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Research article

Global dynamics of an endemic disease model with vaccination: Analysis of the asymptomatic and symptomatic groups in complex networks

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Abstract: In this paper, we analyze the global dynamics of an endemic mathematical model that incorporates direct immunity by vaccination, as well as the shift from the asymptomatic to the symptomatic group in complex networks. By analyzing the Jacobian matrix and constructing suitable Lyapunov functionals, the stability of the disease-free equilibrium and the endemic equilibrium is determined with respect to the basic reproduction number R_0. Numerical simulations in scale-free and Poisson network environments are presented. The results validate the correctness of our theoretical analyses.

Keywords: complex networks; equilibrium; stability; vaccination

1. Introduction

It is well known that many infectious diseases have an incubation period. For example, the average incubation period of HIV is 8–9 years. SARS-CoV-2, which causes COVID-19, also has an incubation period, and people infected by the disease appear as asymptomatic or symptomatic and are able to infect susceptible individuals. To reduce the spread of COVID-19, vaccination is a commonly used control measure. Thus, investigating the impact of vaccines on the dynamics of endemic diseases is crucial.

Many endemic compartmental models for COVID-19 have been proposed to study the transmission dynamics since its outbreak. For example, Ma et al. [1] proposed an SEIR-type epidemic model that considers the contact distance between the susceptible individuals and the asymptomatic or symptomatic infected individuals. They analyzed the stability of each equilibrium and obtained the threshold values for population influx and contact distance. Simulation of the SARS-CoV-2 pandemic in Germany has been performed with ordinary differential equations in MATLAB, as presented in [2].

It revealed vaccination to be an effective means in the fight against the pandemic. Yuan et al. [3] discussed an $SEII_aHR$ model and obtained that the disease-free equilibrium (DFE) is globally asymptotically stable if $R_0 < 1$, and that the endemic equilibrium (EE) is uniformly persistent if $R_0 > 1$. Chen et al. [4] considered both the SEIR model and the SPEIQRD model incorporating two control actions, i.e., vaccination and quarantine, to estimate the spread of the virus and control the number of infected and dead people via controller tuning. Rakshit et al. [5] proposed an SIR model with the asymptomatic and symptomatic groups. They simulated case data from the UK, USA and India with their model. Guo et al. [6] proposed an SEIAQR model for COVID-19. By using the limit system of the model and Lyapunov function method, they proved that COVID-19 would die out if the basic reproduction number $R_c \leq 1$, and that the disease is persistent if $R_c > 1$. Khoshnaw et al. [7] discussed an SIUWR model for COVID-19, and some key computational simulations and sensitivity analysis were investigated. Aguilar et al. [8] considered a mathematical model that takes asymptomatic carriers into account. Peirlinck et al. [9] modeled the epidemiology of COVID-19 by using an SEIIR model. Simulations indicated that encouraging increased hygiene, mandating the use of face masks, restricting travel and maintaining distance could be the most successful strategies to manage the impact of COVID-19 until vaccination and treatment could become available.

Since the mutation of single-stranded RNA viruses is relatively strong, such as those for HIV, HCV, SARS, MERS-CoV, SARS-CoV-2 and so on, multi-strain models have also been intensively proposed to study epidemic dynamics; for examples, see [10–14] and the references therein. Many authors have discussed the stability of equilibria in their studies, which may help us determine the long-term behaviors of disease dynamics.

The studies mentioned above mainly made homogeneous contact assumptions, i.e., that each individual in the population has the same probability of contact with an infected individual. However, the contact among people is heterogeneous in the real world. For this reason, the notion of complex networks is incorporated. Many endemic models based on networks have been proposed. Huang et al. [15] proposed a network-based SIQRS epidemic model. Sun et al. [16] constructed an SIS model on networks. They studied the dynamics of the models, including the basic reproduction number and the global asymptotic stability of the DFE and EE. Meng et al. [17] established an SEIRV model on heterogeneous networks. In addition, they used the epidemic parameters in Wuhan, China for numerical analysis. Lv et al. [18] proposed an SIVS epidemic model based on scale-free networks. Li and Yousef [19] proposed a network-based SIR epidemic model with a saturated treatment function. And, they obtained the threshold values R_0 and $\hat{R}_0$. The global stability of equilibria strongly rely on both of them. Besides, forward or backward bifurcation will occur at $R_0 = 1$. Yao and Zhang [20] developed a two-strain SIS model based on heterogeneous networks. They derived five different threshold values, i.e., R_0, R_1, R_2 , R_{12} and R_{21}, which are closely related to the stability of equilibria. Yang and Li [21] studied a two-strain SIS epidemic model based on complex networks. The DFE E_0 is globally asymptotically stable if the basic reproduction number $R_0 < 1$. Otherwise, there exists either a unique strain 1-only or a strain 2-only boundary equilibrium. Cheng et al. [22] talked about a network-based SIQS infectious disease model with a nonmonotonic incidence rate. Moreover, endemic models with time delay in networks have also been established [23, 24].

As the virus mutates, individuals will become infected and remain asymptomatic. However, some will become seriously ill and require medical attention. More and more people will get vaccinated to prevent infection. Furthermore, the spread of the disease begins exhibiting more heterogeneity.

To consider the impact of these features on the dynamics of disease transmission, motivated by the above discussions, we propose an SEIR-type endemic model that incorporates direct immunity by vaccination, as well as the conversion from the asymptomatic to the symptomatic group in complex networks. Conditions for the extinction and permanence of the disease are investigated in this paper.

The rest of the paper is organized in the following way. In Section 2, we introduce the network model. In Section 3, we investigate the dynamics of the proposed model. Section 4 is devoted to numerical simulations to verify our theoretical results. Finally, we give the discussion in Section 5.

2. Models

As reported by the World Health Organization, COVID-19 affects different people in different ways. Some infected people will be asymptomatic and recover without hospitalization, whereas some will experience fever, coughing, difficulty breathing, shortness of breath, chest pain and other serious symptoms. Both asymptomatic and symptomatic infected individuals are infectious. On average, it takes 5–6 days from when someone is infected with the virus for symptoms to show. According to these COVID-19 transmission characteristics, the total populations are divided into five compartments, namely, the susceptible subclass (S), the exposed subclass (E), the asymptomatic infected subclass (I_a), the symptomatic infected subclass (I) and the removed subclass (R). The transmission process for each individual is shown in Figure 1. Moreover, we make the following assumptions:

1) All infected people have an incubation period. The incidence functions for the asymptomatic and the symptomatic infected people are $\alpha_1 S I_a$ and $\alpha_2 S I$, respectively.
2) The infected people have immunity to COVID-19 after rehabilitation and vaccination.
3) The asymptomatic infected people may become symptomatic or recover from the disease, which is dependent on their physical fitness. We assume that the rate of transition from I_a to I is proportional to their density with the scaling factor θ.

Based on the above assumptions, we propose the following ordinary differential equation endemic model of COVID-19, which incorporates direct immunity by vaccination and the shift from the asymptomatic to symptomatic subclass:

$$\begin{cases} S'(t) = A - \alpha_1 S I_a - \alpha_2 S I - \mu S - \nu S, \\ E'(t) = \alpha_1 S I_a + \alpha_2 S I - \mu E - \eta_1 E - \eta_2 E, \\ I_a'(t) = \eta_1 E - \theta I_a - \mu I_a - \mu_1 I_a - \gamma_1 I_a, \\ I'(t) = \eta_2 E + \theta I_a - \mu I - \mu_2 I - \gamma_2 I, \\ R'(t) = \gamma_1 I_a + \gamma_2 I + \nu S - \mu R. \end{cases} \tag{2.1}$$

System (2.1) is an SEIR-type endemic model with two different manifestations of infection in the population. The parameter A is the recruitment rate of susceptible individuals, μ is the natural mortality rate of all individuals, ν is the vaccination rate of susceptible individuals, α_1 denotes the rate of transmission of susceptible individuals to exposed individuals, as induced by asymptomatic individuals, and α_2 denotes the rate of transmission of susceptible individuals to exposed individuals, as induced by symptomatic individuals. Exposed individuals can become asymptomatic at a rate of

η_1, or symptomatic at a rate of η_2 due to their constitutions. Asymptomatic infected individuals will become symptomatic infected individuals at a rate of θ. μ_1 and μ_2 are the fatality rates of the asymptomatic and the symptomatic individuals, respectively. γ_1 and γ_2 are the recovery rates of asymptomatic individuals and symptomatic individuals, respectively.

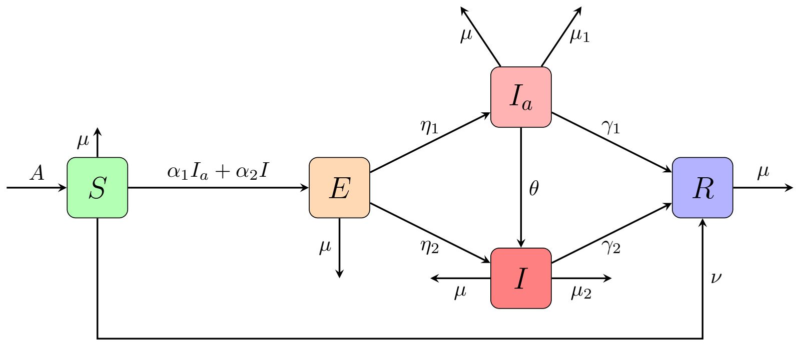

Figure 1. The transmission diagram for system (2.1).

To study the effects of contact heterogeneity on endemic dynamics, we view the entire population as a social network, and each individual corresponds to a network node. Moreover, an edge connecting two individuals denotes a potential contact between both. All nodes of the network are assigned to different groups according to their degrees. Specifically, the kth group has a degree k for $k \in \mathbb{N}_n \triangleq \{1, 2, ..., n\}$, and n is the maximal degree. Let N_k be the number of all individuals that have a degree k. Then, the total number of individuals is $N = N_1 + N_2 + ... + N_n$. To incorporate the property of contact heterogeneity, we consider the network characterized by the degree distribution $P(k)$, which is defined as a randomly selected node with a degree k, i.e., $P(k) = N_k/N$. Based on system (2.1), we construct the following network model with k subsystems for $k \in \mathbb{N}_n$:

$$\begin{cases} S_k'(t) = A - \alpha_1 k S_k \Theta_1 - \alpha_2 k S_k \Theta_2 - \mu S_k - \nu S_k, \\ E_k'(t) = \alpha_1 k S_k \Theta_1 + \alpha_2 k S_k \Theta_2 - \mu E_k - \eta_1 E_k - \eta_2 E_k, \\ J_k'(t) = \eta_1 E_k - \theta J_k - \mu J_k - \mu_1 J_k - \gamma_1 J_k, \\ I_k'(t) = \eta_2 E_k + \theta J_k - \mu I_k - \mu_2 I_k - \gamma_2 I_k, \\ R_k'(t) = \gamma_1 J_k + \gamma_2 I_k + \nu S_k - \mu R_k, \end{cases} \tag{2.2}$$

where S_k represents the number of susceptible individuals with degree k, and E_k represents the number of exposed individuals with degree k. J_k and I_k are the numbers of asymptomatic and symptomatic individuals with degree k, respectively. R_k is the number of recovered individuals with degree k. The parameters for systems (2.1) and (2.2) have the same meanings. Nevertheless, there are slightly different units after dimensionless transformation (see Table 1). And, the transmission diagram for system (2.2) is shown in Figure 2.

Table 1. The units of quantities in systems (2.1) and (2.2).

Quantities	Unit before dimensionless transformation	Unit after dimensionless transformation
$S, E, I_a, I, R,$ S_k, E_k, J_k, I_k, R_k	Number of individuals	Dimensionless
A	Number of individuals/time	1/time
α_1, α_2	1/(time × number of individuals)	1/time
$\mu, \nu, \eta_1, \eta_2, \theta, \mu_1, \mu_2, \gamma_1, \gamma_2$	1/time	1/time

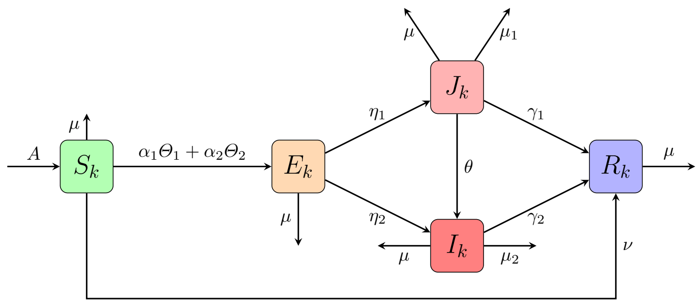

Figure 2. The transmission diagram for system (2.2).

All subsystems of dynamics in system (2.2) are coupled by functions $\Theta_1(t)$ and $\Theta_2(t)$, which are respectively defined by

$$\Theta_1(t) = \sum_{j=1}^{n} P(j|k)J_j \quad \text{and} \quad \Theta_2(t) = \sum_{j=1}^{n} P(j|k)I_j,$$

where $P(j|k)$ represents the conditional possibility of a node with degree k connecting a node with degree j. The possibility of a link connecting to a node with degree j is proportional to $jP(j)$. Hence, $P(j|k) = jP(j)/\langle k\rangle$, where $\langle k\rangle = \sum_{j=1}^{n} jP(j)$ is the mean degree of the network [25]. Then,

$$\Theta_1 = \sum_{j=1}^{n} \frac{jP(j)J_j}{\langle k\rangle}, \qquad \Theta_2 = \sum_{j=1}^{n} \frac{jP(j)I_j}{\langle k\rangle}. \tag{2.3}$$

In this paper, we study the endemic dynamics with the form of relations given by (2.3) for the functions Θ_1 and Θ_2 in uncorrelated networks.

Since the first four equations in system (2.2) are independent of the variable state $R_k(t)$, we can simplify the model as follows:

$$\begin{cases} S_k'(t) = A - \alpha_1 k S_k \Theta_1 - \alpha_2 k S_k \Theta_2 - \mu S_k - \nu S_k, \\ E_k'(t) = \alpha_1 k S_k \Theta_1 + \alpha_2 k S_k \Theta_2 - \mu E_k - \eta_1 E_k - \eta_2 E_k, \\ J_k'(t) = \eta_1 E_k - \theta J_k - \mu J_k - \mu_1 J_k - \gamma_1 J_k, \\ I_k'(t) = \eta_2 E_k + \theta J_k - \mu I_k - \mu_2 I_k - \gamma_2 I_k. \end{cases} \tag{2.4}$$

3. Dynamics

Lemma 3.1. *Let $k \in \mathbb{N}_n$. For any initial data point $S_k(0)$, $E_k(0)$, $J_k(0)$, $I_k(0) > 0$, all of the solutions ($S_k(t)$, $E_k(t)$, $J_k(t)$, $I_k(t)$) of system* (2.4) *are nonnegative for $t > 0$.*

Proof. Assume, on the contrary, that at least one of the subclass values, i.e., $S_k(t)$, $E_k(t)$, $J_k(t)$, $I_k(t)$, is negative with the initial conditions $S_k(0)$, $E_k(0)$, $J_k(0)$, $I_k(0) > 0$ for $k \in \mathbb{N}_n$. According to the continuity, there exists a sufficiently small $\varepsilon > 0$ such that $S_k(t)$, $E_k(t)$, $J_k(t)$, $I_k(t) \geq 0$ for $t \in (0, \varepsilon)$ and $k \in \mathbb{N}_n$. Furthermore, there exist $j \in \mathbb{N}_n$ and the initial time $t_3 \geq \varepsilon > 0$ such that $J_j(t_3) = 0$ and $J'_j(t_3) < 0$ or $I_j(t_3) = 0$ and $I'_j(t_3) < 0$. Otherwise, $J_k(t), I_k(t) \geq 0$ for $t > 0$ and $k \in \mathbb{N}_n$, which means that there is no such t_3. Similarly, there exist $j \in \mathbb{N}_n$ and the initial time points t_1 and t_2 such that $S_j(t_1) = 0$, $S'_j(t_1) \leq 0$ and $E_j(t_2) \leq 0$, $E'_j(t_2) = 0$ for $t_1, t_2 \geq \varepsilon > 0$.

Case 1: $t_1 \leq t_2, t_3$. Here t_2 and t_3 may not exist, but t_3 must exist. We have that $E_j(t_1), J_j(t_1), I_j(t_1), \Theta_1(t_1), \Theta_2(t_1) \geq 0$. Substituting t_1 into the first equation of system (2.4) yields $S'_k(t_1) = A > 0$. Hence, this contradicts the hypothesis.

Case 2: $t_2 \leq t_1, t_3$. Here t_1 and t_3 may not exist, but t_2 must exist. We have that $S_j(t_2), J_j(t_2), I_j(t_2), \Theta_1(t_2), \Theta_2(t_2) \geq 0$. Substituting t_2 into the second equation of system (2.4) yields $E'_j(t_2) = \alpha_1 j S_j(t_2)\Theta_1(t_2) + \alpha_2 j S_j(t_2)\Theta_2(t_2) \geq 0$. Hence, this contradicts the hypothesis.

Case 3: $t_3 \leq t_1, t_2$. Here t_1 and t_2 may not exist, but t_3 must exist. If $J_j(t_3) = 0$, we have that $S_j(t_3), E_j(t_3), I_j(t_3), \Theta_1(t_3), \Theta_2(t_3) \geq 0$. Substituting t_3 into the third equation of system (2.4) yields $J'_j(t_3) = \eta_1 E_j(t_3) \geq 0$. Hence, this contradicts the hypothesis. If $I_j(t_3) = 0$, we have that $S_j(t_3), E_j(t_3), J_j(t_3), \Theta_1(t_3), \Theta_2(t_3) \geq 0$. Substituting t_3 into the last equation of system (2.4) yields $I'(t_3) = \eta_2 E_j(t_3) + \theta J_j(t_3) \geq 0$. Hence, this contradicts the hypothesis.

Similarly, we can prove that $S_k(t) \geq 0$. This completes the proof.

Lemma 3.2. *The solutions of* (2.4) *satisfy that* $0 \leq S_k \leq \frac{A}{\mu}$, $0 \leq E_k \leq \frac{A}{\mu}$, $0 \leq J_k \leq \frac{A}{\mu}$ *and* $0 \leq I_k \leq \frac{A}{\mu}$ *for all $k \in \mathbb{N}_n$.*

Proof. Let $N_k = S_k + E_k + J_k + I_k$ for $k \in \mathbb{N}_n$. Adding the equations in system (2.4), we obtain

$$\begin{aligned} N'_k(t) &= A - \mu N_k - \gamma_1 J_k - \gamma_2 I_k - \nu S_k - \mu_1 J_k - \mu_2 I_k \\ &\leq A - \mu N_k, \quad k \in \mathbb{N}_n. \end{aligned}$$

By integration, it follows that

$$N_k(t) \leq N_k(0)e^{-\mu t} + \frac{A}{\mu}(1 - e^{-\mu t}), \quad k \in \mathbb{N}_n,$$

where $N_k(0)$ represents the total population with degree k at time $t = 0$. Therefore,

$$\limsup_{t \to \infty}(S_k + E_k + J_k + I_k) \leq \frac{A}{\mu}, \quad k \in \mathbb{N}_n.$$

This, together with Lemma 3.1, readily imply the lemma.

System (2.4) always has the DFE

$$P_0 = (S_1^0, E_1^0, J_1^0, I_1^0, S_2^0, E_2^0, J_2^0, I_2^0, \cdots, S_k^0, E_k^0, J_k^0, I_k^0, \cdots, S_n^0, E_n^0, J_n^0, I_n^0),$$

where

$$\begin{aligned} S_1^0 = S_2^0 &= \cdots = S_k^0 &= \cdots = S_n^0 &= \frac{A}{\mu+\nu}, \\ E_1^0 = E_2^0 &= \cdots = E_k^0 &= \cdots = E_n^0 &= 0, \\ J_1^0 = J_2^0 &= \cdots = J_k^0 &= \cdots = J_n^0 &= 0, \\ I_1^0 = I_2^0 &= \cdots = I_k^0 &= \cdots = I_n^0 &= 0. \end{aligned}$$

Assume that there exists the EE

$$P^* = (S_1^*, E_1^*, J_1^*, I_1^*, S_2^*, E_2^*, J_2^*, I_2^*, \cdots, S_k^*, E_k^*, J_k^*, I_k^*, \cdots, S_n^*, E_n^*, J_n^*, I_n^*),$$

where

$$\begin{cases} A - \alpha_1 k S_k^* \Theta_1 - \alpha_2 k S_k^* \Theta_2 - \mu S_k^* - \nu S_k^* = 0, \\ \alpha_1 k S_k^* \Theta_1 + \alpha_2 k S_k^* \Theta_2 - \mu E_k^* - \eta_1 E_k^* - \eta_2 E_k^* = 0, \\ \eta_1 E_k^* - \theta J_k^* - \mu J_k^* - \mu_1 J_k^* - \gamma_1 J_k^* = 0, \\ \eta_2 E_k^* + \theta J_k^* - \mu I_k^* - \mu_2 I_k^* - \gamma_2 I_k^* = 0, \end{cases} \tag{3.1}$$

$k \in \mathbb{N}_n$. Solving system (3.1), we obtain that S_k^*, E_k^*, J_k^* and I_k^* are functions with respect to Θ_1 and Θ_2 for $k \in \mathbb{N}_n$, namely,

$$S_k^* = \frac{A}{k(\alpha_1\Theta_1 + \alpha_2\Theta_2) + \mu + \nu}, \quad E_k^* = \frac{kA(\alpha_1\Theta_1 + \alpha_2\Theta_2)}{[k(\alpha_1\Theta_1 + \alpha_2\Theta_2) + \mu + \nu](\eta_1 + \eta_2 + \mu)}, \tag{3.2}$$

$$\begin{aligned} J_k^* &= \frac{\eta_1 kA(\alpha_1\Theta_1 + \alpha_2\Theta_2)}{[k(\alpha_1\Theta_1 + \alpha_2\Theta_2) + \mu + \nu](\eta_1 + \eta_2 + \mu)(\mu + \mu_1 + \theta + \gamma_1)}, \\ I_k^* &= \frac{kA(\alpha_1\Theta_1 + \alpha_2\Theta_2)[(\mu + \mu_1 + \theta + \gamma_1)\eta_2 + \eta_1\theta]}{[k(\alpha_1\Theta_1 + \alpha_2\Theta_2) + \mu + \nu](\eta_1 + \eta_2 + \mu)(\mu + \mu_1 + \theta + \gamma_1)(\mu + \mu_2 + \gamma_2)}. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (2.3) gives

$$\Theta_1 = \sum_{j=1}^{n} \frac{jP(j)}{\langle k \rangle} \frac{\eta_1 kA(\alpha_1\Theta_1 + \alpha_2\Theta_2)}{[k(\alpha_1\Theta_1 + \alpha_2\Theta_2) + \mu + \nu](\eta_1 + \eta_2 + \mu)(\mu + \mu_1 + \theta + \gamma_1)}, \tag{3.4}$$

$$\Theta_2 = \sum_{j=1}^{n} \frac{jP(j)}{\langle k \rangle} \frac{kA(\alpha_1\Theta_1 + \alpha_2\Theta_2)[(\mu + \mu_1 + \theta + \gamma_1)\eta_2 + \eta_1\theta]}{[k(\alpha_1\Theta_1 + \alpha_2\Theta_2) + \mu + \nu](\eta_1 + \eta_2 + \mu)(\mu + \mu_1 + \theta + \gamma_1)(\mu + \mu_2 + \gamma_2)}. \tag{3.5}$$

From (3.4) and (3.5), we observe

$$\Theta_2 = \frac{[(\mu + \mu_1 + \theta + \gamma_1)\eta_2 + \eta_1\theta]}{\eta_1(\mu + \mu_2 + \gamma_2)}\Theta_1. \tag{3.6}$$

Noting (3.4)–(3.6), we define

$$f_1(\Theta_1) = \sum_{j=1}^{n} \frac{jP(j)}{\langle k \rangle} \frac{\eta_1 kA\left[\alpha_1 + \alpha_2\frac{(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta}{\eta_1(\mu+\mu_2+\gamma_2)}\right]\Theta_1}{\left\{k\left[\alpha_1 + \alpha_2\frac{(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta}{\eta_1(\mu+\mu_2+\gamma_2)}\right]\Theta_1 + \mu + \nu\right\}(\eta_1 + \eta_2 + \mu)(\mu + \mu_1 + \theta + \gamma_1)} - \Theta_1, \tag{3.7}$$

$$f_2(\Theta_2)=\sum_{j=1}^{n}\frac{jP(j)}{\langle k\rangle}\frac{kA\left\{\alpha_1\frac{\eta_1(\mu+\mu_2+\gamma_2)}{[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta]}+\alpha_2\right\}[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta]\Theta_2}{\left[k\left(\alpha_1\frac{\eta_1(\mu+\mu_2+\gamma_2)}{[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta]}+\alpha_2\right)\Theta_2+\mu+\nu\right](\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)(\mu+\mu_2+\gamma_2)}$$
$$-\Theta_2. \tag{3.8}$$

Then, (3.4) and (3.5) are respectively equivalent to

$$f_1(\Theta_1)=0,\qquad f_2(\Theta_2)=0. \tag{3.9}$$

Clearly, $\Theta_1=0$ and $\Theta_2=0$ are solutions for (3.9), which correspond to the DFE P_0. Assume that there exist nontrivial solutions $0<\Theta_1<1$ and $0<\Theta_2<1$ for (3.9). Then, we consider derivatives for the functions (3.7) and (3.8). Hence, we have

$$\frac{d^2f_1}{d\Theta_1^2}<0,\qquad \frac{d^2f_2}{d\Theta_2^2}<0$$

for $\Theta_1>0$ and $\Theta_2>0$. Owing to $f_1(0)=0$ and $f_2(0)=0$, it is guaranteed that the equations given by (3.9) have nontrivial solutions between 0 and 1 if

$$f_1'(0)>0,\quad f_2'(0)>0,\quad f_1(1)<1\quad\text{and}\quad f_2(1)<1. \tag{3.10}$$

Combining (3.10) with (3.7) and (3.8) implies that

$$\begin{aligned}\left.\frac{df_1}{d\Theta_1}\right|_{\Theta_1=0}&=\sum_{k=1}^{n}\frac{kP(k)}{\langle k\rangle}\frac{kA[\alpha_1\eta_1(\mu+\mu_2+\gamma_2)+\alpha_2\eta_2(\mu+\mu_1+\theta+\gamma_1)+\alpha_2\eta_1\theta]}{(\mu+\nu)(\mu+\mu_2+\gamma_2)(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}-1\\&=\frac{\langle k^2\rangle}{\langle k\rangle}\frac{A[\alpha_1\eta_1(\mu+\mu_2+\gamma_2)+\alpha_2\eta_2(\mu+\mu_1+\theta+\gamma_1)+\alpha_2\eta_1\theta]}{(\mu+\nu)(\mu+\mu_2+\gamma_2)(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}-1,\end{aligned}$$

$$\begin{aligned}\left.\frac{df_2}{d\Theta_2}\right|_{\Theta_2=0}&=\sum_{k=1}^{n}\frac{kP(k)}{\langle k\rangle}\frac{kA[\alpha_1\eta_1(\mu+\mu_2+\gamma_2)+\alpha_2\eta_2(\mu+\mu_1+\theta+\gamma_1)+\alpha_2\eta_1\theta]}{(\mu+\nu)(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)(\mu+\mu_2+\gamma_2)}-1\\&=\frac{\langle k^2\rangle}{\langle k\rangle}\frac{A[\alpha_1\eta_1(\mu+\mu_2+\gamma_2)+\alpha_2\eta_2(\mu+\mu_1+\theta+\gamma_1)+\alpha_2\eta_1\theta]}{(\mu+\nu)(\mu+\mu_2+\gamma_2)(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}-1,\end{aligned}$$

$$\begin{aligned}f_1(1)&=\sum_{k=1}^{n}\frac{kP(k)}{\langle k\rangle}\frac{\eta_1kA\left[\alpha_1+\alpha_2\frac{(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta}{\eta_1(\mu+\mu_2+\gamma_2)}\right]}{\left\{k\left[\alpha_1+\alpha_2\frac{(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta}{\eta_1(\mu+\mu_2+\gamma_2)}\right]+\mu+\nu\right\}(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}-1\\&<\sum_{k=1}^{n}\frac{kP(k)}{\langle k\rangle}\frac{\eta_1A\left[\alpha_1+\alpha_2\frac{(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta}{\eta_1(\mu+\mu_2+\gamma_2)}\right]}{\left[\alpha_1+\alpha_2\frac{(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta}{\eta_1(\mu+\mu_2+\gamma_2)}\right](\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}-1\\&=\sum_{k=1}^{n}\frac{kP(k)}{\langle k\rangle}\frac{\eta_1A}{(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}-1\\&=\frac{\eta_1A}{(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}-1\end{aligned}$$

and

$$
\begin{aligned}
f_2(1) &= \sum_{k=1}^{n} \frac{kP(k)}{\langle k\rangle} \frac{kA\left\{\alpha_1 \frac{\eta_1(\mu+\mu_2+\gamma_2)}{\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]} + \alpha_2\right\}\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]}{\left[k\left(\alpha_1 \frac{\eta_1(\mu+\mu_2+\gamma_2)}{\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]} + \alpha_2\right) + \mu + \nu\right](\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)(\mu+\mu_2+\gamma_2)} - 1 \\
&< \sum_{k=1}^{n} \frac{kP(k)}{\langle k\rangle} \frac{A\left\{\alpha_1 \frac{\eta_1(\mu+\mu_2+\gamma_2)}{\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]} + \alpha_2\right\}\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]}{\left\{\alpha_1 \frac{\eta_1(\mu+\mu_2+\gamma_2)}{\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]} + \alpha_2\right\}(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)(\mu+\mu_2+\gamma_2)} - 1 \\
&= \sum_{k=1}^{n} \frac{kP(k)}{\langle k\rangle} \frac{A\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]}{(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)(\mu+\mu_2+\gamma_2)} - 1 \\
&= \frac{A\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]}{(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)(\mu+\mu_2+\gamma_2)} - 1.
\end{aligned}
$$

Define the basic reproduction number of system (2.4) as

$$
R_0 = \frac{\langle k^2\rangle}{\langle k\rangle} \frac{A\left[\alpha_1\eta_1(\mu+\mu_2+\gamma_2) + \alpha_2\eta_2(\mu+\mu_1+\theta+\gamma_1) + \alpha_2\eta_1\theta\right]}{(\mu+\nu)(\mu+\mu_2+\gamma_2)(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}.
$$

Hence, there exists a unique positive EE P^* with $0 < \Theta_1 < 1$ and $0 < \Theta_2 < 1$ for system (2.4) if

$$
R_0 > 1 \text{ and } A \le A_0,
$$

where

$$
A_0 = \min\left\{\frac{(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)}{\eta_1}, \quad \frac{(\eta_1+\eta_2+\mu)(\mu+\mu_1+\theta+\gamma_1)(\mu+\mu_2+\gamma_2)}{\left[(\mu+\mu_1+\theta+\gamma_1)\eta_2+\eta_1\theta\right]}\right\}.
$$

Remark 3.1. *There always exists a DFE P_0 for system* (2.4)*. Besides, given that* $\lim_{\Theta_1\to+\infty} f_1(\Theta_1) = \lim_{\Theta_2\to+\infty} f_2(\Theta_2) = -\infty$*, and considering the continuity of functions $f_1(\Theta_1)$ and $f_2(\Theta_2)$, there exist solutions $\Theta_1 > 0$ and $\Theta_2 > 0$ for* (3.9) *if and only if $R_0 > 1$. Namely, system* (2.4) *has a unique positive EE P^* if and only if $R_0 > 1$.*

Theorem 3.1. *The DFE P_0 for system* (2.4) *is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.*

Proof. The Jacobian matrix for system (2.4) is

$$\begin{pmatrix}
-a & \cdots & 0 & 0 & \cdots & 0 & -\alpha_1 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & -\alpha_1 S_1^0 \frac{nP(n)}{\langle k\rangle} & -\alpha_2 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & -\alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -a & 0 & \cdots & 0 & -\alpha_1 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & -\alpha_1 n S_n^0 \frac{nP(n)}{\langle k\rangle} & -\alpha_2 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & -\alpha_2 n S_n^0 \frac{nP(n)}{\langle k\rangle} \\
0 & \cdots & 0 & -b & \cdots & 0 & \alpha_1 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_1 S_1^0 \frac{nP(n)}{\langle k\rangle} & \alpha_2 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & -b & \alpha_1 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_1 n S_n^0 \frac{nP(n)}{\langle k\rangle} & \alpha_2 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 n S_n^0 \frac{nP(n)}{\langle k\rangle} \\
0 & \cdots & 0 & \eta_1 & \cdots & 0 & -c & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \eta_1 & 0 & \cdots & -c & 0 & \cdots & 0 \\
0 & \cdots & 0 & \eta_2 & \cdots & 0 & \theta & \cdots & 0 & -d & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \eta_2 & 0 & \cdots & \theta & 0 & \cdots & -d
\end{pmatrix}_{4n\times 4n}, \tag{3.11}$$

where $a = \mu + \nu$, $b = \mu + \eta_1 + \eta_2$, $c = \mu + \mu_1 + \gamma_1 + \theta$, $d = \mu + \mu_2 + \gamma_2$, $k \in \mathbb{N}_n$. Obviously, there are n eigenvalues equaling to $-\mu - \nu$ for the matrix (3.11), and the remaining eigenvalues are determined by

$$\begin{pmatrix}
-b & \cdots & 0 & \alpha_1 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_1 S_1^0 \frac{nP(n)}{\langle k\rangle} & \alpha_2 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -b & \alpha_1 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_1 n S_n^0 \frac{nP(n)}{\langle k\rangle} & \alpha_2 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 n S_n^0 \frac{nP(n)}{\langle k\rangle} \\
\eta_1 & \cdots & 0 & -c & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \eta_1 & 0 & \cdots & -c & 0 & \cdots & 0 \\
\eta_2 & \cdots & 0 & \theta & \cdots & 0 & -d & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \eta_2 & 0 & \cdots & \theta & 0 & \cdots & -d
\end{pmatrix}_{3n\times 3n}, \tag{3.12}$$

$k \in \mathbb{N}_n$. For (3.12), we add the column $2n + 1$, multiplied by $-\frac{\alpha_1}{\alpha_2}$, to column $n + 1$; column $2n + 2$, multiplied by $-\frac{\alpha_1}{\alpha_2}$, to column $n+2 \cdots$; and column $3n$, multiplied by $-\frac{\alpha_1}{\alpha_2}$, to column $2n$. Hence, (3.12) becomes

$$\begin{pmatrix}
-b & \cdots & 0 & 0 & \cdots & 0 & \alpha_2 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -b & 0 & \cdots & 0 & \alpha_2 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 n S_n^0 \frac{nP(n)}{\langle k\rangle} \\
\eta_1 & \cdots & 0 & -c & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \eta_1 & 0 & \cdots & -c & 0 & \cdots & 0 \\
\eta_2 & \cdots & 0 & \theta + \frac{\alpha_1}{\alpha_2} d & \cdots & 0 & -d & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \eta_2 & 0 & \cdots & \theta + \frac{\alpha_1}{\alpha_2} d & 0 & \cdots & -d
\end{pmatrix}_{3n\times 3n}, \tag{3.13}$$

$k \in \mathbb{N}_n$. For (3.13), we add the row $n+1$, multiplied by $\frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}$, to row $2n+1$; row $n+2$, multiplied by $\frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}$, to row $2n+2$; $\cdots$ and row $2n$, multiplied by $\frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}$, to row $3n$. Then, (3.13) is converted to

$$\begin{pmatrix} -b & \cdots & 0 & 0 & \cdots & 0 & \alpha_2 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -b & 0 & \cdots & 0 & \alpha_2 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 n S_n^0 \frac{nP(n)}{\langle k\rangle} \\ \eta_1 & \cdots & 0 & -c & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \eta_1 & 0 & \cdots & -c & 0 & \cdots & 0 \\ \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & \cdots & 0 & 0 & \cdots & 0 & -d & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & 0 & \cdots & 0 & 0 & \cdots & -d \end{pmatrix}_{3n\times 3n}, \tag{3.14}$$

$k \in \mathbb{N}_n$. For (3.14), we add the column $n+1$, multiplied by $\frac{\eta_1}{c}$, to column 1; column $n+2$, multiplied by $\frac{\eta_1}{c}$, to column 2; $\cdots$ and column $2n$, multiplied by $\frac{\eta_1}{c}$, to column n. Then, (3.14) is converted to

$$\begin{pmatrix} -b & \cdots & 0 & 0 & \cdots & 0 & \alpha_2 S_1^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -b & 0 & \cdots & 0 & \alpha_2 n S_n^0 \frac{P(1)}{\langle k\rangle} & \cdots & \alpha_2 n S_n^0 \frac{nP(n)}{\langle k\rangle} \\ 0 & \cdots & 0 & -c & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -c & 0 & \cdots & 0 \\ \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & \cdots & 0 & 0 & \cdots & 0 & -d & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & 0 & \cdots & 0 & 0 & \cdots & -d \end{pmatrix}_{3n\times 3n}, \tag{3.15}$$

$k \in \mathbb{N}_n$. Obviously, there are n eigenvalues equaling to $-\mu-\nu_1-\gamma_1-\theta$ for (3.15). The remaining eigenvalues are determined by

$$\begin{pmatrix} -b & 0 & \cdots & 0 & \alpha_2 S_1^0 \frac{P(1)}{\langle k\rangle} & \alpha_2 S_1^0 \frac{2P(2)}{\langle k\rangle} & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\ 0 & -b & \cdots & 0 & \alpha_2 2 S_2^0 \frac{P(1)}{\langle k\rangle} & \alpha_2 2 S_2^0 \frac{2P(2)}{\langle k\rangle} & \cdots & \alpha_2 2 S_2^0 \frac{nP(n)}{\langle k\rangle} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -b & \alpha_2 n S_n^0 \frac{P(1)}{\langle k\rangle} & \alpha_2 n S_n^0 \frac{2P(2)}{\langle k\rangle} & \cdots & \alpha_2 n S_n^0 \frac{nP(n)}{\langle k\rangle} \\ \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & 0 & \cdots & 0 & -d & 0 & \cdots & 0 \\ 0 & \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & \cdots & 0 & 0 & -d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & 0 & 0 & \cdots & -d \end{pmatrix}_{2n\times 2n}, \tag{3.16}$$

$k \in \mathbb{N}_n$. For (3.16), we add the column $n+2$, multiplied by $-\frac{P(1)}{2P(2)}$, to column $n+1$; column $n+3$, multiplied by $-\frac{2P(2)}{3P(3)}$, to column $n+2$; $\cdots$ and column $2n$, multiplied by $-\frac{(n-1)P(n-1)}{nP(n)}$, to column $2n-1$.

Then, (3.16) is converted to

$$\begin{pmatrix} -b & 0 & \cdots & 0 & 0 & 0 & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\ 0 & -b & \cdots & 0 & 0 & 0 & \cdots & \alpha_2 2S_2^0 \frac{nP(n)}{\langle k\rangle} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -b & 0 & 0 & \cdots & \alpha_2 nS_n^0 \frac{nP(n)}{\langle k\rangle} \\ \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & 0 & \cdots & 0 & -d & 0 & \cdots & 0 \\ 0 & \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & \cdots & 0 & \frac{P(1)}{2P(2)}d & -d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_1 d+\alpha_2\theta}{\alpha_2 c}\eta_1+\eta_2 & 0 & 0 & \cdots & -d \end{pmatrix}_{2n\times 2n}, \tag{3.17}$$

$k \in \mathbb{N}_n$. For (3.17), we add the row 1, multiplied by $\frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{\alpha_2 bc}$, to row $n+1$; row 2, multiplied by $\frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{\alpha_2 bc}$, to row $n+2$; $\cdots$ and row n, multiplied by $\frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{\alpha_2 bc}$, to row $2n$. Then, (3.17) is converted to

$$\begin{pmatrix} -b & 0 & \cdots & 0 & 0 & 0 & \cdots & \alpha_2 S_1^0 \frac{nP(n)}{\langle k\rangle} \\ 0 & -b & \cdots & 0 & 0 & 0 & \cdots & \alpha_2 2S_1^0 \frac{nP(n)}{\langle k\rangle} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -b & 0 & 0 & \cdots & \alpha_2 nS_n^0 \frac{nP(n)}{\langle k\rangle} \\ 0 & 0 & \cdots & 0 & -d & 0 & \cdots & \frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} S_1^0 \frac{nP(n)}{\langle k\rangle} \\ 0 & 0 & \cdots & 0 & \frac{P(1)}{2P(2)}d & -d & \cdots & \frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} 2S_2^0 \frac{nP(n)}{\langle k\rangle} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -d+\frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} 2S_n^0 \frac{nP(n)}{\langle k\rangle} \end{pmatrix}_{2n\times 2n}, \tag{3.18}$$

$k \in \mathbb{N}_n$. Obviously, there are n eigenvalues equaling to $-\mu-\eta_1-\eta_2$ for (3.18); the remaining eigenvalues are determined by

$$\begin{pmatrix} -d & 0 & \cdots & \frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} S_1^0 \frac{nP(n)}{\langle k\rangle} \\ \frac{P(1)}{2P(2)}d & -d & \cdots & \frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} 2S_2^0 \frac{nP(n)}{\langle k\rangle} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -d+\frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} 2S_n^0 \frac{nP(n)}{\langle k\rangle} \end{pmatrix}_{n\times n}, \tag{3.19}$$

$k \in \mathbb{N}_n$. For (3.19), we add the row 1, multiplied by $\frac{P(1)}{2P(2)}$, to row 2; row 2, multiplied by $\frac{2P(2)}{3P(3)}$, to row 3; $\cdots$ and row $n-1$, multiplied by $\frac{(n-1)P(n-1)}{nP(n)}$, to row n. Hence, (3.19) is converted to

$$\begin{pmatrix} -d & 0 & \cdots & \frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} S_1^0 \frac{nP(n)}{\langle k\rangle} \\ 0 & -d & \cdots & \frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} \left(\frac{P(1)}{\langle k\rangle} S_1^0 + \frac{2P(2)}{\langle k\rangle} S_2^0\right) \frac{nP(n)}{2P(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -d+\frac{(\alpha_1 d+\alpha_2\theta)\eta_1+\alpha_2 c\eta_2}{bc} \sum_{k=1}^{n} \left(kS_k^0 \frac{kP(k)}{\langle k\rangle}\right) \end{pmatrix}_{n\times n}, \tag{3.20}$$

$k \in \mathbb{N}_n$. Clearly, there are $n-1$ eigenvalues equaling to $-\mu-\mu_2-\gamma_2$ for (3.20), and the last eigenvalue is given by

$$
\begin{aligned}
\lambda =& - d + \frac{(\alpha_1 d + \alpha_2\theta)\eta_1 + \alpha_2 c\eta_2}{bc} \sum_{k=1}^{n} \left(kS_k^0 \frac{kP(k)}{\langle k\rangle} \right) \\
=& - d + \frac{(\alpha_2\theta + \alpha_1 d)\eta_1 + \alpha_2 c\eta_2}{bc} \frac{A}{\mu+\nu} \frac{\langle k^2\rangle}{\langle k\rangle} \\
=& - (\mu + \mu_2 + \gamma_2) + \frac{[\alpha_2\theta + \alpha_1(\mu + \mu_2 + \gamma_2)]\eta_1 + \alpha_2(\mu + \mu_1 + \gamma_1 + \theta)\eta_2}{(\mu + \mu_1 + \gamma_1 + \theta)(\mu + \eta_1 + \eta_2)} \frac{A}{\mu+\nu} \frac{\langle k^2\rangle}{\langle k\rangle} \\
=& (\mu + \mu_2 + \gamma_2) \left\{ \frac{A[\alpha_1(\mu + \mu_2 + \gamma_2)\eta_1 + \alpha_2(\mu + \mu_1 + \gamma_1 + \theta)\eta_2 + \alpha_2\eta_1\theta]}{(\mu + \mu_1 + \gamma_1 + \theta)(\mu + \eta_1 + \eta_2)(\mu + \mu_2 + \gamma_2)(\mu + \nu)} \frac{\langle k^2\rangle}{\langle k\rangle} - 1 \right\} \\
=& (\mu + \mu_2 + \gamma_2)(R_0 - 1).
\end{aligned}
$$

Hence, all eigenvalues are negative if $R_0 < 1$, and there exists a positive eigenvalue if $R_0 > 1$. Therefore, the DFE P_0 for system (2.4) is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.

Remark 3.2. *Define*

$$
A^* = \frac{\langle k\rangle}{\langle k^2\rangle} \frac{(\mu + \nu)(\mu + \mu_2 + \gamma_2)(\eta_1 + \eta_2 + \mu)(\mu + \mu_1 + \theta + \gamma_1)}{[\alpha_1\eta_1(\mu + \mu_2 + \gamma_2) + \alpha_2(\mu + \mu_1 + \theta + \gamma_1)\eta_2 + \alpha_2\eta_1\theta]}.
$$

Then, from Theorem 3.3, we can derive the threshold-type dynamics of the DFE of system (2.4) *with respect to A^*, namely, the DFE is locally asymptotically stable if $A < A^*$, and unstable if $A > A^*$.*

Theorem 3.2. *Let*

$$
A^g = \min \left\{ \frac{(\mu + \nu)(\mu + \mu_1 + \gamma_1)\langle k\rangle}{\alpha_1\langle k^2\rangle}, \frac{(\mu + \nu)(\mu + \mu_2 + \gamma_2)\langle k\rangle}{\alpha_2\langle k^2\rangle} \right\}.
$$

If $A \leq A^g$, then the DFE P_0 for system (2.4) *is globally asymptotically stable.*

Proof. Let $h(x) = x - 1 - \ln x$. It is easy to verify that the function $h(x)$ is nonnegative. We define the Lyapunov function

$$
V = \frac{1}{\langle k\rangle} \sum_{k=1}^{n} kP(k) \left[S_k^0 h\left(\frac{S_k}{S_k^0}\right) + E_k \right] + \Theta_1 + \Theta_2, \quad k \in \mathbb{N}_n.
$$

Obviously, $V \geq 0$, and the equality holds if and only if $E_k = J_k = I_k = 0$ and $S_k = \frac{A}{\mu+\nu}$ for $k \in \mathbb{N}_n$.

Differentiating V with respect to t, we derive

$$\begin{aligned}
V'(t) =& \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)\left[\left(1-\frac{S_k^0}{S_k}\right)S_k' + E_k'\right] + \Theta_1' + \Theta_2' \\
=& \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)\Big[A - \mu S_k - \nu S_k - \mu E_k - \mu J_k - \mu_1 J_k - \gamma_1 J_k - \mu I_k - \mu_2 I_k - \gamma_2 I_k \\
& - \frac{S_k^0}{S_k}(A - \alpha_1 k S_k \Theta_1 - \alpha_2 k S_k \Theta_2 - \mu S_k - \nu S_k)\Big] \\
=& \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)A\left(2 - \frac{S_k}{S_k^0} - \frac{S_k^0}{S_k}\right) + \alpha_1 S_k^0 \frac{\langle k^2\rangle}{\langle k\rangle}\Theta_1 + \alpha_2 S_k^0 \frac{\langle k^2\rangle}{\langle k\rangle}\Theta_2 - \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)\mu E_k \\
& - (\mu+\mu_1+\gamma_1)\Theta_1 - (\mu+\mu_2+\gamma_2)\Theta_2 \\
=& \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)A\left(2 - \frac{S_k}{S_k^0} - \frac{S_k^0}{S_k}\right) + \left[\frac{\alpha_1 A}{\mu+\nu}\frac{\langle k^2\rangle}{\langle k\rangle} - (\mu+\mu_1+\gamma_1)\right]\Theta_1 \\
& + \left[\frac{\alpha_2 A}{\mu+\nu}\frac{\langle k^2\rangle}{\langle k\rangle} - (\mu+\mu_2+\gamma_2)\right]\Theta_2 - \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)\mu E_k.
\end{aligned}$$

Then, $V'(t) \le 0$ is guaranteed if $A \le A^g$, and the equality holds if and only if $E_k = J_k = I_k = 0$ and $S_k = \frac{A}{\mu+\nu}$, $k \in \mathbb{N}_n$. Hence, by LaSalle's invariance principle [26], P_0 is globally asymptotically stable. This completes the proof.

Remark 3.3. *Note that $A < A^* \Leftrightarrow R_0 < 1$. A direct computation yields $A^* > A^g$. Then, we have*

$$A \le A^g \Rightarrow R_0 < 1, \quad R_0 < 1 \not\Rightarrow A \le A^g.$$

Hence, condition $A \le A^g$ is stronger than condition $R_0 < 1$.

For the global asymptotic stability of EE, we assume that

$$\alpha_1 = \alpha_2, \quad \mu_1 = \mu_2, \quad \gamma_1 = \gamma_2. \tag{3.21}$$

Consequently, system (2.4) becomes

$$\begin{cases} S_k'(t) = A - \alpha k S_k \Theta - \mu S_k - \nu S_k, \\ E_k'(t) = \alpha k S_k \Theta - \mu E_k - \eta E_k, \\ L_k'(t) = \eta E_k - \mu L_k - \mu_1 L_k - \gamma_1 L_k, \end{cases} \tag{3.22}$$

where $\alpha = \alpha_1 = \alpha_2$, $\eta = \eta_1 + \eta_2$, $\Theta = \Theta_1 + \Theta_2$, $L_k = I_k + J_k$. And, the invariant set for system (3.22) is

$$\widetilde{\Omega} = \left\{(S_k, E_k, L_k) \in \mathbb{R}_+^{3n} : 0 \le S_k + E_k + L_k \le \frac{A}{\mu}, \quad k \in \mathbb{N}_n\right\}.$$

The EE of system (3.22) is $\widetilde{P}^* = (S_k^*, E_k^*, L_k^*)$, $k \in \mathbb{N}_n$, where

$$S_k^* = \frac{A}{\alpha k\Theta + \mu + \nu}, \; E_k^* = \frac{A\alpha k\Theta}{(\mu+\eta)(\alpha k\Theta + \mu + \nu)}, \; L_k^* = \frac{A\eta\alpha k\Theta}{(\mu+\eta)(\mu+\mu_1+\gamma_1)(\alpha k\Theta + \mu + \nu)}.$$

Owing to (3.21), we obtain the basic reproduction number

$$R_0 = \frac{\langle k^2 \rangle}{\langle k \rangle} \frac{A\alpha\eta}{(\mu+\nu)(\mu+\mu_1+\gamma_1)(\eta+\mu)}.$$

Theorem 3.3. *The EE $\widetilde{P}^*$ of system* (3.22) *is globally asymptotically stable in $\widetilde{\Omega}$ if $R_0 > 1$ and the matrix $\left(\frac{\alpha k k' P(k')}{\langle k \rangle}\right)_{n\times n}$, $k = 1, 2, \cdots, n$, $k' = 1, 2, \cdots, n$ is irreducible.*

Proof. Define

$$V_k(t) = S_k^* h\left(\frac{S_k}{S_k^*}\right) + E_k^* h\left(\frac{E_k}{E_k^*}\right) + \frac{\mu+\eta}{\eta} L_k^* h\left(\frac{L_k}{L_k^*}\right).$$

Hence, $V_k \geq 0$, and the equality holds if and only if $S_k = S_k^*$, $E_k = E_k^*$ and $L_k = L_k^*$. Differentiating V_k, we obtain

$$\begin{aligned}
V_k'(t) =& \left(1 - \frac{S_k^*}{S_k}\right) S_k' + \left(1 - \frac{E_k^*}{E_k}\right) E_k' + \frac{\mu+\eta}{\eta}\left(1 - \frac{L_k^*}{L_k}\right) L_k' \\
=& \left(1 - \frac{S_k^*}{S_k}\right)(A - \alpha k S_k \Theta - \mu S_k - \nu S_k) + \left(1 - \frac{E_k^*}{E_k}\right)(\alpha k S_k \Theta - \mu E_k - \eta E_k) \\
&+ \frac{\mu+\eta}{\eta}\left(1 - \frac{L_k^*}{L_k}\right)(\eta E_k - \mu L_k - \mu_1 L_k - \gamma_1 L_k) \\
=& \left(1 - \frac{S_k^*}{S_k}\right)(\alpha k S_k^* \Theta^* + \mu S_k^* + \nu S_k^* - \alpha k S_k \Theta - \mu S_k - \nu S_k) + \left(1 - \frac{E_k^*}{E_k}\right)(\alpha k S_k \Theta - \frac{\alpha k S_k^* \Theta^*}{E_k^*} E_k) \\
&+ \frac{\mu+\eta}{\eta}\left(1 - \frac{L_k^*}{L_k}\right)\left(\eta E_k - \frac{E_k^*}{L_k^*} L_k\right) \\
=& S_k^*(\mu+\nu)(1 - \frac{S_k}{S_k^*})\left(1 - \frac{S_k^*}{S_k}\right) + \left(1 - \frac{S_k^*}{S_k}\right)(\alpha k S_k^* \Theta^* - \alpha k S_k \Theta) + \left(1 - \frac{E_k^*}{E_k}\right)\left(\alpha k S_k \Theta - \frac{\alpha k S_k^* \Theta^*}{E_k^*} E_k\right) \\
&+ \frac{\alpha k S_k^* \Theta^*}{\eta E_k^*}\left(1 - \frac{L_k^*}{L_k}\right)\left(\eta E_k - \frac{\eta E_k^*}{L_k^*} L_k\right) \\
=& S_k^*(\mu+\nu)\left(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}\right) + \alpha k\left[3 S_k^* \Theta^* - \frac{(S_k^*)^2 \Theta^*}{S_k} + S_k^* \Theta - \frac{S_k E_k^* \Theta}{E_k} - \frac{L_k}{L_k^*} S_k^* \Theta^* - \frac{L_k^* E_k}{L_k E_k^*} S_k^* \Theta^*\right] \\
=& S_k^*(\mu+\nu)\left(2 - \frac{S_k}{S_k^*} - \frac{S_k^*}{S_k}\right) + \alpha k S_k^* \sum_{k'=1}^{n} \frac{k' P(k') L_{k'}^*}{\langle k \rangle}\left(3 - \frac{S_k^*}{S_k} + \frac{L_{k'}}{L_{k'}^*} - \frac{S_k E_k^* L_{k'}}{S_k^* E_k L_{k'}^*} - \frac{L_k}{L_k^*} - \frac{L_k^* E_k}{L_k E_k^*}\right).
\end{aligned}$$

Let $a_{kk'} = \alpha k S_k^* \sum_{k'=1}^{n} \frac{k' P(k') L_{k'}^*}{\langle k \rangle}$, $G_k(L_k) = -\frac{L_k}{L_k^*} + \ln \frac{L_k}{L_k^*}$ and $H(x) = -h(x) = 1 + \ln x - x$. We have

$$F_{kk'} = G_k(L_k) - G_{k'}(L_{k'}) + H\left(\frac{S_k^*}{S_k}\right) + H\left(\frac{E_k L_k^*}{E_k^* L_k}\right) + H\left(\frac{S_k L_{k'} E_k^*}{S_k^* L_{k'}^* E_k}\right) \leq G_k(L_k) - G_{k'}(L_{k'}).$$

Therefore, $a_{kk'}$, G_k, $F_{kk'}$ and V_k satisfy the assumptions of Theorem 3.1 and Corollary 3.3 in Ref. [27]. For the Lyapunov function $V^* = \sum_{k=1}^{n} c_k V_k$, we have that $\frac{dV^*}{dt} \leq 0$ for $(S_1, E_1, L_1, S_2, E_2, L_2, \cdots, S_n, E_n, L_n) \in \widetilde{\Omega}$. We can show that the only compact invariant set $\frac{dV^*}{dt} = 0$ is a singleton $\widetilde{P}^*$. By LaSalle's invariance principle, we conclude that $\widetilde{P}^*$ is globally asymptotically stable in $\widetilde{\Omega}$ if $R_0 > 1$. This completes the proof.

Remark 3.4. *According to Theorem 3.5, the EE will be persistent if $R_0 > 1$ and the matrix $\left(\frac{\alpha k k' P(k')}{\langle k \rangle}\right)_{n\times n}$ is irreducible for $k,\ k' \in \mathbb{N}_n$. To compare the threshold-type dynamics with respect to the reproduction number, we further need the irreducible assumption. This restriction is just a technical requirement in computation.*

4. Simulations

In this section, we present numerical simulations to verify our theoretical results. We set the initial values $S_k(0) \in (0.5, 0.6)$, $E_k(0) \in (0, 0.1)$, $J_k(0) \in (0, 0.1)$ and $I_k(0) \in (0, 0.1)$ for $k \in \mathbb{N}_n$ in all simulations. We consider a scale-free network with a degree distribution of $P(k) = (\gamma - 1)m^{\gamma-1}k^{-\gamma}$, where m is the smallest degree of node in a scale-free network; γ stands for the power law exponent. Here, we applied $m = 2, \gamma = 3$ and $\mathbb{N}_n = \{1, 2, \cdots, 100\}$. Consequently, we obtained that $\langle k \rangle = 6.5399$ and $\langle k^2 \rangle = 20.7495$.

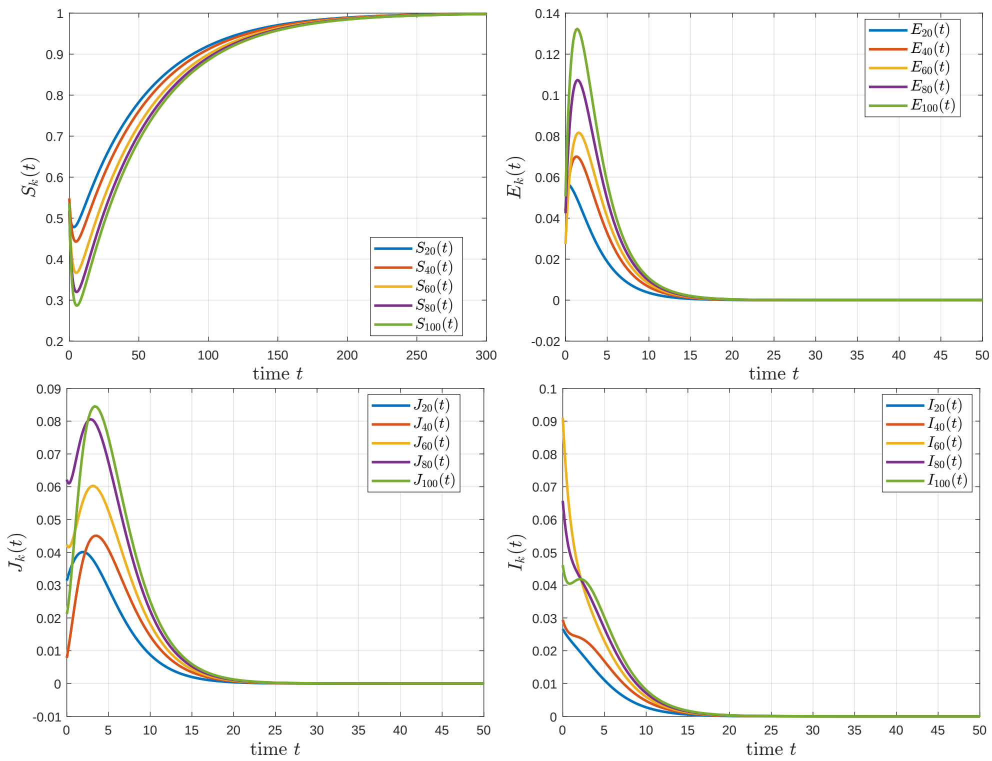

Figure 3. Time evolution of $S_k(t)$, $E_k(t)$, $I_k(t)$ and $J_k(t)$ for $k = 20, 40, 60, 80, 100$ with a fixed recruitment rate.

In Figure 3, the parameters were selected as $A = 0.02$, $\mu_1 = 0.01$, $\mu_2 = 0.02$, $\eta_1 = 0.4$, $\eta_2 = 0.2$, $\gamma_1 = 0.4$, $\gamma_2 = 0.6$, $\nu = 0.01$, $\alpha_1 = 0.02$, $\alpha_2 = 0.04$, $\theta = 0.02$ and $\mu = 0.01$. Subsequently, we have that $R_0 = 0.1666 < 1$, and it follows from Theorem 3.1, i.e., that the DFE is locally asymptotically stable. Moreover, according to Theorem 3.2, we have that $A^g = 0.0993$, which implies that $A < A^g$. This means that the DFE is globally asymptotically stable according to our theoretical analysis. From Figure 3, it can be seen that $S_k(t) \to S_k^0(t) = 1$, and $E_k(t), J_k(t), I_k(t), \Theta_1(t), \Theta_2(t) \to 0$ as $t \to \infty$, where $k = 20, 40, 60, 80, 100$, respectively. This supports the stability of the DFE.

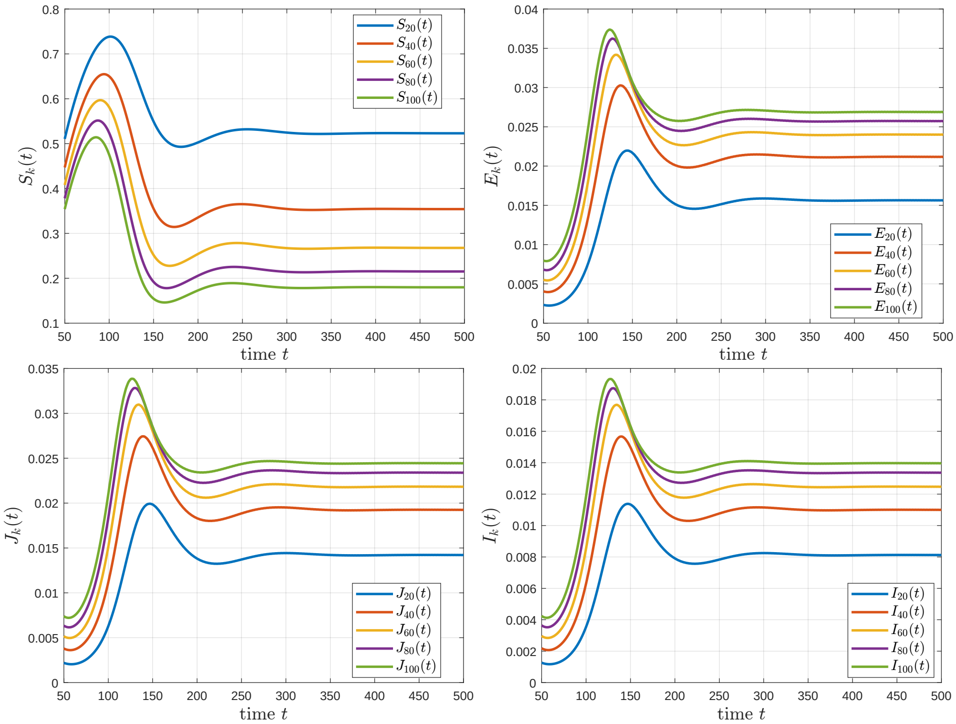

Figure 4. Time evolution of $S_k(t)$, $E_k(t)$, $I_k(t)$ and $J_k(t)$ for $k = 20, 40, 60, 80, 100$ with a fixed recruitment rate.

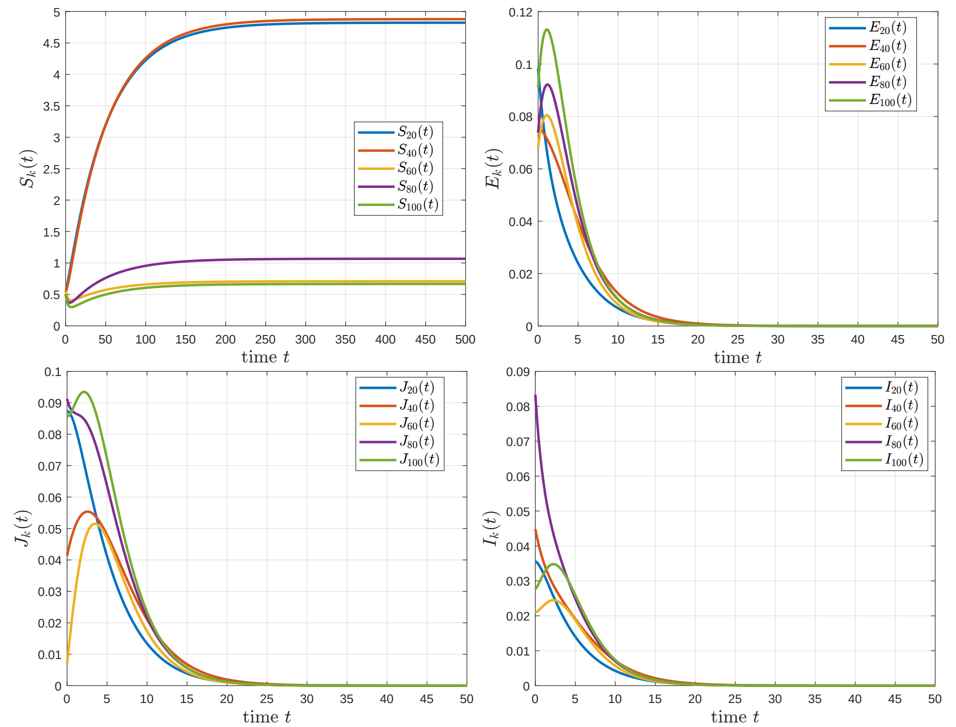

Figure 5. Time evolution of $S_k(t)$, $E_k(t)$, $I_k(t)$ and $J_k(t)$ for $k = 20, 40, 60, 80, 100$ with randomly selected recruitment rates.

In Figure 4, the parameters were selected as $A = 0.02$, $\mu_1 = \mu_2 = 0.01$, $\eta_1 = 0.4$, $\eta_2 = 0.2$, $\gamma_1 = \gamma_2 = 0.4$, $\nu = 0.01$, $\alpha_1 = \alpha_2 = 0.2$, $\theta = 0.02$ and $\mu = 0.01$. Subsequently, we have that $R_0 = 1.4861 > 1$, and it follows from Theorem 3.3 that the endemic equilibrium $\widetilde{P}^*$ of system (3.22) is globally asymptotically stable. Namely, the disease will be endemic. Figure 4 shows that $S_k(t)$, $E_k(t)$, $J_k(t)$, $I_k(t)$, $\Theta_1(t)$ and $\Theta_2(t)$ will persist for a sufficiently large t, where $k = 20, 40, 60, 80, 100$. This supports the stability of the EE.

In Figures 5 and 6, we consider the different recruitment rates in groups with different degrees. Let A_k be the recruitment rate of the group with degree k. The parameters were selected to be the same as those applied in Figures 3, and A_k was randomly selected in $(0, A^g]$ for $k \in \mathbb{N}_n$, where $A^g = 0.0993$. The simulation results are shown in Figures 5. It can be seen that $S_k(t)$ will persist, and $E_k(t)$, $J_k(t)$, $I_k(t)$, $\Theta_1(t)$, $\Theta_2(t) \to 0$ for a sufficiently large t and $k = 20, 40, 60, 80, 100$. Therefore, it seems that, as long as all values of A_k are below the threshold for $k \in \mathbb{N}_n$, the disease will die out.

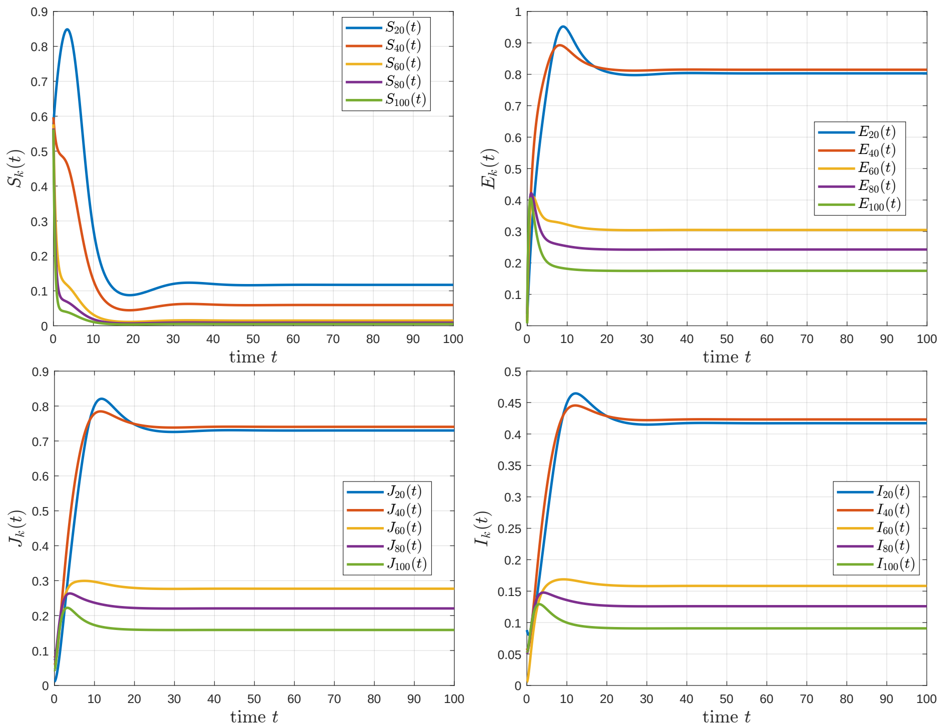

Figure 6. Time evolution of $S_k(t)$, $E_k(t)$, $I_k(t)$ and $J_k(t)$ for $k = 20, 40, 60, 80, 100$ with randomly selected recruitment rates.

In Figures 6, the parameters were selected to be the same as those applied in Figures 4 and A_k was randomly chosen in $(A^*, A_0]$ for $k \in \mathbb{N}_n$, where $A^* = 0.0135$ and $A_0 = 0.6710$. It can be seen that $S_k(t)$, $E_k(t)$, $J_k(t)$, $I_k(t)$, $\Theta_1(t)$ and $\Theta_2(t)$ will persist for a sufficiently large t, where $k = 20, 40, 60, 80, 100$. Therefore, it seems that, as long as all values of A_k are above the threshold for $k \in \mathbb{N}_n$, the disease will also be endemic.

In Figures 7 and 8, we show the time evolution diagrams for Θ_1 and Θ_2 with respect to t under two different population influx patterns, i.e., the fixed population inflow and the random population influx. The fixed population influx pattern means that the recruitment rates of different groups with different degrees are the same, while the random population influx pattern means that the recruitment rates of different groups are different. We applied random values to depict random recruitment rates in the simulations. Note that the stability of the DFE and EE depends on the strength of the relationship between the recruitment rate and threshold values A^g and A^*. Therefore, once the other parameters are fixed, we can obtain the value ranges for A and $A_k, k \in \mathbb{N}_n$ by taking A^g and A^* into account. Figure 7 (left) (Figure 8 (left)) indicates that, if $A \le A^g$ ($A_k \le A^g, k \in \mathbb{N}_n$), the disease will be extinct, and Figure 5 (right) (Figure 8 (right)) indicates, that if $A > A^g$ ($A_k > A^*, k \in \mathbb{N}_n$), the disease will be endemic. This is in line with our expectations.

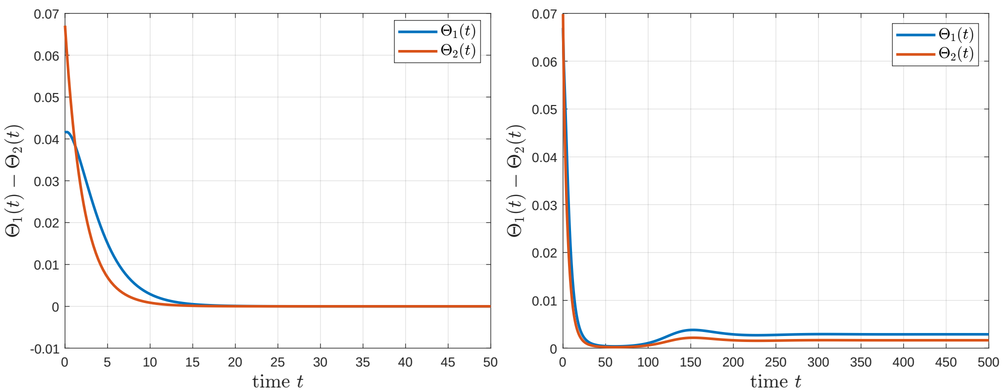

Figure 7. Time evolution of $\Theta_1(t)$ and $\Theta_2(t)$ with a fixed recruitment rate.

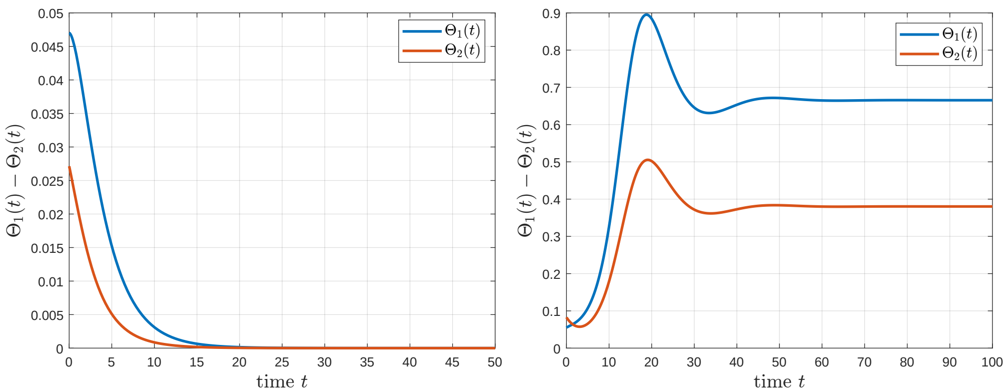

Figure 8. Time evolution of $\Theta_1(t)$ and $\Theta_2(t)$ with randomly selected recruitment rates.

In Figures 9 and 10, we consider the impact of the network structure on the dynamics, and the parameters were fixed as $A = 0.1$, $\mu_1 = \mu_2 = 0.01$, $\eta_1 = 0.4$, $\eta_2 = 0.2$, $\gamma_1 = \gamma_2 = 0.4$, $\nu = 0.01$, $\alpha_1 = \alpha_2 = 0.02$, $\theta = 0.02$ and $\mu = 0.01$. In Figure 9, we consider scale-free networks with the degree distribution $P(k) = (\gamma - 1)m^{\gamma-1}k^{-\gamma}$ for $k \in \mathbb{N}_n$, $m = 2$ and $\gamma = 2.1,\ 2.3,\ 2.5,\ 2.7,\ 2.9$, respectively. In Figure 10, we consider Poisson networks with the degree distribution $P(k) = e^{-c}c^k/k!$ for $k \in \mathbb{N}_n$ and $c = 3,\ 6,\ 9,\ 12,\ 15$, respectively. Obviously, we can observe that $\Theta_1(t)$ and $\Theta_2(t)$ tend to 0 for a sufficiently large t when γ is large in Figure 9. In contrast, $\Theta_1(t)$ and $\Theta_2(t)$ approach 0 for a sufficiently large t when c is small in Figure 10. In summary, the network structure does affect the spreading dynamics.

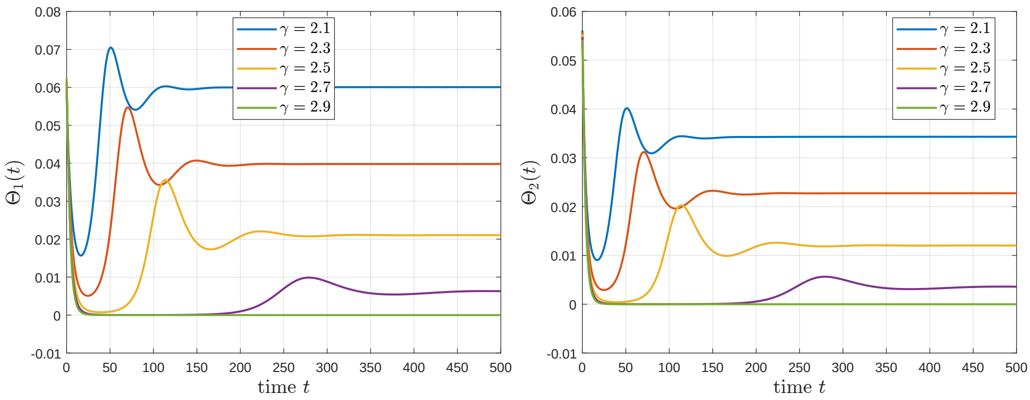

Figure 9. Time evolution of $\Theta_1(t)$ and $\Theta_2(t)$ in scale-free networks with degree distribution $P(k) = (\gamma - 1)m^{\gamma-1}k^{-\gamma}$ for $k \in \mathbb{N}_n$, $m = 2$ and $\gamma = 2.1,\ 2.3,\ 2.5,\ 2.7,\ 2.9$.

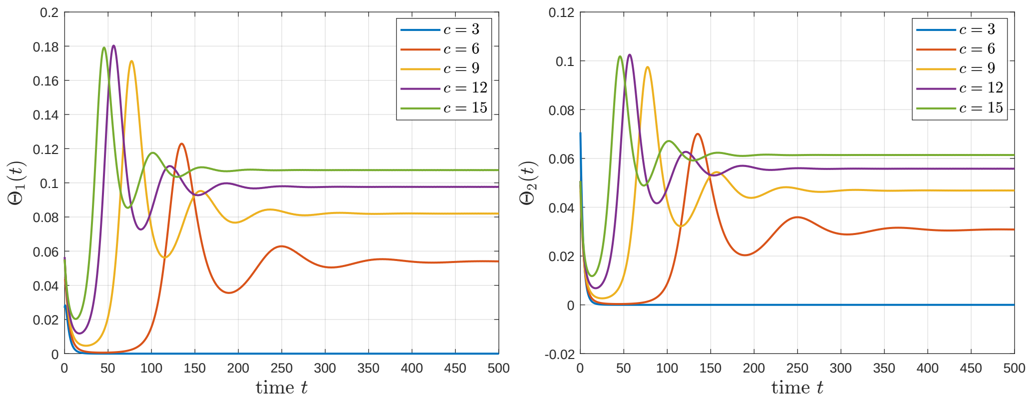

Figure 10. Time evolution of $\Theta_1(t)$ and $\Theta_2(t)$ in Poisson networks with degree distribution $P(k) = e^{-c}c^k/k!$ for $k \in \mathbb{N}_n$ and $c = 3,\ 6,\ 9,\ 12,\ 15$.

5. Conclusions

Complex networks, such as completely random networks [28], small-world networks [29] and scale-free and Poisson networks [30], have been frequently used to model the spread of an epidemic disease. Affected by occupation and geographical location, the contact among the population cannot always be modeled as a uniform collision for COVID-19; also, the epidemic disease transmission is usually heterogeneous. In order to investigate the effect of contact heterogeneity, we proposed an endemic mathematical model that incorporates direct immunity by vaccination, as well as the shift from the asymptomatic to the symptomatic subclass, by applying the idea of a compartmental model in a scale-free network. With the help of eigenvalues of the Jacobian matrix, the Lyapunov function method and LaSalle's invariance principle, we evaluated the dynamics of the proposed model. The results were obtained as follows:

- If $R_0 < 1$ or $A < A^*$, the DFE is locally asymptotically stable.
- If $A \leq A^g$, the DFE is globally asymptotically stable.

• If $R_0 > 1$ and the matrix $\left(\frac{\alpha kk'P(k')}{\langle k\rangle}\right)_{n\times n}$, $k = 1, 2, \cdots, n, k' = 1, 2, \cdots, n$ is irreducible, then the EE of (2.4) with $\alpha_1 = \alpha_2, \mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$ is globally asymptotically stable.

In the past years, ordinary differential equation compartmental models have been widely applied to describe the spread of COVID-19 (see [1–9] for example). Taking contact heterogeneity into account, we have generalized the results of the dynamics for the ordinary differential equations to the networks. We also performed simulations to validate our theoretical results. In the simulations, we mainly show two different scenarios, namely, disease persistence and disease disappearance. So, we selected two different sets of data to simulate. Moreover, we found that different subgroups with different degrees may have different recruitment rates. Again, we selected two more different sets of data to simulate. Based on the result presented in this paper, we know that the recruitment is crucial. Hence, the investigation of the recruitment rates in different groups is an important direction that is significant to better connect with reality.

The paper indicates that the disease may be under control if $R_0 < 1$. So, we can adjust the values of parameters to reduce the value of R_0. However, some values of parameters, including the natural mortality rate, transmission rate, recovery rate, disease mortality rate, incubation period duration and the rate of shift from the asymptomatic to the symptomatic group, are inherent attributes of diseases or nature that cannot be changed in the short term. Once these values are determined, we can control the spread of the disease by adjusting the value of the vaccination rate v and/or the recruitment rate A. From the result of this paper, we see that accelerating vaccination and reducing travel and contact are appropriate strategies for controlling the spread of the disease. Our study can give theoretical perspective that can facilitate understanding of the transmission mechanism for COVID-19, and the result may provide some reasonable suggestions that can be used by the policy-makers to control the disease.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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