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Research article

Energy equality in the isentropic compressible Navier-Stokes-Maxwell equations

Jie Zhang[1], **Gaoli Huang**[1] **and Fan Wu**[2,*]

[1] School of Mathematics and Information Science, Guangzhou University, Guangzhou, Guangdong 510006, China

[2] College of Science, Nanchang Institute of Technology, Nanchang, Jiangxi 330099, China

* **Correspondence:** Email: wufan0319@yeah.net.

Abstract: This paper concerns energy conservation for weak solutions of compressible Navier-Stokes-Maxwell equations. For the energy equality to hold, we provide sufficient conditions on the regularity of weak solutions, even for solutions that may include exist near-vacuum or on a boundary. Our energy conservation result generalizes/extends previous works on compressible Navier-Stokes equations and an incompressible Navier-Stokes-Maxwell system.

Keywords: energy conservation; compressible Navier-Stokes-Maxwell equations; Onsager's conjecture

1. Introduction

In this paper, the following isentropic compressible Navier-Stokes-Maxwell (CNSM) system is considered, which consists of Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The coupling comes from the Lorentz force in the fluid equation and the electric current in the Maxwell following equations:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}u + \nabla P(\rho) = j\times b, \\ \partial_t E - \nabla\times b + j = 0, \ \ j := E + u\times b, \\ \partial_t b + \nabla\times E = 0, \\ \mathrm{div} b = 0, \end{cases} \tag{1.1}$$

with the initial data

$$(\rho, u, E, b)(\cdot, 0) = (\rho_0, u_0, E_0, b_0), \tag{1.2}$$

where ρ is the density, u is the velocity field, and E and b represent electronic and magnetic fields, respectively. The fluid pressure is represented by $P(\rho)$ meets:

$$P(\rho) = a\rho^{\gamma} \quad with \quad a > 0, \quad \gamma > 1, \tag{1.3}$$

where a is a physical constant and γ is the adiabatic exponent. The viscosity coefficients μ and λ are constant and satisfy the physical restrictions $\mu > 0$ and $2\mu + 3\lambda \geq 0$. j is the electric current expressed by the Ohms law. The force term $j \times B$ in Navier-Stokes equations comes from Lorentz force under a quasi-neutrality assumption of the net charge carried by the fluid. If the electric current is ignored (i.e., $j = 0$), (1.1) reduces to the well-known isentropic compressible Navier-Stokes (CNS) system. Equation (1.1) is one of the most important mathematical models in continuum mechanics. Lions [1] and Feireisl [2–4] proved that the CNS system admits a weak solution, as long as the adiabatic exponent $\gamma > \frac{3}{2}$. Due to a lack of regularity of weak solutions, it is not known whether weak solutions satisfy the energy equality for both incompressible and compressible fluids equations. It is a nature problem: how "good" is regularity for weak solutions needed to ensure the energy equality?

For a CNS system, the appearance of ρ makes $\partial_t(\rho u)$ nonlinear, and; therefore, some density regularity is required in when using commutator estimates. Yu [5] used the Lions's commutator estimate to show energy conservation for compressible Navier-Stokes equations with a degenerate viscosity but without vacuum. Nguyen et al. [6] extended Yu's result with a weaker regularity condition in a bounded domain. Liang [7] established a L^pL^s type condition for the energy equality, in particular, there was no need regularly assume the density derivative. Recently, Ye et al. [8] showed that Lions' condition for energy balance is also valid for the weak solutions of isentropic compressible Navier-Stokes equations allowing for vacuum under suitable integral conditions on the density and its derivative. This is a very interesting result.

Comparing with fruitful results for either an incompressible Navier-Stokes system or a compressible case, there are a few results regarding an incompressible/compressible NSM model due to its hyperbolic structure. For the incompressible NSM system, Ma and Wu [9] obtained the Shinbrot type energy conservation criteria for the weak solution. In addition, for the distributional solution, he showed the Lions' energy conservation criteria [10]. To the best of our knowledge, there is no result concerning the energy equality for a CNSM system (1.1).

Motivated by [7–10], the purpose of this paper is to establish the conditional energy conservation of weak solutions to the CNSM system (1.1) allowing for vacuum. By proving an energy conservation/equality, the commutator estimates are required for treating the nonlinear terms. Furthermore, due to the special structure of parabolic hyperbolic coupling, the derivative to the velocity field u needs to be transfered. We state the result in detail in the theorem below.

Theorem 1.1. *Let* $0 \leq \rho < c < \infty$, $\nabla\sqrt{\rho} \in L^4(0,T;L^4(\mathbb{T}^3))$, $u \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^2(0,T;H^1(\mathbb{T}^3))$, $(E,b) \in L^\infty(0,T;L^2(\mathbb{T}^3))$ *and* $j \in L^2(0,T;L^2(\mathbb{T}^3))$ *be a weak solution to system* (1.1)*. In addition, if* $(u,b) \in L^4(0,T;L^4(\mathbb{T}^3))$ *and* $E \in L^2(0,T;L^2(\mathbb{T}^3))$*, then the weak solution satisfies the following energy equality:*

$$\begin{aligned} &\int_{\mathbb{T}^3}(\frac{1}{2}|\sqrt{\rho}u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})dx + \int_0^T\int_{\mathbb{T}^3}\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2 + |j|^2dxds \\ &= \int_{\mathbb{T}^3}(\frac{1}{2}|\sqrt{\rho_0}u_0|^2 + \frac{1}{2}|E_0|^2 + \frac{1}{2}|b_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1})dx. \end{aligned} \tag{1.4}$$

Remark 1.1. *This theorem extends the energy equality of the incompressible NSM to the isentropic compressible equations* (1.1) *with vacuum.*

Remark 1.2. *Since* $u, b \in L^\infty(0,T;L^2(\mathbb{T}^3)) \cap L^p(0,T;L^q(\mathbb{T}^3))$, *for any* $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$, $q\geq 4$, *one can deduce that*

$$\|u\|_{L^4(0,T;L^4(\mathbb{T}^3))} \leq C\|u\|^a_{L^\infty(0,T;L^2(\mathbb{T}^3))}\|u\|^{1-a}_{L^p(0,T;L^q(\mathbb{T}^3))},$$

and

$$\|b\|_{L^4(0,T;L^4(\mathbb{T}^3))} \leq C\|b\|^a_{L^\infty(0,T;L^2(\mathbb{T}^3))}\|b\|^{1-a}_{L^p(0,T;L^q(\mathbb{T}^3))},$$

for some $a \in (0,1)$. *Thus, it is also true that* $L^4(0,T;L^4(\mathbb{T}^3))$ *in Theorem 1.1 is replaced with* $L^p(0,T;L^q(\mathbb{T}^3))$, *for any* $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$, $q\geq 4$.

2. Preliminaries

We will recall some definitions and lemmas that will be used later. First, we denote $\mathcal{D}\left(\mathbb{T}^3\right)$ as the space of indefinitely differentiable with compact support and $\mathcal{D}'\left(\mathbb{T}^3\right)$ as the space of distributions.

Definition 2.1. *The* (ρ, u, E, b) *is called a weak solution to the CNSM systems* (1.1) *and* (1.2) *if* (ρ, u, E, b) *satisfies the following assumptions for any time* $t \in [0,T]$:
• *The problems* (1.1) *and* (1.2) *holds in* $\mathcal{D}'(0,T;\mathbb{T}^3)$*;*
• *The equation* $(1.1)_1$ *is satisfied in the sense of renormalized solutions: for any function* $b \in C^1(\mathbb{R})$ *such that* $b'(x) = 0$ *for* $x \geq M$*), we get in* $\mathcal{D}'(0,T;\mathbb{T}^3)$:

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho))\operatorname{div}u = 0$$

where M is a constant that varies for different functions b.
• *The weak solutions require the following properties:*

$$\begin{aligned}&\sqrt{\rho}u \in L^\infty(0,T;L^2(\mathbb{T}^3)),\ u \in L^2(0,T;W_0^{1,2}(\mathbb{T}^3)),\ E \in L^\infty(0,T;L^2(\mathbb{T}^3)),\\ &\rho \in L^\infty(0,T;L^1\cap L^\gamma(\mathbb{T}^3)),\ b \in L^\infty(0,T;L^2(\mathbb{T}^3)),\ j \in L^2(0,T;L^2(\mathbb{T}^3)),\end{aligned} \tag{2.1}$$

• *The energy inequality for weak solutions holds:*

$$\begin{aligned}&\int_{\mathbb{T}^3}(\frac{1}{2}|\sqrt{\rho}u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})dx + \int_0^T\int_{\mathbb{T}^3}\mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div}u|^2 + |j|^2 dxds\\ \leq &\int_{\mathbb{T}^3}(\frac{1}{2}|\sqrt{\rho_0}u_0|^2 + \frac{1}{2}|E_0|^2 + \frac{1}{2}|b_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1})dx.\end{aligned} \tag{2.2}$$

Let $\eta \in C_c^\infty(\mathbb{R}^d)$ (d is the number of the space dimension) be a standard mollification kernel and set

$$\eta^\varepsilon(x) = \frac{1}{\varepsilon^{d+1}}\eta\left(\frac{x}{\varepsilon}\right),\ w^\epsilon = \eta^\varepsilon * w,\ f^\varepsilon(w) = f(w) * \eta^\varepsilon.$$

We should notice that w^ε is well-defined on $\Omega^\varepsilon = \{x \in \Omega : d(x,\partial\Omega) > \varepsilon\}$. Next, we recall some useful lemmas which will be frequently used throughout the paper.

Lemma 2.1. *[8] Let* $r, s, r_1, r_2, s_1, s_2 \in [1, +\infty)$ *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ *and* $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. *Assume* $f \in L^{r_1}(0,T; L^{s_1}(\mathbb{T}^3))$ *and* $g \in L^{r_2}(0,T; L^{s_2}(\mathbb{T}^3))$. *Then, for any* $\varepsilon > 0$, *there holds*

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^r(0,T;L^s(\mathbb{T}^3))} \to 0, \text{ as } \varepsilon \to 0,$$

and

$$\|(f \times g)^\varepsilon - (f^\varepsilon \times g^\varepsilon)\|_{L^r(0,T;L^s(\mathbb{T}^3))} \to 0, \text{ as } \varepsilon \to 0.$$

Lemma 2.2. *[8, 11, 12] Let* $1 \le r, s, r_1, s_1, r_2, s_2 \le \infty$, *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ *and* $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$. *Let* ∂ *be a partial derivative in space or time; in addition, let* $\partial_t f,\ \nabla f \in L^{r_1}(0,T; L^{s_1}(\mathbb{T}^3))$, $g \in L^{r_2}(0,T; L^{s_2}(\mathbb{T}^3))$. *Then, there holds*

$$\|\partial(fg)^\varepsilon - \partial(f g^\varepsilon)\|_{L^r(0,T;L^s(\mathbb{T}^3))} \le C \left(\|\partial_t f\|_{L^{r_1}(0,T;L^{s_1}(\mathbb{T}^3))} + \|\nabla f\|_{L^{r_1}(0,T;L^{s_1}(\mathbb{T}^3))}\right) \|g\|_{L^{r_2}(0,T;L^{s_2}(\mathbb{T}^3))},$$

or some constant $C > 0$ *independent of* ε, f *and* g. *Moreover, as* $\varepsilon \to 0$ *if* $r_2, s_2 < \infty$,

$$\partial(fg)^\varepsilon - \partial(f g^\varepsilon) \to 0 \text{ in } L^r(0,T; L^s(\mathbb{T}^3)).$$

Lemma 2.3. *[13] Let* $B_0 \hookrightarrow B \hookrightarrow B_1$ *be three Banach spaces with compact embedding* $B_0 \hookrightarrow\hookrightarrow B_1$, *and let there exist* $0 < \delta < 1$ *and* $C > 0$ *such that*

$$\|u\|_B \le C \|u\|_{B_0}^{1-\delta} \|u\|_{B_1}^{\delta} \text{ for all } u \in B_0 \cap B_1.$$

Denote for $T > 0$,

$$W(0,T) = W^{s_0, r_0}(0,T; B_0) \cap W^{s_1, r_1}(0,T; B_1)$$

with

$$s_0, s_1 \in \mathbb{R};\ 0 \le r_0, r_1 \le \infty.$$

$$s_\delta = (1-\delta)s_0 + \delta s_1,\ \frac{1}{r_\delta} = \frac{1-\delta}{r_0} + \frac{\delta}{r_1},\ s^* = s_\delta - \frac{1}{r_\delta}.$$

Assume that $s_\delta > 0$ *and* G *is a bounded set in* $W(0,T)$, *Then, we have the following:*

- *If* $s_* \le 0$, *then* G *is relatively compact in* $L^p(0,T; B)$ *for all* $1 \le p < p^* := -\frac{1}{s^*}$.
- *If* $s_* > 0$, *then* G *is relatively compact in* $C(0,T; B)$.

3. Proof of theorem 1.1

First, we mollify the system (1.1) and obtain

$$\partial_t \rho^\varepsilon + \nabla \cdot (\rho u)^\varepsilon = 0, \tag{3.1}$$

$$\partial_t (\rho u)^\varepsilon + \nabla \cdot (\rho u \otimes u)^\varepsilon - \mu \Delta u^\varepsilon - (\lambda + \mu) \nabla \text{div} u^\varepsilon + \nabla (P(\rho))^\varepsilon = (j \times b)^\varepsilon, \tag{3.2}$$

$$\partial_t E^\varepsilon - (\nabla \times b)^\varepsilon + j^\varepsilon = 0, \tag{3.3}$$

and

$$\partial_t b^\varepsilon + (\nabla \times E)^\varepsilon = 0 \tag{3.4}$$

for any $0 < \varepsilon < 1$.

Next, let $\phi(t)$ be a smooth solution function compactly supported in $(0, +\infty)$. Multiplying (3.2)–(3.4) by $\phi(t)u^\varepsilon$, $\phi(t)E^\varepsilon$, and $\phi(t)b^\varepsilon$, respectively, then integrating over $(0, T) \times \mathbb{T}^3$, one has the following:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon \partial_t(\rho u)^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon \nabla \cdot (\rho u \otimes u)^\varepsilon dxdt - \mu \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon \Delta u^\varepsilon dxdt \\
&- (\lambda + \mu) \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon \nabla \mathrm{div} u^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon \nabla(P(\rho))^\varepsilon dxdt \\
&- \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon (j \times b)^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)E^\varepsilon \partial_t E^\varepsilon dxdt - \int_0^T \int_{\mathbb{T}^3} \phi(t)E^\varepsilon (\nabla \times b)^\varepsilon dxdt \\
&+ \int_0^T \int_{\mathbb{T}^3} \phi(t)E^\varepsilon j^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)b^\varepsilon \partial_t b^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)b^\varepsilon (\nabla \times E)^\varepsilon dxdt = 0.
\end{aligned} \tag{3.5}$$

We use (A)–(H) and (J)–(L) to represent the terms on the left-hand side of (3.5), respectively. We will estimate them as follows.

3.1. Estimate of (A)

By a straightforward computation, we can obtain the following:

$$\begin{aligned}
(A) &= \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon (\partial_t(\rho u)^\varepsilon - \partial_t(\rho u^\varepsilon))dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon \partial_t(\rho u^\varepsilon)dxdt \\
&=: (A_1) + \int_0^T \int_{\mathbb{T}^3} \phi(t)\rho_t |u^\varepsilon|^2 dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)\rho \partial_t \frac{|u^\varepsilon|^2}{2} dxdt \\
&=: (A_1) + (A_2) + (A_3).
\end{aligned}$$

We know that (A_3) is the desire term while (A_2) will be canceled with the term (B_2) later. By Hölder's inequality and Lemma 2.2, it gives that the following:

$$\begin{aligned}
(A_1) &= \int_0^T \int_{\mathbb{T}^3} \phi(t)u^\varepsilon (\partial_t(\rho u)^\varepsilon - \partial_t(\rho u^\varepsilon))dxdt \\
&\le C\|u^\varepsilon\|_{L^4(0,T;L^4(\mathbb{T}^3))} \|\partial_t(\rho u)^\varepsilon - \partial_t(\rho u^\varepsilon)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))} \\
&\le C\|u\|^2_{L^4(0,T;L^4(\mathbb{T}^3))} (\|\partial_t \rho\|_{L^2(0,T;L^2(\mathbb{T}^3))} + \|\nabla \rho\|_{L^2(0,T;L^2(\mathbb{T}^3))}).
\end{aligned}$$

Based on system (1.1), ρ_t and $\nabla\rho$ can be denoted as follows:

$$\rho_t = -2\sqrt{\rho} v \cdot \nabla \sqrt{\rho} - \rho \mathrm{div} u, \ \nabla\rho = 2\sqrt{\rho}\nabla\sqrt{\rho}.$$

We will obtain the estimate of ρ_t and $\nabla\rho$ by using $0 \le \rho < c < \infty$, $(u, \nabla\sqrt{\rho}) \in L^4(0, T; L^4(\mathbb{T}^3))$ and $\nabla u \in L^2(0, T; L^2(\mathbb{T}^3))$ in Theorem 1.1, which implies that

$$\begin{aligned}
\|\rho_t\|_{L^2(0,T;L^2(\mathbb{T}^3))} &\le C\left(\| - 2\sqrt{\rho} u \cdot \nabla\sqrt{\rho}\|_{L^2(0,T;L^2(\mathbb{T}^3))} + \|\rho \mathrm{div} u\|_{L^2(0,T;L^2(\mathbb{T}^3))}\right) \\
&\le C\left(\|u\|_{L^4(0,T;L^4(\mathbb{T}^3))} \|\nabla\sqrt{\rho}\|_{L^4(0,T;L^4(\mathbb{T}^3))} + \|\nabla u\|_{L^2(0,T;L^2(\mathbb{T}^3))}\right),
\end{aligned} \tag{3.6}$$

and

$$\|\nabla\rho\|_{L^2(0,T;L^2(\mathbb{T}^3))} \le C\|\sqrt{\rho}\nabla\sqrt{\rho}\|_{L^2(0,T;L^2(\mathbb{T}^3))} \le C\|\nabla\sqrt{\rho}\|_{L^4(0,T;L^4(\mathbb{T}^3))}. \tag{3.7}$$

Inserting (3.6) and (3.7) into (A_1) yields the following:

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^3}\phi(t)u^\varepsilon[\partial_t(\rho u)^\varepsilon-\partial_t(\rho u^\varepsilon)]dxdt\\ \le& C\|u\|^2_{L^4(0,T;L^4(\mathbb{T}^3))}\left(\left(\|u\|_{L^4(0,T;L^4(\mathbb{T}^3))}+1\right)\|\nabla\sqrt{\rho}\|_{L^4(0,T;L^4(\mathbb{T}^3))}+\|\nabla u\|_{L^2(0,T;L^2(\mathbb{T}^3))}\right)\\ \le& C.\end{aligned}$$

From Lemma 2.2, we get the estimate of (A_1) that

$$\limsup_{\varepsilon\to 0}|(A_1)|=0.$$

3.2. Estimate of (B)

By utilizing integration by parts and the mass equation (1.1), we deduce that

$$\begin{aligned}(B)=&-\int_0^T\int_{\mathbb{T}^3}\phi(t)\nabla u^\varepsilon(\rho u\otimes u)^\varepsilon dxdt\\ =&-\int_0^T\int_{\mathbb{T}^3}\phi(t)\nabla u^\varepsilon[(\rho u\otimes u)^\varepsilon-(\rho u)\otimes u^\varepsilon]dxdt-\int_0^T\int_{\mathbb{T}^3}\phi(t)\nabla u^\varepsilon\cdot((\rho u)\otimes u^\varepsilon)dxdt\\ =&:(B_1)+\int_0^T\int_{\mathbb{T}^3}\phi(t)u^\varepsilon\cdot\mathrm{div}((\rho u)\otimes u^\varepsilon)dxdt\\ =&:(B_1)+\int_0^T\int_{\mathbb{T}^3}\phi(t)[\mathrm{div}(\rho u)|u^\varepsilon|^2+\frac{1}{2}(\rho u)\cdot\nabla|u^\varepsilon|^2]dxdt\\ =&:(B_1)+\int_0^T\int_{\mathbb{T}^3}\phi(t)\mathrm{div}(\rho u)|u^\varepsilon|^2dxdt-\frac{1}{2}\int_0^T\int_{\mathbb{T}^3}\phi(t)\mathrm{div}(\rho u)|u^\varepsilon|^2dxdt\\ =&:(B_1)+(B_2)+\frac{1}{2}\int_0^T\int_{\mathbb{T}^3}\phi(t)\partial_t\rho|u^\varepsilon|^2dxdt\\ =&:(B_1)+(B_2)+(B_3).\end{aligned}$$

Taking the mass equation $(1.1)_1$ into consideration, we know that $(A_2)+(B_2)=0$. The (B3) is the desired term.

$$(A_3)+(B_3)=\frac{1}{2}\int_0^T\int_{\mathbb{T}^3}\phi(t)\partial_t(\rho|u^\varepsilon|^2)dxdt. \tag{3.8}$$

By Hölder's inequality and triangle inequality, we deduce the following:

$$\begin{aligned}(B_1)=&-\int_0^T\int_{\mathbb{T}^3}\phi(t)\nabla u^\varepsilon[(\rho u\otimes u)^\varepsilon-(\rho u)\otimes u^\varepsilon]dxdt\\ \le& C\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))}\|(\rho u\otimes u)^\varepsilon-(\rho u)\otimes u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))}\\ \le& C\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))}(\|(\rho u\otimes u)^\varepsilon-(\rho u)\otimes u\|_{L^2(0,T;L^2(\mathbb{T}^3))}+\|(\rho u)\otimes u-(\rho u)\otimes u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))})\\ \le& C\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))}(\|(\rho u\otimes u)^\varepsilon-(\rho u)\otimes u\|_{L^2(0,T;L^2(\mathbb{T}^3))}+\|\rho u\|_{L^4(0,T;L^4(\mathbb{T}^3))}\|u-u^\varepsilon\|_{L^4(0,T;L^4(\mathbb{T}^3))})\end{aligned}$$

Thanks to the standard properties of mollifiers, we have the following:

$$\limsup_{\varepsilon\to 0}|(B_1)|=0.$$

3.3. Estimates of (C) and (D)

Utilizing integration by parts, we know that the following (C) and (D) are the desired terms, where

$$
\begin{aligned}
(C) =& -\mu \int_0^T \int_{\mathbb{T}^3} \phi(t) u^\varepsilon \Delta u^\varepsilon dxdt \\
=& \mu \int_0^T \int_{\mathbb{T}^3} \phi(t) |\nabla u^\varepsilon|^2 dxdt,
\end{aligned}
$$

and

$$
\begin{aligned}
(D) =& -(\lambda+\mu) \int_0^T \int_{\mathbb{T}^3} \phi(t) u^\varepsilon \nabla \mathrm{div} u^\varepsilon dxdt \\
=& (\lambda+\mu) \int_0^T \int_{\mathbb{T}^3} \phi(t) |\mathrm{div} u^\varepsilon|^2 dxdt.
\end{aligned}
$$

3.4. Estimate of (E)

Utilizing integration by parts and applying (1.1) leads to the following:

$$
\begin{aligned}
(E) =& \int_0^T \int_{\mathbb{T}^3} \phi(t) u^\varepsilon \nabla [(P(\rho))^\varepsilon - P(\rho)] dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t) u^\varepsilon \nabla P(\rho) dxdt \\
=:& (E_1) + \int_0^T \int_{\mathbb{T}^3} \phi(t)(u^\varepsilon - u) \nabla P(\rho) dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t) u \nabla P(\rho) dxdt \\
=:& (E_1) + (E_2) + \int_0^T \int_{\mathbb{T}^3} \phi(t) u \cdot \frac{a\gamma}{\gamma-1} \rho \nabla(\rho^{\gamma-1}) dxdt \\
=:& (E_1) + (E_2) - \int_0^T \int_{\mathbb{T}^3} \phi(t) \mathrm{div}(\rho u) \cdot \frac{a\gamma}{\gamma-1} \rho^{\gamma-1} dxdt \\
=:& (E_1) + (E_2) + \int_0^T \int_{\mathbb{T}^3} \phi(t) \partial_t \rho \cdot \frac{a\gamma}{\gamma-1} \rho^{\gamma-1} dxdt \\
=:& (E_1) + (E_2) + \frac{1}{\gamma-1} \int_0^T \int_{\mathbb{T}^3} \phi(t) \partial_t (a\rho)^\gamma dxdt \\
=:& (E_1) + (E_2) + \frac{1}{\gamma-1} \int_0^T \int_{\mathbb{T}^3} \phi(t) \partial_t P(\rho) dxdt \\
=:& (E_1) + (E_2) + (E_3).
\end{aligned}
$$

The term (E_3) is the desired term, and the estimate of (E_1) and (E_2) will be finished as follows:

$$
\begin{aligned}
(E_1) =& \int_0^T \int_{\mathbb{T}^3} \phi(t) u^\varepsilon \nabla [(P(\rho))^\varepsilon - P(\rho)] dxdt \\
\leq& \|u^\varepsilon\|_{L^4(0,T;L^4(\mathbb{T}^3))} \|\nabla (P(\rho))^\varepsilon - \nabla P(\rho)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))},
\end{aligned}
$$

and

$$
\begin{aligned}
(E_2) =& \int_0^T \int_{\mathbb{T}^3} \phi(t)(u^\varepsilon - u) \cdot \nabla P(\rho) dxdt \\
\leq& C\|u^\varepsilon - u\|_{L^4(0,T;L^4(\mathbb{T}^3))} \|\nabla P(\rho)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))}.
\end{aligned}
$$

By the upper bounded of ρ and Hölder's inequality, we have the following:

$$\|\nabla P(\rho)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))} \le C\|P'(\rho)\nabla\sqrt{\rho}\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))} \le C\|\nabla\sqrt{\rho}\|_{L^4(0,T;L^4(\mathbb{T}^3))}. \tag{3.9}$$

Combining the standard properties of mollifiers and (3.9), we know that

$$\limsup_{\varepsilon\to 0}|(E_1)| = \limsup_{\varepsilon\to 0}|(E_2)| = 0.$$

3.5. Estimates of (F) and (J)

Next, we turn to estimate (F) and (J), of which the proof is inspired by [10], and we include that

$$\begin{aligned}
(F)+(J) &= \int_0^T\int_{\mathbb{T}^3}\phi(t)[-u^\varepsilon(j\times b)^\varepsilon + E^\varepsilon\cdot j^\varepsilon]dxdt\\
&= \int_0^T\int_{\mathbb{T}^3}\phi(t)u^\varepsilon[-(j\times b)^\varepsilon + (j^\varepsilon\times b^\varepsilon) - (j^\varepsilon\times b^\varepsilon)] + \phi(t)E^\varepsilon\cdot j^\varepsilon dxdt\\
&= \int_0^T\int_{\mathbb{T}^3}\phi(t)u^\varepsilon[(j^\varepsilon\times b^\varepsilon) - (j\times b)^\varepsilon)]dxdt + \int_0^T\int_{\mathbb{T}^3}\phi(t)[(u^\varepsilon\times b^\varepsilon)j^\varepsilon + E^\varepsilon\cdot j^\varepsilon]dxdt\\
&=:(FJ)_1 + \int_0^T\int_{\mathbb{T}^3}\phi(t)|j^\varepsilon|^2dxdt + \int_0^T\int_{\mathbb{T}^3}\phi(t)[(u^\varepsilon\times b^\varepsilon) - (u\times b)^\varepsilon]j^\varepsilon dxdt\\
&=:(FJ)_1 + (FJ)_2 + (FJ)_3.
\end{aligned}$$

We see that $(FJ)_2$ is desired term, while the estimates of $(FJ)_1$ and $(FJ)_2$ will be finished. By Hölder's inequality, we can conclude that

$$\begin{aligned}
(FJ)_1 &= \int_0^T\int_{\mathbb{T}^3}\phi(t)u^\varepsilon[(j^\varepsilon\times b^\varepsilon) - (j\times b)^\varepsilon)]dxdt\\
&\le C\|u^\varepsilon\|_{L^4(0,T;L^4(\mathbb{T}^3))}\|(j^\varepsilon\times b^\varepsilon) - (j\times b)^\varepsilon\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))}\\
&\le C\|u\|_{L^4(0,T;L^4(\mathbb{T}^3))}\|(j^\varepsilon b^\varepsilon) - (jb)^\varepsilon\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))},
\end{aligned}$$

and

$$\begin{aligned}
(FJ)_3 &= \int_0^T\int_{\mathbb{T}^3}\phi(t)[(u^\varepsilon\times b^\varepsilon) - (u\times b)^\varepsilon]j^\varepsilon dxdt\\
&\le C\|(u^\varepsilon\times b^\varepsilon) - (u\times b)^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))}\|j^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))}\\
&\le C\|(u^\varepsilon\times b^\varepsilon) - (u\times b)^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))}\|j\|_{L^2(0,T;L^2(\mathbb{T}^3))}.
\end{aligned}$$

However, the following results are valid by using Hölder's inequality:

$$\|j^\varepsilon b^\varepsilon\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\mathbb{T}^3))} \le C\|j\|_{L^2(0,T;L^2(\mathbb{T}^3))}\|b\|_{L^4(0,T;L^4(\mathbb{T}^3))}. \tag{3.10}$$

and

$$\|u^\varepsilon\times b^\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}^3))} \le C\|u\|_{L^4(0,T;L^4(\mathbb{T}^3))}\|b\|_{L^4(0,T;L^4(\mathbb{T}^3))}. \tag{3.11}$$

Therefore, from $(FJ)_1$, $(FJ)_3$, (3.10) and (3.11), with the help of Lemma 2.1, we obtain the following:

$$\limsup_{\varepsilon\to 0}|(FJ)_1| = 0,\ \limsup_{\varepsilon\to 0}|(FJ)_3| = 0.$$

3.6. Estimates of (G), (H), (L) and (K)

The remaining is to estimate (G), (H), (L) and (K). Using a straightforward computation leads to

$$\begin{aligned}(G) =& \int_0^T \int_{\mathbb{T}^3} \phi(t)\partial_t E^\varepsilon \cdot E^\varepsilon dxdt\\ =&\frac{1}{2}\int_0^T \int_{\mathbb{T}^3} \phi(t)\partial_t |E^\varepsilon|^2 dxdt,\end{aligned}$$

and

$$\begin{aligned}(H)+(L) =& \int_0^T \int_{\mathbb{T}^3} \phi(t)[-E^\varepsilon \cdot (\nabla\times b)^\varepsilon + b^\varepsilon\cdot(\nabla\times E)^\varepsilon]dxdt\\ =&-\int_0^T \int_{\mathbb{T}^3} \phi(t)E_i^\varepsilon\cdot\epsilon_{ijk}\partial_j b_k^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)b^\varepsilon\cdot(\nabla\times E)^\varepsilon dxdt\\ =&\int_0^T \int_{\mathbb{T}^3} \phi(t)\epsilon_{ijk}\partial_j E_i^\varepsilon\cdot b_k^\varepsilon dxdt + \int_0^T \int_{T^3} \phi(t)b^\varepsilon\cdot(\nabla\times E)^\varepsilon dxdt\\ =&-\int_0^T \int_{\mathbb{T}^3} \phi(t)\epsilon_{kji}\partial_j E_i^\varepsilon\cdot b_k^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)b^\varepsilon\cdot(\nabla\times E)^\varepsilon dxdt\\ =&-\int_0^T \int_{\mathbb{T}^3} \phi(t)b^\varepsilon\cdot(\nabla\times E)^\varepsilon dxdt + \int_0^T \int_{\mathbb{T}^3} \phi(t)b^\varepsilon\cdot(\nabla\times E)^\varepsilon dxdt\\ =&0,\end{aligned}$$

and

$$\begin{aligned}(K) =& \int_0^T \int_{\mathbb{T}^3} \phi(t)b^\varepsilon\cdot\partial_t b^\varepsilon dxdt\\ =&\frac{1}{2}\int_0^T \int_{\mathbb{T}^3} \phi(t)\partial_t |b^\varepsilon|^2 dxdt.\end{aligned}$$

Then, summarizing all above the aforementioned estimates, putting them into (3.5) and taking the limit as $\varepsilon\to 0$, we obtain the following:

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^3} \phi(t)\partial_t(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})dxdt\\ &+\int_0^T \int_{\mathbb{T}^3} \phi(t)(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2 + |j|^2)dxdt = 0.\end{aligned}$$

We can express it in the following form:

$$\begin{aligned}&-\int_0^T \int_{\mathbb{T}^3} \phi_t(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})dxdt\\ &+\int_0^T \int_{\mathbb{T}^3} \phi(t)(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2 + |j|^2)dxdt = 0.\end{aligned}\tag{3.12}$$

Next, we study a similar method in [5] and shall prove the energy equality up to the initial time $t = 0$. First, we claim that the following results are valid for any $t_0 \geq 0$:

$$\begin{aligned} \lim_{t\to t_0^+} \|E(t)\|_{L^2(\mathbb{T}^3)} = \|E(t_0)\|_{L^2(\mathbb{T}^3)}, \lim_{t\to t_0^+} \|b(t)\|_{L^2(\mathbb{T}^3)} = \|b(t_0)\|_{L^2(\mathbb{T}^3)}, \\ \lim_{t\to t_0^+} \|\sqrt{\rho}u(t)\|_{L^2(\mathbb{T}^3)} = \|\sqrt{\rho}u(t_0)\|_{L^2(\mathbb{T}^3)}, \lim_{t\to t_0^+} \|\rho^\gamma(t)\|_{L^1(\mathbb{T}^3)} = \|\rho^\gamma(t_0)\|_{L^1(\mathbb{T}^3)}. \end{aligned} \tag{3.13}$$

Based on the mass equation (1.1), we can write

$$\partial_t \rho^\gamma = -\gamma\rho^\gamma \mathrm{div} u - 2\gamma\rho^{\gamma-\frac{1}{2}} u \cdot \nabla\sqrt{\rho},$$

and

$$\partial_t(\sqrt{\rho}) = -\frac{\sqrt{\rho}}{2}\mathrm{div} u - u \cdot \nabla\sqrt{\rho},$$

which, together with the assumptions in Theorem 1.1, gives

$$(\partial_t\rho^\gamma, \partial_t\sqrt{\rho}) \in L^2(0, T; L^2(\mathbb{T}^3)),$$

and

$$(\nabla\rho^\gamma, \nabla\sqrt{\rho}) \in L^4(0, T; L^4(\mathbb{T}^3)).$$

Hence, due to Lemma 2.3, it yields that the following:

$$(\rho^\gamma, \sqrt{\rho}) \in C([0, T]; L^2(\mathbb{T}^3)). \tag{3.14}$$

Consequently, for any $t_0 \geq 0$, by the right temporal continuity of ρ^γ in $L^2(\mathbb{T}^3)$ and $L^2(\mathbb{T}^3) \subset L^1(\mathbb{T}^3)$, we deduce that the following:

$$\rho^\gamma(t) \to \rho^\gamma(t_0) \textit{ strongly in } L^1(\mathbb{T}^3) \textit{ as } t \to t_0^+, \tag{3.15}$$

Furthermore, using the momentum equation $(1.1)_2$, we obtain the following:

$$\rho u \in L^\infty(0, T; L^2(\mathbb{T}^3)),\ (\rho u)_t \in L^2(0, T; H^{-1}(\mathbb{T}^3)).$$

Then, because of Lemma 2.3, we have the following:

$$\rho u \in C([0, T]; L^2_{weak}(\mathbb{T}^3)). \tag{3.16}$$

Similarly, from $(1.1)_3$, $(1.1)_4$ and (2.2), we can deduce that the following:

$$\partial_t E \in L^2(0, T; L^2(\mathbb{T}^3)),\ , \partial_t b \in L^\infty(0, T; L^2(\mathbb{T}^3)).$$

On the other hand, the assumptions in Theorem 1.1 implies

$$(E, b) \in L^\infty(0, T; L^2(\mathbb{T}^3)),$$

which can be obtained that leads to the following conclusion:

$$(E, b) \in C([0, T]; L^2(\mathbb{T}^3)). \tag{3.17}$$

Hence, for any $t_0 \geq 0$, from (3.17), we get that the following:

$$\begin{aligned} &E(t) \to E(t_0) \ \textit{strongly in}\ L^2(\mathbb{T}^3)\ \textit{as}\ t \to t_0^+, \\ &b(t) \to b(t_0) \ \textit{strongly in}\ L^2(\mathbb{T}^3)\ \textit{as}\ t \to t_0^+. \end{aligned} \tag{3.18}$$

Meanwhile, utilizing (2.2), (3.14), (3.16), (3.17) and the assumptions in Theorem 1.1 yields to the following:

$$\begin{aligned} 0 \leq &\overline{\lim_{t\to 0}} \int |\sqrt{\rho}u - \sqrt{\rho_0}u_0|^2 dx \\ =&2\overline{\lim_{t\to 0}} \left(\int (\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})dx - \int (\frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|E_0|^2 + \frac{1}{2}|b_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1})dx \right) \\ &+ 2\overline{\lim_{t\to 0}} \left(\int \sqrt{\rho_0}u_0(\sqrt{\rho_0}u_0 - \sqrt{\rho}u)dx + \frac{a}{\gamma-1}\int (\rho_0^\gamma - \rho^\gamma)dx \right) \\ &+ \overline{\lim_{t\to 0}} \left(\int (E_0^2 - E^2) + (b_0^2 - b^2)dx \right) \\ \leq &2\overline{\lim_{t\to 0}} \int \sqrt{\rho_0}u_0(\sqrt{\rho_0}u_0 - \sqrt{\rho}u)dx \\ \leq &2\overline{\lim_{t\to 0}} \int u_0(\rho_0 u_0 - \rho u)dx + 2\overline{\lim_{t\to 0}} \int u_0\sqrt{\rho}u(\sqrt{\rho} - \sqrt{\rho_0})dx = 0, \end{aligned}$$

which implies

$$\sqrt{\rho}u(t) \to \sqrt{\rho}u(0) \ \textit{strongly in}\ L^2(\mathbb{T}^3)\ \textit{as}\ t \to 0^+. \tag{3.19}$$

Similarly, we can establish the right temporal continuity of $\sqrt{\rho}u$ in $L^2(\mathbb{T}^3)$; hence, for any $t_0 \geq 0$, we have the following:

$$\sqrt{\rho}u(t) \to \sqrt{\rho}u(t_0) \ \textit{strongly in}\ L^2(\mathbb{T}^3)\ \textit{as}\ t \to t_0^+. \tag{3.20}$$

Combining (3.15), (3.18) and (3.20), we have now completed the proof of (3.13).

We notice that (3.12) is valid for ϕ belonging to $W^{1,\infty}$ rather than C^1. Therefore, for any $t_0 > 0$, we can use a new test function ϕ_τ to represent ϕ for some positive τ and α such that $\tau + \alpha < t_0$, that is

$$\phi_\tau(t) = \begin{cases} 0, & 0 \leq t \leq \tau, \\ \dfrac{t-\tau}{\alpha}, & \tau \leq t \leq \tau + \alpha, \\ 1, & \tau + \alpha \leq t \leq t_0, \\ \dfrac{t_0 - t}{\alpha}, & t_0 \leq t \leq t_0 + \alpha, \\ 0, & t_0 + \alpha \leq t. \end{cases}$$

Then, substituting this function into (3.12), we have the following:

$$\begin{aligned} &-\int_\tau^{\tau+\alpha} \int_{\mathbb{T}^3} \frac{1}{\alpha}(\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})dxdt \\ &+ \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} \int_{\mathbb{T}^3} (\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})dxdt \\ &+ \frac{1}{\alpha}\int_\tau^{t_0+\alpha} \int_{\mathbb{T}^3} \phi_\tau(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2 + |j|^2)dxdt = 0. \end{aligned} \tag{3.21}$$

Letting $\alpha \to 0$ and using the fact that $\int_0^t \int_{\mathbb{T}^3} \phi_\tau(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2 + |j|^2)dxdt$ is continuous with respect to t and the Lebesgue point Theorem, for all τ and $t_0 \in [0, T]$, we arrive at the following:

$$\begin{aligned}
&-\int_{\mathbb{T}^3} (\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})(\tau)dt \\
&+\int_{\mathbb{T}^3} (\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})(t_0)dt \\
&+\int_\tau^{t_0}\int_{\mathbb{T}^3} (\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2 + |j|^2)dxdt = 0.
\end{aligned} \tag{3.22}$$

Finally, taking $\tau \to 0$, combining the continuity of $\int_0^{t_0} \int_{\mathbb{T}^3} (\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2 + |j|^2)dxdt$ and (3.13), for all $t_0 \in [0, T]$, we can deduce that

$$\begin{aligned}
&\int_{\mathbb{T}^3} (\frac{1}{2}\rho|u|^2 + \frac{1}{2}|E|^2 + \frac{1}{2}|b|^2 + \frac{a\rho^\gamma}{\gamma-1})(t_0)dt + \int_0^{t_0}\int_{\mathbb{T}^3} (\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2 + |j|^2)dxdt \\
&\qquad = \int_{\mathbb{T}^3} (\frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|E_0|^2 + \frac{1}{2}|b_0|^2 + \frac{a\rho_0^\gamma}{\gamma-1})dt.
\end{aligned}$$

This now completes the proof of Theorem 1.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. P. L. Lions, *Mathematical Topics in Fluid Dynamics*, Compressible models Oxford Science Publication, Oxford, **2** (1998).
2. E. Feireisl, A. Novotný, H. Petzeltova, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.*, **3** (2001), 358–392. https://doi.org/10.1007/PL00000976

3. E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford, 2003. https://doi.org/10.1093/acprof:oso/9780198528388.001.0001
4. E. Feireisl, A. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids*, Birkhäuser, Basel, 2009. https://doi.org/10.1007/978-3-7643-8843-0
5. C. Yu, Energy conservation for the weak solutions of the compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.*, **225** (2017), 1073–1087. https://doi.org/10.1007/s00205-017-1121-4
6. Q. H. Nguyen, P. T. Nguyen, B. Q. Tang, Energy equalities for compressible Navier-Stokes equations, *Nonlinearity*, **32** (2019), 4206–4231. https://doi.org/10.1088/1361-6544/ab28ae
7. Z. Liang, Regularity criterion on the energy conservation for the compressible Navier-Stokes equations, in *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, **151** (2021), 1954–1971. https://doi.org/10.1017/prm.2020.87
8. Y. Ye, Y. Wang, W. Wei, Energy equality in the isentropic compressible Navier-Stokes equations allowing vacuum, *J. Differ. Equations*, **338** (2022), 551–571. https://doi.org/10.1016/j.jde.2022.08.013
9. D. Ma, F. Wu, Shinbrot's energy conservation criterion for the 3D Navier-Stokes-Maxwell system, *C.R. Math.*, **361** (2023), 91–96. https://doi.org/10.5802/crmath.379
10. F. Wu, On the Energy equality for distributional solutions to Navier-Stokes-Maxwell system, *J. Math. Fluid Mech.*, **24** (2022), 111. https://doi.org/10.1007/s00021-022-00740-0
11. P. L. Lions, Mathematical topics in fluid mechanics, *Incompressible Models*, in *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, **1** (1996).
12. I. Lacroix-Violet, A. Vasseur, Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit, *J. Math. Pures Appl.*, **114** (2018), 191–210. https://doi.org/10.1016/j.matpur.2017.12.002
13. J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.*, **146** (1986), 65–96. https://doi.org/10.1007/BF01762360

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