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Research article

Superstability of the p-power-radical functional equation related to sine function equation

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Abstract: In this paper, we find solutions and investigate the superstability bounded by a function (Găvruta sense) for the p-power-radical functional equation related to sine function equation:

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y)$$

from an approximation of the p-power-radical functional equation:

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = g(x)h(y),$$

where p is a positive odd integer, and f, g and h are complex valued functions on $\mathbb{R}$. Furthermore, the obtained results are extended to Banach algebras.

Keywords: stability; superstability; sine functional equation; p-radical functional equation; p-power-radical functional equation

1. Introduction

The stability problem for the functional equation was conjectured by Ulam [1] in 1940. In the following year, Hyers [2] presented a partial answer for the case of the additive mapping in this problem: If f satisfies $|f(x+y) - f(x) - f(y)| \leq \varepsilon$ for some fixed $\varepsilon > 0$, then there is an additive mapping g satisfying $g(x+y) = g(x) + g(y)$ and $|f(x) - g(x)| \leq \varepsilon$, which is called the

Hyers-Ulam stability.

Baker et al. [3] announced, in 1979, the new concept for the *superstability* as follows: If f satisfies $|f(x+y) - f(x)f(y)| \leq \varepsilon$ for some fixed $\varepsilon > 0$, then either f is bounded or f satisfies the exponential functional equation $f(x+y) = f(x)f(y)$.

Baker [4] showed the superstability of the cosine (also called d'Alembert) functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y). \tag{C}$$

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y) + f(x-y) = 2f(x)g(y), \tag{W}$$

$$f(x+y) + f(x-y) = 2g(x)f(y), \tag{K_{gf}}$$

$$f(x+y) - f(x-y) = 2f(x)f(y), \tag{T}$$

$$f(x+y) - f(x-y) = 2g(x)f(y), \tag{T_{gf}}$$

$$f(x+y) - f(x-y) = 2g(x)g(y), \tag{T_{gg}}$$

$$f(x+y) - f(x-y) = 2g(x)h(y), \tag{T_{gh}}$$

in which (W) is called the Wilson equation, (K_{gf}) is called the Kim's equation and remaining equations are raised in Kim's papers [5–7].

The superstability of the trigonometric (cosine (C), Wilson (W), Kim ((K_{gf}), (T), (T_{gf}) and (T_{gh})) functional equations were founded in Badora [8], Badora and Ger [9], Kannappan and Kim [10], Kim and Dragomir [11], Kim [5–7] and in papers [12–15].

In 1983, Cholewa [16] proved the superstability of the sine functional equation

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = f(x)f(y) \tag{S}$$

under the stability inequality bounded by constant. This was improved to the condition bounded by a function in Badora and Ger [9]. Their results were also further improved later by Kim [17, 18], who obtained the superstability under the assumption that the stability inequality is bounded by a constant or a function for the generalized sine functional equations:

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = f(x)g(y), \tag{S_{fg}}$$

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = g(x)f(y), \tag{S_{gf}}$$

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = g(x)g(y), \tag{S_{gg}}$$

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = g(x)h(y). \tag{S_{gh}}$$

In 2009, Eshaghi Gordji and Parviz [19] introduced the quadratic-radical functional equation

$$f\big(\sqrt{x^2+y^2}\big) = f(x) + f(y). \tag{R}$$

related to the additive mapping and proved its stability.

Recently, Almahalebi et al. [20], Kim [21, 22] obtained the superstability of p-radical functional equations in relation with Wilson (W), Kim ((K_{gf}), (T_{gf}) and (T_{gh})). In the concept of the p-radical, the sine functional equation (S) and (S)-type's equations (S_{fg}), (S_{gf}), (S_{gg}), (S_{gh}) are expressed as follows:

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y), \tag{S^r}$$

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)g(y), \tag{S^r_{fg}}$$

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = g(x)f(y), \tag{S^r_{gf}}$$

$$f\left(\sqrt[p]{\frac{x^2+y^2}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^2-y^2}{2}}\right)^2 = g(x)g(y), \tag{S^r_{gg}}$$

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = g(x)h(y). \tag{S^r_{gh}}$$

In the above, letting $f(x) = F(x^p)$, then F satisfies (S)-type equations. Hence, in this paper, they will be reasonably called the p-power-radical equation. Since the function $f(x) = \sin x^p$ is the solution of the equation (S^r), it will be called the p-power-radical sine functional equation.

Our aim of this paper is to find solutions and to investigate the superstability bounded by a function (Găvruta sense) for the p-power-radical sine functional equation (S^r) from an approximation of the p-power-radical functional equation (S^r_{gh}).

As a corollary, we obtain the superstability bounded by a constant and a function for the p-power-radical sine functional equation (S^r) from an approximation of the p-power-radical functional equations (S^r), (S^r_{gf}), (S^r_{fg}), (S^r_{gg}). Moreover, the obtained results are extended to Banach algebras.

In this paper, let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{C}$ be the field of complex numbers. We assume that f, g, h are nonzero functions, ε is a nonnegative real number, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a given nonnegative function and p is a positive odd integer.

2. Solutions of the functional equations

Let's recall the trigonometric formula, the p-power-radical functional equation's forms for the functional equations (cosine (d'Alembert) (C), Wilson (W), Kim (T_{gf}), (T_{gg}) and (T_{gh}) are the following:

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x)f(y), \tag{C^r}$$

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x)g(y), \tag{W^r}$$

$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = 2g(x)f(y), \tag{T^r_{gf}}$$

$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = 2g(x)g(y), \tag{T^r_{gg}}$$

$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=2g(x)h(y). \qquad (T^r_{gh})$$

We can confirm that each equation has a solution as follows: $(C^r): f(x)=\cos(x^p)$, $(W^r): f(x)=\sin(x^p)$, $g(x)=\cos(x^p)$, $(T^r_{gf}): f(x)=sin(x^p)$, $g(x)=\cos(x^p)$, $(T^r_{gg}): f(x)=\cos(x^p)$, $g(x)=i\sin(x^p)$, $(T^r_{gh}): f(x)=\cos(x^p)$, $g(x)=\sin(x^p)$, $h(x)=-\sin(x^p)$.

In addition, the solution of each equation can also be found in perspective of the hyperbolic function, exponential function and p-power function, simultaneously.

Letting p=1 in the above paragraph, we know that each original equation ((C), (W), (T_{gf}), (T_{gg}), (T_{gh})) has the corresponding solution of the same form, respectively. They also are represented by the hyperbolic function, exponential function and p-power function, simultaneously.

Now let's consider the functional equations generated by the product of the above equations, then we obtain the target equations: p-power-radical functional equation (S^r) and (S^r)-type's equations (S^r_{fg}), (S^r_{gf}), (S^r_{gg}) and (S^r_{gh}).

1) (S^r) has a solution as the p-power function $f(x)=x^p$:

$$f\left(\sqrt[p]{\frac{x+y}{2}}\right)^2-f\left(\sqrt[p]{\frac{x-y}{2}}\right)^2=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2=xy=f(\sqrt[p]{x})f(\sqrt[p]{y}).$$

2) When (S^r) has a solution as the sine function, it also has simultaneously as an exponential solution as follows:

$$\left(\frac{1}{2i}e^{i\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^p}-\frac{1}{2i}e^{-i\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^p}\right)^2-\left(\frac{1}{2i}e^{i\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^p}-\frac{1}{2i}e^{-i\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^p}\right)^2$$

$$=f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2=\sin\left(\frac{x^p+y^p}{2}\right)^2-\sin\left(\frac{x^p-y^p}{2}\right)^2=\sin(x^p)\sin(y^p)$$

$$=f(x)f(y)=\left(\frac{e^{ix^p}-e^{-ix^p}}{2i}\right)\left(\frac{e^{iy^p}-e^{-iy^p}}{2i}\right).$$

3) When (S^r) has a solution as the hyperbolic sine function, it also has simultaneously as an exponential solution as follows:

$$\left(\frac{1}{2}e^{\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^p}-\frac{1}{2}e^{-\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^p}\right)^2-\left(\frac{1}{2}e^{\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^p}-\frac{1}{2}e^{-\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^p}\right)^2$$

$$=f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2=\sinh\left(\frac{x^p+y^p}{2}\right)^2-\sinh\left(\frac{x^p-y^p}{2}\right)^2$$

$$=\sinh(x^p)\sinh(y^p)=f(x)f(y)=\left(\frac{e^{x^p}-e^{-x^p}}{2}\right)\left(\frac{e^{y^p}-e^{-y^p}}{2}\right).$$

Although the mentioned all functional equations may have arisen from sine or cosine, as shown in the previous, they have solutions as the p-power, the hyperbolic and the exponential function, simultaneously. Hence, they can be considered as the p-power-radical, the p-power-radical exponential and the p-power-radical hyperbolic functional equation, simultaneously.

Letting p = 1 in the above items 1), 2) and 3), then (S^r) arrives (S). Hence, based on the above reasons, the Eq (S) well-known as the sine function equation can also be called as the p-power, the exponential and the hyperbolic functional equation, simultaneously.

In the following lemma, we find the forms of solutions of the p-power-radical functional equations (S^r_{gh}), (S^r_{gg}), (S^r_{fg}).

Lemma 1. *If $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy (S^r_{gh}), then, as one of the solutions of (S^r_{gh}), f, g, h have the forms $f(x) = \cos(x^p)$, $g(x) = \sin(x^p)$ and $h(x) = -\sin(x^p)$ for all $x \in \mathbb{R}$.*

Proof. For all $x, y \in \mathbb{R}$,

$$\begin{aligned}&\left(\cos\frac{x^p+y^p}{2}\right)^2-\left(\cos\frac{x^p-y^p}{2}\right)^2\\ &\quad= f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = g(x)h(y)\\ &\quad= -\sin(x^p)\sin(y^p) = \sin(x^p)(-\sin(y^p)).\end{aligned}$$

In the next lemma, let's find an exponential solution for (S^r_{fg}).

Lemma 2. *If $f, g : \mathbb{R} \to \mathbb{C}$ satisfy (S^r_{fg}), then, as the solutions of (S^r_{fg}), f, g have the following two forms*

(i) $f(x) = e^{x^p}$, $g(x) = e^{x^p} - e^{-x^p}$ for all $x \in \mathbb{R}$,

(ii) $f(x) = e^{x^p}$, $g(x) = 2\sinh(x^p)$ for all $x \in \mathbb{R}$.

Proof. For all $x, y \in \mathbb{R}$,

$$\left(e^{\frac{x^p+y^p}{2}}\right)^2-\left(e^{\frac{x^p-y^p}{2}}\right)^2 = f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)g(y) = \begin{cases}(i)\ e^{x^p}\left(e^{y^p}-e^{-y^p}\right)\\ (ii)\ e^{x^p}2\sinh(y^p).\end{cases}$$

In the next lemma, let's find a hyperbolic and trigonometric solution for (S^r_{gg}).

Lemma 3. *If $f, g : \mathbb{R} \to \mathbb{C}$ satisfy (S^r_{gg}), then, as the solutions of (S^r_{gg}), f, g have the following two forms*

(i) $f(x) = \cosh(x^p)$, $g(x) = \sinh(x^p)$,

(ii) $f(x) = \cos(x^p)$, $g(x) = i\sin(x^p)$.

Proof. For all $x, y \in \mathbb{R}$,

$$(i)\left(\cosh\frac{x^p+y^p}{2}\right)^2-\left(\cosh\frac{x^p-y^p}{2}\right)^2 = f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = g(x)g(y) = \sinh x^p \sinh y^p.$$

$$(ii)\left(\cos\frac{x^p+y^p}{2}\right)^2-\left(\cos\frac{x^p-y^p}{2}\right)^2 = f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = g(x)g(y) = i\sin(x^p)\,i\sin(y^p).$$

3. Superstability of (S^r) from (S^r_{gh})

In Section 3, we study the superstability of the p-power-radical sine functional equation (S^r) from an approximation of the p-power-radical functional equation (S^r_{gh}) related to (S).

Theorem 1. *Assume that $f, g, h : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)h(y) \right| \leq \varphi(y) \tag{3.1}$$

for all $x, y \in \mathbb{R}$, where p is a positive odd integer.

Then, either g is bounded or h satisfies (S^r). Moreover, if g satisfies (C^r), then h and g satisfy p-power-radical equation (T^r_{gf}): $= h(\sqrt[p]{x^p + y^p}) - h(\sqrt[p]{x^p - y^p}) = 2g(x)h(y)$.

Proof. By putting $x = \sqrt[p]{2}x$ and $y = \sqrt[p]{2}y$ in (3.1), it is written equivalently as

$$|f\left(\sqrt[p]{x^p + y^p}\right)^2 - f\left(\sqrt[p]{x^p - y^p}\right)^2 - g(\sqrt[p]{2}x)h(\sqrt[p]{2}y)| \leq \varphi(\sqrt[p]{2}y), \quad \forall\, x, y \in \mathbb{R}. \tag{3.2}$$

Assume that g is unbounded. Then, we can choose a sequence $\{x_n\}$ in $\mathbb{R}$ such that

$$0 \neq |g(\sqrt[p]{2}x_n)| \to \infty, \text{ as } n \to \infty. \tag{3.3}$$

Taking $x = x_n$ in (3.2), we get

$$\left| \frac{f\left(\sqrt[p]{x_n^p + y^p}\right)^2 - f\left(\sqrt[p]{x_n^p - y^p}\right)^2}{g(\sqrt[p]{2}x_n)} - h(\sqrt[p]{2}y) \right| \leq \frac{\varphi(\sqrt[p]{2}y)}{|g(\sqrt[p]{2}x_n)|},$$

and by (3.3), we get

$$h(\sqrt[p]{2}y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p + y^p}\right)^2 - f\left(\sqrt[p]{x_n^p - y^p}\right)^2}{g(\sqrt[p]{2}x_n)}. \tag{3.4}$$

Replacing x by $\sqrt[p]{2x_n^p + x^p}$ and $\sqrt[p]{2x_n^p - x^p}$ in (3.1), we have

$$\begin{aligned}
2\varphi(y) \geq & \left| g(\sqrt[p]{2x_n^p + x^p})h(y) - f\left(\sqrt[p]{\frac{2x_n^p + x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{2x_n^p + x^p - y^p}{2}}\right)^2 \right| \\
& + \left| g(\sqrt[p]{2x_n^p - x^p})h(y) - f\left(\sqrt[p]{\frac{2x_n^p - x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{2x_n^p - x^p - y^p}{2}}\right)^2 \right| \\
\geq & \,\Big| \left(g(\sqrt[p]{2x_n^p + x^p}) + g(\sqrt[p]{2x_n^p - x^p})\right) h(y) \\
& - \left(f\left(\sqrt[p]{x_n^p + \frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p + y^p}{2}}\right)^2 \right) \\
& + \left(f\left(\sqrt[p]{x_n^p + \frac{x^p - y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p - y^p}{2}}\right)^2 \right) \Big|
\end{aligned} \tag{3.5}$$

for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Consequently,

$$\frac{2\varphi(y)}{|g(\sqrt[p]{2}x_n)|} \geq \Bigg| \frac{g(\sqrt[p]{2x_n^p + x^p}) + g(\sqrt[p]{2x_n^p - x^p})}{g(\sqrt[p]{2}x_n)} h(y) - \frac{f\left(\sqrt[p]{x_n^p + \frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p+y^p}{2}}\right)^2}{g(\sqrt[p]{2}x_n)} + \frac{f\left(\sqrt[p]{x_n^p + \frac{x^p-y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p-y^p}{2}}\right)^2}{g(\sqrt[p]{2}x_n)} \Bigg| \quad (3.6)$$

for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Taking $n \to \infty$ in (3.6) and using (3.3) and (3.4), we reach a conclusion that, for every $x \in \mathbb{R}$, there exists the limit function

$$L_1(x) := \lim_{n\to\infty} \frac{g(\sqrt[p]{2x_n^p + x^p}) + g(\sqrt[p]{2x_n^p - x^p})}{g(\sqrt[p]{2}x_n)},$$

where $L_1 : \mathbb{R} \to \mathbb{C}$ satisfies the equation as even

$$h(\sqrt[p]{x^p + y^p}) - h(\sqrt[p]{x^p - y^p}) = L_1(x)h(y), \quad \forall x, y \in \mathbb{R}. \quad (3.7)$$

From the definition of L_1, we obtain the equality $L_1(0) = 2$, which, jointly with (3.7), indicates that h is odd. Keeping this in mind, through (3.7), we deduce the equality

$$\begin{aligned} h(\sqrt[p]{x^p + y^p})^2 - h(\sqrt[p]{x^p - y^p})^2 &= [h(\sqrt[p]{x^p + y^p}) + h(\sqrt[p]{x^p - y^p})]L_1(x)h(y) \quad (3.8) \\ &= [h(\sqrt[p]{2x^p + y^p}) + h(\sqrt[p]{2x^p - y^p})]h(y) \\ &= [h(\sqrt[p]{y^p + 2x^p}) - h(\sqrt[p]{y^p - 2x^p})]h(y) \\ &= L_1(y)h(\sqrt[p]{2}x)h(y). \end{aligned}$$

The oddness of h imposes it to vanish at 0. Putting $x = y$ in (3.7), we conclude with the previous result that

$$h(\sqrt[p]{2}y) = h(y)L_1(y). \quad (3.9)$$

The (3.8) by (3.9) arrives to the equation

$$h(\sqrt[p]{x^p + y^p})^2 - h(\sqrt[p]{x^p - y^p})^2 = h(\sqrt[p]{2}x)h(\sqrt[p]{2}y),$$

for all $x, y \in \mathbb{R}$, which, with $\sqrt[p]{2}$-divisibility of $\mathbb{R}$, states conclusively (S^r).

In addition, if g satisfies (C^r) and L_1 forces $2g$, then (3.7) forces that h and g satisfy (T^r_{gf}).

Theorem 2. *Suppose that $f, g, h : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)h(y) \right| \leq \varphi(x), \quad \forall x, y \in \mathbb{R}, \quad (3.10)$$

which satisfies one of two cases $g(0) = 0$, $f(x)^2 = f(-x)^2$, *where* p *is a positive odd integer.*

Then, either h *is bounded or* g *satisfies* (S^r). *In addition, if* h *satisfies* (C^r), *then* g *and* h *satisfy the* p*-power-radical Wilson type equation* (W^r): $= g\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)h(y)$.

Proof. Let h be unbounded, then we can select a sequence $\{y_n\}$ in $\mathbb{R}$ such that $h(\sqrt[p]{2}y_n)| \to \infty$ as $n \to \infty$. With a minor change of the steps shown in the start part of the proof in Theorem 1, we can get

$$g(\sqrt[p]{2}x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right)^2 - f\left(\sqrt[p]{x^p - y_n^p}\right)^2}{h(\sqrt[p]{2}y_n)}. \tag{3.11}$$

Replacing y by $\sqrt[p]{y^p + 2y_n^p}$ and $\sqrt[p]{-y^p + 2y_n^p}$ in (3.10), the same procedure of (3.5) and (3.6) allows, with (3.11), use to argue the existence of a limit function

$$L_2(y) := \lim_{n\to\infty} \frac{h(\sqrt[p]{y^p + 2y_n^p}) + h(\sqrt[p]{-y^p + 2y_n^p})}{h(\sqrt[p]{2}y_n)},$$

where $L_2 : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$g(\sqrt[p]{x^p + y^p}) + g(\sqrt[p]{x^p - y^p}) = g(x)L_2(y), \quad \forall x, y \in \mathbb{R}. \tag{3.12}$$

Hence, from the definition of L_2, L_2 is even and $L_2(0) = 2$.

Let's start with the case $g(0) = 0$. Then it leads to the conclusion, by (3.12), that g is odd.

Putting $y = x$ in (3.12), we obtain

$$g(\sqrt[p]{2}x) = g(x)L_2(x), \quad \forall x, \in \mathbb{R}. \tag{3.13}$$

From (3.12), the oddness of g and (3.13), we obtain the equation

$$\begin{aligned} g(\sqrt[p]{x^p + y^p})^2 - g(\sqrt[p]{x^p - y^p})^2 &= g(x)L_2(y)[g(\sqrt[p]{x^p + y^p}) - g(\sqrt[p]{x^p - y^p})] \\ &= g(x)[g(\sqrt[p]{x^p + 2y^p}) - g(\sqrt[p]{x^p - 2y^p})] \\ &= g(x)[g(\sqrt[p]{2y^p + x^p}) + g(\sqrt[p]{2y^p - x^p})] \\ &= g(x)g(\sqrt[p]{2}y)L_2(x) \\ &= g(\sqrt[p]{2}x)g(\sqrt[p]{2}y) \end{aligned}$$

for all $x, y \in \mathbb{R}$, which, with $\sqrt[p]{2}$-divisibility of $\mathbb{R}$, states conclusively (S^r).

Second, let's consider the case $f(x)^2 = f(-x)^2$. in this case, it is sufficient to show that $g(0) = 0$.

Suppose that it is not the case. Then, without loss of generality, we may assume in following that $g(0) = c$ (constant).

Taking $x = 0$ in (3.10), from the above assumption, we get the inequality

$$|h(y)| \le \frac{\varphi(0)}{c}, \quad \forall\, y \in \mathbb{R}.$$

The above inequality indicates that h is globally bounded, which is a contradiction due to the assumption of unboundedness. Therefore the claimed $g(0) = 0$ holds, and the proof of the theorem is completed.

In addition, if h satisfies (C^r) and L_2 forces $2h$, then (3.12) forces that g and h satisfy (W^r): $= g(\sqrt[p]{x^p + y^p}) + g(\sqrt[p]{x^p - y^p}) = 2g(x)h(y)$.

The following corollary follows from Theorems 1 and 2, immediately.

Corollary 1. *Assume that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)h(y)\right| \leq \min\{\varphi(x), \varphi(y)\}, \quad \forall x, y \in \mathbb{R}, \tag{3.14}$$

where p is a positive odd integer.

Then

(i) either g is bounded or h satisfies (S^r). Moreover, if g satisfies (C^r), then h and g satisfy (T^r_{gf}): $= h(\sqrt[p]{x^p + y^p}) - h(\sqrt[p]{x^p - y^p}) = g(x)h(y)$.

(ii) either h is bounded, or g satisfies (S^r) under $g(0) = 0$ or $f(x)^2 = f(-x)^2$. Moreover, if h satisfies (C^r), then g and h satisfy the Wilson equation (W^r): $= g(\sqrt[p]{x^p + y^p}) + g(\sqrt[p]{x^p - y^p}) = 2g(x)h(y)$.

4. Application to the superstability of the Eqs (S^r), (S^r_{gg}), (S^r_{gf}) and (S^r_{fg})

In this section, as corollaries, we obtain the superstability of the p-power-radical sine functional equation (S^r) from an approximation of (S^r), and (S^r_{gf}), (S^r_{fg}) and (S^r_{gg}). Their proofs follow from Theorems 1 and 2, and Corollary 1.

4.1. Superstability of the Eq (S_{gg})

Corollary 2. *Assume that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)g(y)\right| \leq \begin{cases} (i)\ \varphi(y) \\ (ii)\ \varphi(x) \\ (iii)\ \min\{\varphi(x), \varphi(y)\} \end{cases} \quad \forall x, y \in \mathbb{R},$$

where p is a positive odd integer.

Then, either g is bounded or g satisfies (S^r), respectively. In particular, the case (ii) holds under the condition $g(0) = 0$ or $f(x)^2 = f(-x)^2$.

4.2. Superstability of the Eq (S_{gf})

Corollary 3. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)f(y)\right| \leq \varphi(y) \tag{4.1}$$

for all $x, y \in \mathbb{R}$, where p is a positive odd integer.

Then, either g is bounded or f satisfy (S^r). Moreover, if g satisfies (C^r), f and g satisfy (T^r_{gf}): $= f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)f(y)$.

Corollary 4. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)f(y)\right| \leq \varphi(x), \quad \forall x, y \in \mathbb{R} \tag{4.2}$$

which satisfies one of the cases $g(0) = 0$, $f(x)^2 = f(-x)^2$, where p is a positive odd integer.

Then, either f is bounded or g satisfies (S^r). Additionally, if f satisfies (C^r), g and f satisfy (W^r): $= g\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)f(y)$.

Corollary 5. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)f(y)\right| \leq \min\{\varphi(x), \varphi(y)\} \tag{4.3}$$

for all $x, y \in \mathbb{R}$, where p is a positive odd integer.

Then

(i) either g is bounded or f and g satisfy (S^r), respectively. Additionally, if g satisfies (C^r), then f and g satisfy (T^r_{gf}): $= f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)f(y)$;

(ii) either f is bounded or g satisfies (S^r). Additionally, if f satisfies (C^r), then g and f satisfy the Wilson equation (W^r): $= g\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)f(y)$.

Proof. It is sufficient to present that either g is bounded or g satisfies (S). The other cases follow from Corollaries 3 and 4, immediately.

The inequality (4.3) can also be presented equivalently as

$$|f\left(\sqrt[p]{x^p + y^p}\right)^2 - f\left(\sqrt[p]{x^p - y^p}\right)^2 - g(\sqrt[p]{2}x)f(\sqrt[p]{2}y)| \leq \min\{\varphi(\sqrt[p]{2}x), \varphi(\sqrt[p]{2}y)\}, \quad \forall\, x, y \in \mathbb{R}. \tag{4.4}$$

First, if f is bounded, then $y_0 \in \mathbb{R}$ can be chosen such that $f(\sqrt[p]{2}y_0) \neq 0$. From this y_0 and (4.4), we get

$$\begin{aligned}
|g(\sqrt[p]{2}x)| - \left|\frac{f\left(\sqrt[p]{x^p + y_0^p}\right)^2 - f\left(\sqrt[p]{x^p - y_0^p}\right)^2}{f(\sqrt[p]{2}y_0)}\right| \\
\leq \left|\frac{f\left(\sqrt[p]{x^p + y_0^p}\right)^2 - f\left(\sqrt[p]{x^p - y_0^p}\right)^2}{f(\sqrt[p]{2}y_0)} - g(\sqrt[p]{2}x)\right| \\
\leq \frac{\min\{\varphi(\sqrt[p]{2}x), \varphi(\sqrt[p]{2}y_0)\}}{f(\sqrt[p]{2}y_0)} \leq \frac{\varphi(\sqrt[p]{2}y_0)}{f(\sqrt[p]{2}y_0)}.
\end{aligned}$$

Thus, it implies that g is also bounded on $\mathbb{R}$. Namely, since an unboundedness of g exacts it of f, let run along the step of Theorem 2.

The process of Theorem 2 gives us the limit (3.11), which, since f satisfies (S^r) by Theorem 1, validates

$$g(\sqrt[p]{2x}) = f(\sqrt[p]{2x}), \qquad \forall x \in \mathbb{R}.$$

By the $\sqrt[p]{2}$-divisibility of $\mathbb{R}$, we obtain $g = f$. Thus, it is true that g also satisfies (S^r).

4.3. Superstability of the Eq (S_{fg})

Corollary 6. *Assume that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - f(\sqrt[p]{x})g(\sqrt[p]{y}) \right| \le \varphi(y), \tag{4.5}$$

where p is a positive odd integer.

Then, either f is bounded or g satisfies (S^r). Additionally, if f satisfies (C^r), then g and f satisfy (T^r_{gf}): $= g\left(\sqrt[p]{x^p + y^p}\right) - g\left(\sqrt[p]{x^p - y^p}\right) = 2f(x)g(y)$.

Corollary 7. *Assume that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - f(\sqrt[p]{x})g(\sqrt[p]{y}) \right| \le \varphi(x),$$

where p is a positive odd integer.

Then, either g is bounded or f satisfies (S^r) under one condition of the cases $f(0) = 0$, $f(x)^2 = f(-x)^2$. In addition, if g satisfies (C^r), then f and g satisfy the Wilson equation (W^r): $= f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = f(x)g(y)$.

Corollary 8. *Assume that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - f(x)g(y) \right| \le \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in \mathbb{R}$, where p is a positive odd integer.

Then

(i) either f is bounded or g satisfies (S^r). In addition, if f satisfies (C^r), then g and f satisfy (T^r_{gf}): $= g\left(\sqrt[p]{x^p + y^p}\right) - g\left(\sqrt[p]{x^p - y^p}\right) = 2f(x)g(y)$;

(ii) either g is bounded or f and g satisfy (S^r), respectively, under one condition of the cases $f(0) = 0$, $f(x)^2 = f(-x)^2$. In addition, if g satisfies (C^r), then f and g satisfy the Wilson equation (W^r): $= f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = 2f(x)g(y)$.

4.4. Superstability of the p-power-radical sine functional equation (S^r)

As a corollary for all the obtained results, we obtain the superstability of the p-power-radical sine functional equation (S^r).

Corollary 9. *Assume that $f : \mathbb{R} \to \mathbb{C}$ satisfies the inequality*

$$\left| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - f(x)f(y) \right| \le \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

where p is a positive odd integer.

Then, either f is bounded or f satisfies (S^r)

Proof. Replacing the functions g and h in Theorems 1 and 2 by f, in the case (ii), the assumption $f(0) = 0$ or $f(x)^2 = f(-x)^2$ can be eliminated (see [9, Theorem 5]).

Remark 1. *(i) Applying $\varphi(x) = \varphi(y) = \varepsilon$ for all results in Sections 3 and 4, then they yield the superstability results bounded by constant (Hyers-sense).*

(ii) Applying '$p = 1$' to all the p-power-radical functional equations (S^r), (S^r_{gf}), (S^r_{fg}), (S^r_{gg}), (S^r_{gh}) in Sections 3 and 4, then they yield the superstability results for all (S)-type functional equations: (S), (S_{gf}), (S_{fg}), (S_{gg}), (S_{gh}).

In addition, for all results of the (S)-types obtained above, applying again (i) $\varphi(x) = \varphi(y) = \varepsilon$, then they yield the additional results (Hyers-sense) for (S)-types.

(iii) Many results obtained for the (S)-types and p-power-radical (S^r)-types in (i) and (ii) are found in Cholewa [16], Badora [8], Badora and Ger [9], Kannappan and Kim [10], Kim and Dragomir [11], Kim [17, 18, 22] and in papers [8, 9, 13–15].

5. Extension of the stability to Banach algebras

All results in Sections 3 and 4 can be expanded to the stability on Banach algebras. The following theorem is based on Theorems 1 and 2, and Corollary 1. The remainder results also are represented as similar type as Theorem 3, respectively, their proofs will skip for the sake of brevity.

Theorem 3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : \mathbb{R} \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)h(y) \right\| \leq \begin{cases} (i)\ \varphi(y), \\ (ii)\ \varphi(x), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

where p is a positive odd integer.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition $x^* \circ g$ is bounded or h satisfies (S^r), In addition, if g satisfies (C^r), then h and g satisfy (T^r_{gf}):= $h\left(\sqrt[p]{x^p + y^p}\right) - h\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)h(y)$;*

(ii) *either the superposition $x^* \circ h$ under the cases $g(0) = 0$ or $f(x)^2 = f(-x)^2$ is bounded or g satisfies (S^r). In addition, if h satisfies (C^r), then g and h satisfy the Wilson equation (W^r):= $g\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)h(y)$;*

(iii) *the above* (i) *and* (ii) *hold. In addition, if the superposition $x^* \circ g$ is unbounded, then g satisfies (S^r)*

Proof. (i) Assume that (i) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known, we have $\|x^*\| = 1$, whence, for every $x, y \in \mathbb{R}$, we have

$$\begin{aligned} \varphi(y) &\geq \left\| g(x)h(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(g(x)h(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2\right) \right| \end{aligned}$$

$$\geq \left| x^*(g(x)) \cdot x^*(h(y)) - x^*\left(f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)\right) + x^*\left(f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)\right)\right|,$$

which states that the superpositions $x^* \circ g$ and $x^* \circ h$ produce a solution of stability inequality (3.1) of Theorem 1. Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 1 forces that the function $x^* \circ h$ is a solution of (S^r), that is,

$$(x^* \circ h)\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - (x^* \circ h)\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 = (x^* \circ h)(x)(x^* \circ h)(y). \tag{5.1}$$

In other presents, by the linear multiplicativity of x^*, for all $x, y \in \mathbb{R}$, the difference $\mathcal{D}S^r : \mathbb{R}\times\mathbb{R} \to E$ defined by

$$\mathcal{D}S^r(x, y) := h\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - h\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - h(x)h(y)$$

falls into the kernel of x^*. Thus, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}S^r(x, y) \in \bigcap\{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in \mathbb{R}$. Since the space E is a semisimple, $\bigcap\{\ker x^* : x^* \in E^*\} = 0$, which means that h satisfies the claimed Eq (S^r).

In addition, if g satisfies (C^r), then it is trivial that h and g satisfy $h\left(\sqrt[p]{x^p + y^p}\right) - h\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)h(y)$.

(ii) By assumption, the superposition $z^* \circ h$ with $g(0) = 0$ or $f(x)^2 = f(-x)^2$ is unbounded, an appeal to Theorem 2 shows that the results hold.

The superposition $x^* \circ g$ satisfies (5.1), that is a solution of the Eq (S^r).

As in (i), a linear multiplicativity of x^* and semisimplicity imply

$$g\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)g(y) \in \bigcap\{\ker x^* : x^* \in E^*\} = 0,$$

which means that g satisfies (S^r). In addition, if h satisfies (C^r), then it is trivial that g and h satisfy $g\left(\sqrt[p]{x^p + y^p}\right) + g\left(\sqrt[p]{x^p - y^p}\right) = 2g(x)h(y)$.

(iii) It follows from the above (i) and (ii), and the additional case of (iii) holds by Corollary 1.

Corollary 10. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \longrightarrow E$ satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)g(y)\right\| \leq \begin{cases} (i)\ \varphi(x), \\ (ii)\ \varphi(y), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

where p is a positive odd integer.

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ g$ is bounded or g satisfies (S^r). In particular, the case (ii) holds under the condition $g(0) = 0$ or $f(x)^2 = f(-x)^2$.

Corollary 11. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f, g : \mathbb{R} \longrightarrow E$ *satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)f(y) \right\| \le \begin{cases} (i)\ \varphi(x), \\ (ii)\ \varphi(y), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

where p is a positive odd integer.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition* $x^* \circ g$ *is bounded or* f *satisfies* (S^r)*, In addition, if* g *satisfies* (C^r)*, then* f *and* g *satisfy* (T^r_{gf})*;*

(ii) *either the superposition* $x^* \circ f$ *under the cases* $g(0) = 0$ *or* $f(x)^2 = f(-x)^2$ *is bounded or* g *satisfies* (S^r)*. In addition, if* f *satisfies* (C^r)*, then* g *and* f *satisfy the Wilson equation* (W^r)*;*

(iii) *the above* (i) *and* (ii) *hold. Also, additionally, if the superposition* $x^* \circ g$ *is unbounded, then* g *satisfies* (S^r)

Corollary 12. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f, g : \mathbb{R} \longrightarrow E$ *satisfy the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - f(x)g(y) \right\| \le \begin{cases} (i)\ \varphi(x), \\ (ii)\ \varphi(y), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

where p is a positive odd integer.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *either the superposition* $x^* \circ f$ *is bounded or* g *satisfy* (S^r)*. In addition, if* f *satisfies* (C^r)*, then* g *and* f *satisfy* (T^r_{gf})*;*

(ii) *either the superposition* $x^* \circ g$ *under the cases* $g(0) = 0$ *or* $f(x)^2 = f(-x)^2$ *is bounded or* f *satisfies* (S^r)*. In addition, if* g *satisfies* (C^r)*, then* f *and* g *satisfy* (W^r)*;*

(iii) *the above* (i) *and* (ii) *hold. Also, additionally, if the superposition* $x^* \circ g$ *is unbounded, then* g *satisfies* (S^r)*,*

Corollary 13. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach algebra. Assume that* $f : \mathbb{R} \longrightarrow E$ *satisfies the inequality*

$$\left\| f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - f(x)f(y) \right\| \le \begin{cases} (i)\ \varphi(x), \\ (ii)\ \varphi(y), \\ (iii)\ \min\{\varphi(x), \varphi(y)\}, \end{cases}$$

where p is a positive odd integer.

For an arbitrary linear multiplicative functional $x^* \in E^*$*, either the superposition* $x^* \circ f$ *is bounded or* f *satisfies* (S^r)*.*

Remark 2. *Follow (i) and (ii) of Remark 1 for all results in Section 5, namely,*

(i) Apply $\varphi(x) = \varphi(y) = \varepsilon$ *in all results.*

(ii) Apply '$p = 1$*' in all results. Next, apply (i) again in the results.*

Then, a number of the results are found in the same papers in (iii) of Remark 1.

6. Conclusions

In this paper, we studied solutions and creating of the p-power-radical functional equations arisen simultaneously from the trigonometric functions, hyperbolic function, exponential function and p-radical function.

We investigated the superstability bounded by a function (Găvruta sense) for the p-power-radical sine functional equation (S^r) from an approximation of the p-power-radical functional equations (S^r_{gh}), and (S^r), (S^r_{gf}), (S^r_{fg}), (S^r_{gg}) with p is a positive odd integer. Furthermore, the obtained results extended to Banach algebras. As a result, we have improved the previous stability results for (S)-type functional equations: (S), (S_{gf}), (S_{fg}), (S_{gg}), (S_{gh}) to that of the p-power-radical equations: (S^r), (S^r_{gf}), (S^r_{fg}), (S^r_{gg}), (S^r_{gh}).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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