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Research article

The S-asymptotically ω-periodic solutions for stochastic fractional differential equations with piecewise constant arguments

Shufen Zhao*

School of Mathematics and Statistics, Taishan University, Taian 271000, China

* **Correspondence:** Email: zsfzx1982@sina.com.

Abstract: In this paper, two kinds of stochastic differential equations with piecewise constant arguments are investigated. Sufficient conditions for the existence of the square-mean S-asymptotically ω-periodic solutions of these two type equations are derived where ω is an integer. Then, the global asymptotic stability for one of them is considered by using the comparative approach. In order to show the theoretical results, we give two examples.

Keywords: square-mean S-asymptotically ω-periodic solution; Lévy noise; piecewise constant argument

1. Introduction

For the applications in many practical dynamical phenomena arising in engineering, physics, economy and science, the differential equations of fractional order have been widely studied by many authors [1, 2]. It is normal for things to be influenced by the past, as the past events are important for the present current behavior. Shah and Wiener [3] considered the differential equations containing piecewise constant argument which is a constant delay of generalized type [4]. This type of system attracts a lot of attention in control theory and neural networks [5, 6]. The differential systems with piecewise constant argument have the structure of continuous dynamical systems within intervals of certain lengths, so these kind of systems have properties of both differential and difference equations. It is difficult to obtain the exact solutions of the differential systems with piecewise constant argument, so the numerical solutions are required in practice, especially, Milošvić [7] investigated the strong convergence and stability of the Euler-Maruyama method for stochastic differential equations with piecewise constant arguments.

The periodic development of things is an important phenomenon in nature, which is valued and deeply studied by the majority of scientific and technological workers, there have been many papers dealing with the properties about almost automorphic, asymptotically almost automorphic, almost periodic and S-asymptotically ω-periodic solutions of various deterministic differential systems according to their different applications in different areas [8, 9]. The papers [10–12] deal with the properties of the S-asymptotically ω-periodic solutions of the determinate systems in finite dimension, and [11, 13] deal the S-asymptotically ω-periodic solutions of the determinate systems in infinite dimension. Especially in [13], Cuevas et al. considered the S-asymptotically ω-periodic solution of the following form,

$$\begin{cases} x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ax(s)\mathrm{d}s + f(t, x_t), \\ x_0 = \psi_0 \in \mathcal{B}, \end{cases}$$

where $\mathcal{B}$ is some abstract phase space, and in [14], Dimbour et al. considered the same question of the differential equations of the form

$$\begin{cases} x'(t) = Ax(t) + A_0 x([t]) + g(t, x(t)), \\ x(0) = c_0. \end{cases}$$

Note that the effect of noise is unavoidable in the study of some natural sciences as well as man-made phenomena such as ecology, biology, finance markets, engineering and other fields. Since the properties of the Lévy processes have useful information in explaining some drastic changes in nature and various scientific fields [15, 16], it is worth studying the stochastic fractional differential equations with Lévy noise [17–19]. It is natural to extend the known results from deterministic systems to stochastic systems. So in this work, the properties of a class of stochastic fractional differential equations driven by Lévy noise with piecewise constant argument of the following form

$$\begin{cases} \mathrm{d}x(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ax(s)\mathrm{d}s\mathrm{d}t + f(t, x([t]), x(t))\mathrm{d}t + g(t, x([t]), x(t))\mathrm{d}w(t) \\ \qquad\qquad + \int_{|u|_U<1} F(t, x(t^-), u)\tilde{N}(\mathrm{d}t, \mathrm{d}u) + \int_{|u|_U\geq 1} G(t, x(t^-), u)N(\mathrm{d}t, \mathrm{d}u), \\ x(0) = c_0, \end{cases} \tag{1.1}$$

where $x(\cdot)$ takes value in a real separable Hilbert space H, $1 < \alpha < 2$ and A is a linear densely defined sectorial operator on H, with domain $D(A)$. The convolution integral in (1.1) is the Riemann-Liouville fractional integral. $w(t)$ is an U-valued Wiener process with covariance operator Q, the so-called Q-Wiener process. $\tilde{N}(\mathrm{d}t, \mathrm{d}u)$ and $N(\mathrm{d}t, \mathrm{d}u)$ are introduced in the second section. Properties about the mild solution and the square-mean S-asymptotically ω-periodic solution of the Cauchy problem (1.1) are presented in this work. Furthermore, we also analyze about the following system

$$\begin{cases} \mathrm{d}x(t) = Ax(t)\mathrm{d}t + f(t, x([t]), x(t))\mathrm{d}t + g(t, x([t]), x(t))\mathrm{d}w(t) \\ \qquad\qquad + \int_{|u|_U<1} F(t, x(t^-), u)\tilde{N}(\mathrm{d}t, \mathrm{d}u) + \int_{|u|_U\geq 1} G(t, x(t^-), u)N(\mathrm{d}t, \mathrm{d}u), \\ x(0) = c_0, \end{cases} \tag{1.2}$$

and the operator $A : D(A) \subset H \to H$ is the generator of a bounded linear operator semigroup $S(t)$ with $S(t) \leq M_0 e^{-\gamma t}$ for any $t \geq 0$ and $M_0 > 0$ is a positive number.

The concept of the Poisson square-mean S-asymptotically ω-periodic process was presented in [18]. In this work, we show the existence results of the square-mean S-asymptotically ω-periodic solution of (1.1) and (1.2) with piecewise constant arguments, and we will show that when $\omega \in \mathbf{Z}^+$ the system (1.1) and (1.2) may have the square-mean S-asymptotically ω-periodic solution.

This article is composed of seven sections. In Section 2, we introduce notation, definitions and preliminary facts. In Section 3, the results about the mild solutions for problems (1.1) and (1.2) are obtainted using the successive approximation. In Section 4, we show the existence and uniqueness of the square-mean S-asymptotically ω-periodic solution for (1.1) and (1.2) using the Banach contraction mapping principle. In Section 5, we give sufficient conditions for the globally asymptotically stable of the square-mean S-asymptotically ω-periodic solution of (1.2). In Section 6, two examples are given to illustrate the theoretical results. Some conclusions are given in the last section.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfy the usual conditions, $(H, |\cdot|)$ and $(U, |\cdot|)$ are real separable Hilbert spaces. $\mathcal{L}(U, H)$ denote the space of all bounded linear operators from U to H which with the usual operator norm $\|\cdot\|_{\mathcal{L}(U,H)}$ is a Banach space. $L^2(P, H)$ is the space of all H-valued random variables X such that $E|X|^2 = \int_\Omega |X|^2 \mathrm{d}P < \infty$ where the expectation E is defined as $Ex = \int_\Omega x(\omega)\mathrm{d}P$ for any random variable x defined on $(\Omega, \mathcal{F}, P)$. For $X \in L^2(P, H)$, let $\|X\| := (\int_\Omega |X|^2 \mathrm{d}P)^{1/2}$, it is well known that $(L^2(P, H), \|\cdot\|)$ is a Hilbert space. We denote by $\mathcal{M}^2([0, T], H)$ for the collection of stochastic processes $x(t) : [0, T] \to L^2(P, H)$ such that $E\int_0^T |x(s)|\mathrm{d}s < \infty$. In the following discussion, we always consider the Lévy processes that are U-valued, and first we recall the definition of Lévy process.

Definition A U-valued stochastic process $L = (L(t), t \geq 0)$ is called Lévy process if

(1) $L(0) = 0$ almost surely;
(2) L has independent and stationary increments;
(3) L is stochastically continuous, i.e. for all $\epsilon > 0$ and all $s > 0$, $\lim_{t\to s} \mathbf{P}(|L(t) - L(s)|_U > \epsilon) = 0$.

Let L is a Lévy process on U, we write $\Delta L(t) = L(t) - L(t^-)$ for all $t \geq 0$. We define a counting Poisson random measure N on $(U - \{0\})$ through

$$N(t, O) = \sharp\{0 \leq s \leq t : \ \Delta L(s)(\omega') \in O\} = \Sigma_{0\leq s\leq t}\chi_O(\Delta L(s)(\omega'))$$

for any Borel set O in $(U - \{0\})$, χ_O is the indicator function. We write $\nu(\cdot) = E(N(1, \cdot))$ and call it the intensity measure associated with L. We say that a Borel set O in $(U - \{0\})$, is bounded below if $0 \in \bar{O}$ where $\bar{O}$ is closure of O. If O is bounded below, then $N(t, O) < \infty$ almost surely for all $t \geq 0$ and $(N(t, O), \ t \geq 0)$ is a Poisson process with intensity $\nu(O)$. So N is called Poisson random measure. For each $t \geq 0$ and O bounded below, the associated compensated Poisson random measure $\tilde{N}$ is defined by $\tilde{N}(t, O) = N(t, O) - t\nu(O)$ (see [15, 20]).

Proposition 2.1. *(see [15]) (Lévy-Itô decomposition). If L is a U-valued Lévy process, then there exist $a \in U$, a U-valued Wiener process w with covariance operator Q, the so-called Q-wiener process, and an independent Poisson random measure N on $\mathbf{R}^+ \times (U - \{0\})$ such that, for each $t \geq 0$,*

$$L(t) = at + w(t) + \int_{|u|_U<1} u\tilde{N}(t, \mathrm{d}u) + \int_{|u|_U\geq 1} uN(t, \mathrm{d}u), \tag{2.1}$$

where the Poisson random measure N has the intensity measure ν which satisfies $\int_U (|y|_U^2 \wedge 1)\nu(\mathrm{d}y) < \infty$ and $\tilde{N}$ is the compensated Poisson random measure of N.

For detailed properties of Lévy process and Q-Wiener processes, we refer the readers to [15,20–22]. In this work, the covariance operator Q of w satisfying $TrQ < \infty$ and the Lévy process L is defined on the filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathbf{R}^+})$, and $b := \int_{|y|_U\geq 1} \nu(\mathrm{d}x)$ throughout the paper.

Definition 2.1. *(see [17]) A stochastic process $x : \mathbf{R} \to L^2(P, H)$ is said to be L^2-continuous if for any $s \in \mathbf{R}$, $\lim_{t\to s} \|x(t) - x(s)\|^2 = 0$. It is L^2-bounded if $\sup_{t\in\mathbf{R}} \|x(t)\| < \infty$.*

Definition 2.2. *(see [18])*

(1) An L^2-continuous stochastic process $x : \mathbf{R}^+ \to L^2(P, H)$ is said to be square-mean S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty} \|x(t + \omega) - x(t)\| = 0$. We denote the collection of such process by $SAP_\omega(L^2(P, H))$.

(2) A function $g : \mathbf{R}^+ \times L^2(P, H) \to \mathcal{L}(U, L^2(P, H))$, $(t, X) \mapsto g(t, X)$ is said to be square-mean S-asymptotically ω-periodic in t for each $X \in L^2(P, H)$ if g satisfies

$$\|g(t, X) - g(t', X')\|^2_{\mathcal{L}(U,L^2(P,H))} \to 0 \text{ as } (t', X') \to (t, X)$$

and

$$\lim_{t\to\infty} \|(g(t + \omega, X) - g(t, X))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))} = 0, \quad \forall X \in L^2(P, H).$$

(3) A function $F : \mathbf{R}^+ \times L^2(P, H) \times U \to L^2(P, H)$, $(t, X, u) \mapsto F(t, X, u)$ with

$$\int_U \|F(t, \phi, u)\|^2 \nu(\mathrm{d}u) < \infty$$

is said to be Poisson square-mean S-asymptotically ω-periodic in t for each $X \in L^2(P, H)$ if F satisfies

$$\int_U \|F(t, X, u) - F(t', X', u)\|^2 \nu(\mathrm{d}u) \to 0 \text{ as } (t', X') \to (t, X)$$

and that

$$\lim_{t\to\infty} \int_U \|F(t + \omega, X, u) - F(t, X, u)\|^2 \nu(\mathrm{d}u) = 0 \quad \forall X \in L^2(P, H).$$

Remark 2.1. *Any square-mean S-asymptotically ω-periodic process $x(t)$ is L^2-bounded and, by [11], $SAP_\omega(L^2(P, H))$ with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbf{R}^+} \|x(t)\| = \sup_{t\in\mathbf{R}^+} (E|x(t)|^2)^{\frac{1}{2}}$$

is a Banach space.

To simplify the text, for definitions about square-mean S-asymptotically ω-periodic functions with parameters we refer the readers to [18]. Now let's recall some definition about sectorial operators, for details, see [23–25].

Definition 2.3. *Let $\mathbf{X}$ be an Banach space, $A : D(A) \subseteq \mathbf{X} \to \mathbf{X}$ is a closed linear operator . A is said to be sectorial operator of type μ and angle θ if there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbf{R}$ such that the resolvent $\rho(A)$ of A exists outside the sector $\mu + S_\theta = \{\mu + \lambda : \lambda \in \mathbf{C}, |arg(-\lambda)| < \theta\}$ and $\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda-\mu|}$, when λ does not belong to $\mu + S_\theta$.*

If A is sectorial of type μ with $1 < \theta < \pi(1-\frac{\alpha}{2})$, then A is the generator of a solution operator given by $S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\mathrm{d}\lambda$, where γ is a suitable path lying outside the sector $\mu + S_\theta$ [26].

We now directly give the definitions of mild solution of (1.1) and (1.2) which are based on the works of [17] and [25].

Definition 2.4. *A stochastic process* $\{x(t),\ t \in [0,T]\}$, $0 \le T < \infty$ *is said to be a mild solution to (1.1) if*

(1) $x(t)$ *is* $\mathcal{F}_t$*-adapted,* $t \ge 0$ *and has Càdlàg paths on* $t \ge 0$ *almost surely and*
(2) $x(t)$ *satisfies the following stochastic integral equation*

$$\begin{aligned} x(t) &= S_\alpha(t)c_0 + \int_0^t S_\alpha(t-s)f(s, x([s]), x(s))\mathrm{d}s \\ &\quad + \int_0^t S_\alpha(t-s)g(s, x([s]), x(s))\mathrm{d}w(s) \\ &\quad + \int_0^t \int_{|u|<1} S_\alpha(t-s)F(s, x(s^-), u)\tilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \int_0^t \int_{|u|\ge 1} S_\alpha(t-s)G(s, x(s^-), u)N(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

Definition 2.5. *A stochastic process* $\{x(t),\ t \in [0,T]\}$, $0 \le T < \infty$ *is said to be a mild solution to (1.2) if*

(1) $x(t)$ *is* $\mathcal{F}_t$*-adapted,* $t \ge 0$ *and has Càdlàg paths on* $t \ge 0$ *almost surely and*
(2) $x(t)$ *satisfies the following stochastic integral equation*

$$\begin{aligned} x(t) &= S(t)c_0 + \int_0^t S(t-s)f(s, x([s]), x(s))\mathrm{d}s \\ &\quad + \int_0^t S(t-s)g(s, x([s]), x(s))\mathrm{d}w(s) \\ &\quad + \int_0^t \int_{|u|<1} S(t-s)F(s, x(s^-), u)\tilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \int_0^t \int_{|u|\ge 1} S(t-s)G(s, x(s^-), u)N(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

3. The mild solutions of (1.1) and (1.2)

In the remainder of this article, the following conditions are considered to hold.

(H1) The operator A in (1.1) is a sectorial operator, Cuesta [27] showed that, if A is a sectorial operator of the type $\mu < 0$, for some $M > 0$ and $0 < \theta < \pi(1-\frac{\pi}{2})$, there is $C > 0$ such that

$$\|S_\alpha(t)\| \le \frac{CM}{1+|\mu|t^\alpha},\ t \ge 0. \tag{3.1}$$

(H2) $f : \mathbf{R}^+ \times L^2(P,H) \times L^2(P,H) \to L^2(P,H)$, $g :\ \mathbf{R}^+ \times L^2(P,H) \times L^2(P,H) \to \mathcal{L}(U, L^2(P,H))$ are jointly measurable and $\mathcal{F}_t$ adapted. $F : \mathbf{R}^+ \times L^2(P,H) \times U \to L^2(P,H)$, $G : \mathbf{R}^+ \times L^2(P,H) \times U \to L^2(P,H)$

are jointly measurable, $\mathcal{F}_t$ adapted and $\mathcal{F}_t$-predictable. $\forall t \in \mathbf{R}^+$, $\exists L > 0$ which is independent of t, such that

$$\begin{aligned}
\|f(t,x,y) - f(t,x_1,y_1)\|^2 &\le L(\|x - x_1\|^2 + \|y - y_1\|^2),\\
\|(g(t,x,y) - g(t,x_1,y_1))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))} &\le L(\|x - x_1\|^2 + \|y - y_1\|^2),\\
\int_{|u|_U<1} \|F(t,x,u) - F(t,z,u)\|^2 \nu(\mathrm{d}u) &\le L\|x - z\|^2,\\
\int_{|u|_U\ge 1} \|G(t,y,u) - G(t,z,u)\|^2 \nu(\mathrm{d}u) &\le L\|y - z\|^2
\end{aligned}$$

and

$$f(t,0,0) = 0,\ g(t,0,0) = 0,\ F(t,0,u) = 0,\ G(t,0,u) = 0. \tag{3.2}$$

Theorem 3.1. *If (H1) and (H2) hold, the system (1.1) has a unique mild solution.*

Proof. Set $x_0 \equiv S_\alpha(t)c_0$, and $n = 1, 2, ...,$ $\forall\ T \in (0, \infty)$, the Picard iterations are defined as follows:

$$\begin{aligned}
x_n(t) \quad = \quad & S_\alpha(t)c_0 + \int_0^t S_\alpha(t-s) f(s, x_{n-1}([s]), x_{n-1}(s))\mathrm{d}s\\
&+ \int_0^t S_\alpha(t-s) g(s, x_{n-1}([s]), x_{n-1}(s))\mathrm{d}w(s)\\
&+ \int_0^t \int_{|u|<1} S_\alpha(t-s) F(s, x_n(s^-), u)\tilde{N}(\mathrm{d}s, \mathrm{d}u)\\
&+ \int_0^t \int_{|u|\ge 1} S_\alpha(t-s) G(s, x_n(s^-), u)N(\mathrm{d}s, \mathrm{d}u)
\end{aligned} \tag{3.3}$$

for $t \in [0, T]$. Obviously, $x_0(\cdot) \in L^2(P, H)$. By using the Cauchy inequality

$$\begin{aligned}
&\|x_n(t)\|^2\\
&= \|S_\alpha(t)c_0 + \int_0^t S_\alpha(t-s) f(s, x_{n-1}([s]), x_{n-1}(s))\mathrm{d}s\\
&+ \int_0^t S_\alpha(t-s) g(s, x_{n-1}([s]), x_{n-1}(s))\mathrm{d}w(s)\\
&\ + \int_0^t \int_{|u|<1} S_\alpha(t-s) F(s, x_{n-1}(s^-), u)\tilde{N}(\mathrm{d}s, \mathrm{d}u)\\
&\ + \int_0^t \int_{|u|\ge 1} S_\alpha(t-s) G(s, x_{n-1}(s^-), u)N(\mathrm{d}s, \mathrm{d}u)\|^2\\
&\le 5(CM)^2\|c_0\|^2 + 5\|\int_0^t S_\alpha(t-s) f(s, x_{n-1}([s]), x_{n-1}(s))\mathrm{d}s\|^2\\
&\ + 5\int_0^t \|S_\alpha(t-s)\|^2 \|g(s, x_{n-1}([s]), x_{n-1}(s))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))}\mathrm{d}s\\
&\ + 5\|\int_0^t \int_{|u|_U<1} S_\alpha(t-s) F(s, x_{n-1}(s^-), u)\tilde{N}(\mathrm{d}s, \mathrm{d}u)\|^2
\end{aligned}$$

$$
\begin{aligned}
&+5\|\int_0^t\int_{|u|\geq 1}S_\alpha(t-s)G(s,x_{n-1}(s^-),u)N(\mathrm{d}s,\mathrm{d}u)\\
&+\int_0^t\int_{|u|\geq 1}S_\alpha(t-s)G(s,x_{n-1}(s^-),u)\nu(\mathrm{d}u)\mathrm{d}s\|^2\\
=\;&\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5
\end{aligned}
$$

Note the inequality (3.1), we get the estimation $\Lambda_1 \leq 5(CM)^2\|c_0\|^2$. Using the Hölder inequality, and the conditions for mapping $f : \mathbf{R}^+ \times L^2(P,H) \times L^2(P,H) \to L^2(P,H)$, we get

$$
\begin{aligned}
\Lambda_2 \quad &\leq 5(CM)^2\int_0^t\frac{1}{1+|\mu|(t-s)^\alpha}\mathrm{d}s\times\int_0^t\frac{1}{1+|\mu|(t-s)^\alpha}L(\|x_{n-1}([s])\|^2+\|x_{n-1}(s)\|^2)\mathrm{d}s\\
&\leq 5(CM)^2\int_0^t\frac{1}{1+|\mu|(t-s)^\alpha}\mathrm{d}s\times\int_0^t\frac{1}{1+|\mu|(t-s)^\alpha}L(\|x_{n-1}([s])\|^2+\|x_{n-1}(s)\|^2)\mathrm{d}s\\
&\leq 5(CM)^2L[\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}]\times\int_0^t(\|x_{n-1}([s])\|^2+\|x_{n-1}(s)\|^2)\mathrm{d}s
\end{aligned}
$$

By using a similar estimate to Λ_2, combined the Itô's isometry, the B-D-G inequality and the properties of integrals for Poisson random measures, we get

$$
\begin{aligned}
&\Lambda_3+\Lambda_4+\Lambda_5\\
&\leq +5(CM)^2\int_0^t\frac{1}{1+|\mu|^2(t-s)^{2\alpha}}L(\|x_{n-1}([s])\|^2+\|x_{n-1}(s)\|^2)\mathrm{d}s\\
&\;+5(CM)^2L\int_0^t\|x_{n-1}(s^-)\|^2\mathrm{d}s+10(CM)^2L\int_0^t\|x_{n-1}(s)\|^2\mathrm{d}s\\
&\;+10(CM)^2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}bL\int_0^t\|x_{n-1}(s^-)\|^2\mathrm{d}s
\end{aligned}
$$

so we get

$$
\begin{aligned}
&\sup_{0\leq s\leq t}\|x_n(s)\|^2\\
&\leq 5(CM)^2\|c_0\|^2+5(CM)^2L(2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\\
&+5+2b\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})\int_0^t\sup_{0\leq\theta\leq s}\|x_{n-1}(\theta)\|^2\mathrm{d}s,
\end{aligned}
$$

then $\forall\, \tilde{k} \in \mathbf{Z}^+$, the following inequality holds.

$$
\begin{aligned}
&\max_{1\leq n\leq\tilde{k}}\sup_{0\leq s\leq t}\|x_n(s)\|^2\\
&\leq 3(CM)^2\|c_0\|^2+5(CM)^2L(2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+5\\
&+2b\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})\times\int_0^t\max_{1\leq n\leq\tilde{k}}\sup_{0\leq\theta\leq s}\|x_{n-1}(\theta)\|^2\mathrm{d}s.
\end{aligned}
$$

If we let $c_1 = 5(CM)^2\|c_0\|^2,\ c_2 = 5(CM)^2L(2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} + 5 + 2b\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})$, then by the Gronwall inequality,

$$\max_{1\le n\le\tilde{k}}\sup_{0\le s\le t}\|x_n(s)\|^2 \le c_1e^{c_2t}.$$

Due to the arbitrariness of $\tilde{k}$, we have

$$\sup_{0\le s\le t}\|x_n(s)\|^2 \le c_1e^{c_2T}. \tag{3.4}$$

For

$$\begin{aligned}
&\|x_1(t) - x_0(t)\|^2 && (3.5)\\
&= \|\int_0^t S_\alpha(t-s)f(s, x_0([s]), x_{n-1}(s))\mathrm{d}s + \int_0^t S_\alpha(t-s)g(s, x_0([s]), x_0(s))\mathrm{d}w(s)\\
&+ \int_0^t\int_{|u|<1} S_\alpha(t-s)F(s, x_0(s^-), u)\tilde{N}(\mathrm{d}s, \mathrm{d}u) && (3.6)\\
&+ \int_0^t\int_{|u|\ge1} S_\alpha(t-s)G(s, x_0(s^-), u)N(\mathrm{d}s, \mathrm{d}u)\|^2\\
&= \|\int_0^t S_\alpha(t-s)[f(s, c_0, c_0) - f(s, 0, 0)]\mathrm{d}s\\
&+ \int_0^t S_\alpha(t-s)[g(s, c_0, c_0) - g(s, 0, 0)]\mathrm{d}w(s)\\
&+ \int_0^t\int_{|u|<1} S_\alpha(t-s)[F(s, c_0, u) - F(s, 0, u)]\tilde{N}(\mathrm{d}s, \mathrm{d}u) && (3.7)\\
&+ \int_0^t\int_{|u|\ge1} S_\alpha(t-s)[G(s, c_0, u) - G(s, 0, u)]N(\mathrm{d}s, \mathrm{d}u)\|^2\\
&\le 4(CM)^2\|c_0\|^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2\\
&+16(CM)^2\|c_0\|^2L(\frac{|\mu|^{-2/\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}) + 8(CM)^2\|c_0\|^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2b, && (3.8)
\end{aligned}$$

and let

$$\begin{aligned}
\tilde{C} \quad = \quad & 4(CM)^2\|c_0\|^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2\\
&+16(CM)^2\|c_0\|^2L(\frac{|\mu|^{-2/\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}) + 8(CM)^2\|c_0\|^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2b
\end{aligned}$$

we claim that for $n \ge 0$,

$$\|x_{n+1}(t) - x_n(t)\|^2 \le \frac{\tilde{C}(\tilde{M}t)^n}{n!},\ \text{for } 0 \le t \le T, \tag{3.9}$$

where $\tilde{M} = 4(CM)^2L[\frac{2|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)} + 5 + 2b\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}]$, we will show this claim by induction. Obviously, (3.9) holds when $n = 0$. We assume that (3.9) holds for some $n > 0$, we now prove that (3.9) still holds for $n + 1$. Note that

$$\|x_{n+2}(t) - x_{n+1}(t)\|^2$$

$$
\begin{aligned}
&\leq 4\|\int_0^t S_\alpha(t-s)[f(s,x_{n+1}([s]),x_{n+1}(s))-f(s,x_n([s]),x_n(s))]\mathrm{d}s\|^2\\
&\quad+4\|\int_0^t S_\alpha(t-s)[g(s,x_{n+1}([s]),x_{n+1}(s))-g(s,x_n([s]),x_n(s))]\mathrm{d}w(s)\|^2\\
&\quad+4\|\int_0^t\int_{|u|<1} S_\alpha[F(s,x_{n+1}(s^-),u)-F(s,x_n(s^-),u)]\tilde{N}(\mathrm{d}s,\mathrm{d}u)\|^2\\
&\quad+4\|\int_0^t\int_{|u|\geq 1} S_\alpha[G(s,x_{n+1}(s^-),u)-G(s,x_n(s^-),u)]N(\mathrm{d}s,\mathrm{d}u)\|^2\\
&\leq 4(CM)^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+1)\times\int_0^t[\|x_{n+1}([s])-x_n([s])\|^2+\|x_{n+1}(s)-x_n(s)\|^2]\mathrm{d}s\\
&\quad+4(CM)^2L\int_0^t\frac{1}{1+|\mu|^2(t-s)^{2\alpha}}\|x_{n+1}(s^-)-x_n(s^-)\|^2\mathrm{d}s\\
&\quad+8\|\int_0^t\int_{|u|\geq 1} S_\alpha(t-s)[G(s,x_{n+1}(s^-),u)-G(s,x_n(s^-),u)]\tilde{N}(\mathrm{d}s,\mathrm{d}u)\|^2\\
&\quad+8\|\int_0^t\int_{|u|\geq 1} S_\alpha(t-s)[G(s,x_{n+1}(s^-),u)-G(s,x_n(s^-),u)]\nu(\mathrm{d}u)\mathrm{d}s\|^2\\
&\leq 4(CM)^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+1)[\int_0^t\frac{\tilde{C}(\tilde{M}[s])^n}{n!}\mathrm{d}s+\int_0^t\frac{\tilde{C}(\tilde{M}s)^n}{n!}\mathrm{d}s]\\
&\quad+12(CM)^2L\int_0^t\frac{1}{1+|\mu|^2(t-s)^{2\alpha}}\frac{\tilde{C}(\tilde{M}s)^n}{n!}\mathrm{d}s\\
&\quad+8(CM)^2b\int_0^t\frac{1}{1+|\mu|(t-s)^{\alpha}}\mathrm{d}s\int_0^t\|x_{n+1}(s^-)-x_n(s^-)\|^2\mathrm{d}s\\
&\leq 8(CM)^2L(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+1)\int_0^t\frac{\tilde{C}(\tilde{M}s)^n}{n!}\mathrm{d}s\\
&\quad+12(CM)^2L\int_0^t\frac{\tilde{C}(\tilde{M}s)^n}{n!}\mathrm{d}s\\
&\quad+8(CM)^2bL\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\int_0^t\frac{\tilde{C}(\tilde{M}s)^n}{n!}\mathrm{d}s\\
&\leq 4(CM)^2L[\frac{2|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+5+2b\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}]\int_0^t\frac{\tilde{C}(\tilde{M}s)^n}{n!}\mathrm{d}s\\
&=\frac{\tilde{C}(\tilde{M}t)^{n+1}}{(n+1)!}.
\end{aligned}
$$

That is, (3.9) holds for $n+1$. By induction, we get that (3.9) holds for all $n \geq 0$. Furthermore, we find that

$$
\begin{aligned}
E\sup_{0\leq t\leq T}|x_{n+1}-x_n(t)|^2 &\leq \tilde{M}\int_0^T\|x_n(s)-x_{n-1}(s)\|^2\mathrm{d}s\\
&\leq 4\tilde{M}\int_0^T\frac{C[\tilde{M}s]^{n-1}}{(n-1)!}\mathrm{d}s\\
&= 4\frac{C[\tilde{M}T]^n}{n!}.
\end{aligned}
$$

Hence

$$P\{\sup_{0\le t\le T}|x_{n+1}(t)-x_n(t)|>\frac{1}{2^n}\}\le 4\frac{\tilde{C}[\tilde{M}T]^n}{n!}.$$

Note that $\sum_{n=0}^{\infty}4\frac{\tilde{C}[\tilde{M}T]^n}{n!}<\infty$, by using the Borel-Cantelli lemma, we can get a stochastic process $x(t)$ on $[0,T]$ such that $x_n(t)$ uniformly converges to $x(t)$ as $n\to\infty$ almost surely. It is easy to check that $x(t)$ is the unique mild solution of (1.1). The proof of the theorem is complete. □

Remark 3.1. *The conclusion of Theorem 3.1 holds for the Cauchy problem (1.2).*

4. The square-mean S-asymptotically ω-periodic solution of (1.1) and (1.2)

Lemma 4.1. *If $x(t)\in SAP_\omega(L^2(P,H))$ and $\omega\in\mathbf{Z}^+$, then $x([t])\in SAP_\omega(L^2(P,H))$.*

The proof process of Lemma 4.1 is similar to that of Lemma 2 of [14].

Proof. Since $x(t)\in SAP_\omega(L^2(P,H))$, then for any $\epsilon>0$, $\exists\ T_\epsilon^0\in\mathbf{R}^+$, such that $\forall\ t>T_\epsilon$, we have $\|x(t+\omega)-x(t)\|<\epsilon$. Let $T_\epsilon=[T_\epsilon^0]+1$. For $t>T_\epsilon$, $[t]\ge T_\epsilon$ for T_ϵ is an integer. Then we deduce that for the above ϵ, $\exists\ T_\epsilon\in\mathbf{R}^+$, such that $\|x([t+\omega])-x([t])\|=\|x([t]+\omega)-x([t])\|<\epsilon$. □

Theorem 4.1. *Assume (H1)-(H2) are satisfied and $f:\mathbf{R}^+\times L^2(P,H)\times L^2(P,H)\to L^2(P,H)$, $g:\mathbf{R}^+\times L^2(P,H)\times L^2(P,H)\to\mathcal{L}(U,L^2(P,H))$ are uniformly square-mean S-asymptotically ω-periodic on bounded sets of $L^2(P,H)\times L^2(P,H)$. Let $\omega\in\mathbf{Z}^+$. Then (1.1) has a unique square-mean S-asymptotically ω-periodic solution if*

$$2CM\{L(2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+5\frac{|\mu|^{-2/\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}+2b(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2)\}^{\frac{1}{2}}<1.$$

Proof. Define an operator $\tilde{\Gamma}:SAP_\omega(L^2(P,H))\mapsto SAP_\omega(L^2(P,H))$

$$\begin{aligned}&(\tilde{\Gamma}x)(t)\\&=S_\alpha(t)c_0+\int_0^t S_\alpha(t-s)f(s,x([s]),x(s))\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,x([s]),x(s))\mathrm{d}w(s)\\&\quad+\int_0^t\int_{|u|<1}S_\alpha(t-s)F(s,x(s^-),u)\tilde{N}(\mathrm{d}u,\mathrm{d}s)\\&\quad+\int_0^t\int_{|u|\ge1}S_\alpha(t-s)G(s,x(s^-),u)N(\mathrm{d}u,\mathrm{d}s).\end{aligned}$$

for every $x\in SAP_\omega(L^2(P,H))$. By (H1) and (3.1), we get that the operator $\tilde{\Gamma}$ is well defined. By Lemma 4.1, and the Lemma 4.2, Lemma 4.3 in [18], we get $(\tilde{\Gamma}x)(t)\in SAP_\omega(L^2(P,H))$. For every $x,y\in SAP_\omega(L^2(P,H))$,

$$\begin{aligned}&\|(\tilde{\Gamma}x)(t)-(\tilde{\Gamma}y)(t)\|^2\\&=\|\int_0^t S_\alpha(t-s)(f(s,x([s]),x(s))-f(s,y([s]),y(s)))\mathrm{d}s\\&\quad+\int_0^t S_\alpha(t-s)(g(s,x([s]),x(s))-g(s,y([s]),y(s)))\mathrm{d}w(s)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{|u|<1} S_\alpha(t-s)(F(s,x(s^-),u)-F(s,y(s^-),u))\tilde{N}(\mathrm{d}u,\mathrm{d}s)\\
&+\int_0^t\int_{|u|<1} S_\alpha(t-s)(G(s,x(s^-),u)-G(s,y(s^-),u))N(\mathrm{d}u,\mathrm{d}s)\|^2\\
\leq\ & 4(CM)^2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\int_0^t\frac{1}{1+|\mu|(t-s)^\alpha}\|[f(s,x([s]),x(s))-f(s,y([s]),y(s))]\|^2\mathrm{d}s\\
&+4\int_0^t\frac{(CM)^2}{1+|\mu|^2(t-s)^{2\alpha}}\|(g(s,x([s]),x(s))-g(s,y([s]),y(s)))Q^{1/2}\|^2_{\mathcal{L}(U,L^2(P,H))}\mathrm{d}s\\
&+4(CM)^2L\int_0^t\int_{|u|_U<1}\frac{1}{1+|\mu|^2(t-s)^{2\alpha}}\|F(s,x(s^-),u)-F(s,y(s^-),u)\|^2\nu(\mathrm{d}u)\mathrm{d}s\\
&+8\|\int_0^t\int_{|u|_U\geq 1} S_\alpha(t-s)(G(s,x(s^-),u)-G(s,y(s^-),u))\tilde{N}(\mathrm{d}u,\mathrm{d}s)\|^2\\
&+8\|\int_0^t\int_{|u|_U\geq 1} S_\alpha(t-s)(G(s,x(s^-),u)-G(s,y(s^-),u))\nu(\mathrm{d}u)\mathrm{d}s\|^2\\
\leq\ & 4(CM)^2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin\pi/\alpha}\int_0^t\frac{1}{1+|\mu|(t-s)^\alpha}L(\|x([s])-y([s])\|^2+\|x(s)-y(s)\|^2)\mathrm{d}s\\
&+4(CM)^2\int_0^t\frac{1}{1+|\mu|^2(t-s)^{2\alpha}}L(\|x([s])-y([s])\|^2+\|x(s)-y(s)\|^2)\mathrm{d}s\\
&+4(CM)^2L\int_0^t\frac{1}{1+|\mu|^2(t-s)^{2\alpha}}\mathrm{d}s\sup_{s\in\mathbf{R}^+}\|x(s)-y(s)\|^2\\
&+8(CM)^2L\int_0^t\frac{1}{1+|\mu|^2(t-s)^{2\alpha}}\mathrm{d}s\sup_{s\in\mathbf{R}^+}\|x(s)-y(s)\|^2\\
&+8(CM)^2b\int_0^t\frac{1}{1+|\mu|(t-s)^\alpha}\mathrm{d}s\\
&\quad\times\int_0^t\int_{|u|\geq 1}\frac{1}{1+|\mu|(t-s)^\alpha}\|G(s,x(s^-),u)-G(s,y(s^-),u)\|^2\nu(\mathrm{d}u)\mathrm{d}s
\end{aligned}
$$

$$
\leq 4(CM)^2L(2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+5\frac{|\mu|^{-2/\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}+2b(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2)\sup_{s\in\mathbf{R}^+}\|x(s)-y(s)\|^2
$$

Since $2CM\{L(2\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}+5\frac{|\mu|^{-2/\alpha}\pi}{2\alpha\sin(\pi/2\alpha)}+2b(\frac{|\mu|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)})^2)\}^{\frac{1}{2}}<1$, we obtain the result by the Banach contraction mapping principle. □

Similar discussion on (1.2), we get the following conclusion.

Theorem 4.2. *Assume (H1)and (H2) hold, the functions* $f:\mathbf{R}^+\times L^2(P,H)\times L^2(P,H)\to L^2(P,H)$, $g:\mathbf{R}^+\times L^2(P,H)\times L^2(P,H)\to\mathcal{L}(U,L^2(P,H))$ *are uniformly square-mean S-asymptotically* ω*-periodic on any bounded set K where* $K\subset L^2(P,H)\times L^2(P,H)$*. Let* $\omega\in\mathbf{Z}^+$*. Then (1.2) has a unique square-mean S-asymptotically* ω*-periodic solution if* $2[L(2\frac{M_0}{\gamma}+5\frac{M_0^2}{2\gamma}+2b\frac{M_0^2}{\gamma^2})]^{\frac{1}{2}}<1$.

5. Stability of solutions

In this section, the stability of the square-mean S-asymptotically ω-periodic solution of (1.2) is discussed. We first give the definition about globally asymptotically stable in the square-mean sense.

Definition 5.1. *$\forall d \in L^2(P,H)$, and let $x_d(t)$ be the mild solution of the cauchy problem (1.2) with initial value d. $x^*(t)$ is the unique square-mean S-asymptotically ω-periodic solution of (1.2). $x^*(t)$ is called globally asymptotically stable in square-mean sense if*

$$\lim_{t\to\infty} \|x_d(t) - x^*(t)\|^2 = 0$$

Theorem 5.1. *Assume the conditions of Theorem 4.1 are satisfied, $-\gamma + 5LM_0^2(\frac{1+2b}{\gamma} + 4) < 0$ and*

$$e^{-\gamma+5LM_0^2(\frac{1+2b}{\gamma}+4)} + (e^{-\gamma+5LM_0^2(\frac{1+2b}{\gamma}+4)} - 1)\frac{5LM_0^2(\frac{1}{\gamma}+1)}{-\gamma+5LM_0^2(\frac{1+2b}{\gamma}+4)} < 1,$$

then the square-mean S-asymptotically ω-periodic solution $x^*(t)$ of (1.1) is globally asymptotically stable in square-mean sense.

Proof. Suppose $x(t)$ is the mild solution of the cauchy problem (1.2) with initial value x_0, then

$$\begin{aligned}
&\|x(t) - x^*(t)\|^2 \\
&\le 5\|S(t)(x_0 - c_0)\|^2 + 5\|\int_0^t S(t-s)[f(s,x([s]),x(s)) - f(s,x^*([s]),x^*(s))]\mathrm{d}s\|^2 \\
&+5\|\int_0^t S(t-s)[g(s,x([s]),x(s)) - g(s,x^*([s]),x^*(s))]\mathrm{d}w(s)\|^2 \\
&+5\|\int_0^t\int_{|u|<1} S(t-s)[F(s,x(s),u) - F(s,x^*(s),u)]\tilde{N}(\mathrm{d}s,\mathrm{d}u)\|^2 \\
&+5\|\int_0^t\int_{|u|\ge 1} S(t-s)[G(s,x(s),u) - G(s,x^*(s),u)]N(\mathrm{d}s,\mathrm{d}u)\|^2 \\
&= 5\sum_{i=1}^{5} I_i
\end{aligned}$$

Obviously $I_1 = \|S(t)(x_0 - c_0)\|^2 \le M_0^2 e^{-2\gamma t}\|x_0 - c_0\|^2$,

$$\begin{aligned}
I_2 &\le \int_0^t S(t-s)\mathrm{d}s \int_0^t S(t-s)\|f(s,x([s]),x(s)) - f(s,x^*([s]),x^*(s))\|^2\mathrm{d}s \\
&\le L\frac{M_0^2}{\gamma}\int_0^t e^{-\gamma(t-s)}[\|x(s) - x^*(s)\|^2 + \|x([s]) - x^*([s])\|^2]\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
I_3 &\le \int_0^t S^2(t-s)\|g(s,x([s]),x(s)) - g(s,x^*([s]),x^*(s))\|^2\mathrm{d}s \\
&\le L\int_0^t S^2(t-s)[\|x(s) - x^*(s)\|^2 + \|x([s]) - x^*([s])\|^2]\mathrm{d}s
\end{aligned}$$

$$\leq \ LM_0^2 \int_0^t e^{-2\gamma(t-s)}[\|x(s) - x^*(s)\|^2 + \|x([s]) - x^*([s])\|^2]\mathrm{d}s,$$

$$I_4 \leq L \int_0^t S^2(t-s)\|x(s) - x^*(s)\|^2\mathrm{d}s \leq LM_0^2 \int_0^t e^{-2\gamma(t-s)}\|x(s) - x^*(s)\|^2\mathrm{d}s,$$

$$\begin{aligned} I_5 \ &\leq \ 2\|\int_0^t \int_{|u|\geq 1} S(t-s)[G(s, x(s), u) - G(s, x^*(s), u)]\tilde{N}(\mathrm{d}s, \mathrm{d}u)\|^2 \\ &\quad + 2\|\int_0^t \int_{|u|\geq 1} S(t-s)[G(s, x(s), u) - G(s, x^*(s), u)]\nu(\mathrm{d}u)\mathrm{d}s\|^2 \\ &\leq \ 2L \int_0^t S^2(t-s)\|x(s) - x^*(s)\|^2\mathrm{d}s + 2b \int_0^t S(t-s)\mathrm{d}s \int_0^t S(t-s)\|x(s) - x^*(s)\|^2\mathrm{d}s \\ &\leq \ 2LM_0^2 \int_0^t e^{-2\gamma(t-s)}\|x(s) - x^*(s)\|^2\mathrm{d}s \\ &\quad +2b\frac{M_0^2}{\gamma} \int_0^t e^{-\gamma(t-s)}\|x(s) - x^*(s)\|^2\mathrm{d}s, \end{aligned}$$

then

$$\begin{aligned} &\|x(t) - x^*(t)\|^2 \\ &\leq 5M_0^2 e^{-\gamma t}\|x_0 - c_0\| + 5LM_0^2(\frac{1+2b}{\gamma} + 4) \int_0^t e^{-\gamma(t-s)}\|x(s) - x^*(s)\|^2\mathrm{d}s \\ &+5LM_0^2(\frac{1}{\gamma} + 1) \int_0^t e^{-\gamma(t-s)}\|x([s]) - x^*([s])\|^2\mathrm{d}s. \end{aligned} \tag{5.1}$$

Set $D(t) = \|x(t) - x^*(t)\|^2$, $a_1 = 5LM_0(\frac{1+2b}{\gamma} + 4)$ and $a_2 = 5LM_0(\frac{1}{\gamma} + 1)$. By the inequality (5.1),

$$D(t) \leq \tilde{D}(t)$$

where $\tilde{D}(t)$ is the solution of the following system

$$\begin{cases} (\tilde{D}(t))' = -\gamma\tilde{D}(t) + a_1\tilde{D}(t) + a_2\tilde{D}([t]), \\ \tilde{D}(0) = 5M_0^2 D(0). \end{cases} \tag{5.2}$$

Let $m_0(t) = e^{-\gamma+a_1}t + (e^{(-\gamma+a_1)t} - 1)\frac{a_2}{-\gamma+a_1}$ and $b_0 = m(1)$. Note that the solution of (5.2) is

$$\tilde{D}(t) = 5M_0^2 D(0)m(\{t\})b_0^{[t]}$$

where $\{t\}$ is the decimal place of t. When $-\gamma + a_1 < 0$ and $e^{-\gamma+a_1} + (e^{-\gamma+a_1} - 1)\frac{a_2}{-\gamma+a_1} < 1$,

$$\lim_{t\to\infty} \tilde{D}(t) = 0,$$

then we get the conclusion. □

6. Applications

Now, we provide two examples to illustrate the results obtained in the previous two sections. In the following, let $H = L^2([0,\pi])$ and we first consider the following initial problem.

$$\begin{cases} dx(t,\xi) = \int_0^t \frac{(t-s)^{\frac{3}{2}-2}}{\Gamma(\frac{3}{2}-1)}(\frac{\partial^2}{\partial\xi^2} - v)x(s,\xi)dsdt \\ +(\sin\ln(t+1) + \cos t)x([t],\xi)dt + (\sin\ln(t+1) + \sin t)x(t,\xi)dw(t) \\ + \int_{|u|<1} x(t,\xi)(\cos t + \frac{1}{t})\tilde{N}(dt,du) + \int_{|u|\geq 1}(\cos t + \frac{1}{t})N(dt,du), \\ (t,\xi) \in (0,+\infty)\times[0,\pi], \\ x(t,\pi) = x(t,0) = 0, x(0,\xi) = \varrho(\xi) \end{cases} \tag{6.1}$$

where $v > 0$ is a positive number, and $w(t)$ is a Q-Wiener process on H. Obviously, the operator $A : H \to H$ is gaven by $A = \frac{\partial^2}{\partial\xi^2} - v$ with domain $D(x) = \{x \in H : x'' \in H, x(0) = x(\pi) = 0\}$, and A satisfies the (H1). Let $H = U$, then

$$f(t, x([t]), x(t)) = (\sin\ln(t+1) + \cos t)x([t],\xi),$$

$$g(t, x([t]), x(t)) = (\sin\ln(t+1) + \sin t)x(t,\xi),$$

$$F(t, x(t^-), u) = G(t, x(t^-), u) = x(t,\xi)(\cos t + \frac{\ln(t+1)}{t}).$$

We take $L = \max\{4, 4\|Q\|_{\mathcal{L}(U,U)}, 4\nu(B_1(0)), 4b\}$ where $B_1(0)$ is the unit ball of U, so f, g, F, G satisfies the (H2). According to Theorem 3.1, problem (6.1) has a unique mild solution on $[0,+\infty)$. Next, let's consider the following Cauchy Problem,

$$\begin{cases} dx(t,\xi) = (\frac{\partial^2}{\partial\xi^2} - v)x(s,\xi)dt \\ +(\sin\ln(t+1) + \cos t)x([t],\xi)dt + (\sin\ln(t+1) + \sin t)x(t,\xi)dw(t) \\ + \int_{|u|<1} x(t,\xi)(\cos t + \frac{1}{t})\tilde{N}(dt,du) \\ + \int_{|u|\geq 1}(\cos t + \frac{1}{t})N(dt,du)\ (t,\xi) \in (0,+\infty)\times[0,\pi], \\ x(t,\pi) = x(t,0) = 0, x(0,\xi) = \varrho(\xi). \end{cases} \tag{6.2}$$

When

$$6L(\frac{8|v|^{-\frac{3}{2}}\pi}{3\sqrt{3}} + \frac{10}{3}|v|^{-\frac{4}{3}\pi} + \frac{32}{27}b|v|^{-\frac{4}{9}}\pi^2) < 1,$$

according to the Theorem 5.1, the square-mean S-asymptotically ω-periodic solution of problem (6.2) is globally asymptotically stable on $[0,+\infty)$in the square-mean sense, where ω is an integer.

Remark 6.1. *To the best of our knowlege, this is the first time that the square-mean S-asymptotically ω-periodic solutions of stochastic systems with piecewise constant arguments have been discussed. In the work [18], the authors obtained sufficient conditions for the existence and the uniqueness of the S-asymptotically ω-periodic solution in distribution for a class of stochastic fractional functional differential equations in an abstract space. However, the stochastic system with piecewise constant arguments is a special stochastic differential delay system, we only study the square-mean S-asymptotically ω-periodic solution in this work. Actually, the theoretical frame work of this paper is able to branch out to other types stochastic systems.*

7. Conclusions

In this paper, we first established the existence and the uniqueness of the mild solution of two kinds of stochastic differential equations driven by Lévy noise with piecewise constant arguments in a Banach space. Then, we give sufficient conditions for the existence and the stability of the square-mean S-asymptotically ω-periodic solution of system (1.1) and system (1.2), where ω is an integer. It would be of great interest to extend these results to the case when the systems with other different noises and we will report our research in future work.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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