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Research article

Fourth power mean values of one kind special Kloosterman's sum

Li Rui[1] **and Nilanjan Bag**[2,*]

[1] Department of Mathematics, Longnan Teachers College, Longnan, Gansu, China
[2] Department of Mathematics, Ramakrishna Mission Vivekananda Educational and Research Institute, Belur Math, Howrah, India

* **Correspondence:** Email: nilanjanb2011@gmail.com.

Abstract: The main purpose of this article is using the analytic methods and properties of classical Gauss sums to study the calculating problem of fourth power mean values of one kind special Kloosterman's sum, and give a sharp asymptotic formula for it. At the same time, the paper also provides a new and effective method for the study of related power mean value problems.

Keywords: special Kloosterman's sums; fourth power mean; analytic method; asymptotic formula
2020 Mathematics Subject Classification: 11L03; 11L05.

1. Introduction

Let $q > 1$ be an integer. For any positive integer k, we define a special Kloosterman's sum $S(m, n, k; q)$ as follows:

$$S(m, n, k; q) = \sum_{a=1}^{q}{}' e\left(\frac{ma^k + n\bar{a}}{q}\right),$$

where m and n be any integers, $\sum_{a=1}^{q}{}'$ denotes the summation over all $1 \le a \le q$ such that $(a, q) = 1$, $\bar{a}$ denotes $a \cdot \bar{a} \equiv 1 \bmod q$, $e(y) = e^{2\pi i y}$ and $i^2 = -1$.

If $k = 1$, then $S(m, n, 1; q) = S(m, n; q)$ becomes the classical Kloosterman sum (see H. D. Kloosterman [1])

$$S(m, n; q) = \sum_{a=1}^{q}{}' e\left(\frac{ma + n\bar{a}}{q}\right),$$

which plays a very important role in analytic number theory. Because of this, many mathematicians have studied various properties of $S(m, n; q)$ and obtained a series of important results. It is well known that, for a prime p,

$$S(1, n; p) = -2\sqrt{p}\cos(\theta(n)),$$

where the angles $\theta(n)$ is equi-distributed in $[0, \pi]$ with respect to the Sato-Tate measure $\frac{2}{\pi}\sin^2(\theta)d\theta$. For more details, see [2]. Thus the moments can be estimated by evaluating the corresponding integral

$$\frac{1}{p-1}\sum_{m=1}^{p-1}|S(1, m; p)|^{2\ell} \approx 2^{2\ell}p^{\ell}\frac{2}{\pi}\int_0^{\pi}\cos^{2\ell}(\theta)\sin^{\ell}(\theta)d\theta, \tag{1.1}$$

for any positive integer ℓ. For example, H. Salié [3] proved that for any odd prime p, we have the identity

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{a + m\bar{a}}{p}\right)\right|^4 = 2p^3 - 3p^2 - 3p.$$

The proofs of this result can also be found in [4].

In 2011, using the elementary methods W. P. Zhang [5] proved a general result. For any integer n with $(n, q) = 1$, he proved the identity

$$\sum_{m=1}^{q}\left|\sum_{a=1}^{q}{}' e\left(\frac{ma + n\bar{a}}{q}\right)\right|^4 = 3^{\omega(q)}q^2\phi(q)\prod_{p\|q}\left(\frac{2}{3} - \frac{1}{3p} - \frac{4}{3p(p-1)}\right),$$

where $\phi(q)$ is Euler function, $\omega(q)$ denotes the number of all different prime divisors of q, $p\|q$ denotes the product over all prime divisors of q with $p \mid q$ and $p^2 \nmid q$.

Perhaps the most essential conclusion is the upper bound estimate of $S(m, n; q)$ (see S. Chowla [6] or T. Estermann [7]). That is,

$$\sum_{a=1}^{q}{}' e\left(\frac{ma + n\bar{a}}{q}\right) \ll (m, n, q)^{\frac{1}{2}} \cdot d(q) \cdot q^{\frac{1}{2}},$$

where $d(q)$ denotes the Dirichlet divisor function, (m, n, q) denotes the greatest common factor of m, n and q. For some other important results related to Kloosterman sums, see [8–13].

It seems that not much have been studied on the properties of $S(m, n, k; q)$. In particular, we are primarily interested in analogous result to (1.1). Here we are interested in evaluating the fourth power mean

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^k + \bar{a}}{p}\right)\right|^4, \tag{1.2}$$

where p is a prime and $k \geq 3$ is an integer. For $k = 2$, W. P. Zhang informed us that he had obtained an exact calculating formula for (1.2) in an unpublished paper. If $k \geq 3$, then we have not seen any related results yet. This problem is important to the study of Kloosterman sums, and it is a further extension of the classical Kloosterman sums problem.

The main purpose of this paper is using the elementary and analytic methods, and the properties of the classical Gauss sums to study the calculating problems of (1.2), and give a sharp asymptotic formula for it with $k = 3$. That is, we prove the following result:

Theorem 1. *Let $p > 3$ be an odd prime, then we have the asymptotic formula*

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^4 = 3p^3 + O\left(p^{5/2}\right).$$

Remark: In Theorem 1, we only obtained an asymptotic formula for (1.2) with $k = 3$. Whether there exists an asymptotic formula for (1.2) with $k > 3$ is still an open problem. In addition, whether there exists an exact calculating formula for (1.2) with $k = 3$ also seems to be an interesting problem.

2. Some Lemmas

In this section, we need to give a few simple lemmas. They are necessary in the proof of our theorem. Hereinafter, we shall use some knowledge of elementary number theory, analytic number theory and the properties of the classical Gauss sums. Many contents can be found in many number theory textbooks, such as [14] and [15]. First we prove the following:

Lemma 1. *Let p be an odd prime. Then we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2 = \begin{cases} p^2-p-1 & \text{if } 3 \nmid (p-1), \\ p^2-3p-1 & \text{if } 3 \mid (p-1). \end{cases}$$

Proof. From the trigonometrical identity

$$\sum_{m=0}^{p-1} e\left(\frac{mn}{p}\right) = \begin{cases} p & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n \end{cases} \tag{2.1}$$

we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2 = \sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2 - 1$$

$$= p\sum_{\substack{a=1\\ a^3\equiv b^3 \bmod p}}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{\overline{a}-\overline{b}}{p}\right) - 1 = p\sum_{\substack{a=1\\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{b\left(\overline{a}-1\right)}{p}\right) - 1. \tag{2.2}$$

If $3 \nmid (p-1)$, then the congruence equation $a^3 \equiv 1 \bmod p$ has one solution $a = 1$. So from (2.1) and (2.2) we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2 = p(p-1)-1 = p^2-p-1. \tag{2.3}$$

If $3 \mid (p-1)$, then the congruence equation $a^3 \equiv 1 \bmod p$ has three distinct solutions, one of them is $a = 1$. So from (2.1) and (2.2) we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2 = p(p-1)-2p-1 = p^2-3p-1. \tag{2.4}$$

Now Lemma 1 follows from (2.3) and (2.4).

□

Lemma 2. *Let p be an odd prime and χ be any non-principal character modulo p. Then we have the identity*

$$\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2$$

$$=\begin{cases} p\cdot\left|\sum\limits_{a=1}^{p-1}\overline{\chi}\left(a^3-1\right)\overline{\chi}^3\left(\overline{a}-1\right)\right|^2 & \text{if } 3\mid(p-1) \text{ and } \chi=\lambda,\\ p^2\cdot\left|\sum\limits_{a=1}^{p-1}\overline{\chi}\left(a^3-1\right)\overline{\chi}^3\left(\overline{a}-1\right)\right|^2 & \text{otherwise,}\end{cases}$$

where λ denotes any character of order three modulo p.

Proof. From the properties of the classical Gauss sums and the reduced residue system modulo p we have

$$\begin{aligned}\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2 &= \tau(\chi)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(a^3-b^3\right)e\left(\frac{\overline{a}-\overline{b}}{p}\right)\\ &= \tau(\chi)\sum_{a=1}^{p-1}\overline{\chi}\left(a^3-1\right)\sum_{b=1}^{p-1}\overline{\chi}^3(b)e\left(\frac{\overline{b}\left(\overline{a}-1\right)}{p}\right)\\ &= \tau(\chi)\tau\left(\chi^3\right)\sum_{a=1}^{p-1}\overline{\chi}\left(a^3-1\right)\overline{\chi}^3\left(\overline{a}-1\right),\end{aligned} \tag{2.5}$$

where $\tau(\chi)$ is the classical Gauss sum, defined be

$$\tau(\chi)=\sum_{t=1}^{p-1}\chi(t)e\left(\frac{t}{p}\right).$$

If $3\mid(p-1)$ and $\chi=\lambda$, then $|\tau\left(\lambda^3\right)|=1$. If $\chi\neq\lambda$, then $|\tau(\chi^3)|=|\tau(\chi)|=\sqrt{p}$. So from (2.5) we can deduce Lemma 2. □

Lemma 3. *Let $f(x,y)$ be a polynomial with rational integer coefficients which is absolutely irreducible. If $N(p)$ denotes the number of solutions of the congruence*

$$f(x,y)\equiv 0 \bmod p,$$

then for large primes p, we have the asymptotic formula

$$N(p)=p+O\left(p^{1/2}\right).$$

Proof. See [16, Theorem 1A] □

Lemma 4. *Let p be an odd prime, and*

$$f(x,y) = y^2(x^2+x+1)^2 + x^2(y^2+y+1)^2 + xy(x^2+x+1)(y^2+y+1) - 3x^2y^2.$$

Then we have the asymptotic formula

$$\sum_{\substack{a=1 \\ f(a,b)\equiv 0 \bmod p}}^{p-1} \sum_{b=1}^{p-1} 1 = p + O\left(p^{1/2}\right).$$

Proof. It is clear that

$$\begin{aligned} f(x,y) &= x^4y^2 + x^2y^4 + x^3y^3 + 3(x^3y^2 + x^2y^3) + x^3y + xy^3 + 4x^2y^2 \\ &\quad +3(x^2y + xy^2) + x^2 + y^2 + xy \end{aligned}$$

is a symmetric polynomial in x and y. Let $f(x,y)$ have some factorization over any finite extension of $\mathbb{F}_p$. If any factorization of $f(x,y)$ occurs, all factors are also symmetric in x and y. Hence the possibilities are

$$f(x,y) = \begin{cases} (G_2+G_3+G_4+G_5)(1+H_1) \\ (G_2+G_3+G_4)(1+H_1+H_2) \\ (G_2+G_3)(1+H_1+H_2+H_3) \\ G_2(1+H_1+H_2+H_3+H_4), \end{cases}$$

where H_i and G_i are symmetric polynomials of two variables x and y of degree i. Let $f(x,y) = (G_2+G_3+G_4+G_5)(1+H_1)$. It is not possible, because then G_5H_1 is the highest degree term $x^4y^2 + x^2y^4 + x^3y^3$. Then the symmetric polynomial of degree 1, H_1 must divide $x^4y^2 + x^2y^4 + x^3y^3$, i.e., $x+y$ divides $x^2y^2(x^2+y^2+xy)$, which is not possible.
Similarly, G_3H_3 is not equal to $x^4y^2 + x^2y^4 + x^3y^3$, as it does not have 3 degree divisors, which is symmetric. Hence $f(x,y) = (G_2+G_3)(1+H_1+H_2+H_3)$ is not possible.
Now let, $f(x,y) = G_2(1+H_1+H_2+H_3+H_4)$. We have $G_2 = x^2+y^2+xy$, which forces $H_4 = x^2y^2$ and $G_2H_3 = 3(x^3y^2+x^2y^3)$. Then x^2+y^2+xy divides $x^3y^2+x^2y^3$, which is not possible for $p > 3$.
Hence the only case remaining is $f(x,y) = (G_2+G_3+G_4)(1+H_1+H_2)$. Then $G_2 = x^2+y^2+xy$, and $G_4H_2 = x^2y^2(x^2+y^2+xy)$.

Case 1: $G_4 = a_1x^2y^2$ and $H_2 = b_1(x^2+y^2+xy)$. We have

$$3(x^3y^2 + x^2y^3) = G_4H_1 + G_3H_2,$$

where $H_1 = a_2(x+y)$, $G_3 = b_2(x^3+y^3) + b_3(xy^2+yx^2)$. Then we have $b_1b_2 = 0$ and $b_1b_3 = 0$. Now $b_1 \neq 0$. Hence $b_2 = b_3 = 0$. Also we have $a_2 = 3a_1^{-1}$. This gives $H_1 = 3a_1^{-1}(x+y)$ and $G_3 = 0$. Hence we have the factorization

$$(x^2+y^2+xy+a_1x^2y^2)(1+3a_1^{-1}(x+y)+b_1(x^2+y^2+xy)) = f(x,y).$$

Then comparing with the degree 3-part of $f(x,y)$, we get a contradiction.

Case 2: $G_4 = a_1xy(x^2 + y^2 + xy)$ and $H_2 = b_1xy$. Then we have

$$H_2G_2 + G_3H_1 + G_4 = x^3y + xy^3 + 4x^2y^2,$$

where $H_1 = a_2(x+y)$, $G_3 = b_2(x^3+y^3)+b_3(xy^2+yx^2)$. Then comparing the coefficients we get $a_2b_2 = 0$, $a_1 + b_1 + a_2b_2 + a_2b_3 = 1$ and $a_1 + b_1 + 2a_2b_3 = 4$. Hence we must have $b_2 = 0$, otherwise we have $a_1 + b_1 = 1$ and $a_1 + b_1 = 4$, which is not possible. Hence we deduce $G_3 = b_3(xy^2 + yx^2)$. Now consider

$$3(x^3y^2 + x^2y^3) = G_4H_1 + G_3H_2,$$

which gives $a_1a_2 = 0$ and $b_1b_3 = 3$, which implies $a_2 = 0$ and $b_3 = 3b_1^{-1}$. Hence we have the factorization

$$(x^2 + y^2 + xy + 3b_1^{-1}(xy^2 + x^2y) + a_1xy(x^2 + y^2 + xy))(1 + b_1xy) = f(x, y).$$

Comparing the coefficients of degree 4-part we get a contradiction.

This completes the proof for irreducibility of $f(x, y)$. Also such computations can be done in any finite extension of $\mathbb{F}_p$. Hence $f(x, y)$ is absolute irreducible for any prime $p > 3$. Hence by Lemma 3, we get the number of $\mathbb{F}_p$-points on $f(x, y) = 0$ is equal to $p + O(p^{1/2})$.

This proves Lemma 4. □

Lemma 5. *Let p be an odd prime, then we have the asymptotic formula*

$$\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \overline{\chi}\left(a^3 - 1\right) \overline{\chi}^3 (\overline{a} - 1) \right|^2 = 3p^2 + O(p^{3/2}).$$

Proof. From the orthogonality of the characters modulo p and Lemma 4 we have

$$\begin{aligned}
&\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \overline{\chi}\left(a^3 - 1\right) \overline{\chi}^3 (\overline{a} - 1) \right|^2 \\
= &\ (p-1) \sum_{\substack{a=1 \\ (a^3-1)(\overline{a}-1)^3 \equiv (b^3-1)(\overline{b}-1)^3 \bmod p}}^{p-1} \sum_{b=1}^{p-1} 1 \\
= &\ (p-1) \sum_{\substack{a=1 \\ (a+\overline{a})^3 - 3(a+\overline{a})^2 \equiv (b+\overline{b})^3 - 3(b+\overline{b})^2 \bmod p}}^{p-1} \sum_{b=1}^{p-1} 1 \\
= &\ (p-1) \sum_{\substack{a=1 \\ (a-b)(ab-1)f(a,b) \equiv 0 \bmod p}}^{p-1} \sum_{b=1}^{p-1} 1 \\
= &\ (p-1)^2 + (p-1)^2 + (p-1) \sum_{\substack{a=1 \\ f(a,b) \equiv 0 \bmod p}}^{p-1} \sum_{b=1}^{p-1} 1 - (p-1) \sum_{\substack{a=1 \\ f(a,a) \equiv 0 \bmod p}}^{p-1} 1
\end{aligned}$$

$$-(p-1)\sum_{\substack{a=1\\ f(a,\overline{a})\equiv 0 \bmod p}}^{p-1} 1+O(p)=3p^2+O(p^{3/2}).$$

This proves Lemma 5.

□

3. Proof of the Theorem 1

In this section, we will provide the proof of our main theorem. First from the orthogonality of the characters modulo p we have

$$\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2=(p-1)\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^4. \tag{3.1}$$

On the other hand, if $3\nmid(p-1)$, then note that for any non-principal character $\chi \bmod p$, we have $|\tau(\chi)|=|\tau(\chi^3)|=\sqrt{p}$, and using Lemma 1, Lemma 2 and Lemma 5 we have

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2=\left|\sum_{m=1}^{p-1}\chi_0(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2\\
&\quad+\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2\\
&= (p^2-p-1)^2+p^2\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left|\sum_{a=1}^{p-1}\overline{\chi}\left(a^3-1\right)\overline{\chi}^3\left(\overline{a}-1\right)\right|^2\\
&= (p^2-p-1)^2+p^2\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\overline{\chi}\left(a^3-1\right)\overline{\chi}^3\left(\overline{a}-1\right)\right|^2-p^2\left|\sum_{a=1}^{p-1}\chi_0\left(a^3-1\right)\right|^2\\
&= (p^2-p-1)^2+3p^3(p-1)-p^2(p-1)^2+O\left(p^{7/2}\right)\\
&= 3p^3(p-1)+O\left(p^{7/2}\right).
\end{aligned}\tag{3.2}$$

If $3\mid(p-1)$, let λ be any three-order character modulo p, then $|\tau\left(\lambda^3\right)|=1$ and

$$\left|\sum_{a=1}^{p-1}\overline{\lambda}\left(a^3-1\right)\overline{\lambda}^3\left(\overline{a}-1\right)\right|\ll\sqrt{p}. \tag{3.3}$$

From estimate (3.3), Lemma 1, Lemma 2, Lemma 5 and the method of proving (3.2) we also have the asymptotic formula

$$\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2=\left|\sum_{m=1}^{p-1}\chi_0(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2$$

$$
\begin{aligned}
&+p\left|\sum_{a=1}^{p-1}\overline{\lambda}\left(a^3-1\right)\overline{\lambda}^3\left(\overline{a}-1\right)\right|^2+p\left|\sum_{a=1}^{p-1}\lambda\left(a^3-1\right)\lambda^3\left(\overline{a}-1\right)\right|^2\\
&+\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0,\lambda,\lambda^2}}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^2\right|^2\\
=\ &(p^2-3p-1)^2+p^2\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0,\lambda,\lambda^2}}\left|\sum_{a=1}^{p-1}\overline{\chi}\left(a^3-1\right)\overline{\chi}^3\left(\overline{a}-1\right)\right|^2+O(p^2)\\
=\ &(p^2-3p-1)^2+3p^3(p-1)-p^2(p-1)^2+O\left(p^{7/2}\right)\\
=\ &3p^3(p-1)+O\left(p^{7/2}\right). \qquad (3.4)
\end{aligned}
$$

Combining (3.1), (3.2) and (3.4) we may immediately deduce the asymptotic formula

$$
\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+\overline{a}}{p}\right)\right|^4=3p^3+O\left(p^{5/2}\right).
$$

This complete the proof of Theorem 1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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