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Research article

Robust equilibrium reinsurance and investment strategy for the insurer and reinsurer under weighted mean-variance criterion

Yiming Su[1], **Haiyan Liu**[1] **and Mi Chen**[1,2,*]

[1] School of Mathematics and Statistics & Fujian Key Laboratory of Analytical Mathematics and Applications, Fujian Normal University, Fuzhou 350117, China

[2] Key Laboratory of Analytical Mathematics and Applications (Ministry of Education) & Fujian Provincial Key Laboratory of Statistics and Artificial Intelligence, Fujian Normal University, Fuzhou 350117, China

* **Correspondence:** Email: chenmi0610@163.com.

Abstract: This paper investigates the time-consistent robust optimal reinsurance problem for the insurer and reinsurer under weighted objective criteria. The joint objective criterion is obtained by weighting the mean-variance objectives of both the insurer and reinsurer. Specifically, we assume that the net claim process is approximated by a diffusion model, and the insurer can purchase proportional reinsurance from the reinsurer. The insurer adopts the loss-dependent premium principle considering historical claims, while the reinsurance contract still uses the expected premium principle due to information asymmetry. Both the insurer and reinsurer can invest in risk-free assets and risky assets, where the risky asset price is described by the constant elasticity of variance model. Additionally, the ambiguity-averse insurer and ambiguity-averse reinsurer worry about the uncertainty of parameter estimation in the model, therefore, we obtain a robust optimization objective through the robust control method. By solving the corresponding extended Hamilton-Jacobi-Bellman equation, we derive the time-consistent robust equilibrium reinsurance and investment strategy and corresponding value function. Finally, we examined the impact of various parameters on the robust equilibrium strategy through numerical examples.

Keywords: the insurer and reinsurer; loss-dependent premium principle; constant elasticity of variance model; weighted mean-variance criterion; ambiguity aversion

1. Introduction

Optimization problems play an important role in actuarial science, and the optimal reinsurance-investment strategies of insurers have been popular topics in financial research in recent years. Reinsurance and investment are critical tools for insurers to diversify risks and increase returns. The primary challenge for insurers is to attain optimal goals through controlling their reinsurance and investment strategies. This problem has been broadly studied with various criteria, such as minimizing the bankruptcy probability (see [1–3]), maximizing the expected utility of terminal wealth (see [4–7]), and the mean-variance optimization (see [8–11]).

In most studies, insurance premiums are typically determined based on future losses and charged using mean (variance) premium principles. However, in practice, current premiums are associated with historical losses. Fung et al. [12] and Niehaus and Terry [13] conducted empirical research to explore the dynamic relationship between premiums and losses. Barberis et al. [14] introduced the extrapolation bias to study a consumption-based asset pricing model. Inspired by extrapolation bias, Chen et al. [15] proposed an extrapolation claim model in which future premiums are determined by both historical and future claims and studied the optimal reinsurance strategy under this model. Hu and Wang [16] further introduced the loss-dependent premium principle and investigated how it affects the insurer's reinsurance strategy. Chen and Yang [17] extended the consideration of reinsurance and investment problems with correlated claims to the robust framework.

In traditional investment-reinsurance models, the ambiguity-neutral insurers (ANI) trust the accuracy of parameter estimation in the model. However, in practice, it is hard to accurately estimate parameters in insurance and financial markets, resulting in so-called model uncertainty. In recent years, model uncertainty has been widely employed in optimal risk control. The main method for solving model uncertainty is the robust control method proposed by Anderson et al. [18], where they studied continuous-time asset pricing models under this method and used the difference between the reference model and the true model as a penalty term to reflect investors' attitudes towards model uncertainty. Maenhout [19] studied optimization problems in intertemporal consumption through dynamic programming and derived closed-form expressions for the optimal strategy under "homothetic robustness". These studies greatly inspired research on model uncertainty in actuarial science. Zhang and Siu [20] utilized game theory to study the investment and reinsurance problem under model uncertainty conditions. Yi et al. [21] investigated the optimal reinsurance-investment strategies when the risk asset price process is described by the Heston model. Yi et al. [22] extended the robust optimal investment-reinsurance problem to the mean-variance framework. Zheng et al. [23] explored the robust optimal strategies under the constant elasticity of variance (CEV) model and terminal utility function. Li et al. [24] considered the problem of optimal excess-of-loss reinsurance and investment under a jump model. Gu et al. [25] explored the optimal excess-of-loss reinsurance contract with fuzzy aversion. Wang et al. [26] studied the robust equilibrium reinsurance-investment strategies of two insurance companies with ambiguity aversion in a robust game framework.

Before (re)insurance contracts are signed, negotiations take place among the participants. Therefore, (re)insurance contracts that consider the interests of multiple parties are more practical and more likely to be accepted. Recently, there have been several studies considering the multiple-party interests. Asimit and Boonen [27], Boonen and Jiang [28] and Zhuang et al. [29] explored (re)insurance contracts that considered multiple-party interests in a one-period (static) model. Moreover, there have been corresponding studies in a continuous-time (dynamic) framework. For instance, Chen and Shen [30] and Yuan et al. [31] considered the interests of both parties within a Stackelberg game framework when reinsurance contracts are signed. Apart from game-theoretic studies, there are two types of approaches that consider joint interests in a continuous-time framework. One approach combines the wealth processes of both parties to form a common wealth process to consider their interests. For instance, Zhao et al. [32] and Guan and Hu [33] considered the maximization of exponential utility criteria and mean-variance criteria, respectively, by weighting the wealth processes of the insurer and reinsurer. Yang [34] quantified the competition between the insurer and the reinsurer by representing their interests through relative wealth processes. Another approach integrates the objective criteria of both parties, which becomes more complex during the solution process due to the retention of the wealth processes of both parties. Huang et al. [35], Zhang [36] and Chen et al. [37] multiplied the objective criteria of the insurer and reinsurer to considered the optimal strategy under the maximization of the product of exponential utilities. On the other hand, Li et al. [38] and Li et al. [39] formed a common objective criterion by weighting the mean-variance criteria of the insurer and reinsurer, where the weight α represents the outcome of negotiations and serves to balance the interests of both parties.

Although there have been numerous studies integrating the aforementioned ways of linking the interests of both parties with robustness, there is still no research on considering the mean-variance weighted criteria of both sides within a robust framework. This paper primarily focuses on this aspect. Specifically, the insurer adopts the loss-dependent premium principle by combining a weighted average of past claim indices and the expectation of future claims, which is an extension of the traditional expected premium principle. Due to the fact that the reinsurer may not have access to historical claims information, the reinsurance contract adopts the expected premium principle. In addition, both the insurer and reinsurer invest their surplus in the financial market, where the risky asset is described by the CEV model. We address the issue of parameter estimation uncertainty in the model using robust control methods and derive the extended Hamilton-Jacobi-Bellman (HJB) equation within a robust framework. Finally, by utilizing stochastic control theory, closed-form expressions for the robust equilibrium strategy and the corresponding value function can be obtained. Furthermore, we also consider several special cases of the model and analyze the impact of model parameters on the strategies through numerical simulations. Different from Yang [34], we incorporate the interests of both the insurer and reinsurer by weighting their respective objective criteria. The reinsurer's involvement in decision-making is enhanced, and we consider the CEV risk model in the investment market. Furthermore, unlike Li et al. [38] and Li et al. [39], we take into account the impact of historical claims from the perspective of the insurer. We derive robust insurance investment strategies within a robust framework, and the numerical analysis reveals different effects of parameters on the strategies.

This paper is structured as follows: In Section 2, we introduce our model from three perspectives. In Section 3, we present a robust optimization problem considering model uncertainty and derive the explicit solutions for the robust equilibrium strategies and the corresponding value function under the mean-variance weighted sum criterion. In Section 4, we illustrate our results through numerical simulations. Section 5 summarizes this paper. The proofs of the theorems are provided in the appendix.

2. Model setting and assumptions

In this paper, we suppose that all investments and assets are infinitely divisible and all assets are tradable continuously over time, without considering transaction costs or taxes. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete, filtered probability space satisfying the usual conditions, where the information flow

$\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by independent random processes and includes all market information available before time t. Here, $T > 0$ is a fixed, finite time horizon.

2.1. Surplus process

We assume that the surplus process of the insurer satisfies the following classical risk model,

$$dR(t) = cdt - d\sum_{i=1}^{N(t)} Z_i,$$

where c is the premium rate, $N(t)$ is a homogeneous Poission process with intensity $\lambda > 0$, $\{Z_i, i \geq 1\}$ is a sequence of positive independent and identically distributed random variables and independent of $N(t)$, and they have a common distribution function of $F(z)$ with finite first and second moments, where $F_Z(z) = 0$ for $z \leq 0$ and $0 < F_Z(z) \leq 1$ for $z > 0$. The process $L(t)$ can be approximated by a diffusion model

$$L(t) \approx \mu dt - \sigma_0 W_0(t),$$

where $\mu = \lambda E(Z)$, $\sigma_0^2 = \lambda E(Z^2)$ and $W_0(t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$.

The traditional premium principle is based on future losses, but in reality, premiums are also related to historical claims. For instance, when renewing the insurance contracts, insurance companies will take into account the claims that have occurred in the recent past. Inspired by Barberis et al. [14], we assume that the insurer is an extrapolator who believes that if claims have recently increased (decreased), they will continue to show an increasing (decreasing) trend in the near future. Then, we introduce the loss-dependent premium principle proposed by Hu and Wang [16], which is constructed by a stochastic volatility model. Firstly, we define the exponential weighted average of historical losses as follows:

$$v(t) = \beta \int_0^t e^{-\beta(t-s)} dL(s - dt), 0 < \beta < 1, \tag{2.1}$$

where $dL(s - dt)$ means the total claims that occurred during the time interval $[s - dt, s]$, and the constant parameter β represents the strength of extrapolation. When β is relatively large, $v(t)$ is primarily determined by recent losses. The differential form of $v(t)$ is

$$\mathrm{d}v(t) = \beta(\mu - v(t))dt - \beta\sigma_0 dW_0(t). \tag{2.2}$$

It should be noted that the total weight of past losses, given by $\beta \int_0^t e^{-\beta(t-s)} ds = 1 - e^{-\beta(t)}$, is less than 1. Therefore, we assign a time-varying weight of $e^{-\beta(t)}$ to the expected future loss. Subsequently, the premium charged by the insurer per unit of time based on the loss-dependent premium principle is as follows:

$$C = (1 + n_1)\left[v(t) + e^{-\beta t}\mu\right],$$

here n_1 represents the safety loading of the insurer. When $\beta = 0$, this premium principle can degenerate to the traditional expected value premium principle.

2.2. Reinsurance and investment

In general, insurers transfer their potential claim risks by purchasing reinsurance and investing in financial markets. We suppose that the insurer chooses to purchase proportional reinsurance in this paper, and the retention level of the insurer is $q(t) \in [0, 1]$. Since the reinsurer may not have access to the insurer's historical claim information, assuming that the reinsurance contract follows the expected premium principle, the insurer should pay the reinsurer a reinsurance premium of $(1 + n_2)(1 - q(t))\mu$ at time t. To exclude the insurer's arbitrage behavior, we require $n_2 > n_1$. Therefore, the surplus process in the presence of the reinsurance of the insurer and reinsurer are respectively given by

$$dR_1(t) = \left[(1 + n_1)(v(t) + e^{-\beta t}\mu)\right] dt - \left[(1 + n_2)(1 - q(t))\mu\right] dt - q(t)\left[\mu dt - \sigma_0 dW_0(t)\right],$$

and

$$dR_2(t) = (1 + n_2)(1 - q(t))\mu dt - (1 - q(t))\left[\mu dt - \sigma_0 dW_0(t)\right].$$

In addition, both the insurer and reinsurer are allowed to invest their surplus in a financial market which consists of two kinds of asset: risk-free asset and risky asset. The price process of the risk-free asset is given by

$$dB(t) = rB(t)dt,\ B(0) = 1,$$

where $r > 0$ is the risk-free interest rate. The price of the risky assets available for the insurer and reinsurer to invest in are described by the CEV model:

$$dS_1(t) = S_1(t)\left[b_1 dt + \sigma_1 S_1^{\delta_1}(t) dW_1(t)\right], S_1(0) = s_{11}, \tag{2.3}$$

$$dS_2(t) = S_2(t)\left[b_2 dt + \sigma_2 S_2^{\delta_2}(t) dW_2(t)\right], S_2(0) = s_{21}, \tag{2.4}$$

where $b_1 > 0$, $b_2 > 0$ are expected instantaneous rates of return of the risky assets. Without any loss of generality, we assume that $b_1 > r$, $b_2 > r$, $\sigma_1 S_1^{\delta_1}(t)$, $\sigma_2 S_2^{\delta_2}(t)$ are instantaneous volatilities, δ_1, δ_2 are elasticity parameters that satisfy the general condition $\delta_1 \geq 0$, $\delta_2 \geq 0$, $W_1(t)$ and $W_2(t)$ are standard Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and they are independent of $W_0(t)$, i.e., $E[W_0(t)W_1(t)] = 0$ and $E[W_0(t)W_2(t)] = 0$.

Remark 2.1. *We denote* $E[W_1(t)W_2(t)] = \rho t$, $\rho \in (-1, 1]$. *When* $W_1(t)$ *and* $W_2(t)$ *are dependent and* $W_1(t) \neq W_2(t)$ *(i.e.,* $0 < |\rho| < 1$*), it is difficult to obtain explicit solutions for the optimal strategies. Therefore, this paper provides analytical results only for the cases of* $\rho = 0$ *and* $\rho = 1$. *For the case of* $\rho = 1$, *which corresponds to both parties investing in the same risky asset* $S_1(t)$*, the solution process is similar to the case of* $\rho = 0$ *but simpler. The analytical results for the case of* $\rho = 1$ *are discussed in Remark 3.4. In the following discussion, we will focus on the case of* $\rho = 0$.

2.3. Wealth process

Let $\pi_1(t)$ denote the amount invested by the insurer in the risky asset $S_1(t)$, and $\pi_2(t)$ denote the amount invested by the reinsurer in the risky asset $S_2(t)$ at time t. Assume that $u(t) := (\pi_1(t), \pi_2(t), q(t))_{t\in[0,T]}$ represents the decision variables of both the insurer and reinsurer at time t, then, the wealth processes of the insurer and reinsurer are respectively described by

$$dX(t) = [rX(t) + (b_1 - r)\pi_1(t) + (1 + n_1)(e^{-\beta t}\mu + v) - (1 + n_2)\mu + q(t)n_2\mu]dt$$

$$+ q(t)\sigma_0 dW_0(t) + \pi_1(t)\sigma_1 S_1^{\delta_1}(t)dW_1(t), \tag{2.5}$$

and

$$dY(t) = [rY(t) + (b_2 - r)\pi_2(t) + n_2(1 - q(t))\mu]dt + (1 - q(t))\sigma_0 dW_0(t) + \pi_2(t)\sigma_2 S_2^{\delta_2}(t)dW_2(t), \tag{2.6}$$

with the initial conditions $X(0) = x_0$ and $Y(0) = y_0$.

Similar to Chen and Yang [17] and Huang et al. [35], we provide the following definition of admissible strategies:

Definition 1. *A strategy $u(t) := (\pi_1(t), \pi_2(t), q(t))_{t\in[0,T]}$ is called a admissible strategy if it satisfies*

(i) $\pi_1(t)$, $\pi_2(t)$ and $q(t)$ are progressively measurable, and $\pi_1(t), \pi_2(t) \in [0, +\infty)$, $q(t) \in [0, 1]$ for any $t \in [0, T]$;

(ii) $E\left[\int_0^T \|u(t)\|^2 dt\right] < \infty$, where $\|u(t)\|^2 = q^2(t) + \pi_1^2(t) + \pi_2^2(t)$;

(iii) $\forall (t, x, y, v, s_1, s_2) \in [0, T]\times\mathbb{R}^2\times\mathbb{R}^+\times\mathbb{R}^2$, the equations (2.5) and (2.6) have unique strong solutions $\{X^u(t)\}_{t\in[0,T]}$ and $\{Y^u(t)\}_{t\in[0,T]}$ respectively, with $E_{t,x,y,v,s_1,s_2}[U(X^u(T))] < \infty$, $E_{t,x,y,v,s_1,s_2}[U(Y^u(T))] < \infty$.

Let $\mathcal{U}$ denote the set of all admissible strategies.

3. Optimization problem

When signing a reinsurance contract, negotiation between both parties is required. The optimal strategy for one party often conflicts with the interests of the other party, therefore, contracts that maximize the common interests of both parties are more likely to be accepted. In this paper, we adopt the mean-variance weighted objective criterion used in Li et al. [38] and Li et al. [39]. This objective criterion considers the optimization problem from the perspectives of both insurers and reinsurers, where both parties aim to maximize the expected terminal wealth and minimize the variance of terminal wealth. The specific form is as follows

$$\sup_{u\in\mathcal{U}} J^u(t, x, y, v, s_1, s_2) := \sup_{u\in\mathcal{U}} \left\{\alpha J_x^u(t, x, y, v, s_1, s_2) + (1 - \alpha)J_y^u(t, x, y, v, s_1, s_2)\right\}, \tag{3.1}$$

where

$$J_x^u(t, x, y, v, s_1, s_2) = \mathrm{E}_{t,x,y,v,s_1,s_2}[X^u(T)] - \frac{\gamma_1}{2}\mathrm{Var}_{t,x,y,v,s_1,s_2}[X^u(T)],$$
$$J_y^u(t, x, y, v, s_1, s_2) = \mathrm{E}_{t,x,y,v,s_1,s_2}[Y^u(T)] - \frac{\gamma_2}{2}\mathrm{Var}_{t,x,y,v,s_1,s_2}[Y^u(T)].$$

The weighting parameter α $(0 \le \alpha \le 1)$ plays a role in balancing the interests of the insurer and reinsurer. The specific value of α can be determined by the insurer and reinsurer through relative weighting of their respective ultimate objectives. In reality, some large financial companies not only own insurance companies but also reinsurers, and these large financial companies may make reinsurance and investment decisions for both. In addition, Golubin [40] discusses methods for determining the value of α. One approach is to rely on exogenous methods provided by experts based on empirical research. Another method is based on cooperative game theory. For further discussion on the determination of α, refer to Golubin [40] and the references therein.

3.1. Optimization problems with model ambiguity

In the given framework, the ambiguity-neutral insurer (ANI) and the ambiguity-neutral reinsurer (ANR) do not doubt the accuracy of the probability distribution $\mathbb{P}$ and its parameter estimation. However, in theory, the parameter model used contains significant uncertainties. These uncertainties mainly come from two aspects, it is difficult for investors to accurately estimate the expected return process of risky assets, and there may also be errors in estimating the drift parameters. On the other hand, there may also be uncertainties in the parameter estimation of the surplus process for the insurer.

To consider the uncertainty of the model, we adopt a systematic and quantitative approach by referring to the methods proposed by Anderson et al. [18]. Therefore, we consider alternative models to obtain robust optimal strategies by broadly defining a class of probability measures $\mathbb{Q}$ that are equivalent to the probability measure $\mathbb{P}$. Let these alternative probability measures belong to set $\mathcal{Q}$, which is defined by

$$\mathcal{Q} := \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P}\}.$$

Next, we introduce a process $\{\theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t)) \mid t \in [0, T]\}$ satisfying

1) $\theta(t)$ is progressively measurable;
2) $\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\theta(t)\|^2 \mathrm{d}t\right)\right] < \infty$, where $\|\theta(t)\|^2 = \theta_0^2(t) + \theta_1^2(t) + \theta_2^2(t)$.

We denote the space of all such processes as Θ. For each $\theta \in \Theta$, we define a new probability measure $\mathbb{Q}$ that is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_T$ and satisfies

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \|\theta(u)\|^2 du\right\},$$

where $W(t) = (W_0(t), W_1(t), W_2(t))'$ is a standard three-dimensional Brownian motion. Therefore, by choosing different processes $\theta \in \Theta$, different probability measures for the diffusion part of the wealth process are obtained. According to the Girsanov's theorem, the Brownian motion under $\mathbb{Q} \in \mathcal{Q}$ can be defined as $dW^{\mathbb{Q}}(t) = dW^{\mathbb{P}}(t) + \theta(t)dt$, i.e.,

$$dW_0^{\mathbb{Q}}(t) = dW_0(t) + \theta_0(t)dt, \quad dW_1^{\mathbb{Q}}(t) = dW_1(t) + \theta_1(t)dt, \quad dW_2^{\mathbb{Q}}(t) = dW_2(t) + \theta_2(t)dt.$$

It can be observed that the main difference between the alternative model and the reference model lies in the drift term. Moreover, since the Brownian motion W_0, W_1, W_2 are mutually independent, they remain independent even after the measure transformation.

Under the probability measure $\mathbb{Q}$, the Eqs (2.5) and (2.6) can be respectively rewritten as follows:

$$\begin{aligned} dX^u(t) &= [rX^u + (b_1 - r)\pi_1(t) + (1 + n_1)(e^{-\beta t}\mu + v) - (1 + n_2)\mu + q(t)n_2\mu \\ &\quad - q(t)\sigma_0\theta_0 - \pi_1(t)\sigma_1\theta_1 S_1^{\delta_1}(t)]dt + q(t)\sigma_0 dW_0^{\mathbb{Q}}(t) + \pi_1(t)\sigma_1 S_1^{\delta_1}(t)dW_1^{\mathbb{Q}}(t), \end{aligned} \tag{3.2}$$

$$\begin{aligned} dY^u(t) &= [rY^u + (b_2 - r)\pi_2(t) + n_2\mu(1 - q(t)) - \sigma_0\theta_0(1 - q(t)) - \pi_2(t)\sigma_2\theta_2 S_2^{\delta_2}(t)]dt \\ &\quad + (1 - q(t))\sigma_0 dW_0^{\mathbb{Q}}(t) + \pi_2(t)\sigma_2 S_2^{\delta_2}(t)dW_2^{\mathbb{Q}}(t). \end{aligned} \tag{3.3}$$

The Eqs (2.3) and (2.4) become

$$dS_1(t) = S_1(t)\left[(b_1 - \sigma_1\theta_1 S_1^{\delta_1}(t))dt + \sigma_1 S_1^{\delta_1}(t)dW_1^{\mathbb{Q}}(t)\right], \tag{3.4}$$

$$dS_2(t) = S_2(t)\left[(b_2 - \sigma_2\theta_2 S_2^{\delta_2}(t))dt + \sigma_2 S_2^{\delta_2}(t)dW_2^{\mathbb{Q}}(t)\right]. \tag{3.5}$$

Correspondingly, the differential of historical loss information v in Eq (2.2) becomes

$$dv(t) = \beta(\mu - v(t) + \sigma_0\theta_0)dt - \beta\sigma_0 dW_0^{\mathbb{Q}}(t). \tag{3.6}$$

The value functions in Eq (3.1) ignore the uncertainty of the model, but the ambiguity-averse insurer (AAI) and ambiguity-averse reinsurer (AAR) are skeptical about the accuracy of the reference model $\mathbb{P}$, and they choose $\mathbb{Q}$ as a probability measure for the alternative model from $\mathcal{Q}$. Actually, the ambiguity-averse policy maker wants to find the worst alternative from the available alternatives to deal with the mean-variance optimization problem. Inspired by Maenhout [19], Yi et al. [21] and Yuan et al. [31], we modify the objective functions of the AAI and AAR as robust optimization problems formulated by the following equations:

$$J_x^{\mathbb{Q},u} = \mathbb{E}_{t,x,y,v,s_1,s_2}^{\mathbb{Q}}[X_{t,x,y,v,s_1,s_2}(T)] - \frac{\gamma_1}{2}\,\mathrm{Var}_{t,x,y,v,s_1,s_2}^{\mathbb{Q}}[X_{t,x,y,v,s_1,s_2}(T)] + \mathbb{E}^{\mathbb{Q}}[h_x(\mathbb{Q}\|\mathbb{P})],$$
$$J_y^{\mathbb{Q},u} = \mathbb{E}_{t,x,y,v,s_1,s_2}^{\mathbb{Q}}[Y_{t,x,y,v,s_1,s_2}(T)] - \frac{\gamma_2}{2}\,\mathrm{Var}_{t,x,y,v,s_1,s_2}^{\mathbb{Q}}[Y_{t,x,y,v,s_1,s_2}(T)] + \mathbb{E}^{\mathbb{Q}}[h_y(\mathbb{Q}\|\mathbb{P})],$$

where $h(\mathbb{Q}\|\mathbb{P})$ is a penalty function that measures the relative entropy between $\mathbb{Q}$ and $\mathbb{P}$, and also reflects the decision maker's confidence in the reference model $\mathbb{P}$. Correspondingly, the weighted sum objective criterion considering model aversion is described by

$$\sup_{u\in\mathcal{U}}\inf_{\mathbb{Q}\in\mathcal{Q}} J^{\mathbb{Q},u}(t,x,y,v,s_1,s_2) = \sup_{u\in\mathcal{U}}\inf_{\mathbb{Q}\in\mathcal{Q}}\left\{\alpha J_x^{\mathbb{Q},u}(t,x,y,v,s_1,s_2) + (1-\alpha)J_y^{\mathbb{Q},u}(t,x,y,v,s_1,s_2)\right\}. \tag{3.7}$$

A smaller penalty term indicates that the decision maker has less trust in the reference model, and the deviation between the worst-case substitute model and the reference model will be greater. When $h(\mathbb{Q}\|\mathbb{P}) = 0$, the penalty term disappears and the decision maker has no information about the true model, and all the alternative models are on the equal footing. When $h(\mathbb{Q}\|\mathbb{P}) \to \infty$, the ambiguity-averse decision maker strongly believes that the reference model $\mathbb{P}$ is the true model, and any substitute model that deviates from $\mathbb{P}$ will be punished infinitely. It should be emphasized that the penalty term depends on the relative entropy generated by diffusion risk. The increase in relative entropy from t to $t + dt$ is equal to $\frac{1}{2}[\theta_0^2(t) + \theta_1^2(t)]dt$ in the insurance model, while it is equal to $\frac{1}{2}[\theta_0^2(t) + \theta_2^2(t)]dt$ in the reinsurance model.

We consider the penalty function of the following form used in Huang et al. [35] and Wang et al. [26],

$$h_x(\mathbb{Q}\|\mathbb{P}) = \int_t^T \Psi_x(s, X^u(s), v(s), \theta(s))ds,$$
$$h_y(\mathbb{Q}\|\mathbb{P}) = \int_t^T \Psi_y(s, Y^u(s), v(s), \theta(s))ds,$$

where

$$\Psi_x(s, X^u(s), v(s), \theta(s)) = \frac{\theta_0^2(s)}{2\phi_0(s, X^u(s), v(s))} + \frac{\theta_1^2(s)}{2\phi_1(s, X^u(s), v(s))},$$

$$\Psi_y\left(s, Y^u(s), v(s), \theta(s)\right) = \frac{\theta_0^2(s)}{2\phi_0\left(s, Y^u(s), v(s)\right)} + \frac{\theta_2^2(s)}{2\phi_2\left(s, Y^u(s), v(s)\right)}.$$

The advantage of this penalty function is that it makes the robustness of the model not dependent on wealth variables X and Y. Based on the approaches of Zeng et al. [10] and Wang et al. [26], we assume that

$$\phi_0(t, X^u(t), v(t)) = \phi_0(t, Y^u(t), v(t)) = m_0, \quad \phi_1(t, X^u(t), v(t)) = m_1, \quad \phi_2(t, Y^u(t), v(t)) = m_2,$$

where $m_i \geq 0, i = 0, 1, 2$, represents the ambiguity-aversion coefficient describing the decision maker's attitude towards diffusion risk. Specifically, we interpret m_0 as the degree of ambiguity aversion in the claim process, and m_1, m_2 as the degree of ambiguity-aversion in the investment market. When $m_i = 0$, the policy maker's attitude towards diffusion risk is ambiguity-neutral. It is worth noting that the optimization problem in Eq (3.7) is time-inconsistent, thus the Bellman optimality principle is invalidated. We use game-theoretic methods from Björk and Murgoci [41] and Björk et al. [42] to solve it and derive the time-consistent equilibrium strategy.

Definition 2. *For an admissible strategy $u^*(t) = \{(\pi_1(t), \pi_2(t), q(t))\}_{t\in[0,T]}$ with any fixed initial state $(t, x, y, v, s_1, s_2) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, we define the following strategy*

$$u_\varepsilon(\lambda) = \begin{cases} \tilde{u}, & t \leq \lambda < t + \varepsilon, \\ u^*(\lambda), & t + \varepsilon \leq \lambda < T, \end{cases} \tag{3.8}$$

where $\tilde{u} = (\tilde{\pi}_1, \tilde{\pi}_2, \tilde{q})$, and $\varepsilon \in \mathbb{R}^+$. If $\forall\ \tilde{u} = (\tilde{\pi}_1, \tilde{\pi}_2, \tilde{q}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, we have

$$\liminf_{\varepsilon\to 0} \frac{J^{u^*}(t, x, y, v, s_1, s_2) - J^{u_\varepsilon}(t, x, y, v, s_1, s_2)}{\varepsilon} \geq 0,$$

then $u*$ is called an equilibrium strategy, and the equilibrium value function is $J^{u^*}(t, x, y, v, s_1, s_2)$.

3.2. Robust equilibrium reinsurance investment strategy

For any $\varphi\left(t, x, y, v, s_1, s_2\right) \in C^{1,2,2,2,2,2}\left([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\right)$, we denote

$$\begin{aligned}
&\mathcal{A}^u\varphi(t, x, y, v, s_1, s_2)\\
&= \varphi_t + [rx + (b_1 - r)\pi_1 + (1 + n_1)(e^{-\beta t}\mu + v) - (1 + n_2)\mu + qn_2\mu - q\sigma_0\theta_0 - \pi_1\sigma_1\theta_1 s_1^{\delta_1}]\varphi_x\\
&\quad + [ry + (b_2 - r)\pi_2 + n_2(1 - q)\mu - (1 - q)\sigma_0\theta_0 - \pi_2\sigma_2\theta_2 s_2^{\delta_2}]\varphi_y + \beta(\mu - v + \sigma_0\theta_0)\varphi_v\\
&\quad + (b_1 - \sigma_1\theta_1 s_1^{\delta_1})s_1\varphi_{s_1} + (b_2 - \sigma_2\theta_2 s_2^{\delta_2})s_2\varphi_{s_2} + \frac{1}{2}(q^2\sigma_0^2 + \pi_1^2\sigma_1^2 s_1^{2\delta_1})\varphi_{xx}\\
&\quad + \frac{1}{2}[(1 - q)^2\sigma_0^2 + \pi_2^2\sigma_2^2 s_2^{2\delta_2}]\varphi_{yy} + \frac{1}{2}\beta^2\sigma_0^2\varphi_{vv} + \frac{1}{2}\sigma_1^2 s_1^{2\delta_1+2}\varphi_{s_1s_1} + \frac{1}{2}\sigma_2^2 s_2^{2\delta_2+2}\varphi_{s_2s_2}\\
&\quad + q(1 - q)\sigma_0^2\varphi_{xy} - q\beta\sigma_0^2\varphi_{xv} - (1 - q)\beta\sigma_0^2\varphi_{yv} + \pi_1\sigma_1^2 s_1^{2\delta_1+1}\varphi_{xs_1} + \pi_2\sigma_2^2 s_2^{2\delta_2+1}\varphi_{ys_2}.
\end{aligned}$$

Similar to the proof of Theorem 4.1 of Björk and Murgoci [41] and Theorem 1 of Kryger and Steffensen [43], we have the following verification theorem:

Theorem 3.1 (Verification Theorem)**.** *For problem* (3.7)*, if there exist real value functions* $V(t,x,y,v,s_1,s_2)$*,* $g_1(t,x,y,v,s_1,s_2)$ *and* $g_2(t,x,y,v,s_1,s_2) \in C^{1,2,2,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+)$ *satisfying the following conditions:* $\forall(t,x,y,v,s_1,s_2)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$,

$$
\begin{aligned}
&\sup_{u\in\mathcal{U}}\inf_{\mathbb{Q}\in\mathcal{Q}}\Big\{\mathcal{A}^u V(t,x,y,v,s_1,s_2)-\alpha\mathcal{A}^u\frac{\gamma_1}{2}(g_1(t,x,y,v,s_1s_2))^2\\
&+\alpha\gamma_1 g_1(t,x,y,v,s_1,s_2)\mathcal{A}^u g_1(t,x,y,v,s_1,s_2)-(1-\alpha)\mathcal{A}^u\frac{\gamma_2}{2}(g_2(t,x,y,v,s_1s_2))^2\\
&\qquad+(1-\alpha)\gamma_2 g_2(t,x,y,v,s_1,s_2)\mathcal{A}^u g_2(t,x,y,v,s_1,s_2)\\
&\qquad+\alpha(\frac{\theta_0^2}{2m_0}+\frac{\theta_1^2}{2m_1})+(1-\alpha)(\frac{\theta_0^2}{2m_0}+\frac{\theta_2^2}{2m_2})\Big\}=0,\\
&\qquad V(T,x,y,v,s_1,s_2)=\alpha x+(1-\alpha)y,
\end{aligned}
\tag{3.9}
$$

$$
\mathcal{A}^{u^*}g_1(t,x,y,v,s_1s_2)=0,\quad g_1(T,x,y,v,s_1s_2)=x, \tag{3.10}
$$

$$
\mathcal{A}^{u^*}g_2(t,x,y,v,s_1s_2)=0,\quad g_2(T,x,y,v,s_1s_2)=y, \tag{3.11}
$$

and

$$
\begin{aligned}
u^* :=\arg&\sup_{u\in\mathcal{U}}\inf_{\mathbb{Q}\in\mathcal{Q}}\{\mathcal{A}^u V(t,x,y,v,s_1,s_2)-\alpha\mathcal{A}^u\frac{\gamma_1}{2}(g_1(t,x,y,v,s_1s_2))^2\\
&+\alpha\gamma_1 g_1(t,x,y,v,s_1,s_2)\mathcal{A}^u g_1(t,x,y,v,s_1,s_2)-(1-\alpha)\mathcal{A}^u\frac{\gamma_2}{2}(g_2(t,x,y,v,s_1s_2))^2\\
&+(1-\alpha)\gamma_2 g_2(t,x,y,v,s_1,s_2)\mathcal{A}^u g_2(t,x,y,v,s_1,s_2)\\
&+\alpha(\frac{\theta_0^2}{2m_0}+\frac{\theta_1^2}{2m_1})+(1-\alpha)(\frac{\theta_0^2}{2m_0}+\frac{\theta_2^2}{2m_2})\},
\end{aligned}
\tag{3.12}
$$

then $J^{u^*}(t,x,y,v,s_1,s_2)=V(t,x,y,v,s_1,s_2)$*,* $E_{t,x,y,v,s_1,s_2}[X^{u^*}(T)]=g_1(t,x,y,v,s_1,s_2)$*,* $E_{t,x,y,v,s_1,s_2}[Y^{u^*}(T)]=g_2(t,x,y,v,s_1,s_2)$ *and* u^* *is a time-consistent robust strategy.*

After giving the verification theorem, we now present the main results in Theorem 3.2.

Theorem 3.2 (Time-consistent robust equilibrium strategy)**.** *Let*

$$
\begin{aligned}
\mathrm{L}_1(t)&=\alpha\gamma_1\beta B_3(t)+(1-\alpha)\gamma_2 e^{r(T-t)}+m_0(2\alpha-1)\left[\frac{n_2\mu}{\sigma_0^2 m_0}+\beta\alpha B_3(t)-(1-\alpha)e^{r(T-t)}\right],\\
\mathrm{L}_2(t)&=\alpha\sigma_2^2[\gamma_1+m_0(2\alpha-1)][e^{r(T-t)}-\beta B_3(t)]-n_2\mu(2\alpha-1).
\end{aligned}
$$

For the robust optimization problem (3.7)*, the robust equilibrium strategies and the corresponding equilibrium value function are given by*

$$
q_1^*(t)=\begin{cases}0, & \mathrm{L}_1(t)\le 0,\\ \tilde{q}_1(t), & \mathrm{L}_1(t)>0 \ \ and\ \ \mathrm{L}_2(t)>0,\\ 1, & \mathrm{L}_1(t)>0\ \ and\ \ \mathrm{L}_2(t)\le 0,\end{cases} \tag{3.13}
$$

where

$$\tilde{q}_1(t) = \frac{n_2\mu(2\alpha-1) + \sigma_0^2\left[(1-\alpha)\gamma_2 e^{r(T-t)} + m_0(2\alpha-1)(\beta A_3(t) - (1-\alpha)e^{r(T-t)}) + \alpha\gamma_1\beta B_3(t)\right]}{\sigma_0^2\left[\alpha\gamma_1 + (1-\alpha)\gamma_2 + m_0(2\alpha-1)^2\right]e^{r(T-t)}},$$

and

$$\pi_1^*(t) = \frac{(b_1 - r) + 2\delta_1\sigma_1^2(\gamma_1 B_4(t) + \frac{m_1}{\alpha}A_4(t))}{\sigma_1^2 s_1^{2\delta_1}(\gamma_1 + m_1)e^{r(T-t)}}, \tag{3.14}$$

$$\pi_2^*(t) = \frac{(b_2 - r) + 2\delta_2\sigma_2^2(\gamma_2 C_5(t) + \frac{m_2}{1-\alpha}A_5(t))}{\sigma_2^2 s_2^{2\delta_2}(\gamma_2 + m_2)e^{r(T-t)}}, \tag{3.15}$$

$$\begin{aligned} V(t,x,y,v,s_1,s_2) = {} & \alpha e^{r(T-t)}x + (1-\alpha)e^{r(T-t)}y + \frac{\alpha(1+n_1)}{r+\beta}(e^{r(T-t)} - e^{-\beta(T-t)})v \\ & + A_4(t)s_1^{-2\delta_1} + A_5(t)s_2^{-2\delta_2} + A_6(t), \end{aligned} \tag{3.16}$$

where $A_4(t), B_4(t), A_5(t), C_5(t)$ and $A_6(t)$ are given by Eqs (A.37), (A.36), (A.42), (A.41) *and* (A.43), *respectively.*

Proof. See Appendix A.

Remark 3.1. *If $\beta = 0$, the loss-dependent premium degenerates to the traditional expected value premium principle, then the robust equilibrium reinsurance strategy under expected value premium is*

$$q_2^*(t) = \frac{n_2\mu(2\alpha-1) + \sigma_0^2(1-\alpha)\left[\gamma_2 - m_0(2\alpha-1)\right]e^{r(T-t)}}{\sigma_0^2\left[\alpha\gamma_1 + (1-\alpha)\gamma_2 + m_0(2\alpha-1)^2\right]e^{r(T-t)}}.$$

The robust equilibrium investment strategies under expected value premium are the same as Eqs (3.14) *and* (3.15). *Since we assumed no correlation between the insurance market and financial market at the outset, the investment strategy is independent of the insurance market parameters.*

Remark 3.2. *If $m_i = 0, i = 0,1,2$, i.e., without considering robustness, the equilibrium optimal reinsurance strategy under loss-dependent premium is*

$$q_3^*(t) = \frac{n_2\mu(2\alpha-1) + \sigma_0^2\left[\alpha\gamma_1\beta B_3 + (1-\alpha)\gamma_2 e^{r(T-t)}\right]}{\sigma_0^2\left[\alpha\gamma_1 + (1-\alpha)\gamma_2\right]e^{r(T-t)}}.$$

The equilibrium optimal investment strategies under loss-dependent premium are

$$\hat{\pi}_1(t) = \frac{b_1 - r}{\gamma_1\sigma_1^2 s_1^{2\delta_1}e^{r(T-t)}}\left[1 + \frac{b_1 - r}{r}(1 - e^{2r\delta_1(t-T)})\right],$$

$$\hat{\pi}_2(t) = \frac{b_2 - r}{\gamma_2\sigma_2^2 s_2^{2\delta_2}e^{r(T-t)}}\left[1 + \frac{b_2 - r}{r}(1 - e^{2r\delta_2(t-T)})\right],$$

which are the same as the investment strategies in Li et al. [38].

Remark 3.3. *If $\beta = 0$ and $m_i = 0, i = 0,1,2$, i.e., without using loss-dependent premium and without considering robustness, the result in Theorem 3.2 reduces to that in Li et al. [38].*

Remark 3.4. *When $\rho = 1$ (i.e.,$W_1(t) = W_2(t)$)), the robust equilibrium reinsurance strategy is the same as Eq* (3.13)*, and the robust equilibrium investment strategy and the corresponding value function are given by the following expressions,*

$$
\begin{aligned}
\pi_1(t) &= \frac{\frac{\gamma_2}{(1-\alpha)m_1+\gamma_2}(b_1 - r) + 2\delta_1\sigma_1^2\left[\gamma_1 B_4(t) + \frac{\gamma_2}{(1-\alpha)m_1+\gamma_2}\left(m_1 A_4(t) - (1-\alpha)m_1 C_4(t)\right)\right]}{\sigma_1^2 s_1^{2\delta_1}\left[\gamma_1 + \frac{\alpha\gamma_2}{(1-\alpha)m_1+\gamma_2}m_1\right]e^{r(T-t)}},\\
\pi_2(t) &= \frac{\frac{\gamma_1}{\alpha m_1+\gamma_1}(b_1 - r) + 2\delta_1\sigma_1^2\left[\gamma_2 C_4(t) + \frac{\gamma_1}{\alpha m_1+\gamma_1}\left(m_1 A_4(t) - \alpha m_1 B_4(t)\right)\right]}{\sigma_1^2 s_1^{2\delta_1}\left[\gamma_2 + \frac{(1-\alpha)\gamma_1}{\alpha m_1+\gamma_1}m_1\right]e^{r(T-t)}},\\
V(t, x, y, s_1) &= \alpha e^{r(T-t)}x + (1-\alpha)e^{r(T-t)}y + A_3(t)v + A_4(t)s_1^{-2\delta_1} + A_5(t).
\end{aligned}
$$

The process of solving for $A_3(t), A_4(t), B_4(t), C_4(t)$ and $A_5(t)$ are similar to that in Appendix A. We omit the detailed derivation here.

4. Numerical analysis

In this section, we present some numerical analysis to study the influencing factors of the robust equilibrium reinsurance-investment strategy and explain the results for better understanding in the economic sense. Unless otherwise specified, the basic parameters are shown in Table 1.

Table 1. Some basic parameters.

Common parameters	r	μ	σ_0	α	β	m_0	t	T
	0.03	0.5	1.5	0.6	0.12	0.8	0	10
Insurer	n_1	γ_1	m_1	b_1	σ_1	δ_1	s_1	
	0.2	0.5	1	0.06	6.16	0.6	36	
Reinsurer	n_2	γ_2	m_2	b_2	σ_2	δ_2	s_2	
	0.25	0.6	1.2	0.05	5.16	0.5	26	

4.1. Sensitivity analysis of the equilibrium reinsurance strategy

In this part, we consider the sensitivity of the equilibrium reinsurance strategy. Figure 1 shows that the robust equilibrium reinsurance strategy $q_1^*(t)$ decreases as α increases. This is due to the increasing decision-making power of the insurer as α increases. Considering the insurer's preference, it aims to purchase more reinsurance to transfer insurance risk to the reinsurer. When $\alpha > 0.5$, more voices are heard from the insurer in the decision, and the decreasing trend of $q_1^*(t)$ over time is attributed to the fact that, under the principle of loss-dependent insurance premium, the premium paid by policyholders is positively correlated with their past claims. Therefore, this premium principle imposes constraints on policyholders' behavior. A decrease in premiums collected by the insurer leads to a reduction in its retention level. On the other hand, when $\alpha < 0.5$, the reinsurer who apply the expected premium principle are given more priority, and $q_1^*(t)$ also decreases over time. This can be attributed to the accumulation of investment returns in financial markets over time, which increases the wealth of the reinsurer and their risk absorption capacity. Therefore, they are more willing to take on more reinsurance business.

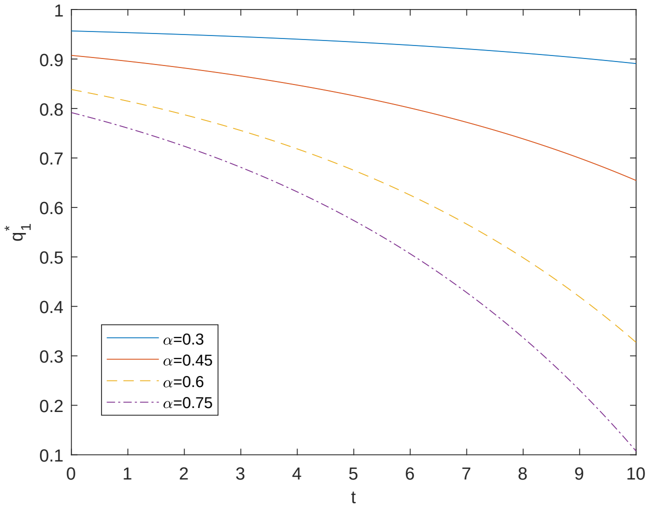
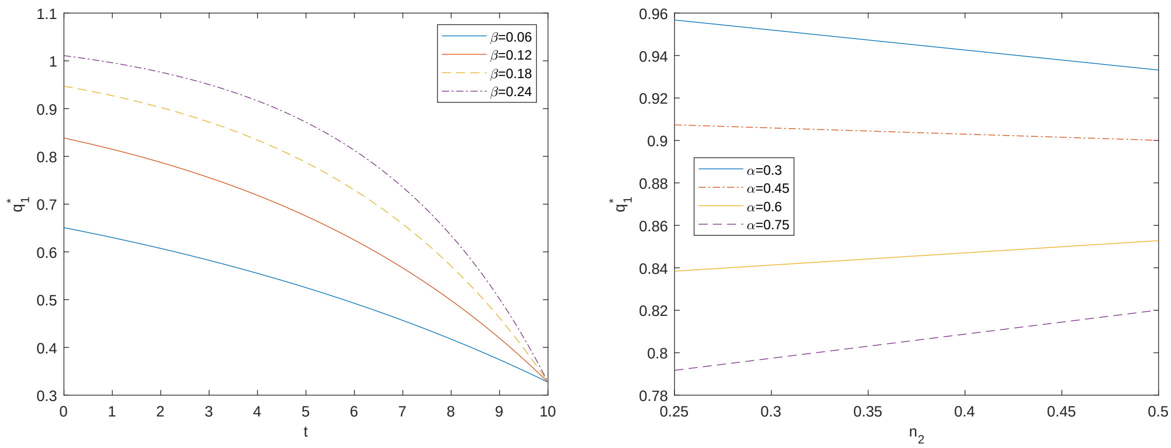


Figure 1. The effects of α and t on $q_1^*(t)$.

Figure 2(a) reveals that the insurer's retention level $q_1^*(t)$ increases with the increase in extrapolation intensity β at the initial stage of decision-making. Moreover, as β becomes larger, $q_1^*(t)$ becomes more sensitive with a larger rate of change. This is attributed to the negative correlation between the dynamic weighted average loss v in Eq (2.2) and the insurer's wealth dynamics in Eq (2.5), which enables risk hedging. As β increases, the insurer's ability to resist risk also increases.



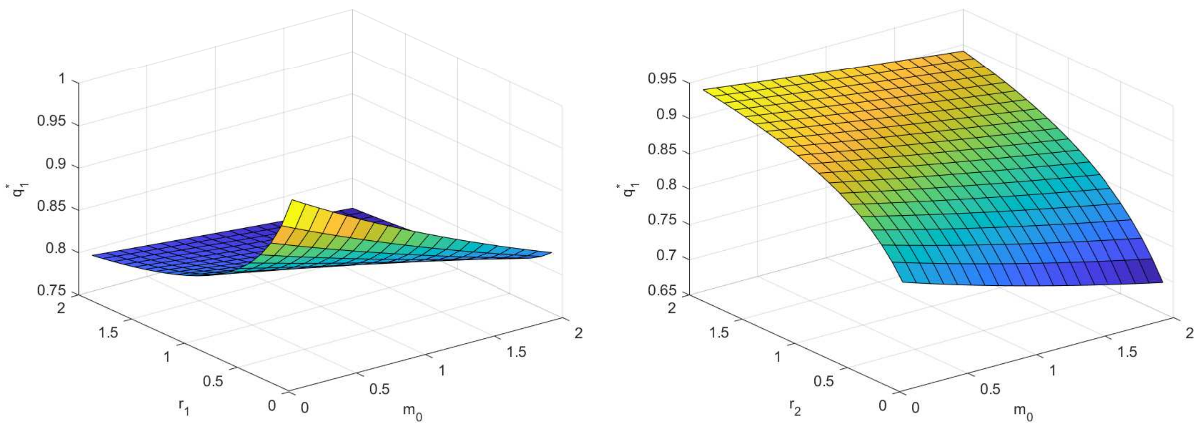
(a) The effects of β and t on $q_1^*(t)$.

(b) The effects of α and n_2 on $q_1^*(t)$.

Figure 2. Effects of β, t, α and n_2 on $q_1^*(t)$.

From Figure 2(b), it is observed that when $\alpha > 0.5$, $q_1^*(t)$ increases with an increase in safety loading n_2, whereas for $\alpha < 0.5$, there is a decreasing trend in $q_1^*(t)$ with an increase in n_2. This can be attributed to the fact that when the insurer dominates, the cost of reinsurance becomes more expensive with an increase in safety loading n_2, and therefore, the insurer is more inclined to purchase less reinsurance to maintain stable income. Conversely, when the reinsurer dominates, he will gain more profit from reinsurance with an increase in n_2, and thus is more willing to accept more reinsurance.

Figure 3 reveals that $q_1^*(t)$ decreases with an increase in the parameter m_0. As m_0 increases, the AAI becomes more uncertain towards the claim distribution, and will be more likely to purchase an increased amount of reinsurance to counteract the impact of model uncertainty. Furthermore, $q_1^*(t)$ is a decreasing function of parameter γ_1. As γ_1 increases, the AAI becomes more risk-averse and will purchase more reinsurance to transfer the risk to the reinsurer. On the other hand, $q_1^*(t)$ is an increasing function of parameter γ_2 . As γ_2 increases, the AAR becomes more risk-averse and thus is more willing to accept less reinsurance.



(a) The effects of γ_1 and m_0 on $q_1^*(t)$.

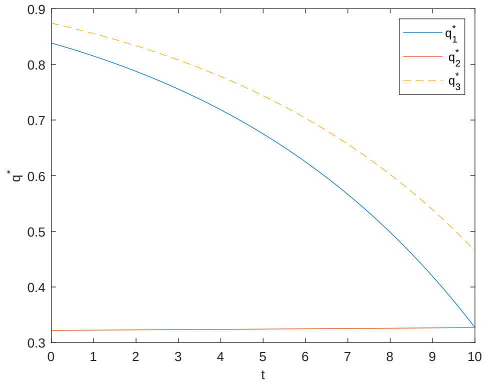
(b) The effects of γ_2 and m_0 on $q_1^*(t)$.

Figure 3. Effects of γ_1, γ_2 and m_0 on $q_1^*(t)$.

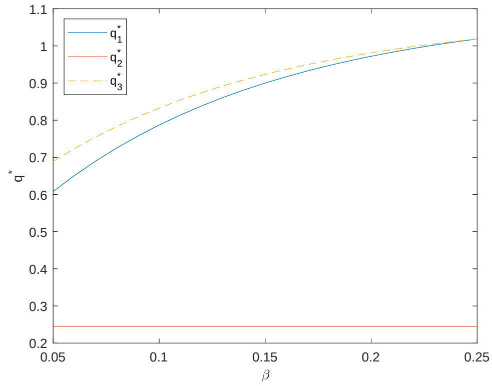
Figures 4–6 illustrate the impact of various variables on the equilibrium reinsurance strategies under three different models. One common observation is that $q_3^*(t) > q_1^*(t) > q_2^*(t)$. The explanation for $q_1^*(t) > q_2^*(t)$ is that, compared to the expected premium principle, the insurer's ability to absorb risk under the loss-dependent premium principle is stronger, reducing the demand for risk transfer through reinsurance. The reason for $q_3^*(t) > q_1^*(t)$ is that, compared to ambiguity-neutral decision makers, the AAI have a greater aversion to model uncertainty, and thus tend to adopt more conservative strategies by transferring more of their risk to the reinsurer, resulting in a higher demand for reinsurance.

In Figure 4(a), it is worth noting that under the expected value premium principle, $q_2^*(t)$ increases with t, while under the loss-dependence premium principle, $q_1^*(t)$ and $q_3^*(t)$ decrease with t. Figure 4(b) shows that as β increases, the change trend of $q_1^*(t)$ and $q_3^*(t)$ are the same, indicating that considering robustness does not affect the correlation between $q^*(t)$ and β.

As is shown in Figure 5(a), the equilibrium reinsurance strategies for all three models increase with the increase of n_2 . From Figure 5(b), we can see that as the ambiguity aversion coefficient m_0 increases, both $q_1^*(t)$ and $q_2^*(t)$ show a decreasing trend, which indicates that the impact of robustness on $q^*(t)$ is similar under the two aforementioned premium principles. Figure 6 illustrates that the correlation between the equilibrium reinsurance strategies of the three models and the risk aversion coefficients γ_1 and γ_2 is the same.

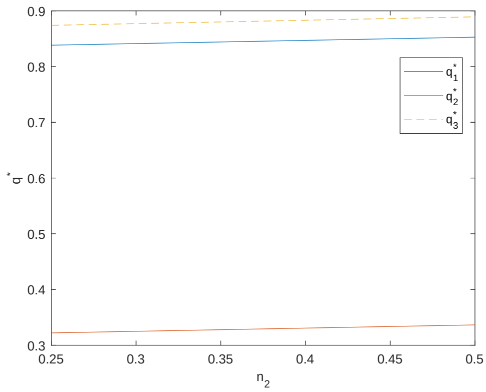


(a) Effect of t on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.

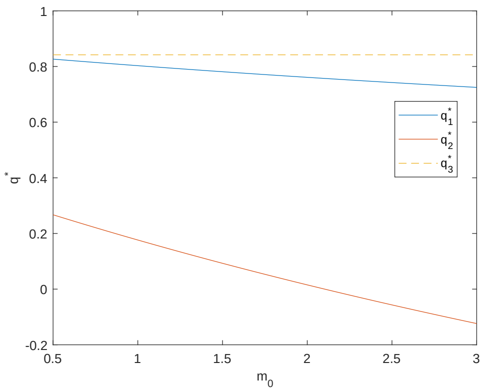


(b) Effect of β on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.

Figure 4. Effects of t and β on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.

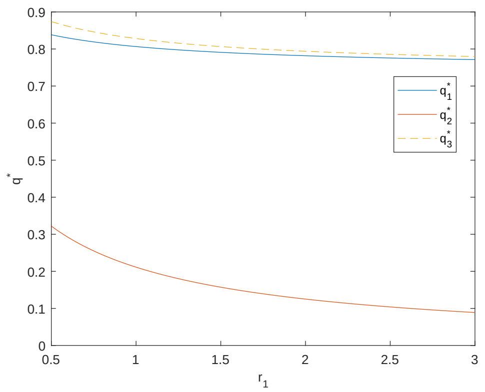


(a) Effect of n_2 on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.

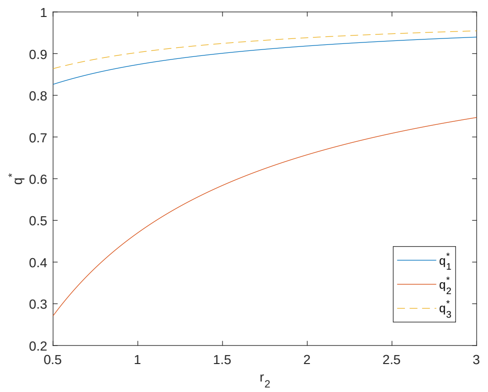


(b) Effect of m_0 on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.

Figure 5. Effects of n_2 and m_0 on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.



(a) Effect of r_1 on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.



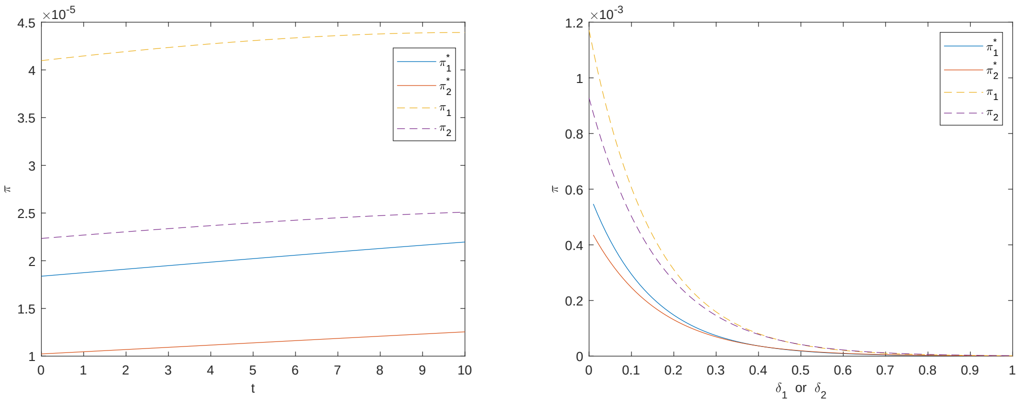
(b) Effect of r_2 on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.

Figure 6. Effects of r_1 and r_2 on $q_1^*(t),\ q_2^*(t),\ q_3^*(t)$.

4.2. Sensitivity analysis of the equilibrium investment strategy

In this part, we discuss the impact of model parameters on the equilibrium investment strategy. Here, π_1^* and π_2^* represent the robust equilibrium investment strategy of the AAI and AAR, respectively, and π_1 and π_2 represent the equilibrium investment strategies of the ANI and ANR.

Figure 7(a) demonstrates the increasing trends of $\pi_1(\pi_1^*)$ and $\pi_2(\pi_2^*)$ as t increases. This phenomenon can be attributed to the fact that over time the insurer and reinsurer enhance their risk-bearing capacity while accumulating wealth, consequently leading to a gradual increase in the allocation of investment towards risk assets. From Figure 7(b), we observe that the robust equilibrium investment strategy $\pi_1(\pi_2)$ decreases as the elasticity coefficient $\delta_1(\delta_2)$ increases. Higher values of δ may lead to a larger decrease in expected volatility and an increased likelihood of significant adverse movements in the risky asset prices. Therefore, with an increase in δ, both the insurer and reinsurer prefer to reduce their investments in the risky asset to mitigate risks.

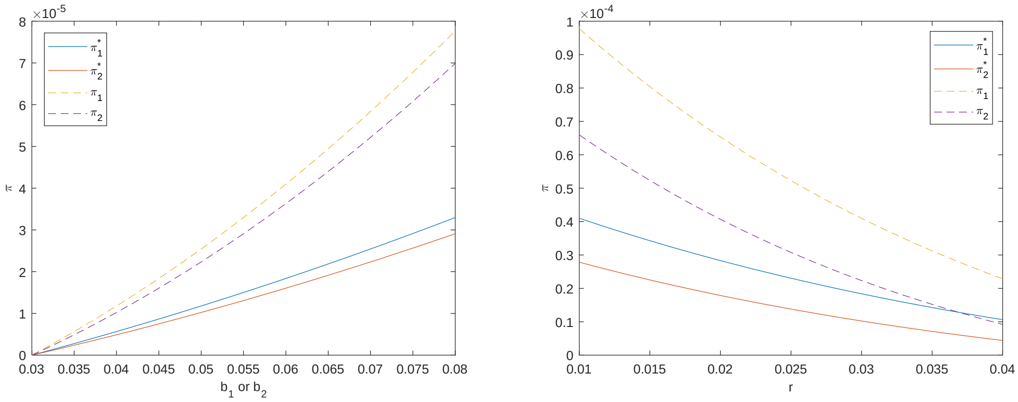


(a) Effect of t on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$. (b) Effect of $\delta_1(\delta_2)$ on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$.

Figure 7. Effects of t and $\delta_1(\delta_2)$ on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$.

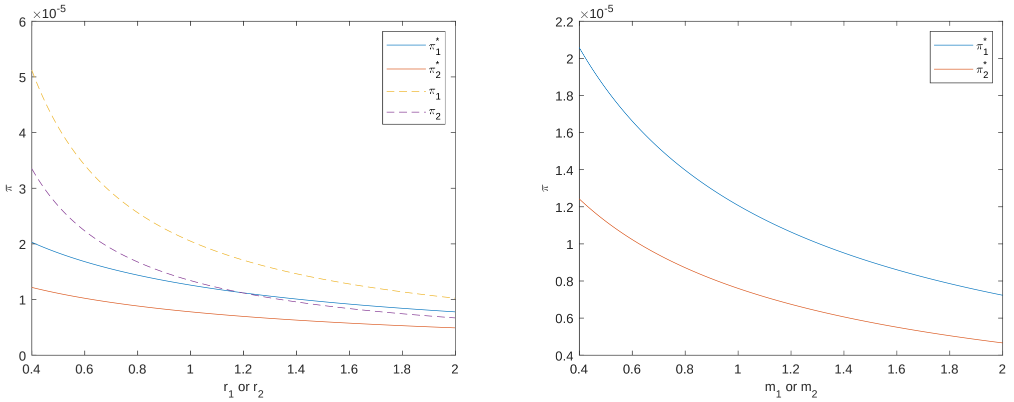
As shown in Figure 8, the robust equilibrium investment strategy for the insurer (reinsurer) is an increasing function of $b_1(b_2)$, and a decreasing function of r. This is in accordance with our intuition. As $b_1(b_2)$ increases, the insurer (reinsurer) will obtain higher returns from investments, leading them to increase their investments in risky assets to gain more profits. Furthermore, as r increases, risk-free assets become more attractive, and the insurer (reinsurer) is willing to invest more funds into risk-free assets. Consequently, the amount of investment in risky assets decreases.

Figure 9(a) reveals that the coefficient of risk aversion $\gamma_1(\gamma_2)$ has a negative effect on the robust equilibrium investment strategy of the insurer (reinsurer). This means that the insurer (reinsurer) with a higher level of risk aversion will reduce her or his investment in risky assets to avoid risks. Figure 9(b) demonstrates that the insurer (reinsurer) reduces her or his investment in the risky market as the ambiguity-aversion coefficient $m_1(m_2)$ increases. As mentioned earlier, the ambiguity-aversion coefficient can describe the decision-maker's attitude towards model uncertainty. Therefore, when $m_1(m_2)$ is larger, the AAI (AAR) is more averse to uncertain risks, and thus is less willing to invest in risky assets.



(a) Effect of $b_1(b_2)$ on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$. (b) Effect of r on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$.

Figure 8. Effects of $b_1(b_2)$ and r on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$.



(a) Effect of $r_1(r_2)$ on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$. (b) Effect of $m_1(m_2)$ on $\pi_1^*(\pi_2^*)$.

Figure 9. Effects of $r_1(r_2)$ and $m_1(m_2)$ on $\pi_1(\pi_2)$, $\pi_1^*(\pi_2^*)$.

Additionally, we can observe the same phenomenon from Figures 7–9: $\pi_1 > \pi_1^*; \pi_2 > \pi_2^*$. Due to the aversion to the uncertainty of estimating parameters in the risky market, the ambiguity-aversion decision-makers adopt more conservative investment strategies, i.e., reducing risk investments to resist ambiguity uncertainty.

5. Conclusions

In this paper, we study the robust equilibrium reinsurance-investment problem for the AAI and the AAR under a mean-variance weighted sum objective criterion. Specifically, it is assumed that the net claims process is approximated by a diffusion process, and the insurer considers the historical claims and adopts the loss-dependent premium principle. However, due to information loss, the reinsurer still employs the traditional expected value premium principle. Both the insurer and reinsurer invest in risk-free and risky assets, where the price process of the risky asset is modeled by the CEV model. After considering the uncertainty of model parameters, we employ robust optimization methods and derive the extended HJB equation. Through dynamic programming theory, we derive closed-form expressions for robust equilibrium reinsurance-investment strategies, as well as their corresponding value functions. We also provide numerical simulations to illustrate the economic implications of our results. We find that the impact of some model parameters on the reinsurance strategy depends on the weighting parameters. In the early stages of decision-making, there is an inverse relationship between extrapolation intensity and reinsurance demand, and employing the loss-dependent premium principle reduces the insurer's demand for reinsurance. Moreover, we find that ambiguity aversion has a significant impact on the reinsurance-investment strategy. As the degree of ambiguity aversion increases, the demand for reinsurance also increases, while the investment in risky assets decreases.

In future research, it may be worthwhile to consider jump risk asset price processes or Ornstein-Uhlenbeck processes. Additionally, robust optimization objectives can be extended to include Alpha-robust mean-variance criteria. These extensions could provide more complex problems and greatly enrich our research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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Appendix A

Proof of Theorem 3.2

In order to solve the extended HJB Eqs (3.9)–(3.11), we postulate the following form of solution,

$$V(t,x,y,v,s_1,s_2) = A_1(t)x + A_2(t)y + A_3(t)v + A_4(t)s_1^{-2\delta_1} + A_5(t)s_2^{-2\delta_2} + A_6(t), \tag{A.1}$$

$$g_1(t,x,y,v,s_1,s_2) = B_1(t)x + B_2(t)y + B_3(t)v + B_4(t)s_1^{-2\delta_1} + B_5(t)s_2^{-2\delta_2} + B_6(t), \tag{A.2}$$

$$g_2(t,x,y,v,s_1,s_2) = C_1(t)x + C_2(t)y + C_3(t)v + C_4(t)s_1^{-2\delta_1} + C_5(t)s_2^{-2\delta_2} + C_6(t), \tag{A.3}$$

with boundary conditions

$$A_1(T) = \alpha,\ A_2(T) = 1-\alpha,\ B_1(T) = C_2(T) = 1,\ \ A_3(T) = A_4(T) = A_5(T) = A_6(T) = 0,$$
$$B_2(T) = B_3(T) = B_4(t) = B_5(T) = B_6(T) = C_1(T) = C_3(T) = C_4(T) = C_5(T) = C_6(T) = 0.$$

The partial derivatives are

$$V_t = A_{1t}x + A_{2t}y + A_{3t}v + A_{4t}s_1^{-2\delta_1} + A_{5t}s_2^{-2\delta_2} + A_{6t}, \quad V_x = A_1,$$

$$
\begin{aligned}
&V_y = A_2, V_v = A_3, \quad V_{s_1} = -2\delta_1 s_1^{-2\delta_1-1} A_4, \quad V_{s_2} = -2\delta_2 s_2^{-2\delta_2-1} A_5,\\
&V_{s_1 s_1} = 2\delta_1(2\delta_1+1) s_1^{-2\delta_1-2} A_4, \quad V_{s_2 s_2} = 2\delta_2(2\delta_2+1) s_2^{-2\delta_2-2} A_5,\\
&g_{1t} = B_{1t}x + B_{2t}y + B_{3t}v + B_{4t}s_1^{-2\delta_1} + B_{5t}s_2^{-2\delta_2} + B_{6t}, \quad g_{1x} = B_1,\\
&g_{1y} = B_2, g_{1v} = B_3, \quad g_{1s_1} = -2\delta_1 s_1^{-2\delta_1-1} B_4, \quad g_{1s_2} = -2\delta_2 s_2^{-2\delta_2-1} B_5,\\
&g_{1s_1 s_1} = 2\delta_1(2\delta_1+1) s_1^{-2\delta_1-2} B_4, \quad g_{1s_2 s_2} = 2\delta_2(2\delta_2+1) s_2^{-2\delta_2-2} B_5,\\
&g_{2t} = C_{1t}x + C_{2t}y + C_{3t}v + C_{4t}s_1^{-2\delta_1} + C_{5t}s_2^{-2\delta_2} + C_{6t}, \quad g_{2x} = C_1,\\
&g_{2y} = C_2, g_{2v} = C_3, \quad g_{2s_1} = -2\delta_1 s_1^{-2\delta_1-1} C_4, \quad g_{2s_2} = -2\delta_2 s_2^{-2\delta_2-1} C_5,\\
&g_{2s_1 s_1} = 2\delta_1(2\delta_1+1) s_1^{-2\delta_1-2} C_4, \quad g_{2s_2 s_2} = 2\delta_2(2\delta_2+1) s_2^{-2\delta_2-2} C_5,\\
&V_{xx} = V_{yy} = V_{vv} = V_{xv} = V_{yv} = V_{xy} = V_{xs_1} = V_{ys_2} = 0,\\
&g_{1xx} = g_{1yy} = g_{1vv} = g_{1xv} = g_{1yv} = g_{1xy} = g_{1xs_1} = g_{1ys_2} = 0,\\
&g_{2xx} = g_{2yy} = g_{2vv} = g_{2xv} = g_{2yv} = g_{2xy} = g_{2xs_1} = g_{2ys_2} = 0,
\end{aligned}
\tag{A.4}
$$

where V, g_i, A_i, B_i and C_i are abbreviations for $V(t,x,y,v,s_1,s_2), g_i(t,x,y,v,s_1,s_2), A_i(t), B_i(t)$ and $C_i(t)$, respectively.

Substituting Eqs (A.1)–(A.4) into Eqs (3.9)–(3.11), we have

$$
\begin{aligned}
&\sup_{u\in\mathcal{U}} \inf_{\mathbb{Q}\in\mathcal{Q}} \{A_{1t}x + A_{2t}y + A_{3t}v + A_{4t}s_1^{-2\delta_1} + A_{5t}s_2^{-2\delta_2} + A_{6t}\\
&+ [rx + (b_1 - r)\pi_1 + (1+n_1)(e^{-\beta t}\mu + v) - (1+n_2)\mu + qn_2\mu - q\sigma_0\theta_0 - \pi_1\sigma_1\theta_1 s_1^{\delta_1}]A_1\\
&+ [ry + (b_2 - r)\pi_2 + n_2(1-q)\mu - (1-q)\sigma_0\theta_0 - \pi_2\sigma_2\theta_2 s_2^{\delta_2}]A_2 + \beta(\mu - v + \sigma_0\theta_0)A_3\\
&- (b_1 - \sigma_1\theta_1 s_1^{\delta_1})2\delta_1 s_1^{-2\delta_1}A_4 - (b_2 - \sigma_2\theta_2 s_2^{\delta_2})2\delta_2 s_2^{-2\delta_2}A_5 + \sigma_1^2\delta_1(2\delta_1+1)A_4 + \sigma_2^2\delta_2(2\delta_2+1)A_5\\
&- \alpha\gamma_1\Big[\ \frac{1}{2}(q^2\sigma_0^2 + \pi_1^2\sigma_1^2 s_1^{2\delta_1})B_1^2 + \frac{1}{2}((1-q)^2\sigma_0^2 + \pi_2^2\sigma_2^2 s_2^{2\delta_2})B_2^2 + \frac{1}{2}\beta^2\sigma_0^2 B_3^2\\
&+ 2\sigma_1^2\delta_1^2 s_1^{-2\delta_1}B_4^2 + 2\sigma_2^2\delta_2^2 s_2^{-2\delta_2}B_5^2 + q(1-q)\sigma_0^2 B_1B_2 - q\beta\sigma_0^2 B_1B_3\\
&-(1-q)\beta\sigma_0^2 B_2B_3 - 2\pi_1\sigma_1^2\delta_1 B_1B_4 - 2\pi_2\sigma_2^2\delta_2 B_2B_5\ \Big]\\
&- (1-\alpha)\gamma_2[\frac{1}{2}(q^2\sigma_0^2 + \pi_1^2\sigma_1^2 s_1^{2\delta_1})C_1^2 + \frac{1}{2}((1-q)^2\sigma_0^2 + \pi_2^2\sigma_2^2 s_2^{2\delta_2})C_2^2 + \frac{1}{2}\beta^2\sigma_0^2 C_3^2\\
&+ 2\sigma_1^2\delta_1^2 s_1^{-2\delta_1}C_4^2 + 2\sigma_2^2\delta_2^2 s_2^{-2\delta_2}C_5^2 + q(1-q)\sigma_0^2 C_1C_2 - q\beta\sigma_0^2 C_1C_3 - (1-q)\beta\sigma_0^2 C_2C_3\\
&- 2\pi_1\sigma_1^2\delta_1 C_1C_4 - 2\pi_2\sigma_2^2\delta_2 C_2C_5] + \frac{\theta_0^2}{2m_0} + \alpha\frac{\theta_1^2}{2m_1} + (1-\alpha)\frac{\theta_2^2}{2m_2}\} = 0,
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
&B_{1t}x + B_{2t}y + B_{3t}v + B_{4t}s_1^{-2\delta_1} + B_{5t}s_2^{-2\delta_2} + B_{6t}\\
&+ [rx + (b_1 - r)\pi_1 + (1+n_1)(e^{-\beta t}\mu + v) - (1+n_2)\mu + qn_2\mu - q\sigma_0\theta_0 - \pi_1\sigma_1\theta_1 s_1^{\delta_1}]B_1\\
&+ [ry + (b_2 - r)\pi_2 + n_2(1-q)\mu - (1-q)\sigma_0\theta_0 - \pi_2\sigma_2\theta_2 s_2^{\delta_2}]B_2 + \beta(\mu - v + \sigma_0\theta_0)B_3\\
&- [b_1 - \sigma_1\theta_1 s_1^{\delta_1}]2\delta_1 s_1^{-2\delta_1}B_4 - [b_2 - \sigma_2\theta_2 s_2^{\delta_2}]2\delta_2 s_2^{-2\delta_2}B_5\\
&+ \sigma_1^2\delta_1(2\delta_1+1)B_4 + \sigma_2^2\delta_2(2\delta_2+1)B_5 = 0,
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
&C_{1t}x + C_{2t}y + C_{3t}v + C_{4t}s_1^{-2\delta_1} + C_{5t}s_2^{-2\delta_2} + C_{6t} \\
&+ [rx + (b_1 - r)\pi_1 + (1 + n_1)(e^{-\beta t}\mu + v) - (1 + n_2)\mu + qn_2\mu - q\sigma_0\theta_0 - \pi_1\sigma_1\theta_1 s_1^{\delta_1}]C_1 \\
&+ [ry + (b_2 - r)\pi_2 + n_2(1 - q)\mu - (1 - q)\sigma_0\theta_0 - \pi_2\sigma_2\theta_2 s_2^{\delta_2}]C_2 + \beta(\mu - v + \sigma_0\theta_0)C_3 \\
&- [b_1 - \sigma_1\theta_1 s_1^{\delta_1}]2\delta_1 s_1^{-2\delta_1}C_4 - [b_2 - \sigma_2\theta_2 s_2^{\delta_2}]2\delta_2 s_2^{-2\delta_2}C_5 \\
&+ \sigma_1^2\delta_1(2\delta_1 + 1)C_4 + \sigma_2^2\delta_2(2\delta_2 + 1)C_5 = 0.
\end{aligned} \tag{A.7}
$$

Based on Eq (A.5), by fixing q, π_1, π_2 and maximizing over θ, we obtain the following first-order condition for the minimum point θ^*,

$$
\begin{aligned}
\theta_0^*(q) &= m_0\left[q\sigma_0 A_1 + (1 - q)\sigma_0 A_2 - \beta\sigma_0 A_3\right], \\
\theta_1^*(\pi_1) &= \frac{m_1\sigma_1}{\alpha}(\pi_1 s_1^{\delta_1}A_1 - 2\delta_1 s_1^{-\delta_1}A_4), \\
\theta_2^*(\pi_2) &= \frac{m_2\sigma_2}{1 - \alpha}(\pi_2 s_2^{\delta_2}A_2 - 2\delta_2 s_2^{-\delta_2}A_5).
\end{aligned} \tag{A.8}
$$

Replacing Eq (A.8) back into Eq (A.5) yields

$$
\begin{aligned}
&A_{1t}x + A_{2t}y + A_{3t}v + A_{4t}s_1^{-2\delta_1} + A_{5t}s_2^{-2\delta_2} + A_{6t} + [rx + (1 + n_1)(e^{-\beta t}\mu + v) \\
&- (1 + n_2)\mu]A_1 + (ry + n_2\mu)A_2 + \beta(\mu - v)A_3 - \frac{1}{2}m_0\sigma_0^2(A_2 - \beta A_3)^2 \\
&- 2b_1\delta_1 s_1^{-2\delta_1}A_4 - 2b_2\delta_2 s_2^{-2\delta_2}A_5 + \sigma_1^2\delta_1(2\delta_1 + 1)A_4 + \sigma_2^2\delta_2(2\delta_2 + 1)A_5 \\
&- \alpha\gamma_1[\frac{1}{2}\sigma_0^2 B_2^2 - \beta\sigma_0^2 B_2 B_3 + \frac{1}{2}\beta^2\sigma_0^2 B_3^2 + 2\sigma_1^2\delta_1^2 s_1^{-2\delta_1}B_4^2 + 2\sigma_2^2\delta_2^2 s_2^{-2\delta_2}B_5^2] \\
&- (1 - \alpha)\gamma_2[\frac{1}{2}\sigma_0^2 C_2^2 - \beta\sigma_0^2 C_2 C_3 + \frac{1}{2}\beta_0^2\sigma_0^2 C_3^2 + 2\sigma_1^2\delta_1^2 s_1^{-2\delta_1}C_4^2 + 2\sigma_2^2\delta_2^2 s_2^{-2\delta_2}C_5^2] \\
&+ \sup_q\{R_0(q)\} + \sup_{\pi_1}\{R_1(\pi_1)\} + \sup_{\pi_2}\{R_2(\pi_2)\} = 0,
\end{aligned} \tag{A.9}
$$

where

$$
\begin{aligned}
R_0(q) =& qn_2\mu(A_1 - A_2) - m_0\sigma_0^2 q(A_1 - A_2)(A_2 - \beta A_3) - \frac{1}{2}m_0\sigma_0^2 q^2(A_1 - A_2)^2 \\
&- \alpha\gamma_1\sigma_0^2[\frac{1}{2}q^2(B_1 - B_2)^2 + q(B_1 - B_2)(B_2 - \beta B_3)] \\
&- (1 - \alpha)\gamma_2\sigma_0^2[\frac{1}{2}q^2(C_1 - C_2)^2 + q(C_1 - C_2)(C_2 - \beta C_3)], \\
R_1(\pi_1) =& (b_1 - r)\pi_1 A_1 - \frac{1}{2}\pi_1^2\sigma_1^2 s_1^{2\delta_1}[\alpha\gamma_1 B_1^2 + (1 - \alpha)\gamma_2 C_1^2] + 2\pi_1\sigma_1^2\delta_1[\alpha\gamma_1 B_1 B_4 \\
&+ (1 - \alpha)\gamma_2 C_1 C_4] - \frac{m_1}{2\alpha}\sigma_1^2(\pi_1 s_1^{\delta_1}A_1 - 2\delta_1 s_1^{-\delta_1}A_4)^2, \\
R_2(\pi_2) =& (b_2 - r)\pi_2 A_2 - \frac{1}{2}\pi_2^2\sigma_2^2 s_2^{2\delta_2}[\alpha\gamma_1 B_2^2 + (1 - \alpha)\gamma_2 C_2^2] + 2\pi_2\sigma_2^2\delta_2[\alpha\gamma_1 B_2 B_5 \\
&+ (1 - \alpha)\gamma_2 C_2 C_5] - \frac{m_2}{2(1 - \alpha)}\sigma_2^2(\pi_2 s_2^{\delta_2}A_2 - 2\delta_2 s_2^{-\delta_2}A_5)^2.
\end{aligned}
$$

Differentiating Eq (A.9) with respect to π_1, π_2 and q, we obtain the following first-order optimality conditions:

$$
\begin{aligned}
q^* =& \frac{n_2\mu(A_1 - A_2)}{\sigma_0^2\left[\alpha\gamma_1(B_1 - B_2)^2 + (1-\alpha)\gamma_2(C_1 - C_2)^2 + m_0(A_1 - A_2)^2\right]} \\
&+ \frac{\alpha\gamma_1(B_1 - B_2)(B_2 - \beta B_3) + (1-\alpha)\gamma_2(C_1 - C_2)(C_2 - \beta C_3) + m_0(A_1 - A_2)(A_2 - \beta A_3)}{\alpha\gamma_1(B_1 - B_2)^2 + (1-\alpha)\gamma_2(C_1 - C_2)^2 + m_0(A_1 - A_2)^2},
\end{aligned} \tag{A.10}
$$

$$
\pi_1^* = \frac{(b_1 - r)A_1 + 2\sigma_1^2\delta_1\left[\alpha\gamma_1 B_1 B_4 + (1-\alpha)\gamma_2 C_1 C_4\right] + \frac{m_1}{\alpha}\sigma_1^2 2\delta_1 A_1 A_4}{\sigma_1^2 s_1^{2\delta_1}\left[\alpha\gamma_1 B_1^2 + (1-\alpha)\gamma_2 C_1^2\right] + \frac{m_1}{\alpha}\sigma_1^2 s_1^{2\delta_1} A_1^2}, \tag{A.11}
$$

$$
\pi_2^* = \frac{(b_2 - r)A_2 + 2\sigma_2^2\delta_2\left[\alpha\gamma_1 B_2 B_5 + (1-\alpha)\gamma_2 C_2 C_5\right] + \frac{m_2}{1-\alpha}\sigma_2^2 2\delta_2 A_2 A_5}{\sigma_2^2 s_2^{2\delta_2}\left[\alpha\gamma_1 B_2^2 + (1-\alpha)\gamma_2 C_2^2\right] + \frac{m_2}{1-\alpha}\sigma_2^2 s_2^{2\delta_2} A_2^2}. \tag{A.12}
$$

Introducing q^*, π_1^*, π_2^* into Eq (A.9) gives

$$
\begin{aligned}
& A_{1t}x + A_{2t}y + A_{3t}v + A_{4t}s_1^{-2\delta_1} + A_{5t}s_2^{-2\delta_2} + A_{6t} \\
& + [rx + (1+n_1)(e^{-\beta t}\mu + v) - (1+n_2)\mu]A_1 + ryA_2 + n_2\mu A_2 + \beta(\mu - v)A_3 \\
& - 2b_1\delta_1 s_1^{-2\delta_1}A_4 - 2b_2\delta_2 s_2^{-2\delta_2}A_5 + \sigma_1^2\delta_1(2\delta_1 + 1)A_4 + \sigma_2^2\delta_2(2\delta_2 + 1)A_5 \\
& - \alpha\gamma_1[\frac{1}{2}\sigma_0^2 B_2^2 - \beta\sigma_0^2 B_2 B_3 + \frac{1}{2}\beta^2\sigma_0^2 B_3^2 + 2\sigma_1^2\delta_1^2 s_1^{-2\delta_1}B_4^2 + 2\sigma_2^2\delta_2^2 s_2^{-2\delta_2}B_5^2] \\
& - (1-\alpha)\gamma_2[\frac{1}{2}\sigma_0^2 C_2^2 - \beta\sigma_0^2 C_2 C_3 + \frac{1}{2}\beta_0^2\sigma_0^2 C_3^2 + 2\sigma_1^2\delta_1^2 s_1^{-2\delta_1}C_4^2 + 2\sigma_2^2\delta_2^2 s_2^{-2\delta_2}C_5^2] \\
& - \frac{1}{2}m_0\sigma_0^2(A_2 - \beta A_3)^2 + R_0(q^*) + R_1(\pi_1^*) + R_2(\pi_2^*) = 0.
\end{aligned} \tag{A.13}
$$

By matching the coefficients of variables x, y, v, s_1 and s_2, we obtain that

$$
\begin{cases}
(A_{1t} + rA_1)x = 0, \\
(A_{2t} + rA_2)y = 0, \\
\left[A_{3t} + (1+n_1)A_1 - \beta A_3\right]v = 0,
\end{cases} \tag{A.14}
$$

$$
\left[A_{4t} - 2b_1\delta_1 A_4 - 2\alpha\gamma_1\sigma_1^2\delta_1^2 B_4^2 - 2(1-\alpha)\gamma_2\sigma_1^2\delta_1^2 C_4^2 + s_1^{2\delta_1}R_1(\pi_1^*)\right]s_1^{-2\delta_1} = 0, \tag{A.15}
$$

$$
\left[A_{5t} - 2b_2\delta_2 A_5 - 2\alpha\gamma_1\sigma_2^2\delta_2^2 B_5^2 - 2(1-\alpha)\gamma_2\sigma_2^2\delta_2^2 C_5^2 + s_2^{2\delta_2}R_2(\pi_2^*)\right]s_2^{-2\delta_2} = 0, \tag{A.16}
$$

and the rest is

$$
\begin{aligned}
& A_{6t} + [(1+n_1)e^{-\beta t}\mu + (1+n_2)\mu]A_1 + n_2\mu A_2 + \beta\mu A_3 + \delta_1\sigma_1^2(2\delta_1 + 1)A_4 \\
& + \delta_2\sigma_2^2(2\delta_2 + 1)A_5 - \frac{1}{2}m_0\sigma_0^2(A_2 - \beta A_3)^2 - \frac{1}{2}\beta^2\sigma_0^2\left[\alpha\gamma_1 B_3^2 + (1-\alpha)\gamma_2 C_3^2\right] \\
& - \frac{1}{2}\sigma_0^2\left[\alpha\gamma_1 B_2^2 + (1-\alpha)\gamma_2 C_2^2\right] + \beta\sigma_0^2\left[\alpha\gamma_1 B_2 B_3 + (1-\alpha)\gamma_2 C_2 C_3\right] + R_0(q^*) = 0.
\end{aligned} \tag{A.17}
$$

By substituting q^*, π_1^*, π_2^* into Eqs (A.6) and (A.7), and then separating the variables x, y, v, s_1 and s_2, we can obtain the following equations:

$$
\begin{cases}
(B_{1t} + rB_1)x = 0, \\
(B_{2t} + rB_2)y = 0, \\
\left[B_{3t} + (1+n_1)B_1 - \beta B_3\right]v = 0,
\end{cases} \tag{A.18}
$$

$$B_{4t}+(b_1-r)\pi_1^*s_1^{2\delta_1}B_1-m_1\sigma_1^2(\pi_1^*s_1^{2\delta_1})^2\frac{A_1}{\alpha}B_1+2m_1\delta_1\sigma_1^2\pi_1^*s_1^{2\delta_1}\frac{A_4}{\alpha}B_1$$
$$-2b_1\delta_1B_4+2\delta_1m_1\pi_1^*s_1^{2\delta_1}\sigma_1^2\frac{A_1}{\alpha}B_4-4m_1\sigma_1^2\delta_1^2\frac{A_4}{\alpha}B_4=0, \quad (A.19)$$
$$B_{5t}+(b_2-r)\pi_2^*s_2^{2\delta_2}B_2-m_2\sigma_2^2(\pi_2^*s_2^{2\delta_2})^2\frac{A_2}{1-\alpha}B_2+2m_2\delta_2\sigma_2^2\pi_2^*s_2^{2\delta_2}\frac{A_5}{1-\alpha}B_2$$
$$-2b_2\delta_2B_5+2m_2\delta_2\sigma_2^2\pi_2^*s_2^{2\delta_2}\frac{A_2}{1-\alpha}B_5-4m_2\delta_2^2\sigma_2^2\frac{A_5}{1-\alpha}B_5=0, \quad (A.20)$$
$$\begin{cases}(C_{1t}+rC_1)x=0,\\(C_{2t}+rC_2)y=0,\\ \left[C_{3t}+(1+n_1)C_1-\beta C_3\right]v=0,\end{cases} \quad (A.21)$$
$$C_{4t}+(b_1-r)\pi_1^*s_1^{2\delta_1}C_1-m_1\sigma_1^2(\pi_1^*s_1^{2\delta_1})^2\frac{A_1}{\alpha}C_1+2m_1\delta_1\sigma_1^2\pi_1^*s_1^{2\delta_1}\frac{A_4}{\alpha}C_1$$
$$-2b_1\delta_1C_4+2m_1\delta_1\sigma_1^2\pi_1^*s_1^2\delta_1\frac{A_1}{\alpha}C_4-4m_1\delta_1^2\sigma_1^2\frac{A_4}{\alpha}C_4=0, \quad (A.22)$$
$$C_{5t}+(b_2-r)\pi_2^*s_2^{2\delta_2}C_2-m_2\sigma_2^2(\pi_2^*s_2^{2\delta_2})^2\frac{A_2}{1-\alpha}C_2+2m_2\delta_2\sigma_2^2\pi_2^*s_2^{2\delta_2}\frac{A_5}{1-\alpha}C_2$$
$$-2b_2\delta_2C_5+2m_2\delta_2\sigma_2^2\pi_2^*s_2^{2\delta_2}\frac{A_2}{1-\alpha}C_5-4m_2\delta_2^2\sigma_2^2\frac{A_5}{1-\alpha}C_5=0, \quad (A.23)$$

and the rest is

$$B_{6t}+\left[(1+n_1)e^{-\beta t}\mu-(1+n_2)\mu\right]B_1+n_2\mu B_2+\beta\mu B_3+q^*n_2\mu(B_1-B_2)+\sigma_1^2\delta_1(2\delta_1+1)B_4$$
$$+\sigma_2^2\delta_2(2\delta_2+1)B_5-m_0\sigma_0^2[q^*(B_1-B_2)+B_2-\beta B_3][q^*(A_1-A_2)+A_2-\beta A_3]=0, \quad (A.24)$$
$$C_{6t}+\left[(1+n_1)e^{-\beta t}\mu-(1+n_2)\mu\right]C_1+n_2\mu C_2+\beta\mu C_3+q^*n_2\mu(C_1-C_2)+\sigma_1^2\delta_1(2\delta_1+1)C_4$$
$$+\sigma_2^2\delta_2(2\delta_2+1)C_5-m_0\sigma_0^2[q^*(C_1-C_2)+C_2-\beta C_3][q^*(A_1-A_2)+A_2-\beta A_3]=0. \quad (A.25)$$

Considering the boundary conditions and solving Eqs (A.14), (A.18) and (A.21), we obtain

$$\begin{cases}A_1(t)=\alpha e^{r(T-t)}, \quad A_2(t)=(1-\alpha)e^{r(T-t)},\\ B_1(t)=C_2(t)=e^{r(T-t)},\\ B_2(t)=C_1(t)=C_3(t)=0,\\ A_3(t)=\alpha\frac{1+n_1}{r+\beta}[e^{r(T-t)}-e^{-\beta(T-t)}],\\ B_3(t)=\frac{1+n_1}{r+\beta}[e^{r(T-t)}-e^{-\beta(T-t)}].\end{cases} \quad (A.26)$$

Inputting Eqs (A.26) into (A.20), (A.22) and simplifying, we have

$$B_{5t}+\left[\frac{2m_2\delta_2\left[(b_2-r)(1-\alpha)+2\sigma_2^2\delta_2(1-\alpha)\gamma_2C_5-2\gamma_2\delta_2\sigma_2^2A_5\right]}{(1-\alpha)(\gamma_2+m_2)}-2b_2\delta_2\right]B_5=0, \quad (A.27)$$
$$C_{4t}+\left[\frac{2m_1\delta_1\left[(b_1-r)\alpha+2\delta_1\sigma_1^2\gamma_1\alpha B_4-2\delta_1\sigma_1^2\gamma_1A_4\right]}{\alpha(\gamma_1+m_1)}-2b_1\delta_1\right]C_4=0. \quad (A.28)$$

With the boundary condition $B_5(T) = 0, C_4(T) = 0$, we can find that $B_5(t)$ and $C_4(t)$ have the following solutions:

$$B_5(t) = 0,\ C_4(t) = 0. \tag{A.29}$$

Substituting the solutions (A.29) into (A.15) and (A.19), we have

$$\begin{aligned} &A_{4t} - 2b_1\delta_1 A_4 + 2\delta_1(b_1 - r)\frac{m_1}{\gamma_1 + m_1}A_4 + 2\alpha\delta_1(b_1 - r)\frac{\gamma_1}{\gamma_1 + m_1}B_4 \\ &- 2\alpha\sigma_1^2\delta_1^2\frac{\gamma_1 m_1}{\gamma_1 + m_1}\left[B_4^2 + (\frac{A_4}{\alpha})^2 - 2B_4\frac{A_4}{\alpha}\right] + \frac{\alpha(b_1 - r)^2}{2\sigma_1^2(\gamma_1 + m_1)} = 0, \end{aligned} \tag{A.30}$$

$$\begin{aligned} &B_{4t} - 2b_1\delta_1 B_4 + 2\delta_1(b_1 - r)\frac{2\gamma_1 m_1}{(\gamma_1 + m_1)^2}\frac{A_4}{\alpha} + 2\delta_1(b_1 - r)\frac{\gamma_1^2 + m_1^2}{(\gamma_1 + m_1)^2}B_4 \\ &+ 4\sigma_1^2\delta_1^2\frac{\gamma_1 m_1^2}{(\gamma_1 + m_1)^2}\left[B_4^2 + (\frac{A_4}{\alpha})^2 - 2B_4\frac{A_4}{\alpha}\right] + \frac{\gamma_1(b_1 - r)^2}{\sigma_1^2(\gamma_1 + m_1)^2} = 0. \end{aligned} \tag{A.31}$$

Denote $I_1(t) := A_4(t) + \frac{\alpha(\gamma_1+m_1)}{2m_1}B_4(t)$, hence $I_{1t} = A_{4t} + \frac{\alpha(\gamma_1+m_1)}{2m_1}B_{4t}$ and $I_1(T) = 0$. Combining Eqs (A.31) and (A.30), we obtain the following equation

$$I_1t - 2\delta_1 r I_1 + \frac{\alpha(b_1 - r)^2}{2m_1\sigma_1^2} = 0. \tag{A.32}$$

Solving Eq (A.32) with $I_1(T) = 0$, we obtain

$$I_1(t) = \frac{\alpha(b_1 - r)^2}{4m_1\delta_1 r\sigma_1^2}\left[1 - e^{-2\delta_1 r(T-t)}\right]. \tag{A.33}$$

Plugging $A_4 = I_1 - \frac{\alpha(r_1+m_1)}{2m_1}B_4$ into Eq (A.31) implies

$$\begin{aligned} &B_{4t} + [2\delta_1(b_1 - r)\frac{m_1(m_1 - \gamma_1)}{(m_1 + \gamma_1)^2} - 2b_1\delta_1 - \delta_1(b_1 - r)^2\frac{(\gamma_1 + 3m_1)\gamma_1}{(\gamma_1 + m_1)^2 r}(1 - e^{-2\delta_1 r(T-t)})]B_4 \\ &+ \sigma_1^2\delta_1^2\gamma_1(\frac{\gamma_1 + 3m_1}{\gamma_1 + m_1})^2 B_4^2 + \frac{(b_1 - r)^2\gamma_1}{(\gamma_1 + m_1)^2\sigma_1^2}\left[\frac{b_1 - r}{2r}(1 - e^{-2\delta_1 r(T-t)}) + 1\right]^2 = 0. \end{aligned} \tag{A.34}$$

Let

$$\begin{aligned} k_1 &= \sigma_1^2\delta_1^2 r_1(\frac{\gamma_1 + 3m_1}{\gamma_1 + m_1})^2, \\ k_2 &= 2\delta_1(b_1 - r)\frac{m_1(m_1 - \gamma_1)}{(m_1 + \gamma_1)^2} - 2b_1\delta_1 - \delta_1(b_1 - r)^2\frac{(\gamma_1 + 3m_1)\gamma_1}{(\gamma_1 + m_1)^2 r}(1 - e^{-2\delta_1 r(T-t)}), \\ k_3 &= \frac{(b_1 - r)^2 r_1}{(\gamma_1 + m_1)^2\sigma_1^2}\left[\frac{b_1 - r}{2r}(1 - e^{-2\delta_1 r(T-t)}) + 1\right]^2. \end{aligned}$$

Then, the Eq (A.34) can be written as

$$B_{4t} + k_1 B_4^2 + k_2 B_4 + k_3 = 0. \tag{A.35}$$

This is a regular Riccati equation satisfying $k_2^2-4k_1k_3>0$, and the solution of the Eq (A.35) with the boundary condition $B_4(T)=0$ is given by

$$B_4(t)=M_1+\frac{e^{tN_1}}{\frac{k_1}{N_1}(e^{tN_1}-e^{TN_1})-\frac{1}{M_1}e^{TN_1}}, \tag{A.36}$$

where

$$N_1=\sqrt{k_2^2-4k_1k_3},\quad M_1=\frac{-k_2-N_1}{2k_1}.$$

Plugging Eq (A.36) into $A_4(t)=I_1(t)-\frac{\alpha(r_1+m_1)}{2m_1}B_4(t)$, we obtain

$$A_4(t)=I_1(t)-\frac{\alpha(r_1+m_1)}{2m_1}\left(M_1+\frac{e^{tN_1}}{\frac{k_1}{N_1}(e^{tN_1}-e^{TN_1})-\frac{1}{M_1}e^{TN_1}}\right). \tag{A.37}$$

Substituting the solutions (A.29) into (A.16) and (A.20), we have

$$\begin{aligned}&A_{5t}-2b_2\delta_2A_5+2\delta_2(b_2-r)\frac{m_2}{\gamma_2+m_2}A_5+2(1-\alpha)\delta_2(b_2-r)\frac{\gamma_2}{\gamma_2+m_2}C_5\\&-2(1-\alpha)\delta_2^2\sigma_2^2\frac{\gamma_2m_2}{\gamma_2+m_2}\left[C_5^2+(\frac{A_5}{1-\alpha})^2-2C_5(\frac{A_5}{1-\alpha})\right]+\frac{(1-\alpha)(b_2-r)^2}{2(\gamma_2+m_2)\sigma_2^2}=0,\end{aligned} \tag{A.38}$$

$$\begin{aligned}&C_{5t}-2b_2\delta_2C_5+2\delta_2(b_2-r)\frac{2\gamma_2m_2}{(\gamma_2+m_2)^2}\frac{A_5}{1-\alpha}+2\delta_2(b_2-r)\frac{\gamma_2^2+m_2^2}{(\gamma_2+m_2)^2}C_5\\&+4\sigma_2^2\delta_2^2\frac{\gamma_2m_2^2}{(\gamma_2+m_2)^2}\left[C_5^2+(\frac{A_5}{1-\alpha})^2-2C_5\frac{A_5}{1-\alpha}\right]+\frac{\gamma_2(b_2-r)^2}{\sigma_2^2(\gamma_2+m_2)^2}=0.\end{aligned} \tag{A.39}$$

Referring to the procedure used to solve for A_4, B_5, we can derive the Riccati equation for C_5 as follows

$$C_{5t}+l_1C_5^2+l_2C_5+l_3=0, \tag{A.40}$$

where

$$\begin{aligned}l_1&=\sigma_2^2\delta_2^2\gamma_2(\frac{\gamma_2+3m_2}{\gamma_2+m_2})^2,\\ l_2&=2\delta_2(b_2-r)\frac{m_2(m_2-\gamma_2)}{(m_2+\gamma_2)^2}-2b_2\delta_2-\delta_2(b_2-r)^2\frac{(\gamma_2+3m_2)\gamma_2}{(\gamma_2+m_2)^2r}(1-e^{-2\delta_2r(T-t)}),\\ l_3&=\frac{(b_2-r)^2\gamma_2}{(\gamma_2+m_2)^2\sigma_2^2}[\frac{b_2-r}{2r}(1-e^{-2\delta_2r(T-t)})+1]^2.\end{aligned}$$

This Riccati equation satisfies $l_2^2-4l_1l_3>0$. Using standard methods, we can obtain the solution of the Eq (A.40) with $C_5(T)=0$ as

$$C_5(t)=M_2+\frac{e^{tN_2}}{\frac{l_1}{N_2}(e^{tN_2}-e^{TN_2})-\frac{1}{M_2}e^{TN_2}}, \tag{A.41}$$

where

$$N_2 = \sqrt{l_2^2 - 4l_1l_3}, \quad M_2 = \frac{-l_2 - N_2}{2l_1}.$$

Correspondingly, we have

$$A_5(t) = I_2(t) - \frac{(1-\alpha)(\gamma_2 + m_2)}{2m_2}\left(M_2 + \frac{e^{tN_2}}{\frac{l_1}{N_2}\left(e^{tN_2} - e^{TN_2}\right) - \frac{1}{M_2}e^{TN_2}}\right), \tag{A.42}$$

where

$$I_2(t) = \frac{(1-\alpha)(b_2 - r)^2}{4m_2\delta_2 r\sigma_2^2}\left[1 - e^{-2\delta_2 r(T-t)}\right].$$

By plugging the aforementioned results into Eqs (A.10), (A.11) and (A.12), the robust equilibrium strategy as described in Theorem 3.2 can be obtained.

Subsequently, by substituting the aforementioned results into Eqs (A.17), (A.24) and (A.25), and incorporating the boundary conditions $A_6(T) = B_6(T) = C_6(T) = 0$, we can derive the following solutions:

$$\begin{aligned} A_6(t) = &\int_t^T \left[(1+n_1)e^{-\beta s}\mu + (1+n_2)\mu\right] A_1(s)ds + \int_t^T n_2\mu A_2(s) + \beta\mu A_3(s)ds \\ &+ \delta_1\sigma_1^2(2\delta_1 + 1)\int_t^T A_4(s)ds + \delta_2\sigma_2^2(2\delta_2+1)\int_t^T A_5(s)ds - \frac{1}{2}m_0\sigma_0^2\int_t^T (A_2(s) - \beta A_3(s))^2 ds \\ &- \frac{1}{2}\beta^2\sigma_0^2\alpha\gamma_1\int_t^T B_3^2(s)ds - \frac{1}{2}\sigma_0^2(1-\alpha)\gamma_2\int_t^T C_2^2(s)ds + \int_t^T R_0\left[q^*(s)\right]ds, \end{aligned} \tag{A.43}$$

$$\begin{aligned} B_6(t) = &\int_t^T [(1+n_1)e^{-\beta s}\mu - (1+n_2)\mu + n_2\mu q^*(s)]B_1(s)ds + \int_t^T \beta\mu B_3(s) + \sigma_1^2\delta_1(2\delta_1+1)B_4(s)\,ds \\ &+ m_0\sigma_0^2\int_t^T [\beta B_3(s) - q^*(s)B_1(s)]\left[q^*(s)\left(A_1(s) - A_2(s)\right) + A_2(s) - \beta A_3(s)\right]ds, \end{aligned} \tag{A.44}$$

$$\begin{aligned} C_6(t) = &\int_t^T n_2\mu C_2(s)\left(1 - q^*(s)\right)ds + \sigma_2^2\delta_2(2\delta_2+1)\int_t^T C_5(s)ds \\ &- m_0\sigma_0^2\int_t^T C_2(s)(1 - q^*(s))\left[q^*(s)\left(A_1(s) - A_2(s)\right) + A_2(s) - \beta A_3(s)\right]ds. \end{aligned} \tag{A.45}$$

Above all, the proof of Theorem 3.2 is completed.