\geq 0} e^{(\beta-1)t}\|v_0\|_{H^s} \leq \|v_0\|_{H^s}. \tag{4.8}$$

For the Duhamel integral terms, applying Lemma 4.2, one can get

$$\|B(u_\omega, v) + B(v, u_\omega)\|_{H^s} \leq 4C_1C_3e^{-\beta t}\varepsilon\|v\|_Y, \tag{4.9}$$

and

$$\|B(v, v)\|_{H^s} \leq C_3e^{-\beta t}\|v\|_Y^2. \tag{4.10}$$

Combining (4.8)–(4.10) yields

$$\|\mathcal{M}v\|_Y \leq E_1 + 4C_1C_3\varepsilon\|v\|_Y + C_3\|v\|_Y^2.$$

Taking $R = 4E_1$, one can obtain

$$\|\mathcal{M}v\|_Y \leq 2E_1 \leq R, \tag{4.11}$$

provided that $E_1 \leq 1/(32C_3)$ and $\varepsilon \leq \min\{1/(16C_1C_2), 1/(8C_1C_3)\}$. For any $v_1, v_2 \in Y_R$, let $v_\delta = v_1 - v_2$, and we get

$$\begin{aligned}
\|\mathcal{M}(v_1) - \mathcal{M}(v_2)\|_{H^s} &\leq \|\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u_\omega \otimes v_\delta + v_\delta \otimes u_\omega)(\tau)d\tau\|_{H^s} \\
&\quad + \|\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(v_1 \otimes v_\delta + v_\delta \otimes v_2)(\tau)d\tau\|_{H^s} \\
&\leq \|B(u_\omega, v_\delta)\|_{H^s} + \|B(v_\delta, u_\omega)\|_{H^s} + \|B(v_1, v_\delta)\|_{H^s} + \|B(v_\delta, v_1)\|_{H^s}.
\end{aligned}$$

Applying Lemma 4.2 again, one can get

$$\|\mathcal{M}(v_1) - \mathcal{M}(v_2)\|_Y \leq 4C_1C_3\varepsilon\|v_1 - v_2\|_Y + 2C_3R\|v_1 - v_2\|_Y \leq \frac{3}{4}\|v_1 - v_2\|_Y, \tag{4.12}$$

provided that $E_1 \leq 1/(32C_3)$ and $\varepsilon \leq \min\{1/(16C_1C_2), 1/(8C_1C_3)\}$. Combining estimates (4.11) and (4.12), one can see that map $\mathcal{M}$ is a strict contraction. We then obtain that there exists a unique fixed point v of the map $\mathcal{M}$ in Y_R, which is the unique solution of problems (1.10) and (1.11). Therefore, we conclude that the quasi-periodic solution in Theorem 1.1 is asymptotically stable. Moreover,

$$\|u - u_\omega\|_{H^s} = \|v\|_{H^s} \leq CE_1e^{-\beta t}.$$

This completes the proof of Theorem 1.4.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported partially by NSFC Grants (12225103, 12071065 and 11871140) and the National Key Research and Development Program of China (2020YFA0713602 and 2020YFC1808301).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. J. R. Mercado, E. P. Guido, A. J. Sánchez-Sesma, M. Íñiguez, A. González, Analysis of the Blasius's formula and the Navier-Stokes fractional equation, in *Fluid Dynamics in Physics, Engineering and Environmental Applications*, Springer, Berlin, Heidelberg, (2013), 475–480. https://doi.org/10.1007/978-3-642-27723-8_44
2. X. C. Zhang, Stochastic Lagrangian particle approach to fractal Navier-Stokes equations, *Commun. Math. Phys.*, **311** (2012), 133–155. https://doi.org/10.1007/s00220-012-1414-2
3. W. A. Woyczyński, Lévy processes in the physical sciences, in *Lévy Processes*, Birkhäuser, Boston, MA, (2001), 241–266. https://doi.org/10.1007/978-1-4612-0197-7_11
4. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires* (French), Dunod, Paris, 1969.
5. J. C. Mattingly, Y. G. Sinai, An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations, *Commun. Contemp. Math.*, **1** (1999), 497–516. https://doi.org/10.1142/S0219199799000183
6. L. Tang, Y. Yu, Partial regularity of suitable weak solutions to the fractional Navier-Stokes equations, *Commun. Math. Phys.*, **334** (2015), 1455–1482. https://doi.org/10.1007/s00220-014-2149-z
7. B. Lai, C. Miao, X. Zheng, Forward self-similar solutions of the fractional Navier-Stokes equations, *Adv. Math.*, **352** (2019), 981–1043. https://doi.org/10.1016/j.aim.2019.06.021
8. M. Colombo, C. De Lellis, L. De Rosa, Ill-posedness of Leray solutions for the hypodissipative Navier-Stokes equations, *Commun. Math. Phys.*, **362** (2018), 659–688. https://doi.org/10.1007/s00220-018-3177-x
9. L. De Rosa, Infinitely many Leray-Hopf solutions for the fractional Navier-Stokes equations, *Commun. Partial Differ. Equations*, **44** (2019), 335–365. https://doi.org/10.1080/03605302.2018.1547745
10. J. Wu, Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces, *Commun. Math. Phys.*, **263** (2006), 803–831. https://doi.org/10.1007/s00220-005-1483-6
11. J. Serrin, A note on the existence of periodic solutions of the Navier-Stokes equations, *Arch. Ration. Mech. Anal.*, **3** (1959), 120–122. https://doi.org/10.1007/BF00284169
12. V. I. Yudovich, Periodic motions of a viscous incompressible fluid, *Sov. Math., Dokl.*, **1** (1960), 168–172.
13. G. Prodi, Qualche risultato riquardo alle equazioni di Navier-Stokes nel caso bidimensionale, *Rend. Semin. Mat. Univ. Padova*, **30** (1960), 1–15.
14. G. Prouse, Soluzioni periodiche dell' equazione di Navier-Stokes, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.*, **35** (1963), 443–447.

15. O. Vejvoda, L. Herrmann, V. Lovicar, M. Sova, I. Straškraba, M. Štědrý, *Partial Differential Equations: Time-Periodic Solutions*, Martinus Nijhoff Publishers, The Hague, 1981.
16. H. Kozono, M. Nakao, Periodic solutions of the Navier-Stokes equations in unbounded domains, *Tohoku Math. J.*, **48** (1996), 33–50. https://doi.org/10.2748/tmj/1178225411
17. M. Kyed, *Time-periodic Solutions to the Navier-Stokes Equations*, Habilitation thesis, Technische Universität in Darmstadt, 2012.
18. M. Kyed, The existence and regularity of time-periodic solutions to the three dimensional Navier-Stokes equations in the whole space, *Nonlinearity*, **27** (2014), 2909–2935. https://doi.org/10.1088/0951-7715/27/12/2909
19. G. P. Galdi, Viscous flow past a body translating by time-periodic motion with zero average, *Arch. Ration. Mech. Anal.*, **237** (2020), 1237–1269. https://doi.org/10.1007/s00205-020-01530-6
20. G. P. Galdi, A. L. Silvestre, Existence of time-periodic solutions to the Navier-Stokes equations around a moving body, *Pac. J. Math.*, **223** (2006), 251–267. https://doi.org/10.2140/pjm.2006.223.251
21. G. P. Galdi, H. Sohr, Existence and uniqueness of time-periodic physically reasonable Navier-Stokes flow past a body, *Arch. Ration. Mech. Anal.*, **172** (2004), 363–406. https://doi.org/10.1007/s00205-004-0306-9
22. M. Yamazaki, The Navier-Stokes equations in the weak-L^n space with time-dependent external force, *Math. Ann.*, **317** (2000), 635–675. https://doi.org/10.1007/PL00004418
23. R. Montalto, The Navier-Stokes equation with time quasi-periodic external force: existence and stability of quasi-periodic solutions, *J. Dyn. Differ. Equations*, **33** (2021), 1341–1362. https://doi.org/10.1007/s10884-021-09944-w
24. P. Baldi, R. Montalto, Quasi-periodic incompressible Euler flows in 3D, *Adv. Math.*, **384** (2021), 107730. https://doi.org/10.1016/j.aim.2021.107730
25. L. Franzoi, R. Montalto, A KAM approach to the inviscid limit for the 2D Navier-Stokes equations, preprint, arXiv:2207.11008.
26. M. Berti, Z. Hassainia, N. Masmoudi, Time quasi-periodic vortex patches of Euler equation in the plane, *Invent. Math.*, **233** (2023), 1279–1391. https://doi.org/10.1007/s00222-023-01195-4
27. N. Crouseilles, E. Faou, Quasi-periodic solutions of the 2D Euler equation, *Asymptotic Anal.*, **81** (2013), 31–34. https://doi.org/10.3233/ASY-2012-1117
28. A. Enciso, D. Peralta-Salas, F. T. de Lizaur, Quasi-periodic solutions to the incompressible Euler equations in dimensions two and higher, *J. Differ. Equations*, **354** (2023), 170–182. https://doi.org/10.1016/j.jde.2023.01.013
29. L. Grafakos, *Classical Fourier Analysis*, Springer, New York, 2014. https://doi.org/10.1007/978-1-4939-1194-3