



Theory article

Dynamic analysis of a delayed differential equation for *Tropidothorax elegans* pests

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Abstract: In this paper, we establish an infectious disease model of *Tropidothorax elegans* to study the impact of them on plants. Our model involves the time delay for *Tropidothorax elegans* to hatch eggs, which is influenced by temperature. Second, we theoretically analyze the existence and the stability of the equilibrium and the normal form near the Hopf bifurcating critical point. Next, we choose three groups of parameters for numerical simulations to verify theoretical analysis of our model. Then, based on numerical simulations, we give bioanalysis which are consistent with the patterns of *Tropidothorax elegans* pests, such as dying off in large numbers of adults during the winter and one or two generations a year.

Keywords: infectious disease model; plant diseases and insect pests; time delay; Hopf bifurcation; normal form; environment

1. Introduction

In the 21st century, the trend of global warming is becoming more and more pronounced, and industrial gases such as carbon dioxide are a major culprit in global warming. Forests play a key role in carbon sequestration to mitigate the process of global warming, and the capacity of forests to sequester carbon is assessed in [1, 2]. To achieve the double carbon goal, one aspect is to protect forests from forest diseases and pests. Therefore, many scholars are studying ways to protect the forest to avoid the damage of forest diseases and pests. The key is to master the patterns of forest pest and disease outbreaks.

To find the patterns of forest pest and disease, studying the effects of factors such as meteorological factors and human intervention on the outbreaks of forest diseases and pests is significant. Various models of forest diseases and pests were studied [3–6]. Many scholars provided theoretical knowledge for modeling analysis [7–9]. In the work of Yao et al. [4], the relation of environment meteorological conditions with the area of forest diseases and pest was clearly studied by using the method of abnormal

index analysis. Kiewra et al. [5] established the weather research and forecasting model to show the influence of different meteorological variables. Likewise, Wang et al. [6] used a MaxEnt model and incorporated annual meteorological and human activity factors to study the outbreaks of pine wilt disease. Obviously, meteorological condition is an important factor in outbreaks of forest pest and disease.

In addition, human intervention is also a key factor affecting forest pests and diseases. In References [10–13], scholars studied many managements to control forest pest and disease, such as using the GIS-based system which is a geographic information system to project the outbreaks of pest, using the skill of genetically engineered baculoviruses for pest, using chemical control and biological control and so on. These are efforts made by people to control pests.

In this paper, we focus on pests caused by the *Tropidothorax elegans*. This is a class of *Lygaeidae* with an extremely wide distribution. The appearance and biological habits of *Lygaeidae* are studied [14, 15]. All pests of *Lygaeidae* are a class of stinging pests and harm plants through their mouthparts. They reproduce one or two generations per year, get through the winter as adults and lay eggs in the following year. During the period of damage, both *Tropidothorax elegans* nymphs and adults suck the leaves, stem and roots of plants. The sucked parts of the leaves will cause the plant to be infected with sooty blotch under the action of some spores, forming a black spot on the leaves, which will greatly affect the photosynthesis of the plant and prevent the plant from absorbing nutrients. This mode of transmission is similar to infectious diseases, and many scholars studied the transmission patterns of many types of infectious diseases and predicted the development of the infectious disease by establishing the model of infectious diseases [16–18]. Some other scholars gave different measures based on the model analysis to block the spread of the virus [19–22].

After gathering preliminary knowledge of *Tropidothorax elegans* and the current research status, we establish a model about *Tropidothorax elegans* and arrange the rest of the context as follows. In Section 2, according to the effect of *Tropidothorax elegans* on plants, we establish an infectious disease model on *Tropidothorax elegans*. In Section 3, we discuss the stability of the equilibrium point of the model and the existence of the Hopf bifurcation. In Section 4, we calculate the normal form of the Hopf bifurcation. In Section 5, we use numerical simulations to verify the correctness of bioanalysis. Finally, we draw the corresponding conclusions in Section 6.

2. Mathematical modeling

Based on the stinging characteristics of *Tropidothorax elegans*, we assume that a susceptible plant becomes an infected plant when it is sucked by *Tropidothorax elegans*. Therefore, we divide all plants into two groups. One group is susceptible plants ($S(t)$, 10^4 plants), which have not been harmed by *Tropidothorax elegans*. The other group is infected plants ($I(t)$, 10^4 plants), i.e., from susceptible plants that are sucked by *Tropidothorax elegans* ($S(t) \rightarrow I(t)$). The infected plants have two types of changes: some become healthy after treatment in time, and the others will die without timely treatment. Besides, the recovered plants can be reinfected again. We denote the number of *Tropidothorax elegans* as $X(t)$ (10^6 pests). Based on above information, the relationship between the variables can be given in Figure 1.

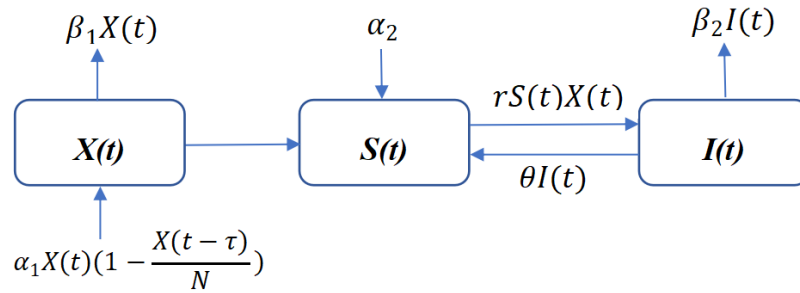


Figure 1. *SIX* Model diagram.

In addition, the hatching time of *Tropidothorax elegans* is also an important factor. We make the hatching time of *Tropidothorax elegans* as the time delay in our study. Subject to resource constraints, we view the reproductive capacity of *Tropidothorax elegans* as a decreasing function related to the number of pre-hatch *Tropidothorax elegans*. Thus, we get $\alpha(t) = \alpha_1(1 - \frac{X(t-\tau)}{N})$, where N is the environmental capacity of *Tropidothorax elegans* and τ is the time delay of the pest hatching period (year). We establish the model as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \alpha_2 - \gamma X(t)S(t) + \theta I(t), \\ \frac{dI(t)}{dt} = \gamma X(t)S(t) - \theta I(t) - \beta_2 I(t), \\ \frac{dX(t)}{dt} = \alpha_1 X(t)(1 - \frac{X(t-\tau)}{N}) - \beta_1 X(t), \end{cases} \quad (2.1)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \theta, N$ are nonnegative parameters. The specific descriptions are given in Table 1. In this table, all variables and parameters are nonnegative.

Table 1. The descriptions of variables and parameters.

Symbol	Descriptions
S	Number of susceptible plants
I	Number of infected plants
X	Number of <i>Tropidothorax elegans</i>
α_1	Reproductive ability of <i>Tropidothorax elegans</i>
α_2	Replenishment rate of plants
β_1	Mortality rate of <i>Tropidothorax elegans</i>
β_2	Mortality rate of infective plants
γ	Transition rate from S to I
θ	Transition rate from I to S
N	Environmental capacity of <i>Tropidothorax elegans</i> in the area

3. Dynamical analysis

In this section, the system (2.1) is considered. When $\alpha_2 = 0$, we calculate a nonisolated boundary solution $E_0 = (S_0, 0, 0)$, where S_0 is an arbitrary value. It means that there is no local infestation of *Tropidothorax elegans*. Therefore, there are no infected plants. However, we focus on the presence of *Tropidothorax elegans*. Thus, We consider the following assumption:

$$(A1) \quad \alpha_1 - \beta_1 > 0.$$

When (A1) holds, we can obtain the unique and positive equilibrium of system (2.1):

$$E = (S^*, I^*, X^*), \quad (3.1)$$

where $S^* = \frac{\alpha_1 \alpha_2 (\beta_2 + \theta)}{\gamma N (\alpha_1 - \beta_1) \beta_2}$, $I^* = \frac{\alpha_2}{\beta_2}$, $X^* = \frac{(\alpha_1 - \beta_1) N}{\alpha_1}$.

Since α_1 represents the reproductive capacity of *Tropidothorax elegans* and β_1 represents the mortality rate of *Tropidothorax elegans*, (A1) implies that the reproductive capacity of insects is much greater than the mortality rate of insects in fact.

3.1. Nonnegativity of solution

In this section, we consider the nonnegativity of system (2.1).

Theorem 3.1. *If $S(\theta) \geq 0$, $I(\theta) \geq 0$, $X(\theta) \geq 0$ ($\theta \in [-\tau, 0]$), the solution $S(t)$, $I(t)$, $X(t)$ of system (2.1) with $\tau \geq 0$ is nonnegative when $t > 0$.*

Proof. First, we prove $X(t) \geq 0$ when $t \geq 0$ under the initial condition of system (2.1). We assume that $X(t) \geq 0$ is not always nonnegative for $t \geq 0$ and make t_1 be the time that $X(t_1) = 0$ and $X'(t_1) < 0$ hold.

According to the third equation of system (2.1), we can obtain $X'(t_1) = 0$. The two conclusions are contradictory. Therefore, $X(t) \geq 0$ when $t > 0$.

Then, we make t_2 be the time that $I(t_2) = 0$ and $I'(t_2) < 0$ hold. According to the second equation of system (2.1), we can obtain $I'(t_2) = \gamma X(t_2) S(t_2)$. If $I'(t_2) < 0$ holds, then $S(t_2) < 0$ and $t_3 < t_2$. Let $S(t_3) = 0$, $S'(t_3) = 0$.

According to the first equation of system (2.1), we can obtain $S'(t_3) = \alpha_2 + \theta I(t_3)$. Because $I(t_3) > 0$, we obtain $S'(t_3) > 0$. The two conclusions are contradictory. So $S(t_3) \geq 0$ when $t > 0$. Thus, the solution to system (2.1) is nonnegative when $t > 0$.

3.2. Stability analysis of equilibrium and the existence of Hopf bifurcation

In this section, we consider the stability and the existence of Hopf bifurcation of E . We can obtain the characteristic equation of system (2.1) at E as follows:

$$[\lambda + (\alpha_1 - \beta_1)e^{-\lambda\tau}][\lambda^2 + (\theta + \beta_2 + \frac{\gamma N(\alpha_1 - \beta_1)}{\alpha_1})\lambda + \frac{\gamma(\alpha_1 - \beta_1)\beta_2}{\alpha_1}] = 0. \quad (3.2)$$

When $\tau = 0$, it becomes:

$$[\lambda + (\alpha_1 - \beta_1)][\lambda^2 + (\theta + \beta_2 + \frac{\gamma N(\alpha_1 - \beta_1)}{\alpha_1})\lambda + \frac{\gamma(\alpha_1 - \beta_1)\beta_2}{\alpha_1}] = 0. \quad (3.3)$$

Equation (3.3) has three roots: let λ_1, λ_2 be the roots of the part of quadratic equation in Eq (3.3), $\lambda_1 + \lambda_2 = -(\theta + \beta_2 + \frac{\gamma N(\alpha_1 - \beta_1)}{\alpha_1})$, $\lambda_1 \cdot \lambda_2 = \frac{\gamma(\alpha_1 - \beta_1)\beta_2}{\alpha_1}$ and $\lambda_3 = \beta_1 - \alpha_1$. (A1) holds and other parameters are positive. Thus, Eq (3.3) has three negative roots and E is locally asymptotically stable for $\tau = 0$.

Next, we consider the existence of bifurcation periodic solutions near the equilibrium E for $\tau > 0$. Let $\lambda = i\omega$ ($\omega > 0$) be a root of characteristic Eq (3.2). We only consider the equation as follows:

$$\lambda + (\alpha_1 - \beta_1)e^{-\lambda\tau} = 0. \quad (3.4)$$

Then, substituting $\lambda = i\omega$ ($\omega > 0$) into Eq (3.4) and separating the real and imaginary parts, we obtain:

$$\begin{aligned} \cos(\omega\tau) &= 0, \\ \sin(\omega\tau) &= \frac{\omega}{\alpha_1 - \beta_1}. \end{aligned} \quad (3.5)$$

Adding the square of the two equations in Eq (3.5), we obtain $\omega = \alpha_1 - \beta_1$, and the expression of τ :

$$\tau^{(j)} = \frac{1}{(\alpha_1 - \beta_1)} \left(\frac{\pi}{2} + 2j\pi \right), j = 0, 1, 2, \dots. \quad (3.6)$$

When $\tau = \tau^{(j)}$ ($j = 0, 1, 2, \dots$), characteristic Eq (3.4) have a pair of pure imaginary roots $\lambda = \pm i\omega$. Calculating the transversality conditions, we obtain:

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau^{(j)}} = \frac{1}{(\alpha_1 - \beta_1)^2} > 0, j = 0, 1, 2, \dots.$$

Theorem 3.2. When (A1) holds, the unique equilibrium E exists and is positive, and system (2.1) undergoes Hopf bifurcation when $\tau = \tau^{(j)}$ ($j = 0, 1, 2, \dots$) near E , where $\tau^{(j)}$ is given by Eq (3.6). Then, when $\tau \in [0, \tau^{(0)})$, the equilibrium E is locally asymptotically stable and unstable when $\tau > \tau^{(0)}$.

4. Normal form of Hopf bifurcation

In this section, we derive the normal form of Hopf bifurcation for the system (2.1) by using the multiple time scales method. To reflect the actual situation, we focus on the delay in hatching and study the patterns of *Tropidothorax elegans* pests. We consider the time-delay τ as a bifurcation parameter. Let $\tau = \tau_c + \varepsilon\mu$, where $\tau_c = \tau^{(j)}$ ($j = 0, 1, 2, \dots$) is the Hopf bifurcation critical value, μ is perturbation parameter and ε is dimensionless scale parameter. When $\tau = \tau_c$, the Eq (3.4) has eigenvalue $\lambda = i\omega$ ($\omega > 0$), at which system (2.1) undergoes a Hopf bifurcation at equilibrium E .

To translate the equilibrium point $E = (S^*, I^*, X^*)$ to the origin and simplify the expression, we write \tilde{S} as S , \tilde{I} and \tilde{X} are the same. At the same time, to normalize the delay, we let $t \rightarrow \frac{t}{\tau}$ rescale the time. Then, we obtain following equations:

$$\begin{cases} \dot{S} = \tau[-\gamma(XS + XS^* + SX^*) + \theta I], \\ \dot{I} = \tau[\gamma(XS + XS^* + SX^*) - (\theta + \beta_2)I], \\ \dot{X} = \tau[\alpha_1 X - \frac{\alpha_1}{N}(XX(t-1) + XX^* - X(t-1)X^*)]. \end{cases} \quad (4.1)$$

We write $Z(t) = (S(t), I(t), X(t))^T$, $Z(t-1) = (S(t-1), I(t-1), X(t-1))^T$. The Eq (4.1) can be written as:

$$\dot{Z} = \tau AZ + \tau BZ(t-1) + \tau F(Z, Z(t-1)), \quad (4.2)$$

where $F(Z, Z(t-1)) = (-\gamma SX, \gamma SX, \frac{\alpha_1}{N} XX(t-1))^T$,

$$A = \begin{pmatrix} -\frac{\gamma N(\alpha_1 - \beta_1)}{\alpha_1} & \theta & -\frac{\alpha_1 \alpha_2 (\theta + \beta_2)}{N(\alpha_1 - \beta_1) \beta_2} \\ \frac{\gamma N \alpha_1 - \beta_1}{\alpha_1} & -(\theta + \beta_2) & \frac{\alpha_1 \alpha_2 (\theta + \beta_2)}{N(\alpha_1 - \beta_1) \beta_2} \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_1 - \alpha_1 \end{pmatrix}.$$

Then, we make h the eigenvector corresponding to eigenvalue $\lambda = i\omega\tau$ of the linear section of Eq (4.2) and h^* the eigenvector corresponding to eigenvalue $\lambda = -i\omega\tau$ of adjoint matrix of the linear section of Eq (4.2). Then, h and h^* should satisfy $\langle h^*, h \rangle = \bar{h}^{*T} h = 1$. According to the above conditions, we obtain:

$$h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} -Vm \\ \omega m \\ UV + Cm \end{pmatrix}, \quad h^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.3)$$

where $m = \frac{\alpha_1^2 \alpha_2 (\beta_2 + \theta)}{N(\alpha_1 - \beta_1) \beta_2}$, $U = \gamma(\alpha_1 - \beta_1)N$, $V = -\beta_2 i + \omega$, $C = (i\omega + \theta + \beta_2)\alpha_1$.

The solution of Eq (4.2) is assumed as follows:

$$Z(t) = Z(T_0, T_1, T_2, \dots) = \sum_{k=1}^{+\infty} \varepsilon^k Z_k(T_0, T_1, T_2, \dots), \quad (4.4)$$

where

$$Z(t) = Z(T_0, T_1, T_2, \dots) = (S(T_0, T_1, T_2, \dots), I(T_0, T_1, T_2, \dots), X(T_0, T_1, T_2, \dots))^T,$$

$$Z_k(t) = Z_k(T_0, T_1, T_2, \dots) = (S_k(T_0, T_1, T_2, \dots), I_k(T_0, T_1, T_2, \dots), X_k(T_0, T_1, T_2, \dots))^T.$$

The derivative, with regard to t , is transformed into

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \dots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots,$$

where D_i is differential operator, and

$$D_i = \frac{\partial}{\partial T_i}, i = 0, 1, 2, \dots.$$

From Eq (4.4), we can obtain

$$\dot{Z} = \varepsilon D_0 Z_1 + \varepsilon^2 D_1 Z_1 + \varepsilon^3 D_2 Z_1 + \varepsilon^2 D_1 Z_2 + \varepsilon^3 D_0 Z_3 + \dots \quad (4.5)$$

Then, using the Taylor expansion to expand $X(T_0-1, \varepsilon(T_0-1), \varepsilon^2(T_0-1), \dots)$ at $X(T_0-1, T_1, T_2, \dots)$, we obtain

$$X(t-1) = \varepsilon X_{1,\tau_c} + \varepsilon^2 X_{2,\tau_c} + \varepsilon^3 X_{3,\tau_c} - \varepsilon^2 D_1 X_{1,\tau_c} - \varepsilon^3 D_2 X_{1,\tau_c} - \varepsilon^3 D_1 X_{2,\tau_c} + \dots, \quad (4.6)$$

where $X_{j,\tau_c} = X_j(T_0-1, T_1, T_2, \dots)$, $j = 1, 2, \dots$.

Combing in Eqs (4.4)–(4.6) and separating the coefficients of ε , ε^2 and ε^3 on both sides of the equation, the expressions of the coefficient before ε , ε^2 and ε^3 are shown from (4.7)–(4.9):

$$\begin{cases} D_0 S_1 + \tau_c(\gamma S^* X_1 + \gamma X^* S_1 - \theta I_1) = 0, \\ D_0 I_1 - \tau_c[\gamma S^* X_1 + \gamma X^* S_1 - (\theta + \beta_2) I_1] = 0, \\ D_0 X_1 + \tau_c[\alpha_1 X_1 - \frac{\alpha_1 X^*}{N}(X_1 + X_{1,\tau_c}) - \beta_1 X_1] = 0. \end{cases} \quad (4.7)$$

$$\begin{cases} D_0 S_2 + \tau_c(\gamma S^* X_2 + \gamma X^* S_2 - \theta I_2) \\ = -D_1 S_1 - \gamma \tau_c X_1 S_1 - \mu(\gamma S^* X_1 + \gamma X^* S_1 - \theta I_1), \\ D_0 I_2 - \tau_c[\gamma S^* X_2 + \gamma X^* S_2 - (\theta + \beta_2) I_2] \\ = -D_1 I_1 + \gamma \tau_c X_1 S_1 + \mu[\gamma S^* X_1 + \gamma X^* S_1 - (\theta + \beta_2) I_1], \\ D_0 X_2 - \tau_c[\alpha_1 X_2 - \frac{\alpha_1 X^*}{N}(X_2 + X_{2,\tau_c}) - \beta_1 X_2] \\ = -D_1 X_1 + \frac{\alpha_1 \tau_c}{N}(X^* D_1 X_{1,\tau_c} - X_1 X_{1,\tau_c}) + \mu[\alpha_1 X_1 - \frac{\alpha_1 X^*}{N}(X_1 + X_{1,\tau_c}) - \beta_1 X_1]. \end{cases} \quad (4.8)$$

$$\begin{cases} D_0 S_3 + \tau_c(\gamma S^* X_3 + \gamma X^* S_3 - \theta I_3) \\ = -D_2 S_1 - D_1 S_2 - \gamma \tau_c(X_1 S_2 + X_2 S_1) - \mu\gamma(S^* X_2 + X^* S_2 + X_1 S_1) + \mu\theta I_2, \\ D_0 I_3 - \tau_c[\gamma S^* X_3 + \gamma X^* S_3 - (\theta + \beta_2) I_3] \\ = -D_2 I_1 - D_1 I_2 + \gamma \tau_c(X_1 S_2 + X_2 S_1) + \mu\gamma(S^* X_2 + X^* S_2 + X_1 S_1) + \mu(\theta + \beta_2) I_2, \\ D_0 X_3 - \tau_c[\alpha_1 X_3 - \frac{\alpha_1 X^*}{N}(X_3 + X_{3,\tau_c}) - \beta_1 X_3] \\ = -D_2 X_1 - D_1 X_2 - \frac{\alpha_1 \tau_c}{N}[X_1 X_{2,\tau_c} - X_1 D_1 X_{1,\tau_c} + X_2 X_{1,\tau_c} - X^*(D_2 X_{1,\tau_c} + D_1 X_{2,\tau_c})] \\ + \mu[\alpha_1 X_2 - \frac{\alpha_1}{N}(X_1 X_{1,\tau_c} + X^*(X_2 + X_{2,\tau_c} - D_1 X_{1,\tau_c})) - \beta_1 X_2]. \end{cases} \quad (4.9)$$

The solution of Eq (4.7) can be assumed as:

$$Z_1 = G h e^{i\omega\tau_c T_0} + \bar{G} \bar{h} e^{-i\omega\tau_c T_0}, \quad (4.10)$$

where h is given by Eq (4.3). We substitute Eq (4.10) into the right side of the Eq (4.7) and write the coefficient of $e^{i\omega\tau_c T_0}$ as m_1 . According to the solvability condition $\langle h^*, m_1 \rangle = 0$, the expression of $\frac{\partial G}{\partial T_1}$ is obtained as follows:

$$\frac{\partial G}{\partial T_1} = M \mu G, \quad (4.11)$$

where

$$M = a[\frac{\alpha_1 X^*}{N}(1 + e^{-i\omega\tau_c}) - (\alpha_1 - \beta_1)],$$

with

$$a = (\tau_c \frac{\alpha_1 X^*}{N} e^{-i\omega\tau_c} - 1)^{-1}.$$

Since μ is a perturbation term, its effect on higher order is smaller. Thus, we only consider its effect on the linear part and ignore its effect on higher order. We assume that the solution of Eq (4.8) is as follows:

$$\begin{aligned} S_2 &= g_1 e^{2i\omega\tau T_0} G^2 + \bar{g}_1 e^{-2i\omega\tau T_0} \bar{G}^2 + l_1 G \bar{G}, \\ I_2 &= g_2 e^{2i\omega\tau T_0} G^2 + \bar{g}_2 e^{-2i\omega\tau T_0} \bar{G}^2 + l_2 G \bar{G}, \\ X_2 &= g_3 e^{2i\omega\tau T_0} G^2 + \bar{g}_3 e^{-2i\omega\tau T_0} \bar{G}^2 + l_3 G \bar{G}, \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} g_1 &= -k_1 k_2, \\ g_2 &= \frac{2i\omega}{2i\omega + \beta_2} k_1 k_2, \\ g_3 &= \frac{-\alpha_1 h_3^2 e^{-i\omega\tau_c}}{N(2i\omega - \alpha_1 + \beta_1) - \alpha_1 X^* (1 + e^{-2i\omega\tau_c})}, \\ l_1 &= \frac{-2\theta k_3 + (\gamma S^* k_4 - k_3)\beta_2}{\gamma X^* \beta_2}, \\ l_2 &= -\frac{2k_3}{\beta_2}, \\ l_3 &= -k_4, \end{aligned}$$

with

$$\begin{aligned} k_1 &= \gamma S^* g_3 + \gamma h_1 h_3, & k_2 &= (2i\omega + \gamma X^* + \frac{2\theta\omega i}{2i\omega + \beta_2})^{-1}, \\ k_3 &= \gamma(h_3 \bar{h}_1 + \bar{h}_3 h_1), & k_4 &= \frac{\alpha_1 h_3 \bar{h}_3 (e^{-i\omega\tau_c} + e^{i\omega\tau_c})}{(\alpha_1 - \beta_1)N - 2\alpha_1 X^*}, \end{aligned}$$

and h_1, h_2, h_3 is given by Eq (4.3).

Then we substitute Eq (4.12) into the right side of Eq (4.9), and the coefficient vector of $e^{i\omega\tau_c T_0}$ is denoted by m_2 . According to the solvability condition $\langle h^*, m_2 \rangle = 0$, the expression of $\frac{\partial G}{\partial T_2}$ can be obtained as follows:

$$\frac{\partial G}{\partial T_2} = \alpha_1 \tau_c H G^2 \bar{G}, \quad (4.13)$$

where

$$H = d[h_3 l_3 (1 + e^{-i\omega\tau_c}) + \bar{h}_3 g_3 (e^{-2i\omega\tau_c} + e^{-i\omega\tau_c})],$$

with

$$d = [h_3 (\tau_c \alpha_1 X^* e^{-i\omega\tau_c} - N)]^{-1},$$

and h_3, g_3, l_3 are given by Eqs (4.3) and (4.12).

Let $G \rightarrow G/\varepsilon$. Then, the deduced third-order normal form of Hopf bifurcation of system (2.1) is:

$$\dot{G} = M\mu G + \alpha_1 \tau_c H G^2 \bar{G}, \quad (4.14)$$

where M is given in Eq (4.11) and H is given in Eq (4.13).

Substituting $G = re^{i\theta}$ into Eq (4.14), the following normal form of Hopf bifurcation in polar coordinates is obtained:

$$\begin{cases} \dot{r} = \operatorname{Re}(M)\mu r + \operatorname{Re}(H)\alpha_1\tau_c r^3, \\ \dot{\theta} = \operatorname{Im}(M)\mu + \operatorname{Im}(H)\alpha_1\tau_c r^2. \end{cases} \quad (4.15)$$

Based on the normal form of Hopf bifurcation in polar coordinates, we only need to consider the first equation in system (4.15). Thus, the following theorem holds:

Theorem 4.1. When $\frac{\operatorname{Re}(M)\mu}{\operatorname{Re}(H)\alpha_1\tau_c} < 0$, the system (4.15) exits a semitrivial fixed point $r = \sqrt{-\frac{\operatorname{Re}(M)\mu}{\operatorname{Re}(H)\alpha_1\tau_c}}$, and the system (2.1) has a periodic solution.

- 1) If $\operatorname{Re}(M)\mu < 0$, the periodic solution reduced on the center manifold is unstable.
- 2) If $\operatorname{Re}(M)\mu > 0$, the periodic solution reduced on the center manifold is stable.

5. Numerical simulations and bioanalysis

In this section, we choose three groups of parameters to simulate the effects of temperature changes and artificial interventions on the model. Then, we give the corresponding biological explanations based on the simulations. Finally, we discuss what humans should do to reduce the harm caused by *Tropidothorax elegans*.

5.1. Determination of parameter values

First, we choose the first group of parameters to describe the situation that *Tropidothorax elegans* grow freely. Second, considering the context of global warming, we choose the second group of parameters to simulate the effects of global warming on *Tropidothorax elegans* pests. At higher latitudes, the increase in temperature will lead to a decrease in the number of adults that freeze to death in winter, and the best temperature for the survival of *Tropidothorax elegans* can be reached more easily, so the incubation time can be shortened [23]. At lower latitudes, increased temperatures will lead to a decrease in the mortality rate of adults, so there will be more *Tropidothorax elegans* laying eggs in next year. Thus, globally higher temperature leads to lower mortality and higher reproduction of *Tropidothorax elegans* around the world. Finally, we take the third group of parameters to simulate the effect of timely artificial control on *Tropidothorax elegans* pests. Timely artificial control can increase the mortality rate of *Tropidothorax elegans* [24]. Thus, increasing the recovery rate of infected plants and reducing the mortality rate of infected plants.

In addition, α_2 is the replenishment rate of plants, β_2 is the mortality rate of infective plants and θ is the transition rate from $I(t)$ to $S(t)$. In fact, because people find plants dead and promptly replenish them with the same number of plants, we consider the replenishment rate to be the same as the mortality rate of plants. The infected plants have two types of changes: some become healthy after treatment in time and the others will die without timely treatment. Thus, we consider $\alpha_2 + \theta = 1$.

Based on the above analyses, three reasonable groups of parameters are taken to describe the above three situations as follows:

- 1) $N = 4.5$, $\alpha_1 = 15$, $\alpha_2 = 0.05$, $\theta = 0.95$, $\beta_1 = 0.8$, $\beta_2 = 0.05$, $\gamma = 0.29$;
- 2) $N = 4.5$, $\alpha_1 = 20$, $\alpha_2 = 0.10$, $\theta = 0.90$, $\beta_1 = 0.6$, $\beta_2 = 0.02$, $\gamma = 0.29$;
- 3) $N = 4.5$, $\alpha_1 = 15$, $\alpha_2 = 0.02$, $\theta = 0.98$, $\beta_1 = 0.9$, $\beta_2 = 0.02$, $\gamma = 0.29$;

5.2. Numerical simulations

First, we show the simulated results under the first group of parameters:

$$N = 4.5, \alpha_1 = 15, \alpha_2 = 0.05, \theta = 0.95, \beta_1 = 0.8, \beta_2 = 0.05, \gamma = 0.29.$$

We calculate the equilibrium $E = (S^*, I^*, X^*) = (0.815195239, 1, 0.94)$ and critical $\tau_c = 0.11135$ of system (2.1). Thus, for the initial functions $[S(\theta), I(\theta), X(\theta)] = [0.81, 0.99, 0.93]$ ($\theta \in [-\tau_c, 0]$), we choose $\tau = 0$, $\tau = 0.08 < \tau_c$ and $\tau = 0.18 > \tau_c$ for simulations. Based on Theorem 3.2, the equilibrium E is locally asymptotically stable when $\tau = 0$ and $\tau = 0.08 < \tau_c$. The result of simulation with $\tau = 0.08$ is shown in Figure 2, which is consistent with theoretical analyses.

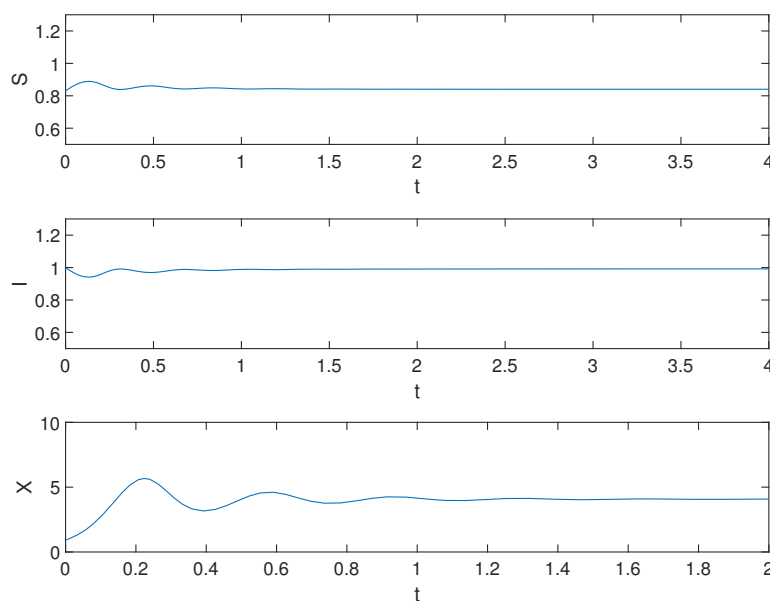


Figure 2. Equilibrium E of the system (2.1) is locally asymptotically stable when $\tau = 0.08$.

In Figure 2, S , I , X become stable after a short period of variations. In the early stage, the number of *Tropidothorax elegans* increases rapidly, and peak after a period of time. Since then, the number of *Tropidothorax elegans* begins to decline. We suspect this is because limited resources act as a constraint on their growth. After a short period of intraspecific competition, the number of *Tropidothorax elegans* becomes relatively stable. This also means that when the incubation time of the *Tropidothorax elegans* is small enough, the *Tropidothorax elegans* will remain locally forever. This may happen at low latitude regions where temperature is high enough.

For $\tau = 0.18 > \tau_c$, we calculate the normal form of Hopf bifurcation and obtain $\text{Re}(M) > 0$, $\text{Re}(H) < 0$ from Eq (4.15). According to the Theorem 4.1, the periodic solution is stable when $\mu > 0$. The solution is shown in Figure 3.

Clearly, S, I, X all form the stable periodic solutions in Figure 3. This means that this pest has become an endemic disease and is difficult to eradicate with $\tau = 0.18$. In Figure 3(c), the infestation period of *Tropidothorax elegans* is about one month and outbreaks once a year which is consistent with the habits of the *Tropidothorax elegans* (http://museum.ioz.ac.cn/topic_detail.aspx?id=7358).

In addition, the number of *Tropidothorax elegans* tends to zero for a period of time. Then, it gradually increases again, repeating the trend of the first cycle. The biological meaning is that *Tropidothorax elegans* spend the winter as adults, the number of *Tropidothorax elegans* decreases considerably due to the low temperature. However, since *Tropidothorax elegans* go through the winter everywhere, such as near plants, on the backs of plant leaves, under rocks or in soil burrows, it is difficult to kill. Therefore, the number of the *Tropidothorax elegans* will increase again in next year because of their strong reproductive ability, repeating the experience of the previous year. Besides, the external environment is stable, so the environmental capacity rarely changes and the peak value remains constant.

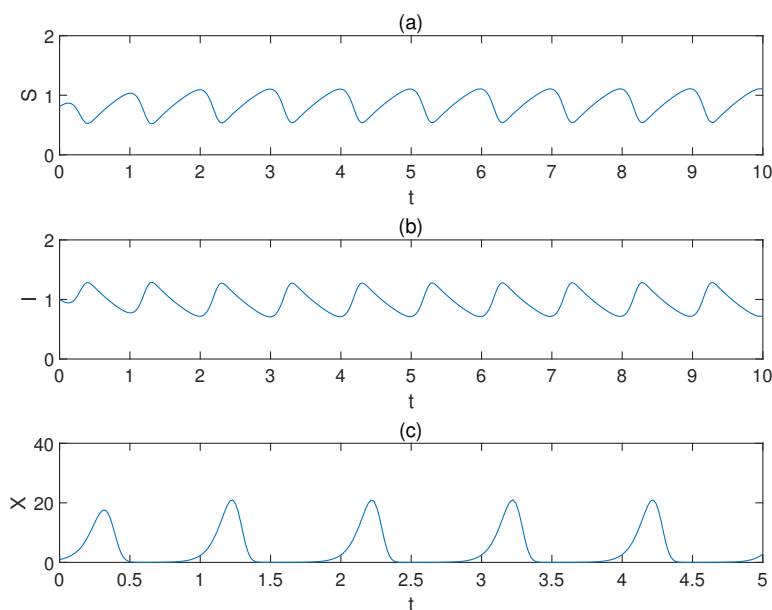


Figure 3. When $\tau = 0.18$, there exists a stable periodic solution near equilibrium E .

For the group of parameters (2):

$$N = 4.5, \alpha_1 = 20, \alpha_2 = 0.10, \theta = 0.90, \beta_1 = 0.6, \beta_2 = 0.02, \gamma = 0.29.$$

We calculate the equilibrium $E = (S^*, I^*, X^*) = (0.789983015, 1, 0.97)$ and $\tau_c = 0.08093$. We know that E is locally asymptotically stable for $\tau \in (0, \tau_c)$ and unstable for $\tau > \tau_c$. Similar to the solution of the first group of parameters when $\tau = 0.18$, the periodic solution of the second group of parameters with $\tau = 0.12 > \tau_c$ is also stable when $\mu > 0$. We choose the initial functions $[S(\theta), I(\theta), X(\theta)] = [0.78, 0.99, 0.96]$ ($\theta \in [-\tau_c, 0]$) for simulation and the result of simulation is shown in Figure 4.

The overall trend is the same as Figure 3, the actual situation of these parameters is: under the background of global warming, temperature has increased. This leads to a lower natural mortality rate and a higher reproduction capacity of *Tropidothorax elegans*, which are more suitable for their survival. The time of the number of *Tropidothorax elegans* tends to zero is significantly shorter in Figure 4(c). That means that the frequency of infestation is significantly higher and the damage to the forest is greater. Thus, the increase of temperature will lead to the increase of damage frequency and damage degree of *Tropidothorax elegans*.

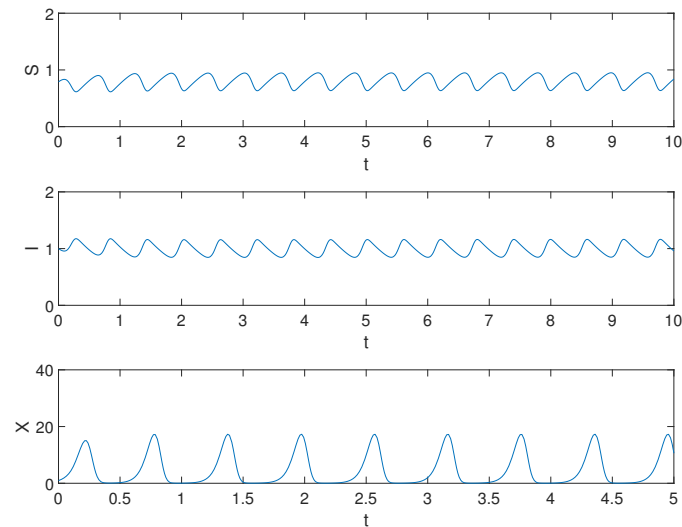


Figure 4. When $\tau = 0.12$, there exists a stable periodic solution near equilibrium E .

Finally, we simulate the third group of parameters:

$$N = 4.5, \alpha_1 = 15, \alpha_2 = 0.02, \theta = 0.98, \beta_1 = 0.9, \beta_2 = 0.02, \gamma = 0.29.$$

We calculate the equilibrium $E = (S^*, I^*, X^*) = (0.81520, 1, 0.94)$ and $\tau_c = 0.11135$ of system (2.1). Compared with the first group of parameters, we choose $\tau = 0.18 > \tau_c$ and the initial functions $[S(\theta), I(\theta), X(\theta)] = [0.81, 0.99, 0.93]$ ($\theta \in [-\tau_c, 0]$) for simulation. The result of simulation is shown in Figure 5.

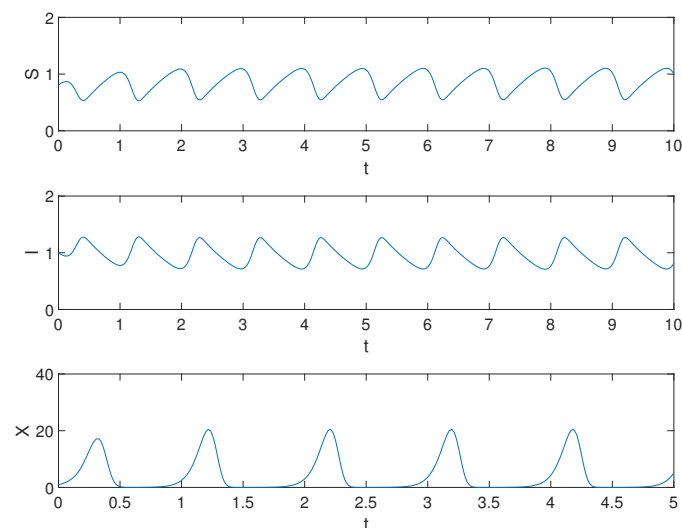


Figure 5. When $\tau = 0.18$, there exists a stable periodic solution near equilibrium E .

The Figure 5 corresponds to the actual situation: after people found this pest, people will kill pests

in time. Obviously, the more timely the manual control, the longer it takes the number of *Tropidothorax elegans* to tend to zero, which shows that the manual control has suppressed the harm of the *Tropidothorax elegans* to the plants. When we choose the prevention methods, we should choose the optimal control method according to the different characteristics of the pest outbreak respectively.

Based on the research [11, 12, 27], the advantage of biological control is that it is highly selective and safer to the environment. However, its killing effect is slow and the economic cost is high. Compared with it, the killing effect of chemical control is faster and the economic cost is low, but it will bring more environmental pollution and other hidden problems. For example, when chemical control is carried out, pests and their natural enemies are killed in large quantities, at which time the effectiveness of biological control is greatly reduced, but it does achieve the purpose of eliminating pests. With time going by, the remaining chemical reagents are reduced, and the ability to kill pests is weakened. Then, biological control is relied on to kill pests [28]. Therefore, in the case of sudden outbreak of *Tropidothorax elegans*, we can use chemical control to prevent and control the bug urgently, and in the later stage of pest management, we can use biological control technology to reduce the pollution.

In addition, we compare the curves above three groups together as shown in Figure 6:

We can see the similarities of these curves:

1) The trend of the change of the number of infected plants and the number of susceptible plants is opposite, this is because that the total number of plants is a constant. which is consistent with the actual situation.

2) In the stage of increasing number of *Tropidothorax elegans*, after the number of *Tropidothorax elegans* peaking, the number of susceptible and the number of infected plants peak. There is a delay between the time which the number of *Tropidothorax elegans* peaks with the time which the number of susceptible plants and infected plants peak. We analyze that this is because the replenishment rate of the plants is smaller than the reproductive capacity of *Tropidothorax elegans* and the mortality rate of the plants is also smaller than the mortality rate of *Tropidothorax elegans*, so the change of the number of plants is smaller and slower, while the change of *Tropidothorax elegans* will be more flexible.

3) In the first and third groups of parameters, we choose different parameters and the same $\tau = 0.18$. In the Figure 6, we find that the two curves almost coincide. However, the time-delay of the second group of parameters is different from other groups, and the curves are different. This means that the effect of parameters variations on the patterns of *Tropidothorax elegans* pests is less than the effect of temperature on the patterns of *Tropidothorax elegans* pests. Therefore, we believe that the speed of reproduction determines the outbreak of *Tropidothorax elegans*.

Some differences in the Figure 6:

1) When the time-delay is the same, we compare the curves for the first group of parameters with the curves for the third group of parameters. The peak of the curves of the third group is lower than the peak of the curves of the first group in Figure 6(a),(b). In fact, this represents a reduction in the number of mortality plants through manual control during the outbreak. In addition, we compare the curves of the first and third group of parameters in Figure 6(c), we find that the breakout period of the curve of the artificial intervention is later than the breakout period of the first set of parameters. This means that artificial intervention can effectively postpone the outbreak of the pest. Therefore, we should do a good job of patrolling in the forest in time, so that the insect pests can be detected and treated in time to ensure the safety of the plants.

2) When the time-delay is different, we compare the first group of parameters with $\tau = 0.18$ and the second group of parameters with $\tau = 0.12$ in Figure 6, and find that the curves of the second group of parameters change more fastly. As the temperature rises, the incubation time of *Tropidothorax elegans* will be shortened and the number of infected plants and susceptible plants will also change rapidly. Due to resource constraints and more intense intraspecific competition, the maximum number of *Tropidothorax elegans* in the region has decreased.

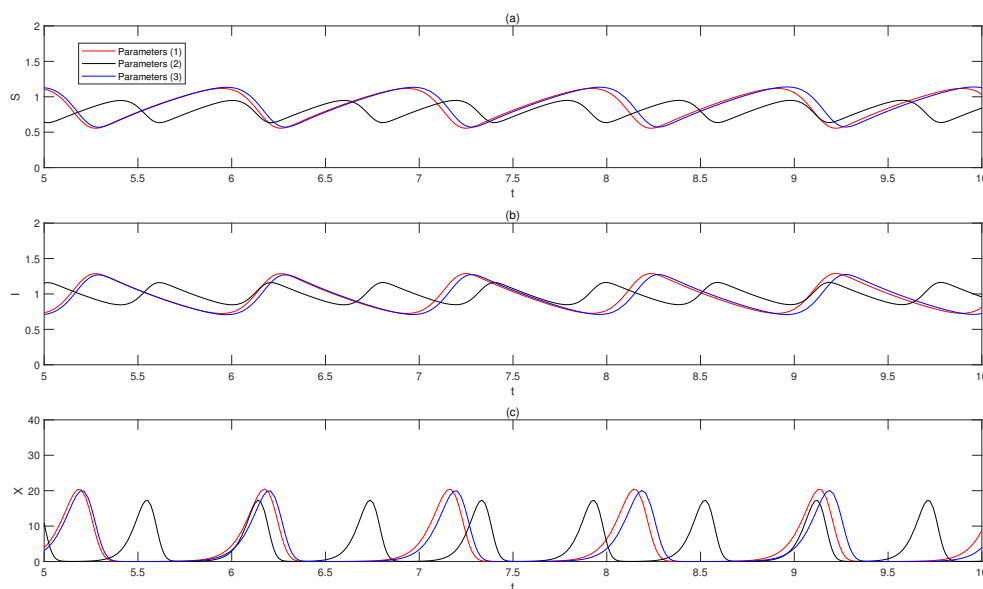


Figure 6. Comparison of three sets of parameters.

6. Conclusions

In this paper, considering the characteristics of *Tropidothorax elegans* to sting plants, we have constructed a *SIX* model with a time delay for *Tropidothorax elegans* to hatch eggs, which is influenced by temperature and manual control. We have studied the stability of the equilibrium and the existence of Hopf bifurcation. Then, we have analyzed the stability and bifurcating direction of the Hopf bifurcating periodic solution by calculating the normal form with the multiple time scales method.

Based on three groups of parameters, we have conducted numerical simulations. First, we have simulated the change of temperature through the first and the second groups of parameters to verify that *Tropidothorax elegans* pests become more frequent with higher temperature and that the laws of outbreaks is once or twice a year, which is consistent with actual situation. Second, we have simulated the effects of human intervention through the first and third groups of parameters and have compared them to conclude that human intervention can delay the outbreak of *Tropidothorax elegans* pests and reduce the number of *Tropidothorax elegans* at the peak of the outbreak. Further more, based on the analyzed characteristics of *Tropidothorax elegans*, we could choose different control methods and apply the right medicine to the right situation. In the future, we will further break down the effects of different control methods on this pest, focusing on two methods: pesticide control and natural enemy control, so that we could remove pests more efficiently and protect the environment at the same time.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This study was funded by Fundamental Research Funds for the Central Universities of China (No. 2572022DJ06) and College Students Innovations Special Project funded by Northeast Forestry University of China (No. DC-2023179) .

Conflict of interest

The authors declare that they have no competing interests.

References

1. W. Yang, Z. Min, M. X. Yang, J. Yan, Exploration of the implementation of carbon neutralization in the field of natural resources under the background of sustainable development-an overview, *Int. J. Environ. Res. Public Health*, **19** (2022), 14109. <https://doi.org/10.3390/ijerph192114109>
2. H. Chen, C. C. Tie, Y. Teng, Z. Geng, J. M. Zhu, Research on forest carbon sequestration assessment and forest management plan based on TOPSIS method and grey correlation analysis, *Int. J. Environ. Pollut.*, **70** (2021), 130–147. <https://doi.org/10.1504/IJEP.2021.132005>
3. B. Yang, W. Q. Cao, C. M. Tian, Visual analysis of impact factors of forest pests and diseases, *J. Visualization*, **22** (2019), 1257–1280. <https://doi.org/10.1007/s12650-019-00593-2>
4. X. H. Yao, Y. P. Xu, B. S. Yuan, L. Y. Han, X. L. Yao, H. H. Han, Study of meteorological assessment techniques of forest diseases and pests in Xiaolong mountain (in Chinese), *J. Northwest A, F Univ. (Nat. Sci. Ed.)*, **10** (2012), 64–68. <https://doi.org/10.13207/j.cnki.jnwafu.2012.10.018>
5. D. Kiewra, M. Kryza, M. Szymanowski, Influence of selected meteorological variables on the questing activity of *Ixodes ricinus* ticks in Lower Silesia, SW Poland, *J. Vector Ecol.*, **39** (2014), 138–145. <https://doi.org/10.1111/j.1948-7134.2014.12080.x>
6. W. Wang, W. T. Peng, X. Y. Liu, G. He, Y. L. Cai, Spatiotemporal dynamics and factors driving the distributions of pine wilt disease-damaged forests in China, *Forests*, **13** (2022), 261. <https://doi.org/10.3390/f13020261>
7. R. Castro, S. Lanucara, V. Piccione, G. Pioggia, G. Modica, M. A. Ragusa, MEDALUS model evolutions and prospects case study Sicily, in *International Conference on Computational Science and Its Applications*, Springer Nature Switzerland, (2023), 310–326. https://doi.org/10.1007/978-3-031-37114-1_21
8. X. M. Chen, D. M. Luo, Dynamical analysis of an almost periodic multispecies mutualism system with impulsive effects and time delays, *Filomat*, **37** (2023), 551–565. <https://doi.org/10.2298/FIL2302551C>
9. Z. Khan, J. Khan, S. Saifullah, A. Ali, Dynamics of hidden attractors in fourdimensional dynamical systems with power law, *J. Funct. Spaces*, **2022** (2022), 3675076. <https://doi.org/10.1155/2022/3675076>

10. H. A. Wood, R. R. Granados, Genetically engineered baculoviruses as agents for pest control, *Annu. Rev. Microbiol.*, **45** (1991), 69–87. <https://doi.org/10.1146/annurev.mi.45.100191.000441>
11. V. Mondello, A. Songy, E. Battison, C. Pinto, C. Coppin, P. Trozel-Aziz, et al., Grapevine trunk diseases: A review of fifteen years of trials for their control with chemicals and biocontrol agents, *Plant Dis.*, **102** (2018), 1189–1217. <https://doi.org/10.1094/PDIS-08-17-1181-FE>
12. T. Shimales, E. Mendesil, B. Zewdie, B. Ayalew, K. Hylander, A. J. M. Tack, Management intensity affects insect pests and natural pest control on Arabica coffee in its native range, *J. Appl. Ecol.*, **60** (2023), 911–922. <https://doi.org/10.1111/1365-2664.14410>
13. W. X. Yao, S. Guo, J. Wang, C. L. Chen, F. H. Yu, X. Li, et al., Droplet deposition and pest control efficacy on pine trees from aerial application, *Pest. Manage. Sci.*, **78** (2022), 3324–3336. <https://doi.org/10.1002/ps.6959>
14. Z. Arslangundogdu, E. Hizal, S. Acer, First record of oxycarenus lavaterae (Fabricius, 1787) (*Heteroptera, Lygaeidae*) in Turkey, *Appl. Ecol. Environ. Res.*, **16** (2018), 1305–1311. https://doi.org/10.15666/aer/1602_13051311
15. C. Q. Yang, Y. Fan, C. G. Du, A preliminary study on the *Tropidothorax elegans* (in Chinese), *Insect Knowl.*, **25** (1988), 36–38.
16. W. S. Avusuglo, R. Mosleh, T. Ramaj, A. Li, S. S. Sharbayta, A. A. Fall, et al., Workplace absenteeism due to COVID-19 and influenza across Canada: a mathematical model, *J. Theor. Biol.*, **572** (2023), 111559. <https://doi.org/10.1016/j.jtbi.2023.111559>
17. L. L. Chang, W. Gong, Z. Jin, G. Q. Sun, Sparse optimal control of pattern formations for an SIR reaction-diffusion epidemic model, *SIAM J. Appl. Math.*, **82** (2022), 1764–1790. <https://doi.org/10.1137/22M1472127>
18. Y. Tatsukawa, M. R. Arefin, S. Utsumi, K. Kuga, J. Tanimoto, Stochasticity of disease spreading derived from the microscopic simulation approach for various physical contact networks, *Appl. Math. Comput.*, **431** (2022), 127328. <https://doi.org/10.1016/j.amc.2022.127328>
19. G. Q. Sun, X. Ma, Z. Z. Zhang, Q. H. Liu, B. L. Li, What is the role of aerosol transmission in SARS-Cov-2 Omicron spread in Shanghai, *BMC Infect. Dis.*, **22** (2022), 880. <https://doi.org/10.1186/s12879-022-07876-4>
20. X. Ma, G. Q. Sun, Z. H. Wang, Y. M. Chu, Z. Jin, B. L. Li, Transmission dynamics of brucellosis in Jilin province, China: Effects of different control measures, *Commun. Nonlinear Sci.*, **114** (2022), 106702. <https://doi.org/10.1016/j.cnsns.2022.106702>
21. G. Q. Sun, H. T. Zhang, L. L. Chang, Z. Jin, H. Wang, S. Q. Ruan, On the dynamics of a diffusive foot-and-mouth disease model with nonlocal infections, *SIAM J. Appl. Math.*, **82** (2022), 1587–1610. <https://doi.org/10.1137/21M1412992>
22. T. A. Biala, Y. O. Afolabi, A. Q. M. Khaliq, How efficient is contact tracing in mitigating the spread of COVID-19? A mathematical modeling approach, *Appl. Math. Model.*, **103** (2022), 714–730. <https://doi.org/10.1016/j.apm.2021.11.011>
23. C. A. Deutsch, J. J. Tewksbury, M. Tigchelaar, D. S. Battisti, S. C. Merrill, R. B. Huey, et al., Increase in crop losses to insect pests in a warming climate, *Science*, **361** (2018), 916–919. <https://doi.org/10.1126/science.aat3466>
24. H. Puebla, P. K. Roy, A. Velasco-Perez, M. M. Gonzalez-Brambila, Biological pest control using a model-based robust feedback, *IET Syst. Biol.*, **12** (2018), 233–240. <https://doi.org/10.1049/iet-syb.2018.5010>

25. L. James, F. D. Hugh, Interspecific hybridization and genetic divergence in Milkweed Bugs (*Oncopeltus: Hemiptera: Lygaeidae*), *Evolution*, **37** (1983), 583–591. <https://doi.org/10.1111/j.1558-5646.1983.tb05575.x>
26. A. Duarte, N. Borralho, P. Cabral, M. Caetano, Recent advances in forest insect pests and diseases monitoring using UAV-Based data: A systematic review, *Forests*, **13** (2022), 911. <https://doi.org/10.3390/f13060911>
27. L. Z. Ji, Z. Wang, X. W. Wang, Forest insect pest management and forest management in China: An overview, *Environ. Manage.*, **48** (2011), 1107–1121. <https://doi.org/10.1007/s00267-011-9697-1>
28. J. N. Liu, Q. Qi, B. Liu, S. J. Gao, Pest control switching models with instantaneous and non-instantaneous impulsive effects, *Math. Comput. Simul.*, **205** (2023), 926–938. <https://doi.org/10.1016/j.matcom.2022.10.027>



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