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*Research article*

# Positive periodic solutions for discrete Nicholson system with multiple time-varying delays

**Xinning Niu, Huixin Liu, Dan Li and Yan Yan***

Department of Mathematics, Northeast Forestry University, Harbin 150040, China

* **Correspondence:** Email: yyanmath@163.com; Tel: +8613244580259.

**Abstract:** Fly communities exhibit rich ecological dynamics, and one of the important influencing factors is the interaction between species. A discrete Nicholson-type system with multiple time varying delays which considers the mutualism relationship between two fly species is investigated in this paper. Sufficient conditions for the existence of positive periodic solutions are elucidated. The result is obtained by the well-known continuation theorem of coincidence degree theory. An example is attached to illustrate our result. Moreover, the actual biological descriptions obtained from our main result are explained.

**Keywords:** discrete Nicholson system; mutualism relationship; positive periodic solution; time-varying delay; continuation theorem of coincidence degree theory

## 1. Introduction

Flies are complete metamorphosis insects that contain various species, including Muscidae (houseflies), Calliphoridae (blowflflies) Drosophilae (fruitflies) and Scrcophagidae (fleshflies), etc. The life history of flies can be divided into egg, larva , pre-pupa, pupa and adult stages. Although the life span of flies is only about one month, they are very fertile and multiply rapidly in a short period [1]. The feeding habits of flies are very complex. They can feed on a variety of substances, such as human food, animal waste, kitchen scraps and other refuses. It is known to us that flies transmit various pathogens from filth to humans and cause many diseases [2–4]. On the other hand, flies are also beneficial to medical research, ecosystem food chain and pollen dispersal. Considering medical research, for example, fruit fly Drosophila is of great significance in studying the pathogenesis and therapy of human diseases. The nervous system of Drosophila is much simpler than that of human beings, but it also exhibits complex behavioral characteristics similar to humans [5, 6]. Therefore, studying fly population dynamics is of crucial importance to both nature and human society.

The study of biological population growth model promotes the development of human society to a great extent. It has important applications in population control, social resource allocation, ecological environment improvement, species protection and human life and health [7–9]. To understand the population dynamics of the Australian sheep blowfly, Gurney et al. [10] constructed the autonomous delay differential equation

$$x'(t) = -\delta x(t) + Px(t-\tau)e^{-\gamma x(t-\tau)}$$

based on experimental data [11, 12]. In this model, $x$ is the density of mature blowflies, $\delta$ is the daily mortality rate of adult blowflies, $P$ is the maximum daily spawning rate of female blowflies, $\tau$ is the time required for a blowfly to mature from an egg to an adult, $1/\gamma$ is the blowfly population size at which the production function $f(u) = ue^{-\gamma u}$ reaches the maximum value. Subsequently, this model and its modified extensions were continually used to describe rich fly dynamics.

Environmental changes play an important role in biological systems. The influence of a periodically changing environment on the system is different from that of a constant environment, and it can better facilitate system evolution. Moreover, delay is one of the important factors which can change the dynamical properties and result in more rich and complex dynamics in biological systems [13, 14]. Many researchers have assumed periodic coefficients and time delays in the system to combine with the periodic changes of the environment [15–18]. For related literature, we refer to [19, 20]. However, considering the fact that adult flies number is a discrete value that varies daily and the situations where population numbers are small and individual effects are important or dominate, a discrete model would indeed be more realistic to describe the population evolution in discrete time-steps [21–23].

Interactions between different species are extremely important for maintaining ecological balance. Such interactions are typically direct or indirect between multiple species, including positive interactions and negative interactions. Among them, the positive interactions can be divided into three categories according to the degree of action: commensalism, protocooperation and mutualism [24, 25]. In the paper [9], a delay differential Nicholson-type system concerning the mutualism effects with constant coefficients was proposed. The existence, global stability and instability of positive equilibrium were obtained. Based on this system, Zhou [26] and Amster [27] considered periodic Nicholson-type system combined with nonlinear harvesting terms. The main research theme is the existence of positive periodic solutions. Recently, Ossandóna et al. [28] presented a Nicholson-type system with nonlinear density-dependent mortality to describe the dynamics of multiple species, the uniqueness and local exponential stability of the periodic solution are established. However, relatively few studies on discrete dynamical systems have explored the mutualism of flies. In this paper, we consider the mutualism relationship between two fly species and establish a two-dimensional discrete Nicholson system with multiple time-varying delays

$$\begin{cases} \Delta x_1(k) = -a_1(k)x_1(k) + b_1(k)x_2(k) + \sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))}, \\ \Delta x_2(k) = -a_2(k)x_2(k) + b_2(k)x_1(k) + \sum_{j=1}^{n} c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))}. \end{cases} \tag{1.1}$$

We assume that $a_i : \mathbb{Z} \to (0, 1)$, $b_i : \mathbb{Z} \to (0, \infty)$, $c_{ij} : \mathbb{Z} \to (0, \infty)$, $\tau_{ij} : \mathbb{Z} \to \mathbb{Z}^+$ and $\gamma_{ij} : \mathbb{Z} \to (0, \infty)$ are $\omega$-periodic discrete functions for $1 \le i \le 2$ and $1 \le j \le n$. The period $\omega$ is a positive integer. Moreover, the interaction rate of second fly specie on first fly species and that of first fly specie on second fly species are represented by $b_1$ and $b_2$, respectively.

Because $\tau_{ij}$ $(1 \le i \le 2)$ have $\omega$-periodicity, we can find the maximum values

$$\overline{\tau}_i = \max_{1\le j\le n}\left\{\max_{1\le k\le\omega}\tau_{ij}(k)\right\} \in \mathbb{Z}^+$$

of $\{\tau_{i1}(k)\}, \{\tau_{i2}(k)\}, \ldots, \{\tau_{in}(k)\}$ for $i = 1, 2$. Note that $0 < a_i(k) < 1$ for $k \in \mathbb{Z}$. Then, the solution $x(\cdot, \phi) = (x_1(\cdot, \phi_1), x_2(\cdot, \phi_2))^T$ of system (1.1) that satisfies the initial condition

$$x_i(s) = \phi_i(s) > 0 \quad \text{for} \ \ s \in [-\overline{\tau}_i, 0] \cap \mathbb{Z} \tag{1.2}$$

is a positive solution. The purpose of this paper is to present sufficient conditions for the existence of positive $\omega$-periodic solution of (1.1).

## 2. Priori bounds for parametric system and auxiliary lemma

We discuss the parametric delay difference system

$$\begin{cases} \Delta x_1(k) = -\lambda a_1(k)x_1(k) + \lambda b_1(k)x_2(k) + \lambda\sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))}, \\ \Delta x_2(k) = -\lambda a_2(k)x_2(k) + \lambda b_2(k)x_1(k) + \lambda\sum_{j=1}^{n} c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))} \end{cases} \tag{2.1}$$

for each parameter $\lambda \in (0, 1)$. Let $\underline{a}_i = \min_{1\le k\le\omega} a_i(k)$ and $\overline{b}_i = \max_{1\le k\le\omega} b_i(k)$ for $i = 1, 2$. Then, an estimation of upper and lower bounds of positive $\omega$-periodic solution of (2.1) can be conducted.

**Proposition 2.1.** *Suppose that*

$$\underline{a}_1\underline{a}_2 - \overline{b}_1\overline{b}_2 > 0 \tag{2.2}$$

*and there exists a constant* $\gamma > 1$ *such that*

$$\sum_{j=1}^{n} c_{ij}(k) > \gamma\, a_i(k) \quad \textit{for} \ \ k = 1, 2, \ldots, \omega \ \ \textit{and} \ \ 1 \le i \le 2. \tag{2.3}$$

*Then, every positive $\omega$-periodic solution $x = (x_1, x_2)^T$ of (2.1) is bounded. Specifically,*

$$A_1 < x_1(k) \le B_1 \quad \textit{and} \quad A_2 < x_2(k) \le B_2 \quad \textit{for} \ \ k = 1, 2, \ldots, \omega,$$

*where*

$$A_1 \le \min\left\{\frac{\ln\gamma}{\overline{\gamma}_1},\ \gamma B_1 e^{-\overline{\gamma}_1 B_1}\right\} \quad \textit{and} \quad B_1 = \frac{\underline{a}_2}{(\underline{a}_1\underline{a}_2 - \overline{b}_1\overline{b}_2)e}\left(\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}} + \frac{\overline{b}_1}{\underline{a}_2}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}}\right),$$

$$A_2 \le \min\left\{\frac{\ln\gamma}{\overline{\gamma}_2},\ \gamma B_2 e^{-\overline{\gamma}_2 B_2}\right\} \quad \textit{and} \quad B_2 = \frac{\underline{a}_1}{(\underline{a}_1\underline{a}_2 - \overline{b}_1\overline{b}_2)e}\left(\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}} + \frac{\overline{b}_2}{\underline{a}_1}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}\right),$$

*in which* $\underline{\gamma}_{1j} = \min_{1\le k\le\omega}\gamma_{1j}(k)$, $\underline{\gamma}_{2j} = \min_{1\le k\le\omega}\gamma_{2j}(k)$, $\overline{c}_{1j} = \max_{1\le k\le\omega} c_{1j}(k)$, $\overline{c}_{2j} = \max_{1\le k\le\omega} c_{2j}(k)$, $\overline{\gamma}_1 = \max_{1\le j\le n}\{\max_{1\le k\le\omega}\gamma_{1j}(k)\}$ *and* $\overline{\gamma}_2 = \max_{1\le j\le n}\{\max_{1\le k\le\omega}\gamma_{2j}(k)\}$.

*Remark* 1. Note that $A_i$ and $B_i$ are the lower bound and upper bound of $x_i$, respectively. We can verify the fact that $A_i < B_i$ for $i = 1, 2$. From the definitions of $A_1$ and $A_2$, we see that

$$A_1 \le \gamma B_1 e^{-\overline{\gamma}_1 B_1} \le \frac{\gamma}{e\overline{\gamma}_1} \quad and \quad A_2 \le \gamma B_2 e^{-\overline{\gamma}_2 B_2} \le \frac{\gamma}{e\overline{\gamma}_2}.$$

Hence, we obtain

$$\begin{aligned} B_1 &> \frac{\underline{a}_2}{(\underline{a}_1\underline{a}_2 - \overline{b}_1\overline{b}_2)e} \sum_{j=1}^{n} \frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}} = 1 \Bigg/ \left(1 - \frac{\overline{b}_1\overline{b}_2}{\underline{a}_1\underline{a}_2}\right) \times \frac{1}{\underline{a}_1 e} \sum_{j=1}^{n} \frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}} \\ &> \frac{\sum_{j=1}^{n} \overline{c}_{1j}}{\underline{a}_1} \frac{1}{e\overline{\gamma}_1} > \frac{\gamma}{e\overline{\gamma}_1} \ge A_1. \end{aligned}$$

Similarly, it follows that

$$B_2 > \frac{\underline{a}_1}{(\underline{a}_1\underline{a}_2 - \overline{b}_1\overline{b}_2)e} \sum_{j=1}^{n} \frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}} > \frac{\gamma}{e\overline{\gamma}_2} \ge A_2.$$

*Proof.* Let $x = (x_1, x_2)^T$ be arbitrary positive $\omega$-periodic solution of (2.1) under the initial condition (1.2). For $i = 1, 2$, we define

$$\overline{x}_i = \max_{1\le k\le\omega} x_i(k) \quad \text{and} \quad \underline{x}_i = \min_{1\le k\le\omega} x_i(k).$$

Then $\underline{x}_i \le x_i(k) \le \overline{x}_i$ for $k \in \mathbb{Z}^+$. We can rewrite system (2.1) into

$$\begin{cases} x_1(k+1) = (1-\lambda a_1(k))x_1(k) + \lambda b_1(k)x_2(k) + \lambda \sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))}, \\ x_2(k+1) = (1-\lambda a_2(k))x_2(k) + \lambda b_2(k)x_1(k) + \lambda \sum_{j=1}^{n} c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))}. \end{cases} \tag{2.4}$$

Taking the maximum on both sides of the first equation of (2.4) in one period, we have

$$\begin{aligned} \overline{x}_1 &= \max_{1\le k\le\omega} \{x_1(k+1)\} \\ &\le \max_{1\le k\le\omega} \{(1-\lambda a_1(k))x_1(k)\} + \lambda \max_{1\le k\le\omega} \{b_1(k)x_2(k)\} \\ &\quad + \lambda \max_{1\le k\le\omega} \left\{ \sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))} \right\} \\ &\le \max_{1\le k\le\omega} \{(1-\lambda a_1(k))\} \max_{1\le k\le\omega} \{x_1(k)\} + \lambda \max_{1\le k\le\omega} \{b_1(k)\} \max_{1\le k\le\omega} \{x_2(k)\} \\ &\quad + \lambda \max_{1\le k\le\omega} \left\{ \sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))} \right\} \\ &\le (1-\lambda\underline{a}_1)\overline{x}_1 + \lambda\overline{b}_1\overline{x}_2 + \lambda \max_{1\le k\le\omega} \left\{ \sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))} \right\}. \end{aligned}$$

Similarly, we obtain

$$\overline{x}_2 \le (1-\lambda\underline{a}_2)\overline{x}_2 + \lambda\overline{b}_2\overline{x}_1 + \lambda \max_{1\le k\le\omega} \left\{ \sum_{j=1}^{n} c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))} \right\}.$$

Hence, it leads to

$$\begin{aligned}\overline{x}_1 &\leq \frac{\overline{b}_1}{\underline{a}_1}\overline{x}_2 + \frac{1}{\underline{a}_1}\max_{1\leq k\leq \omega}\left\{\sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))}\right\} \\ &\leq \frac{\overline{b}_1}{\underline{a}_1}\overline{x}_2 + \frac{1}{\underline{a}_1 e}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\overline{x}_2 &\leq \frac{\overline{b}_2}{\underline{a}_2}\overline{x}_1 + \frac{1}{\underline{a}_2}\max_{1\leq k\leq \omega}\left\{\sum_{j=1}^{n} c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))}\right\} \\ &\leq \frac{\overline{b}_1}{\underline{a}_2}\overline{x}_1 + \frac{1}{\underline{a}_2 e}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}}. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6), basic computations show that

$$\overline{x}_1 \leq 1\Bigg/\left(1-\frac{\overline{b}_1\overline{b}_2}{\underline{a}_1\underline{a}_2}\right)\times\left(\frac{1}{\underline{a}_1 e}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{r}_{1j}} + \frac{\overline{b}_1}{\underline{a}_1\underline{a}_2 e}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}}\right) = \frac{\underline{a}_2}{(\underline{a}_1\underline{a}_2-\overline{b}_1\overline{b}_2)e}\left(\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}} + \frac{\overline{b}_1}{\underline{a}_2}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}}\right) = B_1,$$

$$\overline{x}_2 \leq 1\Bigg/\left(1-\frac{\overline{b}_1\overline{b}_2}{\underline{a}_1\underline{a}_2}\right)\times\left(\frac{1}{\underline{a}_2 e}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{r}_{2j}} + \frac{\overline{b}_2}{\underline{a}_1\underline{a}_2 e}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}\right) = \frac{\underline{a}_1}{(\underline{a}_1\underline{a}_2-\overline{b}_1\overline{b}_2)e}\left(\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}} + \frac{\overline{b}_2}{\underline{a}_1}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}\right) = B_2.$$

Note that $1-\lambda a_i(k) > 0$ for all $k \in \mathbb{Z}$ and $i = 1, 2$. Multiplying both sides of the two equation of (2.1) by $\prod_{r=0}^{k} 1/(1-\lambda a_1(r))$ and $\prod_{r=0}^{k} 1/(1-\lambda a_2(r))$ respectively, we have

$$\begin{aligned} x_1(k+1)\prod_{r=0}^{k}\frac{1}{1-\lambda a_1(r)} - x_1(k)\prod_{r=0}^{k-1}\frac{1}{1-\lambda a_1(r)} - \lambda b_1(k)x_2(k)\prod_{r=0}^{k}\frac{1}{1-\lambda a_1(r)} \\ = \lambda\sum_{j=1}^{n} c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))}\prod_{r=0}^{k}\frac{1}{1-\lambda a_1(r)}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} x_2(k+1)\prod_{r=0}^{k}\frac{1}{1-\lambda a_2(r)} - x_2(k)\prod_{r=0}^{k-1}\frac{1}{1-\lambda a_2(r)} - \lambda b_2(k)x_1(k)\prod_{r=0}^{k}\frac{1}{1-\lambda a_2(r)} \\ = \lambda\sum_{j=1}^{n} c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))}\prod_{r=0}^{k}\frac{1}{1-\lambda a_2(r)}. \end{aligned} \tag{2.8}$$

Choosing natural numbers $k_1$ and $k_2$ such that

$$\overline{\tau}_1 \leq k_1 \leq \overline{\tau}_1 + \omega - 1 \quad \text{and} \quad x_1(k_1) = \underline{x}_1,$$

$$\overline{\tau}_2 \leq k_2 \leq \overline{\tau}_2 + \omega - 1 \quad \text{and} \quad x_2(k_2) = \underline{x}_2.$$

Summing both sides of (2.7) and (2.8) over $k$ ranging from $k_1$ to $k_1 + \omega - 1$ and $k_2$ to $k_2 + \omega - 1$ respectively, by using $x_i(k_i + \omega) = x_i(k_i) = \underline{x}_i$, we obtain

$$\underline{x}_1 \prod_{r=0}^{k_1-1} \frac{1}{1-\lambda a_1(r)} \left( \prod_{r=k_1}^{k_1+\omega-1} \frac{1}{1-\lambda a_1(r)} - 1 \right) = \lambda \sum_{s=k_1}^{k_1+\omega-1} \left( \left( b_1(s)x_2(s) + \sum_{j=1}^{n} c_{1j}(s)x_1(s-\tau_{1j}(s))e^{-\gamma_{1j}(s)x_1(s-\tau_{1j}(s))} \right) \prod_{r=0}^{s} \frac{1}{1-\lambda a_1(r)} \right),$$

and

$$\underline{x}_2 \prod_{r=0}^{k_2-1} \frac{1}{1-\lambda a_2(r)} \left( \prod_{r=k_2}^{k_2+\omega-1} \frac{1}{1-\lambda a_2(r)} - 1 \right) = \lambda \sum_{s=k_2}^{k_2+\omega-1} \left( \left( b_2(s)x_1(s) + \sum_{j=1}^{n} c_{2j}(s)x_2(s-\tau_{2j}(s))e^{-\gamma_{2j}(s)x_2(s-\tau_{2j}(s))} \right) \prod_{r=0}^{s} \frac{1}{1-\lambda a_2(r)} \right).$$

Note that $a_i$ $(i = 1, 2)$ is positive $\omega$-periodic. It follws that

$$\prod_{r=k_i}^{k_i+\omega-1} (1-\lambda a_i(r)) = \prod_{r=0}^{\omega-1} (1-\lambda a_i(r)). \tag{2.9}$$

Hence, we obtain

$$\begin{aligned}\underline{x}_1 &= \frac{\lambda \prod_{r=0}^{k_1+\omega-1}(1-\lambda a_1(r))}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \left( \sum_{s=k_1}^{k_1+\omega-1} \left( b_1(s)x_2(s) + \sum_{j=1}^{n} c_{1j}(s)x_1(s-\tau_{1j}(s))e^{-\gamma_{1j}(s)x_1(s-\tau_{1j}(s))} \right) \prod_{r=0}^{s} \frac{1}{1-\lambda a_1(r)} \right) \\ &= \frac{\lambda}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \sum_{s=k_1}^{k_1+\omega-1} \left( \left( b_1(s)x_2(s) + \sum_{j=1}^{n} c_{1j}(s)x_1(s-\tau_{1j}(s))e^{-\gamma_{1j}(s)x_1(s-\tau_{1j}(s))} \right) \prod_{r=s+1}^{k_1+\omega-1} (1-\lambda a_1(r)) \right),\end{aligned} \tag{2.10}$$

and

$$\underline{x}_2 = \frac{\lambda}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_2(r))} \sum_{s=k_2}^{k_2+\omega-1} \left( \left( b_2(s)x_1(s) + \sum_{j=1}^{n} c_{2j}(s)x_1(s-\tau_{2j}(s))e^{-\gamma_{2j}(s)x_1(s-\tau_{2j}(s))} \right) \prod_{r=s+1}^{k_1+\omega-1} (1-\lambda a_2(r)) \right). \tag{2.11}$$

Recall that $\overline{\gamma}_i = \max_{1\le j\le n}\{\max_{1\le k\le \omega-1} \gamma_{ij}(k)\}$ for $i = 1, 2$. We define $f_1(u) = ue^{-\overline{\gamma}_1 u}$ and $f_2(u) = ue^{-\overline{\gamma}_2 u}$ for $u \ge 0$. Since $\underline{x}_i \le x_i(k) \le \overline{x}_i$ for all $k \in \mathbb{Z}^+$, it turns out that

$$x_i(s-\tau_{ij}(s))e^{-\gamma_{ij}(s)x_i(s-\tau_{ij}(s))} \ge \min\left\{ f_i(\underline{x}_i),\ f_i(\overline{x}_i) \right\} \quad \text{for} \quad s \ge \overline{\tau}_{ij} \quad \text{for} \quad i = 1, 2.$$

Note that $k_1 \geq \overline{\tau}_1$. By using (2.3) and (2.10), we have

$$\begin{aligned}
\underline{x}_1 &\geq \frac{\lambda \min\left\{f_1(\underline{x}_1),\ f_1(\overline{x}_1)\right\}}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \sum_{s=k_1}^{k_1+\omega-1} \left( \sum_{j=1}^{n} c_{1j}(s) \prod_{r=s+1}^{k_1+\omega-1} (1-\lambda a_1(r)) \right) \\
&> \frac{\lambda \min\left\{f_1(\underline{x}_1),\ f_1(\overline{x}_1)\right\}}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \sum_{s=k_1}^{k_1+\omega-1} \left( \gamma a_1(s) \prod_{r=s+1}^{k_1+\omega-1} (1-\lambda a_1(r)) \right) \\
&= \frac{\gamma \min\left\{f_1(\underline{x}_1),\ f_1(\overline{x}_1)\right\}}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \sum_{s=k_1}^{k_1+\omega-1} \left( \lambda a_1(s) \prod_{r=s+1}^{k_1+\omega-1} (1-\lambda a_1(r)) \right) \\
&= \frac{\gamma \min\left\{f_1(\underline{x}_1),\ f_1(\overline{x}_1)\right\}}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \sum_{s=k_1}^{k_1+\omega-1} \left( \Big(1-(1-\lambda a_1(s))\Big) \prod_{r=s+1}^{k_1+\omega-1} (1-\lambda a_1(r)) \right) \\
&= \frac{\gamma \min\left\{f_1(\underline{x}_1),\ f_1(\overline{x}_1)\right\}}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \sum_{s=k_1}^{k_1+\omega-1} \left( \prod_{r=s+1}^{k_1+\omega-1} (1-\lambda a_1(r)) - \prod_{r=s}^{k_1+\omega-1} (1-\lambda a_1(r)) \right) \\
&= \frac{\gamma \min\left\{f_1(\underline{x}_1),\ f_1(\overline{x}_1)\right\}}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_1(r))} \left( \prod_{r=k_1+\omega}^{k_1+\omega-1} (1-\lambda a_1(r)) - \prod_{r=k_1}^{k_1+\omega-1} (1-\lambda a_1(r)) \right).
\end{aligned}$$

Calculating by the same way, from (2.3) and (2.11), we obtain

$$\underline{x}_2 = \frac{\gamma \min\left\{f_2(\underline{x}_2),\ f_2(\overline{x}_2)\right\}}{1-\prod_{r=0}^{\omega-1}(1-\lambda a_2(r))} \left( \prod_{r=k_2+\omega}^{k_2+\omega-1} (1-\lambda a_2(r)) - \prod_{r=k_2}^{k_2+\omega-1} (1-\lambda a_2(r)) \right).$$

Then, it follows from (2.9) that

$$\underline{x}_i > \gamma \min\left\{f_i(\underline{x}_i),\ f_i(\overline{x}_i)\right\} \quad \text{for} \quad i = 1, 2. \tag{2.12}$$

It is natural to divide the argument into two cases: (i) $f_i(\underline{x}_i) \leq f_i(\overline{x}_i)$; (ii) $f_i(\underline{x}_i) > f_i(\overline{x}_i)$.

Case (i): It follows from (2.12) that $\underline{x}_i > \gamma f_i(\underline{x}_i)$. Specifically, we have

$$\underline{x}_1 > \gamma f_1(\underline{x}_1) = \frac{\gamma \underline{x}_1}{e^{\overline{\gamma}_1} \underline{x}_1} \quad \text{and} \quad \underline{x}_2 > \gamma f_2(\underline{x}_2) = \frac{\gamma \underline{x}_2}{e^{\overline{\gamma}_2} \underline{x}_2},$$

which imply that $\underline{x}_1 > \ln \gamma / \overline{\gamma}_1$ and $\underline{x}_2 > \ln \gamma / \overline{\gamma}_2$.

Case (ii): Function $f_i$ is unimodal and takes the only peak value at $1/\overline{\gamma}_i$. Also, $f_i$ monotonically increases on $[0, 1/\overline{\gamma}_i]$ and monotonically decreases on $[1/\overline{\gamma}_i, \infty)$. If $\overline{x}_i \leq 1/1/\overline{\gamma}_i$, then we see that $f_i(\underline{x}_i) \leq f_i(\overline{x}_i) \leq f_i(1/\overline{\gamma}_i)$, which is a contradiction. Hence, it follows that $\overline{x}_i > 1/\overline{\gamma}_i$. Note that $\overline{x}_i \leq B_i$. From (2.12), we obtain

$$\underline{x}_1 > \gamma f_1(\overline{x}_1) \geq \gamma f_1(B_1) = \gamma B_1 e^{-\overline{\gamma}_1 B_1}$$

and

$$\underline{x}_2 > \gamma f_2(\overline{x}_2) \geq \gamma f_2(B_2) = \gamma B_2 e^{-\overline{\gamma}_2 B_2}.$$

Thus, we estimate

$$\underline{x}_1 > \min\left\{\frac{\ln \gamma}{\overline{\gamma}_1},\ \gamma B_1 e_1^{-\overline{\gamma}_1 B_1}\right\} \geq A_1$$

and

$$\underline{x}_2 > \min\left\{\frac{\ln\gamma}{\overline{\gamma}_2},\ \gamma B_2 e_2^{-\overline{\gamma}_2 B_2}\right\} \geq A_2.$$

Now, it can be concluded that each positive $\omega$-periodic solution $x = (x_1, x_2)^T$ of (2.1) satisfies

$$A_1 < \underline{x}_1 \leq x_1(k) \leq \overline{x}_1 \leq B_1$$

and

$$A_2 < \underline{x}_2 \leq x_2(k) \leq \overline{x}_1 \leq B_2$$

for $k \in \mathbb{Z}^+$. The proof is complete.

Suppose that $X$ is a Banach space and $L\colon \operatorname{Dom} L \subset X \to X$ is a linear operator. The operator $L$ is called a Fredholm operator of index zero if

(i) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$,

(ii) $\operatorname{Im} L$ is closed in $X$.

If $L$ is a Fredholm operator of index zero and $P$, $Q\colon X \to X$ are continuous projectors satisfying

$$\operatorname{Im} P = \operatorname{Ker} L \quad \text{and} \quad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q),$$

where $I$ is the identity operator from $X$ to $X$, then the restriction $L_P\colon \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible and has the inverse $K_P\colon \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$.

Let $N\colon X \to X$ be a continuous operator and $\Omega$ an open bounded subset of $X$. The operator $N$ is $L$-compact on $\overline{\Omega}$ if

(i) $QN(\overline{\Omega})$ is bounded,

(ii) $K_P(I - Q)N\colon \overline{\Omega} \to X$ is compact.

We present the continuation theorem of coincidence degree theory (for example, see [29, 30]) as follows:

**Lemma 2.2.** *Let $L\colon \operatorname{Dom} L \subset X \to X$ be a Fredholm operator of index zero and let $N\colon X \to X$ be $L$-compact on $\overline{\Omega}$. Suppose that*

(i) *every solution $x$ of $Lx = \lambda N x$ satisfies $x \notin \partial\Omega$ for $\lambda \in (0, 1)$;*

(ii) *$QNx \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$ and*

$$\deg\{QN,\ \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0.$$

*Then, $Lx = Nx$ has at least one solution in $X \cap \overline{\Omega}$.*

## 3. Positive $\omega$-periodic solution

**Theorem 3.1.** *Suppose that* (2.2) *and* (2.3) *hold. If*

$$\frac{\sum_{k=1}^{\omega}\sum_{j=1}^{n}(c_{ij}(k)}{\sum_{k=1}^{\omega}(a_i(k)-b_i(k))} > 1 \quad \textit{for } i=1,2, \tag{3.1}$$

*then system* (1.1) *has at least one positive ω-periodic solution* $x^*$.

*Proof.* Let $X$ be a set of $\omega$-periodic functions $x=(x_1,x_2)^T$ defined on $\mathbb{Z}^+$ and denote the maximum norm $\|x\| = \max\{\max_{1\le k\le\omega}|x_1(k)|, \max_{1\le k\le\omega}|x_2(k)|\}$ for any $x\in X$. Then, $X$ is a Banach space. Moreover, we define

$$Lx = \begin{pmatrix}(Lx)_1(k)\\(Lx)_2(k)\end{pmatrix} = \begin{pmatrix}x_1(k+1)-x_1(k)\\x_2(k+1)-x_2(k)\end{pmatrix},$$

and

$$Nx = \begin{pmatrix}(Nx)_1(k)\\(Nx)_2(k)\end{pmatrix} = \begin{pmatrix}-a_1(k)x_1(k)+b_1(k)x_2(k)+\sum_{j=1}^{n}c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))}\\-a_2(k)x_2(k)+b_2(k)x_1(k)+\sum_{j=1}^{n}c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))}\end{pmatrix}.$$

It is not difficult to show that $L$ is a linear operator from $X$ to $X$ and $N$ is a continuous operator from $X$ to $X$.

From the definition of $L$, we see that

$$\operatorname{Ker} L = \{x\in X\colon (x_1(k),x_2(k))^T \equiv (c_1,c_2)^T \in \mathbb{R}^2\}$$

and

$$\operatorname{Im} L = \left\{x\in X\colon \sum_{k=1}^{\omega}x_1(k) = \sum_{k=1}^{\omega}x_2(k) = 0\right\}.$$

It turns out that $\dim\operatorname{Ker} L = 2 = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $X$. Thus, $L$ is a Fredholm operator of index zero.

We define $P\colon X\to X$ by

$$Px = \begin{pmatrix}(Px)_1\\(Px)_2\end{pmatrix} = \begin{pmatrix}\dfrac{1}{\omega}\displaystyle\sum_{k=1}^{\omega}x_1(k)\\[2ex]\dfrac{1}{\omega}\displaystyle\sum_{k=1}^{\omega}x_2(k)\end{pmatrix}$$

and let $Q=P$. Then, $P$ and $Q$ are two continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q)$.

It can be shown that the restriction $L_P\colon \operatorname{Dom} L\cap\operatorname{Ker} P \to \operatorname{Im} L$ has the inverse $K_P\colon \operatorname{Im} L\to \operatorname{Dom} L\cap \operatorname{Ker} P$ given by

$$K_Px = \begin{pmatrix}(K_Px)_1\\(K_Px)_2\end{pmatrix} = \begin{pmatrix}\displaystyle\sum_{s=0}^{k-1}x_1(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_1(r)\\[2ex]\displaystyle\sum_{s=0}^{k-1}x_2(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_2(r)\end{pmatrix}$$

for $x=(x_1,x_2)^T\in\operatorname{Im}L$. In fact, for $i=1,2$, since

$$
\begin{aligned}
(K_Px)_i(k+\omega)-(K_Px)_i(k) &= \sum_{s=0}^{k+\omega-1}x_i(s)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_i(r)-\sum_{s=0}^{k-1}x_i(s)+\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_i(r)\\
&= \sum_{s=k}^{k+\omega-1}x_i(s)=\sum_{s=0}^{\omega-1}x_i(s)=0
\end{aligned}
$$

for all $k\in\mathbb{Z}^+$, we see that $K_Px\in\operatorname{Dom}L$ . Moreover, it follows that

$$
\begin{aligned}
(PK_Px)_i &= \frac{1}{\omega}\sum_{k=1}^{\omega}K_Px_i(k)=\frac{1}{\omega}\sum_{k=1}^{\omega}\left(\sum_{s=0}^{k-1}x_i(s)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_i(r)\right)\\
&= \frac{1}{\omega}\left(\sum_{k=1}^{\omega}\sum_{s=0}^{k-1}x_i(s)-\frac{\omega}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_i(r)\right)=\frac{1}{\omega}\left(\sum_{k=1}^{\omega}\sum_{s=0}^{k-1}x_i(s)-\sum_{k=1}^{\omega}\sum_{r=0}^{k-1}x_i(r)\right)=0.
\end{aligned}
$$

Hence, $K_Px\in\operatorname{Ker}P$.

For any $x\in\operatorname{Im}L$, one has

$$
\begin{aligned}
(L_PK_Px)_i &= (K_Px)_i(k+1)-(K_Px)_i(k)\\
&= \sum_{s=0}^{k}x_i(s)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_i(r)-\sum_{s=0}^{k-1}x_i(s)+\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}x_i(r)\\
&= x_i(k)=(Ix)_i.
\end{aligned}
$$

Furthermore, for any $x\in\operatorname{Dom}L\cap\operatorname{Ker}P$, one has

$$
\begin{aligned}
(K_PL_Px)_i &= K_P\big(x_i(k+1)-x_i(k)\big)\\
&= \sum_{s=0}^{k-1}\big(x_i(s+1)-x_i(s)\big)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}\big(x_i(r+1)-x_i(r)\big)\\
&= x_i(k)-x_i(0)-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\big(x_i(s+1)-x_i(0)\big)=x_i(k)-\frac{1}{\omega}\sum_{s=1}^{\omega}x_i(s).
\end{aligned}
$$

Since $x\in\operatorname{Ker}P=\operatorname{Ker}Q=\operatorname{Im}L$, we see that $\sum_{s=1}^{\omega}x_i(s)=0$. Hence, $(K_PL_Px)_i=x_i(k)=(Ix)_i$. We therefore conclude that $K_P=L_P^{-1}$.

We define

$$
\Omega=\left\{x=(x_1,x_2)^T\in X:A_1<x_1(k)<B_1+1,\ \ A_2<x_2(k)<B_2+1\right\}
$$

and prove that the operator $N$ defined above is $L$-compact on $\overline{\Omega}$. We first check that $QN(\overline{\Omega})$ is bounded.

Since $x_1(k)<B_1+1$ and $x_2(k)<B_2+1$ for $k\in\mathbb{Z}^+$, we obtain

$$
(QNx)_1=\frac{1}{\omega}\sum_{k=1}^{\omega}\left(-a_1(k)x_1(k)+b_1(k)x_2(k)+\sum_{j=1}^{n}c_{1j}(k)x_1(k-\tau_{1j}(k))e^{-\gamma_{1j}(k)x_1(k-\tau_{1j}(k))}\right)
$$

$$< \frac{1}{\omega}\sum_{k=1}^{\omega}\left(\overline{b}_1(B_2+1)+\frac{1}{e}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}\right)$$
$$= \left(\overline{b}_1(B_2+1)+\frac{1}{e}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}\right),$$

and

$$\begin{aligned}(QNx)_2 &= \frac{1}{\omega}\sum_{k=1}^{\omega}\left(-a_2(k)x_2(k)+b_2(k)x_1(k)+\sum_{j=1}^{n}c_{2j}(k)x_2(k-\tau_{2j}(k))e^{-\gamma_{2j}(k)x_2(k-\tau_{2j}(k))}\right)\\ &< \frac{1}{\omega}\sum_{k=1}^{\omega}\left(\overline{b}_2(B_1+1)+\frac{1}{e}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}}\right)\\ &= \left(\overline{b}_2(B_1+1)+\frac{1}{e}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}}\right)\end{aligned}$$

for $x \in \overline{\Omega}$. Hence, the operator $QN$ is bounded on $\overline{\Omega}$.

We next show that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. From the definitions of $N$, $QN$ and $K_p$, we obtain

$$\begin{aligned}(K_p(I-Q)Nx)_1 &= \sum_{s=0}^{k-1}(-a_1(s)x_1(s)+b_1(s)x_2(s))\\ &+\sum_{s=0}^{k-1}\left(\sum_{j=1}^{n}c_{1j}(s)x_1(s-\tau_{1j}(s))e^{-\gamma_{1j}(s)x_1(s-\tau_{1j}(s))}\right)\\ &-\left(\frac{k}{\omega}-\frac{\omega+1}{2\omega}\right)\sum_{s=1}^{\omega}(-a_1(s)x_1(s)+b_1(s)x_2(s))\\ &-\left(\frac{k}{\omega}-\frac{\omega+1}{2\omega}\right)\sum_{s=1}^{\omega}\left(\sum_{j=1}^{n}c_{1j}(s)x_1(s-\tau_{1j}(s))e^{-\gamma_{1j}(s)x_1(s-\tau_{1j}(s))}\right)\\ &-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}(-a_1(r)x_1(r)+b_1(r)x_2(r))\\ &-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}\left(\sum_{j=1}^{n}c_{1j}(r)x_1(r-\tau_{1j}(r))e^{-\gamma_{1j}(r)x_1(r-\tau_{1j}(r))}\right).\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}(K_p(I-Q)Nx)_2 &= \sum_{s=0}^{k-1}(-a_2(s)x_2(s)+b_2(s)x_1(s))\\ &+\sum_{s=0}^{k-1}\left(\sum_{j=1}^{n}c_{2j}(s)x_2(s-\tau_{2j}(s))e^{-\gamma_{2j}(s)x_2(s-\tau_{2j}(s))}\right)\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{k}{\omega}-\frac{\omega+1}{2\omega}\right)\sum_{s=1}^{\omega}(-\,a_2(s)x_2(s)+b_2(s)x_1(s))\\
&-\left(\frac{k}{\omega}-\frac{\omega+1}{2\omega}\right)\sum_{s=1}^{\omega}\left(\sum_{j=1}^{n}c_{2j}(s)x_2(s-\tau_{2j}(s))e^{-\gamma_{2j}(s)x_2(s-\tau_{2j}(s))}\right)\\
&-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}(-\,a_2(r)x_2(r)+b_2(r)x_1(r))\\
&-\frac{1}{\omega}\sum_{s=0}^{\omega-1}\sum_{r=0}^{s}\left(\sum_{j=1}^{n}c_{2j}(r)x_2(r-\tau_{2j}(r))e^{-\gamma_{2j}(r)x_2(r-\tau_{2j}(r))}\right)
\end{aligned}
$$

for $x \in X$. For any bounded subset $E \subset \overline{\Omega} \subset X$, it is a subspace of a finite dimensional Banach space $X$. Hence, $E$ is closed, and therefore $E$ is compact. By a straightforward calculation, it can be proven that $K_P(I-Q)N(E)$ is relatively compact.

An arbitrary $\omega$-periodic solution of (2.1) corresponds one-to-one to a solution of $Lx = \lambda Nx$ with parameter $\lambda \in (0,1)$. Proposition 2.1 displays that each positive solution $x = (x_1, x_2)^T$ of $Lx = \lambda Nx$ satisfies that $A_1 < x_1 \le B_1$ and $A_2 < x_2 \le B_2$. It is obvious that if $y = (y_1, y_2)^T \in \partial\Omega$, then $y$ is never a solution of $Lx = \lambda Nx$. Hence, the condition (i) of Lemma 2.2 holds. If $x = (x_1, x_2)^T \in \partial\Omega \cap \operatorname{Ker} L$, then there are four cases to be considered: (1) $x = (A_1, x_2)^T$, (2) $x = (B_1 + 1, x_2)^T$, (3) $x = (x_1, A_2)^T$, (4) $x = (x_1, B_2 + 1)^T$.

Case (1): It follows from $x_1 \equiv A_1$ that

$$
\begin{aligned}
(QNx)_1 &= \frac{1}{\omega}\sum_{k=1}^{\omega}\left(-A_1a_1(k)+b_1(k)x_2(k)+\sum_{j=1}^{n}c_{ij}(k)A_1e^{-\gamma_{1j}(k)A_1}\right)\\
&\ge \frac{A_1}{\omega}\sum_{k=1}^{\omega}\left(-a_1(k)+\frac{1}{e^{A_1\overline{\gamma}_1}}\sum_{j=1}^{n}c_{ij}(k)\right)\\
&> \frac{A_1}{\omega}\sum_{k=1}^{\omega}\left(-a_1(k)+\frac{\gamma}{e^{A_1\overline{\gamma}_1}}a_1(k)\right)\\
&= \frac{A_1}{\omega}\left(\frac{\gamma}{e^{A_1\overline{\gamma}_1}}-1\right)\sum_{k=1}^{\omega}a_1(k).
\end{aligned}
$$

Since $A_1 \le \ln\gamma/\overline{\gamma}_1$, we see that $e^{A_1\overline{\gamma}_1} \le \gamma$. Hence, $(QNx)_1 > 0$.

Case (2): Because of $x_1 \equiv B_1 + 1$, we have

$$
\begin{aligned}
(QNx)_1 &= \frac{1}{\omega}\sum_{k=1}^{\omega}\left(-\,(B_1+1)a_1(k)+b_1(k)x_2(k)+\sum_{j=1}^{n}c_{ij}(k)(B_1+1)e^{-\gamma_{1j}(k)(B_1+1)}\right)\\
&\le \frac{1}{\omega}\sum_{k=1}^{\omega}\left(-\underline{a}_1(B_1+1)+\overline{b}_1B_2+\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{e\underline{\gamma}_{1j}}\right)\\
&= -\underline{a}_1(B_1+1)+\overline{b}_1B_2+\frac{1}{e}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}
\end{aligned}
$$

$$
\begin{aligned}
&= -\underline{a}_1 - \frac{\underline{a}_1\underline{a}_2}{(\underline{a}_1\underline{a}_2 - \overline{b}_1\overline{b}_2)e}\left(\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}} + \frac{\overline{b}_1}{\underline{a}_2}\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}}\right) \\
&\quad + \frac{\underline{a}_1\overline{b}_1}{(\underline{a}_1\underline{a}_2 - \overline{b}_1\overline{b}_2)e}\left(\sum_{j=1}^{n}\frac{\overline{c}_{2j}}{\underline{\gamma}_{2j}} + \frac{\overline{b}_2}{\underline{a}_1}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}}\right) + \frac{1}{e}\sum_{j=1}^{n}\frac{\overline{c}_{1j}}{\underline{\gamma}_{1j}} \\
&= -\underline{a}_1 < 0.
\end{aligned}
$$

Similarly, we can show that $(QNx)_2 > 0$ in Case (3) and $(QNx)_2 < 0$ in Case (4). We therefore conclude that $QNx = ((QNx)_1, (QNx)_2)^T \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$.

Define a continuous operator $H\colon \Omega \cap \operatorname{Ker} L \times [0,1] \to X$ by

$$
H(x,\mu) = \begin{pmatrix} H_1(x,\mu) \\ H_2(x,\mu) \end{pmatrix} = \begin{pmatrix} -\mu\left(Ix_1 - \dfrac{A_1 + B_1}{2}\right) + (1-\mu)(QNx)_1 \\[2ex] -\mu\left(Ix_2 - \dfrac{A_2 + B_2}{2}\right) + (1-\mu)(QNx)_2 \end{pmatrix}.
$$

Recall that the elements of $\partial\Omega \cap \operatorname{Ker} L$ are vectors satisfying $x = (A_1, x_2)^T$, $y = (B_1+1, y_2)^T$, $z = (z_1, A_2)^T$ and $w = (w_1, B_2 + 1)^T$. For $x = (A_1, x_2)^T$, we can check that

$$
H_1(x,\mu) = -\mu\left(A_1 - \frac{A_1 + B_1}{2}\right) + (1-\mu)(QNx)_1 = -\mu\left(\frac{A_1 - B_1}{2}\right) + (1-\mu)(QNx)_1 > 0.
$$

Moreover,

$$
H_1(y,\mu) = -\mu\left(B_1 + 1 - \frac{A_1 + B_1}{2}\right) + (1-\mu)(QNy)_1 = -\mu\left(\frac{A_1 - B_1 + 2}{2}\right) + (1-\mu)(QNy)_1 < 0
$$

for $y = (B_1+1, y_2)^T$. Hence, $H(x,\mu) \neq 0$ and $H(y,\mu) \neq 0$. By similar computations, we have $H(z,\mu) \neq 0$ and $H(w,\mu) \neq 0$. Therefore, we see that $H(x,\mu) \neq 0$ for $(x,\mu) \in \partial\Omega \cap \operatorname{Ker} L \times [0,1]$. Thus, $H$ is a homotopic mapping. Using the homotopy invariance, we have

$$
\deg\{QN,\ \Omega \cap \operatorname{Ker} L,\ 0\} = \deg\left\{\begin{pmatrix} -Ix_1 + \dfrac{A_1 + B_1}{2} \\[2ex] -Ix_2 + \dfrac{A_2 + B_2}{2} \end{pmatrix},\ \Omega \cap \operatorname{Ker} L,\ 0\right\} = 1 \neq 0.
$$

Hence, the condition (ii) of Lemma 2.2 holds. Therefore, the equation $Lx = Nx$ has at least one solution located in $X \cap \overline{\Omega}$. Thus, from Lemma 2.2, we obtain that there is a positive $\omega$-periodic solution of system (1.1). The proof is now complete.

## 4. Existence of positive 4-periodic solution

Consider the delay difference system

$$
\begin{cases}
\Delta x_1(k) = -a_1(k)x_1(k) + b_1(k)x_2(k) + c_{11}(k)x_1(k-1)e^{-\gamma_{11}(k)x_1(k-1)} + c_{12}(k)x_1(k-1)e^{-\gamma_{12}(k)x_1(k-1)}, \\
\Delta x_2(k) = -a_2(k)x_2(k) + b_2(k)x_1(k) + c_{21}(k)x_2(k-4)e^{-\gamma_{21}(k)x_2(k-4)} + c_{22}(k)x_2(k-4)e^{-\gamma_{22}(k)x_2(k-4)}.
\end{cases}
$$

Here, we assume that

$$a_1(k)=\begin{cases}1/2 & \text{if } k=1,\\ 2/5 & \text{if } k=2,\\ 1/4 & \text{if } k=3,\\ 1/5 & \text{if } k=4,\end{cases} \qquad a_2(k)=\begin{cases}3/4 & \text{if } k=1,\\ 3/5 & \text{if } k=2,\\ 1/2 & \text{if } k=3,\\ 5/6 & \text{if } k=4,\end{cases}$$

$$b_1(k)=\begin{cases}1/5 & \text{if } k=1,\\ 1/4 & \text{if } k=2,\\ 1/7 & \text{if } k=3,\\ 1/6 & \text{if } k=4,\end{cases} \qquad b_2(k)=\begin{cases}1/20 & \text{if } k=1,\\ 1/12 & \text{if } k=2,\\ 1/24 & \text{if } k=3,\\ 1/18 & \text{if } k=4,\end{cases}$$

$$c_{11}(k)=\begin{cases}1/2 & \text{if } k=1,\\ 3/4 & \text{if } k=2,\\ 1/3 & \text{if } k=3,\\ 2/3 & \text{if } k=4,\end{cases} \qquad c_{12}(k)=\begin{cases}5/6 & \text{if } k=1,\\ 4/5 & \text{if } k=2,\\ 2/5 & \text{if } k=3,\\ 1/6 & \text{if } k=4,\end{cases}$$

$$c_{21}(k)=\begin{cases}7/8 & \text{if } k=1,\\ 4/5 & \text{if } k=2,\\ 2/3 & \text{if } k=3,\\ 6/7 & \text{if } k=4,\end{cases} \qquad c_{22}(k)=\begin{cases}1/4 & \text{if } k=1,\\ 1/2 & \text{if } k=2,\\ 1/10 & \text{if } k=3,\\ 20/21 & \text{if } k=4,\end{cases}$$

$$\gamma_{11}(k)=\begin{cases}3 & \text{if } k=1,\\ 1 & \text{if } k=2,\\ 1.5 & \text{if } k=3,\\ 2 & \text{if } k=4,\end{cases} \qquad \gamma_{12}(k)=\begin{cases}10 & \text{if } k=1,\\ 4 & \text{if } k=2,\\ 3 & \text{if } k=3,\\ 5 & \text{if } k=4,\end{cases}$$

$$\gamma_{21}(k)=\begin{cases}5 & \text{if } k=1,\\ 2 & \text{if } k=2,\\ 1 & \text{if } k=3,\\ 2.5 & \text{if } k=4,\end{cases} \qquad \gamma_{22}(k)=\begin{cases}2 & \text{if } k=1,\\ 1.5 & \text{if } k=2,\\ 8 & \text{if } k=3,\\ 3 & \text{if } k=4.\end{cases}$$

In addition, $a_i(k)=a_i(k+4)$, $b_i(k)=b_i(k+4)$, $c_{ij}(k)=c_{ij}(k+4)$ and $\gamma_{ij}(k)=\gamma_{ij}(k+4)$ for $k\in\mathbb{Z}$, $i=1,2$ and $j=1,2$. Theorem 3.1 shows that the system has at least one positive 4-periodic solution.

It is clear that $\omega=4$, $a_i$, $b_i$, $c_{ij}$, $\gamma_{ij}$ and $\tau_{ij}$ $(1\le i\le 2,\ 1\le j\le 2)$ are $\omega$-periodic discrete functions satisfying $0<a_i(k)<1$, $0<b_i(k)<1$, $c_{ij}(k)>0$ and $\gamma_{ij}(k)>0$ for $k\in\mathbb{Z}^+$. Since $\underline{a}_1=1/5$, $\underline{a}_1=1/2$, $\overline{b}_1=1/4$ and $\overline{b}_2=1/12$, we see that

$$\underline{a}_1\underline{a}_1-\overline{b}_1\overline{b}_2=\frac{1}{5}\times\frac{1}{2}-\frac{1}{4}\times\frac{1}{12}=\frac{19}{240}>0.$$

Hence, condition (2.2) is satisfied. Let $\gamma=11/10>1$. Then, we can easily check condition (2.3)

$$(c_{11}(k)+c_{12}(k))>\gamma a_1(k)\quad\text{and}\quad(c_{21}(k)+c_{22}(k))>\gamma a_2(k)$$

for $k = 1, 2, 3, 4$. Moreover, it can be calculated that

$$\frac{\sum_{k=1}^{4}(c_{11}(k) + c_{12}(k))}{\sum_{k=1}^{4}(a_1(k) - b_1(k))} = \frac{1869}{248} > 1 \quad \text{and} \quad \frac{\sum_{k=1}^{4}(c_{21}(k) + c_{22}(k))}{\sum_{k=1}^{4}(a_2(k) - b_2(k))} = \frac{22110}{6181} > 1.$$

Namely, condition (3.1) holds. Therefore, from Theorem 3.1, it turns out that the system has at least one positive 4-periodic solution.

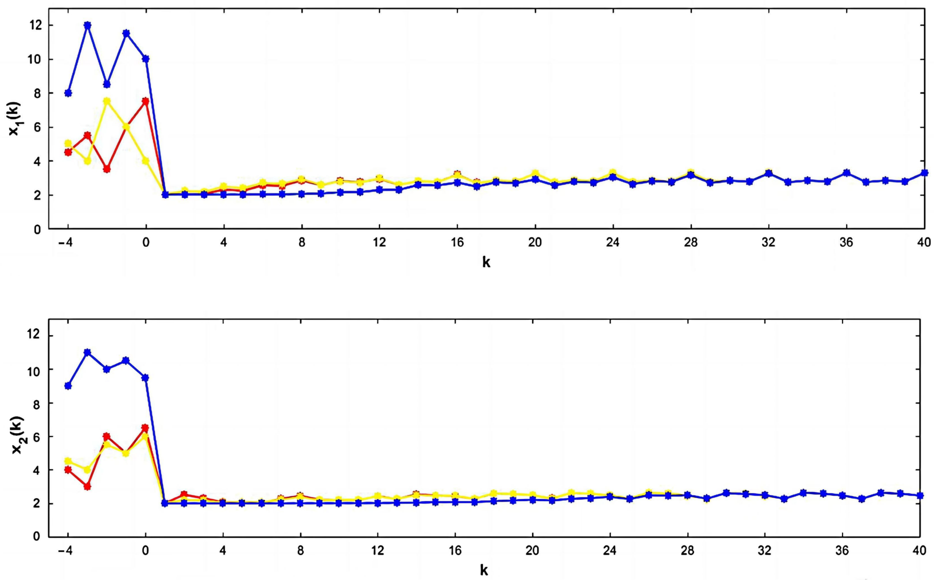


**Figure 1.** Graphs of three arbitrary positive solutions of system. The numerical simulations show that there is a positive 4-periodic solution and this positive 4-periodic solution is locally asymptotically stable.

## 5. Conclusions

A discrete Nicholson system that describles the dynamics of two fly species is studied in this paper. The system considers the mutualism effect between fly species. Continuation theorem of coincidence degree theory is used effectively to seek sufficient conditions for the existence of a positive periodic solution. It is easy to check whether these sufficient conditions hold or not by using coefficients. The positive periodic solution indicates a cycle change in the adult fly populations. From the obtained result, we found that mutualistic interactions between species plays an important role in adult flies populations. But the increase in the flies populations resulting from maximum cumulative mutualism effect only should be less than the death of the flies populations because there is the natural generation of flies populations. Moreover, to avoid species extinction and maintain the coexistence of two fly species in a mutually beneficial environment, we see that (i) the adult fly population produced by maximum daily spawning should exceed a constant multiple of dead fly population for each fly species, and the multiple is greater than constant 1 and (ii) the total population growth must be maintained more than the population loss for each fly species. In fact, the third sufficient condition (3.1) of Theorem 3.1

can be rewritten into the form

$$\sum_{k=1}^{\omega}\left(\sum_{j=1}^{n}(c_{1j}(k)+b_1(k)\right)>\sum_{k=1}^{\omega}a_1(k) \quad \text{and} \quad \sum_{k=1}^{\omega}\left(\sum_{j=1}^{n}(c_{2j}(k)+b_2(k)\right)>\sum_{k=1}^{\omega}a_2(k).$$

The left side of each inequality represents the production of one fly species in a period under the mutualism influence of another, and the right side represents the death of that species in a period. Hence, statement (ii).

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.


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## Conflict of interest

The authors declare that there is no conflicts of interest.

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