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http://www.aimspress.com/journal/era

ERA, 31(11): 6898–6912.
DOI: 10.3934/era.2023349
Received: 26 December 2022
Revised: 07 September 2023
Accepted: 11 October 2023
Published: 26 October 2023

Research article

Additivity of nonlinear higher anti-derivable mappings on generalized matrix algebras

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Abstract: In this article, we proved that each nonlinear higher anti-derivable mapping on generalized matrix algebras is automatically additive. As for its applications, we find a similar conclusion on triangular algebras, full matrix algebras, unital prime rings with a nontrivial idempotent, unital standard operator algebras and factor von Neumann algebras respectively.

Keywords: nonlinear anti-derivable mapping; nonlinear higher anti-derivable mapping; nonlinear derivable mapping; generalized matrix algebras

1. Introduction

In past decades, research of derivations and nonlinear derivable mappings on algebras has attracted the attention of many mathematicians.

Definition 1.1 Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ a unital algebra over $\mathcal{R}$, and $\mathbb{N}$ the set of non-negative integers, $i, j, k, n \in \mathbb{N}$.

(1) If Δ is an additive mapping such that

$$\Delta(XY) = \Delta(X)Y + X\Delta(Y) \tag{1.1}$$

for all $X, Y \in \mathcal{A}$, then Δ is said to be a derivation. If Δ is not necessarily additive and Eq (1.1) hold for all $X, Y \in \mathcal{A}$, then Δ is said to be a nonlinear derivable mapping.

(2) For any $X, Y \in \mathcal{A}$, call $X \circ Y := XY + YX$ the Jordan product of X, Y. If Δ is an additive mapping such that

$$\Delta(X \circ Y) = \Delta(X) \circ Y + X \circ \Delta(Y)$$

for all $X, Y \in \mathcal{A}$, then Δ is said to be a Jordan derivation. Nonlinear Jordan derivation is defined similarly to the nonlinear derivable mapping.

(3) Let $D = \{d_n\}_{n\in\mathbb{N}}$ be a sequence of additive mappings (resp., without assumption of additivity) on $\mathcal{A}$ with $d_0 = id_{\mathcal{A}}$ the identity mapping on $\mathcal{A}$ such that

$$d_n(XY) = \sum_{i+j=n} d_i(X)d_j(Y)$$

for all $n \in \mathbb{N}$, and $X, Y \in \mathcal{A}$, then D is said to be a higher derivation (resp., nonlinear higher derivable mapping).

Obviously, every additive derivation is an additive Jordan derivation, and every additive higher derivation is an additive Jordan higher derivation. However, the inverse statement is not true in general. It is natural to ask the following two questions:

Problem 1 Under what conditions is a Jordan (higher) derivation is a (higher) derivation?

Problem 2 Under what conditions is a nonlinear (Jordan, higher) derivation is a (Jordan, higher) derivation?

There are many works that consider Problem 1. For example, see [1–5]. In this paper, we focus on Problem 2. Rickart [6] proved that, under certain conditions, any one-to-one and multiplicative mapping from a ring into another ring is necessarily additive. Martindale [7] obtained the result that each multiplicative bijective mapping on an arbitrary algebra which contains a nontrivial idempotent is automatically additive. For other similar results about additivity of multiplicative mappings on rings or algebras, we refer the readers to [8–11] and references therein for more details. Daif [12] showed that, under certain conditions, any multiplicative derivation is additive. Later, Daif [13] extended this result to the case of multiplicative generalized derivation. Lu [14] proved that, under some conditions, every multiplicative Jordan derivation on a prime ring is an additive derivation. For more similar results about additivity of nonlinear Jordan derivable on rings or algebras, see [15, 16] and references therein. Fu and Xiao [17] and Ashraf and Jabeen [18] showed that all nonlinear Jordan higher derivable mappings and nonlinear Jordan higher triple derivable mappings on triangular algebras is an additive higher derivation, respectively.

In [1], Benkovič defined anti-derivations on algebras as the following.

Definition 1.2 Let C be a commutative ring with unity, A an algebra over C and M an A-bimodule. Let $\delta : A \to M$ be a linear map. If

$$\delta(ab) = \delta(b)a + b\delta(a)$$

for all $a, b \in A$, then δ is said an anti-derivation. For more results about anti-derivation on rings or algebras, see [19, 20] and references therein.

Motivated by the above definition, we introduce the following higher anti-derivation.

Definition 1.3 Let C be a commutative ring with unity, and A be an algebra over C. Let $D = \{\delta_n\}_{n\in\mathbb{N}}$ be a sequence of additive maps from A into itself with $\delta_0 = id_A$. If

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(b)\delta_j(a)$$

for all $a, b \in A$ and all $n \in \mathbb{N}$, then D is called a higher anti-derivation. If δ_n is not necessarily additive, then D is called a non-linear higher anti-derivable mapping.

Our main purpose in this paper is to show that every nonlinear higher anti-derivable mapping on a generalized matrix algebra is additive. In the following section, we introduce some basic concepts and the properties of generalized matrix algebras we require. Generalized matrix algebra is a particular structure of generalized n-matrix rings (see for example [21]), if we do not consider the scalar multiplication.

2. Generalized matrix algebras

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ and $\mathcal{B}$ be two unital $\mathcal{R}$-algebras, and $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be the unit elements of $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule (i.e., for any $A \in \mathcal{A}, B \in \mathcal{B}$, if $A\mathcal{M} = 0$, then $A = 0$; if $\mathcal{M}B = 0$, then $B = 0$), and $\mathcal{N}$ be a (not necessarily faithful) $(\mathcal{B}, \mathcal{A})$-bimodule. Suppose that there are two bimodule homomorphisms $\Phi_{\mathcal{MN}} : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \mapsto \mathcal{A}$ and $\Psi_{\mathcal{NM}} : \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \mapsto \mathcal{B}$ satisfying the following associativity conditions: $(MN)M' = M(NM')$ and $(NM)N' = N(MN')$ for all $M, M' \in \mathcal{M}, N, N' \in \mathcal{N}$, where $MN = \Phi_{\mathcal{MN}}(M \otimes_{\mathcal{B}} N)$ and $NM = \Psi_{\mathcal{NM}}(N \otimes_{\mathcal{A}} M)$. Then

$$\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B}) = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix} = \left\{ \begin{pmatrix} A & M \\ N & B \end{pmatrix} : A \in \mathcal{A}, M \in \mathcal{M}, N \in \mathcal{N}, B \in \mathcal{B} \right\}$$

is an $\mathcal{R}$-algebra under the usual matrix-like addition, and the following multiplication:

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \begin{pmatrix} A' & M' \\ N' & B' \end{pmatrix} = \begin{pmatrix} AA' + \Phi_{\mathcal{MN}}(M \otimes N') & AM' + MB' \\ NA' + BN' & BB' + \Psi_{\mathcal{NM}}(N \otimes M') \end{pmatrix}$$

for all $A, A' \in \mathcal{A}, M, M' \in \mathcal{M}, N, N' \in \mathcal{N}$ and $B, B' \in \mathcal{B}$, where at least one of the two bimodules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. Such an $\mathcal{R}$-algebra is called a generalized matrix algebra. This type of algebra was first introduced by Morita [22]. In the following, we simply write $\mathcal{G}(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ as $\mathcal{G}$. For any associative algebra $\mathcal{A}$, if $\mathcal{A}$ is unital with the identity $1_{\mathcal{A}}$, and has a non-trivial idempotent P ($P^2 = P$, $P \neq 0$ and $P \neq 1_{\mathcal{A}}$), then the Peirce decomposition of $\mathcal{A}$ corresponding to P is $\mathcal{A} = P\mathcal{A}P + P\mathcal{A}Q + Q\mathcal{A}P + Q\mathcal{G}Q$, where $Q = 1_{\mathcal{A}} - P$. With respect to this decomposition, $\mathcal{A}$ is a generalized matrix algebra, and we then know that any an associative algebra containing a non-trivial idempotent is a generalized algebra.

Consider a generalized matrix algebra $\mathcal{G}$, let 1 be the unit of $\mathcal{G}$. Set

$$P_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}, \ P_2 = 1 - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$$

and $\mathcal{G}_{ij} = P_i \mathcal{G} P_j (1 \leq i, j \leq 2)$. Then, $\mathcal{G}$ can be represented as

$$\mathcal{G} = \mathcal{G}_{11} + \mathcal{G}_{12} + \mathcal{G}_{21} + \mathcal{G}_{22},$$

where $\mathcal{G}_{11}$ is a subalgebra of $\mathcal{G}$ isomorphic to $\mathcal{A}$, $\mathcal{G}_{22}$ is a subalgebra of $\mathcal{G}$ isomorphic to $\mathcal{B}$, $\mathcal{G}_{12}$ is a $(\mathcal{G}_{11}, \mathcal{G}_{22})$-bimodule isomorphic to $\mathcal{M}$, and $\mathcal{G}_{21}$ is a $(\mathcal{G}_{22}, \mathcal{G}_{11})$-bimodule isomorphic to $\mathcal{N}$. Thus, $\mathcal{G}_{12}$ is a faithful $(\mathcal{G}_{11}, \mathcal{G}_{22})$-bimodule. Furthermore, for any $A \in \mathcal{G}$, A can be represented as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{G}_{ij}$ $(1 \leq i \leq j \leq 2)$.

In this section, our main result is the following Theorem 2.1. In [12], Daif proved every multiplicative derivation on a ring having an idempotent element which satisfies some conditions is additive. It is not hard to see an anti-derivation on a generalized matrix algebra $\mathcal{G}$ is a derivation from $\mathcal{G}$ into its anti-algebra. However, Theorem 2.1 is not a direct corollary of the theorem in [12]. This is because there are no idempotent elements in a generalized matrix algebra satisfying the conditions in [12]. Further, in [23], Ferreira and Sandhu showed that multiplicative anti-derivations are additive on generalized n-matrix rings. When $n = 2$, a generalized n-matrix ring is just the generalized matrix ring. However, these results are not the same as the following results.

Theorem 2.1 Let $\mathcal{G}$ be a generalized matrix algebra, and φ be a mapping of $\mathcal{G}$ (without assumption of additivity). If φ satisfies

$$\varphi(XY) = \varphi(Y)X + Y\varphi(X) \tag{2.1}$$

for all $X, Y \in \mathcal{G}$, then φ is additive.

In order to prove Theorem 2.1, we introduce Lemmas 2.1–2.4, and then prove that Lemmas 2.1–2.4 hold.

Lemma 2.1 If φ is an nonlinear anti-derivable mapping on $\mathcal{G}$, then

(i) $\varphi(0) = 0$;

(ii) $\varphi(P_1) = P_1\varphi(P_1)P_2 + P_2\varphi(P_1)P_1$;

(iii) $\varphi(P_2) = P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1$;

(iv) $\varphi(P_1) = -\varphi(P_2)$.

Proof (i) Taking $X = Y = 0$ in Eq (2.1), we have $\varphi(0) = \varphi(0)0 + 0\varphi(0) = 0$, and so $\varphi(0) = 0$.

(ii) Taking $X = P_1, Y = P_1$ in Eq (2.1), we get $\varphi(P_1) = \varphi(P_1)P_1 + P_1\varphi(P_1)$, which implies that $P_1\varphi(P_1)P_1 = P_2\varphi(P_1)P_2 = 0$. Hence, we obtain that $\varphi(P_1) = P_1\varphi(P_1)P_2 + P_2\varphi(P_1)P_1$. Similarly, we can show (iii) holds.

(iv) Taking $X = P_1, Y = P_2$ in Eq (2.1), we get

$$0 = \varphi(P_1P_2) = \varphi(P_2)P_1 + P_2\varphi(P_1) = P_2\varphi(P_2)P_1 + P_2\varphi(P_1)P_1.$$

Similarly, we get

$$0 = \varphi(P_2P_1) = \varphi(P_1)P_2 + P_1\varphi(P_2) = P_1\varphi(P_1)P_2 + P_1\varphi(P_2)P_2.$$

Adding the above two equations, it follows from Lemma 2.1 (ii) and (iii) that

$$\begin{aligned} 0 &= P_1\varphi(P_1)P_2 + P_2\varphi(P_1)P_1 + P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1 \\ &= \varphi(P_1) + \varphi(P_2). \end{aligned}$$

Therefore, $\varphi(P_1) = -\varphi(P_2)$. The proof is completed.

Lemma 2.2 If φ is an nonlinear anti-derivable mapping on $\mathcal{G}$, then for all $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}, A_{22} \in \mathcal{G}_{22}$,

(i) $\varphi(A_{12}) = P_2\varphi(A_{12})P_1$;

(ii) $\varphi(A_{21}) = P_1\varphi(A_{21})P_2$;

(iii) $\varphi(A_{11}) = P_1\varphi(A_{11})P_2 + P_2\varphi(A_{11})P_1$;

(iv) $\varphi(A_{22}) = P_1\varphi(A_{22})P_2 + P_2\varphi(A_{22})P_1$;

(v) $\varphi(P_1)A_{12} = \varphi(P_2)A_{12} = A_{12}\varphi(P_1) = A_{12}\varphi(P_2) = 0$;

(vi) $\varphi(P_1)A_{21} = \varphi(P_2)A_{21} = A_{21}\varphi(P_1) = A_{21}\varphi(P_2) = 0$.

Proof (i) For any $A_{12} \in \mathcal{G}_{12}$, taking $X = P_1, Y = A_{12}$ in Eq (2.1), we have

$$\varphi(A_{12}) = \varphi(P_1A_{12}) = \varphi(A_{12})P_1 + A_{12}\varphi(P_1). \tag{2.2}$$

This yields from $P_2\varphi(P_1)P_2 = 0$ that

$$P_1\varphi(A_{12})P_2 = P_2\varphi(A_{12})P_2 = 0.$$

Similarly, we have

$$\varphi(A_{12}) = \varphi(A_{12}P_2) = \varphi(P_2)A_{12} + P_2\varphi(A_{12}). \tag{2.3}$$

This implies that

$$P_1\varphi(A_{12})P_1 = 0.$$

Therefore, we get $\varphi(A_{12}) = P_2 d_1(A_{12})P_1$. Similarly, we can show that (ii) holds.

(iii) For any $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}$, taking $X = A_{12}, Y = A_{11}$ in Eq (2.1), then by Lemma 2.2 (i), we have

$$0 = \varphi(A_{12}A_{11}) = \varphi(A_{11})A_{12} + A_{11}\varphi(A_{12}) = \varphi(A_{11})A_{12}.$$

This yields from the faithfulness of $\mathcal{G}_{12}$ that

$$P_1\varphi(A_{11})P_1 = 0.$$

Similarly, taking $X = P_2, Y = A_{11}$ in Eq (2.1), we have

$$0 = \varphi(P_2A_{11}) = \varphi(A_{11})P_2 + A_{11}\varphi(P_2).$$

This implies that

$$P_2\varphi(A_{11})P_2 = 0.$$

Therefore, we obtain that $\varphi(A_{11}) = P_1\varphi(A_{11})P_2 + P_2\varphi(A_{11})P_1$. Similarly, we can show that (iv) holds.

(v) For any $A_{12} \in \mathcal{G}_{12}$, it follows from Eqs (2.2)–(2.3) and $\varphi(A_{12}) = P_2\varphi(A_{12})P_1$ that

$$P_1\varphi(A_{12}) = 0 = A_{12}\varphi(P_1) \text{ and } \varphi(A_{12})P_2 = 0 = \varphi(P_2)A_{12}.$$

Therefore, we obtain from $\varphi(P_1) = -\varphi(P_2)$ that $\varphi(P_1)A_{12} = \varphi(P_2)A_{12} = A_{12}\varphi(P_1) = A_{12}\varphi(P_2) = 0$. Similarly, we can show that (vi) holds.

Lemma 2.3 If φ is a nonlinear anti-derivable mapping on $\mathcal{G}$, then for all $A_{11}, B_{11} \in \mathcal{G}_{11}, A_{12}, B_{12} \in \mathcal{G}_{12}, A_{21}, B_{21} \in \mathcal{G}_{21}, A_{22}, B_{22} \in \mathcal{G}_{22}$,

(i) $\varphi(A_{11} + B_{11}) = \varphi(A_{11}) + \varphi(B_{11})$;
(ii) $\varphi(A_{22} + B_{22}) = \varphi(A_{22}) + \varphi(B_{22})$;
(iii) $\varphi(A_{11} + A_{12}) = \varphi(A_{11}) + \varphi(A_{12})$;
(iv) $\varphi(A_{12} + A_{22}) = \varphi(A_{12}) + \varphi(A_{22})$;
(v) $\varphi(A_{21} + A_{22}) = \varphi(A_{21}) + \varphi(A_{22})$;
(vi) $\varphi(A_{12} + B_{12}) = \varphi(A_{12}) + \varphi(B_{12})$;
(vii) $\varphi(A_{21} + B_{21}) = \varphi(A_{21}) + \varphi(B_{21})$.

Proof (i) For any $A_{11}, B_{11} \in \mathcal{G}_{11}$, taking $X = A_{11}, Y = P_1$ in Eq (2.1), we have

$$P_2\varphi(A_{11})P_1 = \varphi(P_1)A_{11}.$$

Taking $X = P_2, Y = A_{11}$ in Eq (2.1), we have $0 = \varphi(P_2 A_{11}) = \varphi(A_{11})P_2 + A_{11}\varphi(P_2)$, this yields from $\varphi(P_1) = -\varphi(P_2)$ that

$$P_1\varphi(A_{11})P_2 = -A_{11}\varphi(P_2) = A_{11}\varphi(P_1).$$

Hence, we can get from above two equations and $\varphi(A_{11}) = P_1\varphi(A_{11})P_2 + P_2\varphi(A_{11})P_1$ that

$$\varphi(A_{11}) = A_{11}\varphi(P_1) + \varphi(P_1)A_{11}.$$

Similarly, we get

$$\varphi(B_{11}) = B_{11}\varphi(P_1) + \varphi(P_1)B_{11}.$$

And,

$$\varphi(A_{11} + B_{11}) = (A_{11} + B_{11})\varphi(P_1) + \varphi(P_1)(A_{11} + B_{11}).$$

Therefore, it follows from above three equations that $\varphi(A_{11} + B_{11}) = \varphi(A_{11}) + \varphi(B_{11})$. Similarly, we can show (ii) holds.

(iii) For any $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}$, taking $X = A_{11} + A_{12}, Y = P_1$ in Eq (2.1), we get that

$$\varphi(A_{11}) = \varphi((A_{11} + A_{12})P_1) = \varphi(P_1)(A_{11} + A_{12}) + P_1\varphi(A_{11} + A_{12}) = \varphi(P_1)A_{11} + P_1\varphi(A_{11} + A_{12}).$$

Similarly, taking $X = A_{11} + A_{12}, Y = P_2$ in Eq (2.1), by Lemma 2.2 (v), we have

$$\varphi(A_{12}) = \varphi((A_{11} + A_{12})P_2) = \varphi(P_2)(A_{11} + A_{12}) + P_2\varphi(A_{11} + A_{12}) = \varphi(P_2)A_{11} + P_2\varphi(A_{11} + A_{12}).$$

Adding the above two equations, we then obtain from $\varphi(P_1) = -\varphi(P_2)$ that $\varphi(A_{11} + A_{12}) = \varphi(A_{11}) + \varphi(A_{12})$. Similarly, we can show (iv) and (v) hold.

(vi) For any $A_{12}, B_{12} \in \mathcal{G}_{12}$, taking $X = A_{12}, Y = B_{12}$ in Eq (2.1), it follows from $A_{12}B_{12} = 0$ that

$$0 = \varphi(A_{12}B_{12}) = \varphi(B_{12})A_{12} + B_{12}\varphi(A_{12}). \tag{2.4}$$

Since $A_{12} + B_{12} = (P_1 + A_{12})(P_2 + B_{12})$, we take $X = P_1 + A_{12}, Y = P_2 + B_{12}$ in Eq (2.1), and then we get from Lemma 2.2, Lemma 2.3(i),(v), Lemma 2.4(iii)-(iv) and Eq (2.4) that

$$\begin{aligned}
\varphi(A_{12} + B_{12}) &= \varphi((P_1 + A_{12})(P_2 + B_{12})) \\
&= \varphi(P_2 + B_{12})(P_1 + A_{12}) + (P_2 + B_{12})\varphi(P_1 + A_{12}) \\
&= (\varphi(P_2) + \varphi(B_{12}))(P_1 + A_{12}) + (P_2 + B_{12})(\varphi(P_1) + \varphi(A_{12})) \\
&= \varphi(P_2)P_1 + \varphi(P_2)A_{12} + \varphi(B_{12})P_1 + \varphi(B_{12})A_{12} \\
&+ P_2\varphi(P_1) + P_2\varphi(A_{12}) + B_{12}\varphi(P_1) + B_{12}\varphi(A_{12}) \\
&= P_2\varphi(A_{12}) + \varphi(B_{12})P_1 \\
&= P_2\varphi(A_{12})P_1 + P_2\varphi(B_{12})P_1 \\
&= \varphi(A_{12}) + \varphi(B_{12}).
\end{aligned}$$

Similarly, we can show (vii) holds. The proof is completed.

Lemma 2.4 If φ is a nonlinear anti-derivable mapping on $\mathcal{G}$, then $\varphi(A_{11} + A_{12} + A_{21} + A_{22}) = \varphi(A_{11}) + \varphi(A_{12}) + \varphi(A_{21}) + \varphi(A_{22})$ for all $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}$ and $A_{22} \in \mathcal{G}_{22}$.

Proof For any $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}, A_{22} \in \mathcal{G}_{22}$, taking $X = P_1, Y = A_{11} + A_{12} + A_{21} + A_{22}$ in Eq (2.1), we get from Lemma 2.3 (iii),(v) and Lemma 2.2(v)-(vi) that

$$\begin{aligned}\varphi(A_{11}) + \varphi(A_{12}) &= \varphi(P_1(A_{11} + A_{12} + A_{21} + A_{22}))\\ &= \varphi(A_{11} + A_{12} + A_{21} + A_{22})P_1 + (A_{11} + A_{12} + A_{21} + A_{22})\varphi(P_1)\\ &= \varphi(A_{11} + A_{12} + A_{21} + A_{22})P_1 + (A_{11} + A_{22})\varphi(P_1).\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}\varphi(A_{21}) + \varphi(A_{22}) &= \varphi(P_2(A_{11} + A_{12} + A_{21} + A_{22}))\\ &= \varphi(A_{11} + A_{12} + A_{21} + A_{22})P_2 + (A_{11} + A_{12} + A_{21} + A_{22})\varphi(P_2)\\ &= \varphi(A_{11} + A_{12} + A_{21} + A_{22})P_2 + (A_{11} + A_{22})\varphi(P_2).\end{aligned}$$

Adding the above two equations, using $\varphi(P_1) = -\varphi(P_2)$, we get $\varphi(A_{11} + A_{12} + A_{21} + A_{22}) = \varphi(A_{11}) + \varphi(A_{12}) + \varphi(A_{21}) + \varphi(A_{22})$. The proof is completed.

Now, we complete the proof of Theorem 2.1.

Proof of Theorem 2.1 For any $X, Y \in \mathcal{G}$, set $X = A_{11} + A_{12} + A_{21} + A_{22}$ and $Y = B_{11} + B_{12} + B_{21} + B_{22}$, where $A_{ij}, B_{ij} \in \mathcal{G}_{ij} (1 \leq i \leq j \leq 2)$, then by Lemmas 2.3 and 2.4, we obtain that

$$\begin{aligned}\varphi(X + Y) &= \varphi((A_{11} + A_{12} + A_{21} + A_{22}) + (B_{11} + B_{12} + B_{21} + B_{22}))\\ &= \varphi((A_{11} + B_{11}) + (A_{12} + B_{12}) + (A_{21} + B_{21}) + (A_{22} + B_{22}))\\ &= \varphi(A_{11} + B_{11}) + \varphi(A_{12} + B_{12}) + \varphi(A_{21} + B_{21}) + \varphi(A_{22} + B_{22})\\ &= \varphi(A_{11}) + \varphi(B_{11}) + \varphi(A_{12}) + \varphi(B_{12}) + \varphi(A_{21}) + \varphi(B_{21}) + \varphi(A_{22}) + \varphi(B_{22})\\ &= \varphi(A_{11} + A_{12} + A_{21} + A_{22}) + \varphi(B_{11} + B_{12} + B_{21} + B_{22})\\ &= \varphi(X) + \varphi(Y).\end{aligned}$$

Therefore, φ is an additive mapping on $\mathcal{G}$. The proof is completed.

Next, we will give the second main result.

3. Additivity of nonlinear higher anti-derivable mappings on generalized matrix algebras

Theorem 3.1 Let $\mathcal{G}$ be a generalized matrix algebra and $D = \{d_n\}_{n\in\mathbb{N}}$ be a sequence mapping from $\mathcal{G}$ into itself (without assumption of additivity) such that

$$d_n(XY) = \sum_{i+j=n} d_i(Y)d_j(X) \tag{3.1}$$

for any $n \in \mathbb{N}, X, Y \in \mathcal{G}$, then D is an additive mapping on $\mathcal{G}$.

In the following, to prove Theorem 3.1, we will introduce Lemmas 3.1–3.3, and then use mathematical induction to prove that Lemmas 3.1–3.3 hold. We assume that $\mathcal{G}$ is a generalized matrix algebra, and $D = \{d_n\}_{n\in\mathbb{N}}$ is a higher anti-derivable mapping on $\mathcal{G}$. Let $\mathbb{N}$ be the set of non-negative integers, $\mathbb{N}^+$ be the set of positive integers, and $i, j, k, p, q, n \in \mathbb{N}$. For any $X, Y \in \mathcal{G}$,

$A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}, A_{22} \in \mathcal{G}_{22}$. We say a map $f : \mathcal{G} \to \mathcal{G}$ satisfies the set of properties $\mathcal{L}$, if

(i) $f(X + Y) = f(X) + f(Y)$;
(ii) $f(0) = 0, f(P_1) = -f(P_2) \in \mathcal{M} + \mathcal{N}$;
(iii) $f(A_{12}) = P_2 f(A_{12}) P_1$;
(iv) $f(P_1)A_{12} = f(P_2)A_{12} = A_{12}f(P_1) = A_{12}f(P_2) = 0$;
(v) $f(A_{21}) = P_1 f(A_{21}) P_2$;
(vi) $f(P_1)A_{21} = f(P_2)A_{21} = A_{21}f(P_1) = A_{21}f(P_2) = 0$;
(vii) $f(A_{11}) = P_1 f(A_{11}) P_2 + P_2 f(A_{11}) P_1$;
(viii) $f(A_{22}) = P_1 f(A_{22}) P_2 + P_2 f(A_{22}) P_1$.

It is known from Theorem 2.1 that d_1 satisfies the set of properties $\mathcal{L}$. Now, for any $X, Y \in \mathcal{G}$, $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}, A_{22} \in \mathcal{G}_{22}$, we assume that $d_k(1 \leqslant k < n)$ satisfies the set of properties $\mathcal{L}$. In the following, we show d_n satisfies the set of properties $\mathcal{L}$.

Lemma 3.1 For any $n \in \mathbb{N}^+, A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}, A_{22} \in \mathcal{G}_{22}$, d_n satisfies the set of properties $\mathcal{L}$.

Proof (i) For any $n \in \mathbb{N}^+$, taking $X = Y = 0$ in Eq (3.1), it follows from the set of properties $\mathcal{L}$(ii) that

$$d_n(0) = \sum_{i+j=n} d_i(0)d_j(0) = \sum_{i+j=n, 1\leq i,j} d_i(0)d_j(0) + d_n(0)0 + 0d_n(0) = 0.$$

For any $n, i, j \in \mathbb{N}^+(i, j < n)$, since $d_i(P_1), d_j(P_1), d_i(P_2), d_j(P_2) \in \mathcal{M}+\mathcal{N}$, and so by the set of properties $\mathcal{L}$ (iv) and (vi), we get that

$$d_i(P_1)d_j(P_1) = d_i(P_2)d_j(P_2) = d_i(P_1)d_j(P_2) = d_i(P_2)d_j(P_1) = 0. \tag{3.2}$$

Taking $X = P_1, Y = P_1$ in Eq (3.1), by Eq (3.2), we get

$$\begin{aligned} d_n(P_1) &= \sum_{i+j=n} d_i(P_1)d_j(P_1) \\ &= \sum_{i+j=n,1\leq i,j} d_i(P_1)d_j(P_1) + d_n(P_1)P_1 + P_1 d_n(P_1) \\ &= d_n(P_1)P_1 + P_1 d_n(P_1). \end{aligned}$$

This implies that

$$P_1 d_n(P_1) P_1 = P_2 d_n(P_1) P_2 = 0. \tag{3.3}$$

Similarly, we have

$$P_1 d_n(P_2) P_1 = P_2 d_n(P_2) P_2 = 0. \tag{3.4}$$

Taking $X = P_1, Y = P_2$ in Eq (3.1), by Eq (3.2), we get

$$0 = \sum_{i+j=n} d_i(P_2)d_j(P_1)$$

$$
\begin{aligned}
&= \sum_{i+j=n,1\leq i,j} d_i(P_1)d_j(P_2) + d_n(P_2)P_1 + P_2d_n(P_1) \\
&= d_n(P_2)P_1 + P_2d_n(P_1).
\end{aligned}
$$

Therefore, we get

$$P_2d_n(P_2)P_1 = -P_2d_n(P_1)P_1. \tag{3.5}$$

Similarly, we obtain that

$$P_1d_n(P_2)P_2 = -P_1d_n(P_1)P_2. \tag{3.6}$$

Therefore, by Eqs (3.3)–(3.6), we get that $d_n(P_1) = -d_n(P_2) \in \mathcal{M} + \mathcal{N}$.

(ii)-(iii) For any $n \in \mathbb{N}^+$, $A_{12} \in \mathcal{G}_{12}$, taking $X = P_1, Y = A_{12}$ in Eq (3.1), it follows from (iii) and (vi) of the set of properties $\mathcal{L}$ that

$$
\begin{aligned}
d_n(A_{12}) &= d_n(P_1A_{12}) \\
&= \sum_{i+j=n} d_i(A_{12})d_j(P_1) \\
&= \sum_{i+j=n,1\leq i,j} P_2d_i(A_{12})P_1d_j(P_1) + d_n(A_{12})P_1 + A_{12}d_n(P_1) \\
&= d_n(A_{12})P_1 + A_{12}d_n(P_1).
\end{aligned}
$$

This yields from $P_2d_n(P_1)P_2 = 0$ that

$$P_2d_n(A_{12})P_2 = P_1d_n(A_{12})P_2 = A_{12}d_n(P_1) = 0. \tag{3.7}$$

Similarly, we get

$$
\begin{aligned}
d_n(A_{12}) &= d_n(A_{12}P_2) \\
&= \sum_{i+j=n} d_i(P_2)d_j(A_{12}) \\
&= \sum_{i+j=n,1\leq i,j} d_i(P_2)P_2d_j(A_{12})P_1 + d_n(P_2)A_{12} + P_2d_n(A_{12}) \\
&= d_n(P_2)A_{12} + P_2d_n(A_{12}).
\end{aligned}
$$

This yields that

$$P_1d_n(A_{12})P_1 = d_n(P_2)A_{12} = 0. \tag{3.8}$$

Therefore, by $d_n(P_1) = -d_n(P_2)$ and Eqs (3.7) and (3.8), we get (ii) and (iii). Similarly, we can show that (iv) and (v) hold.

(vi) For any $n \in \mathbb{N}^+, A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}$, taking $X = A_{11}, Y = A_{12}$ in Eq (3.1), it follows from the set of properties $\mathcal{L}$ (ii) and Lemma 3.1 (ii) that

$$0 = d_n(A_{12}A_{11})$$

$$
\begin{aligned}
&= \sum_{i+j=n} d_i(A_{11})d_j(A_{12}) \\
&= \sum_{i+j=n,1\leq i,j} d_i(A_{11})(P_2d_j(A_{12})P_1) + d_n(A_{11})A_{12} + A_{11}d_n(A_{12}) \\
&= \sum_{i+j=n,1\leq i,j} P_1d_i(A_{11})P_2d_j(A_{12})P_1 + d_n(A_{11})A_{12}.
\end{aligned}
$$

This implies that $P_1d_n(A_{11})P_1A_{12} = 0$, and so by the faithfulness of $\mathcal{G}_{12}$, we get

$$P_1d_n(A_{11})P_1 = 0 \tag{3.9}$$

Taking $X = A_{11}, Y = P_1$ in Eq (3.1), we get from (iv), (vi) and (vii) of the set of properties $\mathcal{L}$ and Lemma 3.1 (vii) that

$$
\begin{aligned}
d_n(A_{11}) &= d_n(A_{11}P_1) \\
&= \sum_{i+j=n} d_i(P_1)d_j(A_{11}) \\
&= \sum_{i+j=n,1\leq i,j} d_i(P_1)(P_1d_j(A_{11})P_2 + P_2d_j(A_{11})P_1) \\
&+ d_n(P_1)A_{11} + P_1d_n(A_{11}) \\
&= d_n(P_1)A_{11} + P_1d_n(A_{11}).
\end{aligned}
$$

This yields that

$$P_2d_n(A_{11})P_2 = 0 \ \text{ and } \ P_2d_n(A_{11})P_1 = d_n(P_1)A_{11}. \tag{3.10}$$

Similarly, taking $X = P_2, Y = A_{11}$ in Eq (3.1), we get from (iv), (vi) and (vii) of the set of properties $\mathcal{L}$ and Lemma 3.1 (vii) that

$$
\begin{aligned}
0 &= d_n(P_2A_{11}) \\
&= \sum_{i+j=n} d_i(A_{11})d_j(P_2) \\
&= \sum_{i+j=n,1\leq i,j} (P_1d_i(A_{11})P_2 + P_2d_i(A_{11})P_1)d_j(P_2) \\
&\quad +d_n(A_{11})P_2 + A_{11}d_n(P_2) \\
&= d_n(A_{11})P_2 + A_{11}d_n(P_2).
\end{aligned}
$$

This yields that

$$P_1d_n(A_{11})P_2 = -A_{11}d_n(P_2) = A_{11}d_n(P_1). \tag{3.11}$$

Therefore, we get from Eqs (3.9)–(3.11) that

$$d_n(A_{11}) = P_1d_n(A_{11})P_2 + P_2d_n(A_{11})P_1 = A_{11}d_n(P_1) + d_n(P_1)A_{11}. \tag{3.12}$$

Similarly, we can show that (viii) holds. The proof is completed.

Lemma 3.2 For any $n \in \mathbb{N}^+$, $A_{11}, B_{11} \in \mathcal{G}_{11}, A_{12}, B_{12} \in \mathcal{G}_{12}, A_{21}, B_{21} \in \mathcal{G}_{21}, A_{22}, B_{22} \in \mathcal{G}_{22}$, then
(i) $d_n(A_{11} + B_{11}) = d_n(A_{11}) + d_n(B_{11})$;
(ii) $d_n(A_{22} + B_{22}) = d_n(A_{22}) + d_n(B_{22})$;
(iii) $d_n(A_{11} + A_{12}) = d_n(A_{11}) + d_n(A_{12})$;
(iv) $d_n(A_{12} + A_{22}) = d_n(A_{12}) + d_n(A_{22})$;
(v) $d_n(A_{21} + A_{22}) = d_n(A_{21}) + d_n(A_{22})$;
(vi) $d_n(A_{12} + B_{12}) = d_n(A_{12}) + d_n(B_{12})$;
(vii) $d_n(A_{21} + B_{21}) = d_n(A_{21}) + d_n(B_{21})$.

Proof (i) For any $n \in \mathbb{N}^+, A_{11}, B_{11} \in \mathcal{G}_{11}$, we get from Eq (3.12) that

$$\begin{aligned} d_n(A_{11} + B_{11}) &= (A_{11} + B_{11})d_n(P_1) + d_n(P_1)(A_{11} + B_{11}) \\ &= (A_{11}d_n(P_1) + d_n(P_1)A_{11}) + (B_{11}d_n(P_1) + d_n(P_1)B_{11}) \\ &= d_n(A_{11}) + d_n(B_{11}) \end{aligned}$$

Similarly, we show that (ii) holds.

(iii) For any $n \in \mathbb{N}^+, A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}$, taking $X = A_{11} + A_{12}, Y = P_1$ in Eq (3.11), we get from the set of properties $\mathcal{L}$ (i) and Lemma 3.1 that

$$\begin{aligned} d_n(A_{11}) &= d_n((A_{11} + A_{12})P_1) \\ &= \sum_{i+j=n} d_i(P_1)d_j(A_{11} + A_{12}) \\ &= \sum_{i+j=n,1\le i,j} d_i(P_1)(d_j(A_{11}) + d_j(A_{12})) + d_n(P_1)(A_{11} + A_{12}) + P_1d_n(A_{11} + A_{12}) \\ &= \sum_{i+j=n,1\le i,j} d_i(P_1)d_j(A_{11}) + d_n(P_1)A_{11} + P_1d_n(A_{11} + A_{12}) \\ &= \sum_{i+j=n,1\le i,j} d_i(P_1)(P_1d_j(A_{11})P_2 + P_2d_j(A_{11})P_1) + d_n(P_1)A_{11} + P_1d_n(A_{11} + A_{12}) \\ &= d_n(P_1)A_{11} + P_1d_n(A_{11} + A_{12}). \end{aligned}$$

Thus, we get

$$d_n(A_{11}) = d_n(P_1)A_{11} + P_1d_n(A_{11} + A_{12}).$$

Similarly, taking $X = A_{11} + A_{12}, Y = P_2$ in Eq (3.1), we obtain that

$$d_n(A_{12}) = d_n(P_2)A_{11} + P_2d_n(A_{11} + A_{12}).$$

Adding the above two equations, we obtain from $d_n(P_1) = -d_n(P_2)$ that $d_n(A_{11} + A_{12}) = d_n(A_{11}) + d_n(A_{12})$. Similarly, we can show (iv) and (v) hold.

(vi) For any $A_{12}, B_{12} \in \mathcal{G}_{12}$, taking $X = A_{12}, Y = B_{12}$ in Eq (3.1), then it follows from $A_{12}B_{12} = 0$ that

$$0 = d_n(A_{12}B_{12}) = \sum_{i+j=n} d_i(B_{12})d_j(A_{12}). \tag{3.13}$$

Since $A_{12} + B_{12} = (P_1 + A_{12})(P_2 + B_{12})$, we take $X = P_1 + A_{12}, Y = P_2 + B_{12}$ in Eq (3.1), and then we get from Lemma 3.1, Lemma 3.2(iii)-(iv), and Eq (3.13) that

$$\begin{aligned}
d_n(A_{12} + B_{12}) &= d_n((P_1 + A_{12})(P_2 + B_{12})) \\
&= \sum_{i+j=n} d_i(P_2 + B_{12})d_j(P_1 + A_{12}) \\
&= \sum_{i+j=n,1\le i,j} d_i(P_2 + B_{12})d_j(P_1 + A_{12}) \\
&+ d_n(P_2 + B_{12})(P_1 + A_{12}) + (P_2 + B_{12})d_n(P_1 + A_{12}) \\
&= \sum_{i+j=n,1\le i,j} d_i(P_2)(d_j(P_1) + d_j(A_{12})) + \sum_{i+j=n,1\le i,j} d_i(B_{12})(d_j(P_1) + d_j(A_{12})) \\
&+ (d_n(P_2) + d_n(B_{12}))(P_1 + A_{12}) + (P_2 + B_{12})(d_n(P_1) + d_n(A_{12})) \\
&= \sum_{i+j=n,1\le i,j} d_i(B_{12})d_j(A_{12}) \\
&+ d_n(P_2)P_1 + d_n(B_{12})P_1 + d_n(B_{12})A_{12} + P_2d_n(P_1) + P_2d_n(A_{12}) + B_{12}d_n(A_{12}) \\
&= \sum_{i+j=n} d_i(B_{12})d_j(A_{12}) - P_2d_n(P_1)P_1 + P_2d_n(P_1)P_1 \\
&+ P_2d_n(A_{12})P_1 + P_2d_n(B_{12})P_1 \\
&= d_n(A_{12}) + d_n(B_{12}).
\end{aligned}$$

Similarly, we can show (vii) holds. The proof is completed.

Lemma 3.3 For any $n \in \mathbb{N}^+$, $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}$ and $A_{22} \in \mathcal{G}_{22}$, then $d_n(A_{11} + A_{12} + A_{21} + A_{22}) = d_n(A_{11}) + d_n(A_{12}) + d_n(A_{21}) + d_n(A_{22})$.

Proof For any $n \in \mathbb{N}^+$, $A_{11} \in \mathcal{G}_{11}, A_{12} \in \mathcal{G}_{12}, A_{21} \in \mathcal{G}_{21}, A_{22} \in \mathcal{G}_{22}$, taking $X = P_1, Y = A_{11} + A_{12} + A_{21} + A_{22}$ in Eq (3.1), we obtain from the set of properties $\mathcal{L}$ (i), Lemma 3.2 (iii) and (v) that

$$\begin{aligned}
d_n(A_{11}) + d_n(A_{12}) &= d_n(A_{11} + A_{12}) \\
&= d_n(P_1(A_{11} + A_{12} + A_{21} + A_{22})) \\
&= \sum_{i+j=n} d_i(A_{11} + A_{12} + A_{21} + A_{22})d_j(P_1) \\
&= \sum_{i+j=n,1\le i,j} d_i(A_{11} + A_{12} + A_{21} + A_{22})d_j(P_1) \\
&+ d_n(A_{11} + A_{12} + A_{21} + A_{22})P_1 \\
&+ (A_{11} + A_{12} + A_{21} + A_{22})d_n(P_1).
\end{aligned}$$

Similarly, we take $X = P_2, Y = A_{11} + A_{12} + A_{21} + A_{22}$ in Eq (3.1), then we obtain that

$$\begin{aligned}
d_n(A_{21}) + d_n(A_{22}) &= d_n(A_{21} + A_{22}) \\
&= d_n(P_2(A_{11} + A_{12} + A_{21} + A_{22})) \\
&= \sum_{i+j=n} d_i(A_{11} + A_{12} + A_{21} + A_{22})d_j(P_2) \\
&= \sum_{i+j=n,1\le i,j} d_i(A_{11} + A_{12} + A_{21} + A_{22})d_j(P_2)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad d_n(A_{11} + A_{12} + A_{21} + A_{22})P_2 \\
&+ \quad (A_{11} + A_{12} + A_{21} + A_{22})d_n(P_2).
\end{aligned}
$$

Adding the above two equations, by $d_n(P_1) + d_n(P_2) = 0$, we get that $d_n(A_{11} + A_{12} + A_{21} + A_{22}) = d_n(A_{11}) + d_n(A_{12}) + d_n(A_{21}) + d_n(A_{22})$. The proof is completed.

Now, we complete the proof of Theorem 3.1.

Proof of Theorem 3.1 For any $n \in \mathbb{N}^+$, $X, Y \in \mathcal{G}$, set $X = A_{11} + A_{12} + A_{21} + A_{22}$ and $Y = B_{11} + B_{12} + B_{21} + B_{22}$, where $A_{ij}, B_{ij} \in \mathcal{G}_{ij} (1 \le i \le j \le 2)$, then, by Lemmas 3.2 and 3.3, we can obtain that

$$
\begin{aligned}
d_n(X+Y) &= d_n((A_{11} + A_{12} + A_{21} + A_{22}) + (B_{11} + B_{12} + B_{21} + B_{22})) \\
&= d_n((A_{11} + B_{11}) + (A_{12} + B_{12}) + (A_{21} + B_{21}) + (A_{22} + B_{22})) \\
&= d_n(A_{11} + B_{11}) + d_n(A_{12} + B_{12}) + d_n(A_{21} + B_{21}) + d_n(A_{22} + B_{22}) \\
&= d_n(A_{11}) + d_n(B_{11}) + d_n(A_{12}) + d_n(B_{12}) + d_n(A_{21}) + d_n(B_{21}) + d_n(A_{22}) + d_n(B_{22}) \\
&= d_n(A_{11} + A_{12} + A_{21} + A_{22}) + d_n(B_{11} + B_{12} + B_{21} + B_{22}) \\
&= d_n(X) + d_n(Y).
\end{aligned}
$$

Therefore, $D = \{d_n\}_{n\in\mathbb{N}}$ is an additive mapping on $\mathcal{G}$. The proof is completed.

In the following, we give some applications of Theorem 3.1.

Because triangular algebras and full matrix algebras are two special classes of generalized matrix algebras, we can get Corollaries 3.1–3.5 immediately. For the definition of triangular algebra, we refer readers to [24]. It is worth pointing out that, in [25], Ferreira showed that under certain conditions, every m-multiplicative derivation on a triangular n-matrix ring is additive.

Corollary 3.1 Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras, $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module, and $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$ be a triangular algebra. If $D = \{d_n\}_{n\in\mathbb{N}}$ is a nonlinear higher anti-derivable mapping on $\mathcal{U}$, then $D = \{d_n\}_{n\in\mathbb{N}}$ is additive.

Corollary 3.2 Let $\mathcal{A}$ be an unital algebra, and $\mathcal{M}_n(\mathcal{A})(2 \le n)$ be the full matrix algebras of all $n \times n$ matrices over $\mathcal{A}$. If $D = \{d_n\}_{n\in\mathbb{N}}$ is a nonlinear higher anti-derivable mapping on $\mathcal{M}_n(\mathcal{A})$, then $D = \{d_n\}_{n\in\mathbb{N}}$ is additive.

Corollary 3.3 Let $\mathcal{R}$ be a unital prime ring with a nontrivial idempotent P, and I be the unit of $\mathcal{R}$. If $D = \{d_n\}_{n\in\mathbb{N}}$ is a nonlinear higher anti-derivable mapping on $\mathcal{U}$, then $D = \{d_n\}_{n\in\mathbb{N}}$ is additive.

Proof of Corollary 3.3 Suppose $Q = I - P$. Since $\mathcal{R}$ a prime ring, it follows that $P\mathcal{R}Q$ is a faithful $(P\mathcal{R}P, Q\mathcal{R}Q)$-bimodule. Then, $\mathcal{R}$ is isomorphic to the generalized matrix algebra

$$
\begin{pmatrix} P\mathcal{R}P & P\mathcal{R}Q \\ Q\mathcal{R}P & Q\mathcal{R}Q \end{pmatrix}
$$

Therefore, by Theorem 3.1, we know that $D = \{d_n\}_{n\in\mathbb{N}}$ is additive.

Since standard operator algebras and factor von Neumann algebras are prime algebras with nontrivial idempotents, by Corollary 3.3, we obtain Corollary 3.4 and Corollary 3.5 as follows.

Corollary 3.4 Let $\mathcal{X}$ be a Banach space over number field $\mathcal{F}$, and $\mathcal{A}(\mathcal{X})$ be an unital standard operator algebra over $\mathcal{X}$. If $D = \{d_n\}_{n\in\mathbb{N}}$ is a nonlinear higher anti-derivable mapping on $\mathcal{A}(\mathcal{X})$, then $D = \{d_n\}_{n\in\mathbb{N}}$ is additive.

Corollary 3.5 Let $\mathcal{H}$ be a Hilbert space over number field $\mathcal{F}$, and $\mathcal{V}$ be a factor von Neumann algebra over $\mathcal{H}$. If $D = \{d_n\}_{n\in\mathbb{N}}$ is a nonlinear higher anti-derivable mapping on $\mathcal{V}$, then $D = \{d_n\}_{n\in\mathbb{N}}$ is additive.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by Talent Project Foundation of Yunnan Province (No.202105AC160089), Natural Science Foundation of Yunnan Province(No.202101BA070001198), and Basic Research Foundation of Yunnan Education Department(Nos.2020J0748,2021J0635).

Conflict of interest

The authors declare there is no conflicts of interest.

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