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Research article

Decision making in railway interlocking systems based on calculating the remainder of dividing a polynomial by a set of polynomials

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Abstract: Decision-making in a railway station regarding the compatibility of the positions of the switches of the turnouts and the indications (proceed/stop) of the railway colour light signals is a safety-critical issue that is considered very labor-intensive. Different authors have proposed alternative solutions to automate its supervision, which is performed by the so-called railway interlocking systems. The classic railway interlocking systems are route-based and their compatibility is predetermined (usually by human experts): only some chosen routes are simultaneously allowed. Some modern railway interlocking systems are geographical and make decisions on the fly, but are unsuitable if the station is very large and the number of trains is high. In this paper, we present a completely new algebraic model for decision-making in railway interlocking systems, based on other computer algebra techniques, that bypasses the disadvantages of the approaches mentioned above (its performance does not depend on the number of trains in the railway station and can be used in large railway stations). The main goal of this work is to provide a mathematical solution to the interlocking problems. We prove that our approach solves it in linear time. Although our approach is interesting from a theoretical perspective, it has a significant limitation: it can hardly be adopted in an actual interlocking implementation, mainly due to the heavy certification requirements for this kind of safety-critical application. Nevertheless, the results may be useful for simulations that do not require certification credit.

Keywords: railway interlocking system; computer algebra; decision making; Groebner bases; commutative algebra

1. Introduction

Rail transportation is an important mode of transportation around the world. It is a reliable and efficient way to transport goods and people over long distances. According to a literature review on resilience in railway transport systems, rail transportation is a critical infrastructure that plays a vital role in the economy and society [1, 2].

Railway interlocking is a safety-critical system that ensures trains do not collide on the tracks. It is a complex problem that has been studied extensively in the literature. One source that provides an overview of railway signalling principles is Pachl [3]. Recent research has explored the use of artificial intelligence to detect faults in railway signal interlocking systems [4]. Other researchers have compared different methods for verifying the safety of railway interlockings [5]. There has also been work on developing formal model-based methodologies to support railway engineers in specifying and verifying interlocking systems [6]. Overall, the railway interlocking problem is an important and active area of research with many different approaches being explored.

Ensuring the compatibility of switch positions and signal indications at a railway station is crucial for safety. Interlocking is a safety measure that prevents improper changes to traffic signals and turnout switches. A railway station consists of sections connected by traffic signals and turnouts, which define the possible movement of trains. A route is a sequence of connected sections that a train can travel along partitioned into several blocks. To prevent collisions and dangerous situations, two trains may never be in the same block of a route. Two intersection routes (or the relevant blocks thereof) must be allocated to trains at the same time, to prevent collisions. Once a route is set and a train receives the signal to proceed, all switches and signals along the route are locked until the train has passed through.

A possible classification of computer-based railway interlocking systems could be as follows:

- Route-table based: For every route request, an algorithm checks its feasibility using a "control table". The software consists of a single algorithm, independent of the topology.
- Geographical: The interlocking program is made up of instances of software objects that mimic the behavior of physical objects. The configuration of this program depends on the topology. Within this category, we can distinguish between:
  - Route-based: Routes are defined a priori, and their definitions constitute data shared by the instantiated objects.
  - On-demand route definition: There is no a priori definition of routes. Instead, a train's request to the interlocking system is given only as the final destination that the train must reach. Instantiated software objects are responsible for exploring possible routes to the destination and choosing one of them. Our approach lies in this category.

Classical railway interlocking systems are route-based and the compatibility of routes is decided (usually by human experts) in advance [7] (although unacceptable errors have been found in real railway interlocking systems [8]).

Some modern railway interlocking systems can make decisions on the fly. This makes them more flexible as routes are not predetermined. However, before authorizing any changes to signals or switch positions, it is necessary to check if two trains would travel on intersecting routes (including the same route) and potentially collide. If this is the case, changes are not made and it may be necessary to wait for a train to leave the station.

The first railway interlocking systems were developed in the nineteenth century (they were complex mechanical devices with levers and interacting bars). Typical railway interlocking systems of the mid twentieth century were based on the use of electric relays and needed complicated electric circuits translating the connectivity of the railway station. In the 1980's, the first computer-controlled railway interlocking systems are installed [9–12]. The first geographical railway interlocking system installed in Spain is dated on 1993 [13].

All modern railway interlocking systems are computer based (either geographical or route-based). Naive implementations of geographical algorithms may run into exponential complexity problems concerning the running time needed for finding safe routes through the railway network. Efficient data validation for geographical interlocking systems has also been studied [14]. Another approach to verifying geographically distributed interlocking systems is through model checking using UMC [15]. The approach presented here describes the configuration of the railway station through a single algebraic structure in which safety can be checked by a single algebraic operation. The main goal of our work is mainly theoretical. Nevertheless, it has the limitation that it cannot be certified and applied in practice. However we think that our approach can also be used for simulations that do not require certification credit and it is interesting from a theoretical point of view. Indeed, our approach represents a step on our research on proposing algorithms to solve railway interlocking problems for any railway station using an algebraic model similar to those used in Artificial Intelligence for implementing expert systems based on polynomials, ideals, and Groebner basis [16]. Our proposal offers significant advantages over previous approaches. The motivation for our work is as follows:

- Our approach relates two seemingly different fields: computational algebra and interlocking problems. Indeed the model we propose here proves that an interlocking problem can be solved by applying a division algorithm. This relation is not only interesting and inspiring from a theoretical point of view, but also has certain practical benefits: a possible proposal for improvement in the division algorithm in the field of computational algebra directly results in faster performance in solving interlocking problems.
- By expressing the interlocking problem as an algebraic system similar to those used in expert systems, we can address problems related to expert systems in railway stations. We believe that once the problem is described in algebraic terms, it will be possible to develop expert systems based on these algebraic systems to solve more complex problems in the future. For example, such systems could detect which signals and switches cannot change the state due to safety concerns and provide recommendations for changing the configuration of switches and signals to allow trains to move safely to specific locations.
- Unlike our previously proposed models that used algebraic systems [17–19], the model we propose now has great advantages over the others:
  - It eliminates the need for calculating Groebner bases, a task that is known to be very slow due to its exponential complexity. Indeed, in contrast to our previous approaches, the present model guarantees a linear complexity.
  - As a result, our new model is much faster than our previous models and has been shown to have linear complexity.
  - In contrast to our previous approaches, we use different polynomials to represent the static and dynamic topology of the railway station. The static topology is represented by a set of polynomials while the dynamic topology is represented by a monomial. Another monomial is

used to represent the position of trains. This allows for immediate updates to the polynomials representing the problem when there are changes in train positions or the configuration of the railway station.

The paper is structured as follows. In Section 2 we discuss other approaches related to the one proposed here. In Section 3 we propose a possible formalisation of the concepts related to a railway interlocking system. In Section 4 we describe our method (as a black box) for determining the safety of a proposed situation in a railway interlocking system. In Section 5 the worst case complexity of the new model in calculated. In Section 6 the extension of the model to trains occupying more than one section of the layout is detailed. In Section 7 we show that the new model is much faster than previous ones. Finally, in Section 8 we set our conclusions.

2. Related works

An overview of different existing approaches to decision making in a railway interlocking system is given afterwards.

The paths (lists of consecutive sections) along the railway station are called *routes*. An example can be a route from an entrance of a station to a given track beside a certain platform. Establishing a route requires setting the turnouts and colour light signals along the route appropriately.

Many approaches have been applied to decision making in a railway interlocking system. Although out of date, [20] includes an interesting annotated bibliography.

In the classic approaches, the routes of the railway interlocking system are predefined. Moreover, the compatibility of routes is determined in advance (traditionally by human experts). They are usually denoted *tabular railway interlocking systems.*

The railway interlocking system is usually denoted *geographical* if the problem is translated into computational notation and decisions are made on the fly.

Geographical approaches do not depend on the track layout of the railway station. An inference engine extracts knowledge in a rule-based expert system and is independent of the given rules and facts stated as true. Similarly, a geographical railway interlocking system can decide upon the safety of any proposed situation in any given railway station.

The [21] uses a theorem prover implemented in a higher-order logic to decide on the safety of a situation. This work is revisited using an annotated logic program with temporal reasoning in [22].

Meanwhile, [7] uses ordered binary decision diagrams to model railway interlocking systems.

A specific work for the Slovak National Railways is [23], that uses Z notation.

Another specific work, in this case for the Danish State Railways is [24], that uses the Vienna Development Method (VDM).

The [25] presents an early but sophisticated formal model that can deal with complex topologies including reversing loops and reversing triangles. It uses Petri nets and graphs and is implemented in *Objective-C* and *PROLOG*.

Another model (component-based) is used in the interesting approach [26–28]. The railway station is abstracted as a set of connected components.

Another example is [29], where CAD, RailML, and logic programming are used. It is applied to a Norwegian railways station.

A very interesting approach that also uses RailML to describe the topology of the railway station and UML class diagrams and is applied to a Dutch station is [30].

We will describe afterwards some models for decision making in railway interlocking systems designed and implemented by the authors. All these models are topologically-independent and do not have restrictions on the direction of the trains.

- Model based on the use of graphs [31]: Our approach is based on graph theory and Boolean matrices (adjacency matrices). This approach is slow and can only be applied to small railways stations because, although the matrices are sparse, their number of rows and columns is the number of sections of the network.
- Algebraic model [17, 32]: Here, our approach is based on algebraic terms which requires calculating a Groebner basis [33, 34] of a polynomial ideal. Since the algorithm for calculating this Groebner basis involves a long time, this approach is not suitable for large stations.
- Model based on Boolean propositional logic [35]: This approach is based on propositional logic: sections are represented by Boolean propositional variables and connectivity between sections are represented by a Boolean propositional formula. The safety of a railway station is detected by solving a SAT problem for each section in the railway station. All techniques, like DPLL [36], to solve a SAT problem involve exponential complexity. However, the high number of SAT problems required in this approach makes it impossible for large stations.
- Logic-algebraic model [18]: The advantage with respect to the algebraic model mentioned above is that in this case, the polynomial ring where computations take place is Boolean and consequently, Groebner bases and normal forms calculations required are much faster. In fact, this model is much faster than the three previous ones. However, the algorithm for calculating a Groebner basis involves high complexity computations and therefore, this approach is not suitable for large stations.
- ASP model [37]: This approach is based on the answer set programming (ASP) paradigm. Connectivity and safety relations are defined by relations and derived relations through logic programming. This approach is further more efficient than previous ones and may be suitable for larger railway stations. However, this is not a polynomial complexity approach and consequently, it is not scalable.
- Model based on preprocess and Boolean Polynomials: An algebraic approach that represents the proposed configuration of the railway station and the position of the trains using Boolean polynomials and Groebner bases is detailed in [19]. The set of polynomials for which the Groebner basis is computed does not depend on the position of the trains. A new Groebner basis has to be computed if any changes in the aspect of the colour light signals or the position switches take place. Although faster than previous approaches, it can still be slow if the railway station is really large.

As may be seen, many of these approaches are based on translating the topology of the railway station into polynomials with several variables and require calculating Groebner bases since they are usually required to solve many algebraic problems involving polynomials with several variables. Unfortunately, calculating a Groebner basis requires a very long time, and consequently, all these approaches are not suitable for very large railway stations with many trains since they need a long time to make a decision upon the safety of the proposed situation.

In this paper, we propose a completely new algebraic approach that does not require computing

Groebner bases and is much faster than our previous ones. In our approach, the computational complexity of deciding upon the safety of a proposed situation is linear with respect to the number of sections and trains in the railway station (see Section 5). Therefore, it is completely suitable for very large railway stations (even if there are many trains involved).

3. A formalisation of the safety of a proposed situation in a railway station

In this section we will define some formal concepts that allow us to prove the validity of our approach.

A railway station is a set of sections $\{S_1 \dots S_N\}$ and a binary relation defining the connection between sections by means of colour light signals and turnouts. There may be trains placed in the sections of a railway station. In this paper, we will initially consider that each train is placed in just one section. Nevertheless, we will generalize our results for the general case that a train can be placed in different sections in Section 6.

A colour light signal is defined as (S_i, S_j), a pair of connected sections. If the indication of the colour light signal is *proceed*, then a train may pass from section S_i to section S_j. If the indication of the colour light signal is *stop*, then trains are not allowed to pass from section S_i to section S_j.

A turnout is defined as (S_i, S_j, S_k), an array of three connected sections. If the switch of the turnout is in the *direct track position*, then a train may pass from section S_i to section S_j (and conversely). If the switch of the turnout is in the *diverted track position*, then a train may pass from section S_i to S_k (and conversely).

The potential connectivity of a railway station derives from the situation of the colour light signals and the situation of the turnouts (that is, from the so called *topology* of the station). We define the relation E for describing the potential connectivity between sections. Formally, E is a set of pair of sections (i, j) indicating that the section S_i is connected to section S_j (by means of a colour light signal and/or a turnout).

Definition 3.1. *We define the set $E \subset \mathbb{Z} \times \mathbb{Z}$ as:*

$$E = \{(i, j) | S_i \textit{ is connected to } S_j \textit{ or } S_j \textit{ is connected to } S_i \\ \textit{by means of a colour light signal or a turnout}\}$$

Consequently, the relation E is symmetric.

Remark 3.1. *In the case that there are a series of n turnouts, we assume the presence of intermediate sections between them. Consequently, in a railway station, each section is connected to a maximum of three other sections on each extreme through a turnout. This means that each section can be connected to a maximum of six other sections. Therefore, the size of E is less than or equal to $6 \cdot N$ where N is the number of sections. As a result, the number of elements in E, is $O(N)$.*

The indications of the colour light signals and the position of the switches of the turnouts (that is, the configuration of the railway station) define if a section is reachable from another. A configuration of the railway station is defined by a subset $P \subseteq E$. $(S_i, S_j) \in P$ if and only if it is possible to pass from the section S_i to the section S_j according to the indications of the colour light signals and the position of the switches of the turnouts (obviously a necessary condition is that section S_i is connected to section S_j).

Remark 3.2. *In this model of railway interlocking systems we assume that if the switch of the turnout* (S_i, S_j, S_k) *is in the direct track position, a train is not allowed to pass from* S_k *to* S_i*. Similarly, if the switch of the turnout* (S_i, S_j, S_k) *is in the diverted track position, a train should not try to pass from* S_j *to* S_i*. We will consider that turnouts are always properly protected by colour light signals.*

Remark 3.3. *Two sections are connected by means of either a colour light signal or a turnout, but not both. If the latter was the case, we introduce a section in between (like* S_9 *between* S_{10} *and* (S_2, S_3, S_9) *in Figure 3.2). Similarly, we include sections between two semaphores or turnouts.*

Remark 3.4. *Note that, despite the fact that turnouts can be long apparatuses, they are not considered sections in this model, but connections between sections.*

The set P determines the possible paths of the railway station. A path is a list of sections $[u_1 \dots u_n]$ such that for every i $(u_i, u_{i+1}) \in P$. The set P fulfills the following important property: For every pair of sections i and j, there is at most one path in P from section i to section j.

Remark 3.5. *For the property in the previous paragraph to hold we are considering paths defined by P (not by E). Besides, we are considering in this model only "usual" railway terminus stations or railway overtaking stations (unlike toy trains where one side of the overtaking station is connected to the other side). In these "usual" stations, we have that for every pair of sections i and j, there is at most one path in P from section i to section j.*

Example 3.1. *Let us consider the railway station of Figure 3.1.*

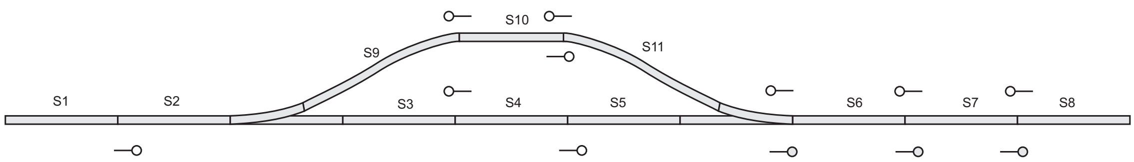

Figure 3.1. Track layout of a very simple railway station.

As may be seen, the railway station is divided into sections $S_1 \dots S_{11}$ *and there are seven colour light signals (for example, there is a colour light signal between sections* S_1 *and* S_2*) and two turnouts (for example, a turnout connecting sections* S_2*,* S_3 *and* S_9*). Note that signalling is on the right hand side of the track.*

According to Definition 3.1, the set E is:

$$E = \{(1,2),(2,9),(9,10),(10,11),(11,6),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8)\\ (2,1),(9,2),(10,9),(11,10),(6,11),(3,2),(4,3),(5,4),(6,5),(7,6),(8,7)\}$$

Note that the set E is defined independently from the specific configuration of the elements of the railway station (from the indications of the colour light signals and the positions of the switches of the turnouts).

Let us consider the configuration depicted in Figure 3.2. The position of the switches of the turnouts is represented by:

- *a small segment, if the switch is in the direct track position (see for instance the turnout between sections* S_2 *and* S_3, S_9 *in Figure 3.2), or*

- *a small angle, if the switch is in the diverted track position (see for instance the turnout between sections S_6 and S_5, S_{11} in Figure 3.2).*

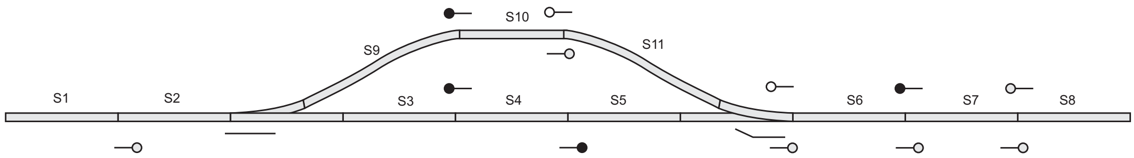

Figure 3.2. A possible configuration of the railway station of Figure 3.1.

In view that the figures are printed in black and white, the aspect of the colour light signals will be represented by a black circle (indication stop*) or white circle (indication* proceed*).*

Now, we will determine the subset $P \subseteq E$ given by the indications of the colour light signals and the position of the switches of the turnouts in Figure 3.2:

$$P = \{(1,2),(9,10),(10,11),(2,3),(3,4),(11,6),(6,7),(7,8),(2,1),\\ (11,10),(5,4),(3,2),(6,11),(8,7)\}$$

As may be seen, $(10,9) \notin P$ because the indication of the colour light signal connecting sections S_{10} and S_9 is stop. *In the same way, $(2,9) \notin P$ because the switch of the turnout (S_2, S_3, S_9) is in direct track position.*

We would like to emphasize here that remark 3.5 holds in Figure 3.2. Although it is apparent that there are sections connected by more than one path, there are no loops once P is defined. That is to say, for every sections i and j there is at most one path in P from i to j.

Let us formally define the set P.

Definition 3.2. *We define the subset $P \subseteq E$ as the set of $(i, j) \in E$ such that exactly one of these conditions holds (with the conditions imposed in this model the conditions are mutually exclusive):*

- *It is always possible to pass from section S_i to section S_j (regardless of the configuration of the railway station).*
  For example, $(2,1) \in P$ in the configuration of Figure 3.2 since it is always possible to pass from section S_2 to section S_1.
- *There is a colour light signal (S_i, S_j) and its indication is* proceed.
  For example, $(1,2) \in P$ in the configuration of Figure 3.2 since the indication of the colour light signal (S_1, S_2) (controlling the pass from section S_1 to section S_2) is proceed.
- *There is a turnout (S_i, S_j, S_k) or a turnout (S_j, S_i, S_k) and its switch is in the direct track position.*
  For example, $(2,3) \in P$ and $(3,2) \in P$ in the configuration of Figure 3.2 since the switch of the turnout (S_2, S_3, S_9) is in the direct track position.
- *There is a turnout (S_i, S_k, S_j) or a turnout (S_j, S_k, S_i) and its switch is in the diverted track position.*
  For example, $(6,11) \in P$ and $(11,6) \in P$ in the configuration of Figure 3.2 since the switch of the turnout (S_6, S_5, S_{11}) is in the diverted track position.

Definition 3.3. *We define the multiset Q as the set of sections in which a train is placed: the number of times that element i appears in Q represents the number of trains located in section S_i.*

We will consider that Q is a multiset instead of a set, since Q will have repeated elements in case that two different trains are placed in the same section (which is dangerous).

Definition 3.4. *Let us consider a railway station and let the relation E describe its potential connectivity (see Definition 3.1). A* railway interlocking problem *is an ordered pair* (P, Q)*, where P is the set of Definition 3.2, describing a certain configuration of the elements of the railway station (indications of the colour light signals and positions of the switches of the turnouts) and Q is the multiset of Definition 3.3, describing the positions of the trains.*

Definition 3.5. *We will say that the railway interlocking problem* (P, Q) *is in a dangerous situation if and only if there are two lists of integers* $[u_1, \dots u_n]$ *and* $[v_1, \dots v_m]$ *such that all the following conditions hold:*

(1) For all $0 < i < n$*, we have that* $(u_i, u_{i+1}) \in P$*. That is to say, consecutive sections in the list* $[u_1, \dots u_n]$ *must be connected.*
(2) For all $0 < j < m$*, we have that* $(v_j, v_{j+1}) \in P$*. That is to say, consecutive sections in the list* $[v_1, \dots v_m]$ *must be connected.*
(3) $\{u_1, v_1\} \subseteq Q$*. That is to say, there must be a train in* u_1 *and* v_1*.*
(4) $u_n = v_m$*. That is to say, both paths reach the same section (there is a possible collision).*
(5) For all $1 < i < n$ $u_i \notin Q$ *and for all* $1 < j < m$ $v_j \notin Q$*. That is to say, intermediate sections in the list must be free of trains so that trains from from section* u_1 *and section* v_1 *may reach section* $u_n = v_m$*.*
(6) For all $1 < i < n$ *and for all* $1 < j < m$ *we have* $u_i \neq v_j$*. That is to say, the possible collision happens at the end of the paths.*
(7) For all $i \neq j$ *we have* $u_i \neq u_j$*. That is to say, the path* $[u_1, \dots u_n]$ *does not contain cycles.*
(8) For all $i \neq j$ *we have* $v_i \neq v_j$*. That is to say, the path* $[v_1, \dots v_m]$ *does not contain cycles.*

We will say that the railway interlocking problem (P, Q) *is in a safe situation if and only if* (P, Q) *is not in a dangerous situation.*

Example 3.2. *Let us recall the railway station of Figure 3.1 and its possible configuration shown in Figure 3.2. Figure 3.3 depicts the placement of one train in section* S_1 *and another train in section* S_{10}*. Therefore, in this case:* $Q = \{1, 10\}$

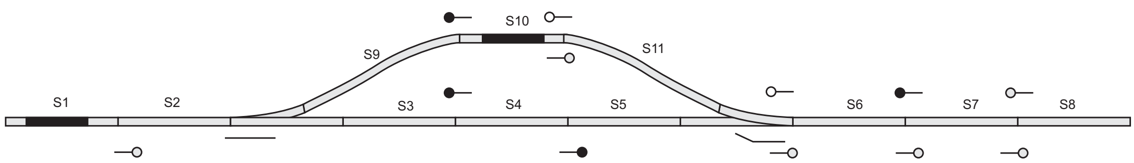

Figure 3.3. A possible placement of trains in the configuration of the railway station of Figure 3.1 shown in Figure 3.2.

Clearly, this situation is safe:

- *the train located in section* S_1 *could stay in section* S_1 *or move to sections* S_1, S_2, S_3, S_4*,*
- *the train located in section* S_{10} *could stay in section* S_{10} *or move to sections* S_{11}, S_6, S_7, S_8*.*

Example 3.3. *However, if a new train were allocated in section S_8, that is to say $Q = \{1, 8, 10\}$ (Figure 3.4), the situation would turn into a dangerous one: the trains situated in sections S_{10} and S_8 could collide in sections S_7 and S_8. Following the formalization of Definition 3.5:*

- *lists* $[10, 11, 6, 7]$ *and* $[8, 7]$ *fulfil the conditions Definition 3.5 (representing a possible collision in section S_7),*
- *lists* $[10, 11, 6, 7, 8]$ *and* $[8]$ *fulfil the conditions Definition 3.5 (representing a possible collision in section S_8).*

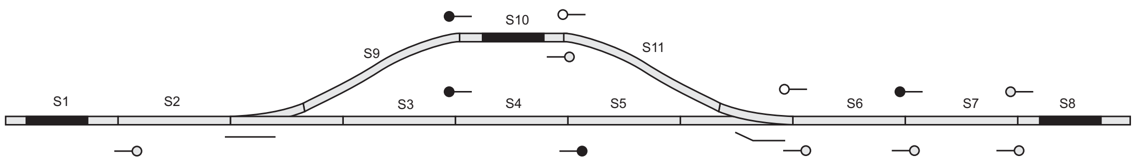

Figure 3.4. Another possible placement of trains in the configuration of the railway station of Figure 3.1. The proposed situation is dangerous.

Example 3.4. *In case the aspect of the colour light signal between section S_{10} and section S_{11} were changed, now indicating* stop*, the situation would be safe again (Figure 3.5).*

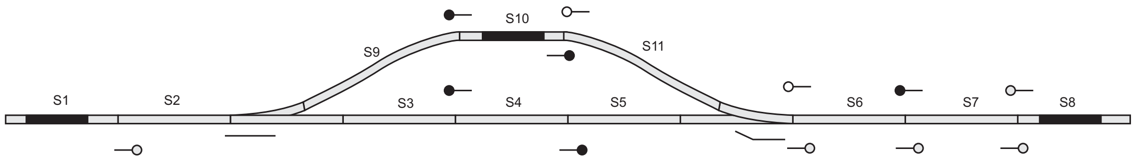

Figure 3.5. Another possible configuration of the railway station of Figure 3.1 with the placement of trains proposed in Figure 3.4. The proposed situation is safe.

4. Overview of the new approach

In this section we will extensively describe our algebraic approach based on the use of polynomials. As we will see, the potential connectivity of a railway station is represented by a set of polynomials with coefficients in the field $\mathbb{Z}_2$. A specific configuration of the railway station is represented by a polynomial with coefficients in the same field. The same happens with a specific placement of trains in the railway station. In order to detect whether a proposed situation is dangerous or not, we only need to check if the remainder of dividing a polynomial by a set of polynomials is zero or not.

4.1. The algebraic model

Our approach is based on defining a set of polynomials in the variables t_i, l_{ij} and m_{ij}, where:

- t_i: a variable t_i is considered for each section S_i in the railway station.
- l_{ij}, m_{ij}: two variables, l_{ij} and m_{ij}, are considered for each pair of sections S_i and S_j when $(i, j) \in E$. That is to say, we consider the variables l_{ij} and m_{ij} if the topology of the station allows to pass from section S_i to section S_j (in some configuration of the railway station).

We will work hereinafter in the polynomial ring

$$\mathcal{A} = \mathbb{Z}_2[l_{ij}, \ldots, m_{ij}, \ldots, t_i, \ldots]$$

and we will use the lexicographical order given by $l_{ij} > m_{ij} > t_i$. The set of polynomials representing the potential connectivity of the railway station, the polynomial describing the specific configuration of the railway station and the polynomial describing the specific placement of trains in the railway station are constructed as follows.

The list of polynomials $\mathcal{E}$ representing the railway station. Through the set E (see Definition 3.1, the set of ordered pairs of integer numbers describing the potential connectivity of a railway station, we will define $\mathcal{E}$ as the list of polynomials of $\mathcal{A}$ formed by:

- $\forall(i, j) \in E$, the two polynomials:

$$g_{ij} = l_{ij}l_{ji}t_i + m_{ij}m_{ji}t_it_j$$

$$g'_{ij} = l_{ij}m_{ji}t_i + m_{ij}m_{ji}t_it_j$$

- For each variable t_i:

$$t_i^2$$

As may be seen, the list of polynomials $\mathcal{E}$ depends only on E and, therefore, it is defined independently from the specific configuration of the elements of the railway station (from indications of the colour light signals and the position of the switches of the turnouts).

The polynomial p representing a given configuration of the railway station. For each set of ordered pairs of integers $P \subseteq E$ representing a given configuration of the railway station (see Definition 3.2), we will consider the monomial $p \in \mathcal{A}$:

$$p = \prod_{(i,j)\in P} l_{ij} \prod_{(i,j)\in E-P} m_{ij}$$

As may be seen, this monomial is defined according to the configuration of the railway station in the following way (note that the symbol | represents the relation *is a divisor of* or *divides*, for example, $xz|xyz$ because xz divides xyz):

- If it is always possible to pass from section S_i to section S_j (regardless of the configuration of the railway station), then $l_{ij}|p$.
- If there is a colour light signal between section S_i and section S_j, then:
  - if the indication of the colour light signal is *proceed*, then $l_{ij}|p$.
  - if the indication of the colour light signal is *stop*, then $m_{ij}|p$.
- If there is a turnout connecting section S_i to either section S_j or section S_k then:
  - if the switch is in the straight track position, then $l_{ij}l_{ji}m_{ik}m_{ki}|p$.
  - if the switch is in the diverted track position, then $l_{ik}l_{ki}m_{ij}m_{ji}|p$.

As may be seen, a variable l_{ij} in p represents that it is possible to pass from section S_i to section S_j. In the same way, a variable m_{ij} in p represents that it is not possible to pass from section S_i to section S_j

The polynomial q representing the placement of the trains. Let us consider that there are trains placed in sections $S_{i_1}, \ldots, S_{i_m}$. We define the monomial $q \in \mathcal{A}$:

$$q = \prod_{i \in Q}^{m} t_i$$

As may be seen, a variable t_i in q represents that there is a train in section S_i.

4.2. Decision making in the algebraic model

Here we will summarize the steps required to solve a railway interlocking problem.

Step 1. Given a railway station by the set describing its potential connectivity, E, we obtain the list $\mathcal{E}$. We would like to emphasize that this list $\mathcal{E}$ is calculated only once for every railway interlocking problem associated to E. We do not need to recalculate this set if there are movements of trains or if there are changes in the configuration of the railway station. Besides, this set only contains $2K + N$ polynomials (where K is the size of E and N is the number of sections).

Step 2. Given a configuration of the railway station, P, we calculate the monomial p.
As may be seen, for every change of the configuration in the railway station, we need to calculate p, but this monomial contains K variables where K is $O(N)$.

Step 3. Given a placement of the trains, Q, we need to calculate the monomial q.
As may be seen, for every change of the placement of the trains, we need to calculate q, but this monomial contains at most N variables.

Step 4. In order to solve the (P, Q) railway interlocking problem we need to compute:

$$\mathrm{NR}(p \cdot q, \mathcal{E})$$

where NR represents the remainder of the monomial pq respect to the list $\mathcal{E}$. If this value is 0, the railway interlocking problem is in a dangerous situation, otherwise it is in a safe situation (see Theorem A.11). This may be calculated with any Computer Algebra System (CAS). In this paper we will use the CAS *CoCoA 5.2* [38, 39].

As we will see in Section 7, all these steps can be completed in less than 1 s for very large railway stations using a conventional computer.

4.3. Examples of application of the algebraic method proposed

We will use the CAS *CoCoA* in order to show how to solve some railway interlocking problems for the railway station of Example 3.1 (depicted in Figure 3.1) using the algebraic approach proposed in this paper.

Example 4.1. *Let us recall the potential connectivity of the railway station of Example 3.1. Let us remember that in this case:*

$$\begin{aligned} E = \{&(1,2),(2,9),(9,10),(10,11),(11,6),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8)\\ &(2,1),(9,2),(10,9),(11,10),(6,11),(3,2),(4,3),(5,4),(6,5),(7,6),(8,7)\}\end{aligned}$$

The process starts as follows:

Step 1. We define the polynomial ring and the list $\mathcal{E}$ *corresponding to the topology of this railway station as described in Section 4.1. The variables of the polynomial ring are:*

- *variables:* $t_1 \dots t_{11}$ *(since this railway station contains 11 sections)* *and*
- *variables:* $l_{1,2}, l_{2,9}, l_{9,10}, l_{10,11}, l_{11,6}, l_{2,3}, l_{3,4}, l_{4,5}, l_{5,6}, l_{6,7}, l_{7,8}$, $l_{2,1}, l_{9,2}, l_{10,9}, l_{11,10}, l_{6,11}, l_{3,2}, l_{4,3}, l_{5,4}, l_{6,5}, l_{7,6}, l_{8,7}$, $m_{1,2}, m_{2,9}, m_{9,10}, m_{10,11}, m_{11,6}, m_{2,3}, m_{3,4}, m_{4,5}, m_{5,6}, m_{6,7}, m_{7,8}$, $m_{2,1}, m_{9,2}, m_{10,9}, m_{11,10}, m_{6,11}, m_{3,2}, m_{4,3}, m_{5,4}, m_{6,5}, m_{7,6}, m_{8,7}$

In CoCoA *syntax:*

```
use ZZ/(2)[l1_2, l2_9, l9_10, l10_11, l11_6, l2_3, l3_4, l4_5, l5_6, l6_7,
l7_8, l2_1, l9_2, l10_9, l11_10, l6_11, l3_2, l4_3, l5_4, l6_5, l7_6, l8_7,
m1_2, m2_9, m9_10, m10_11, m11_6, m2_3, m3_4, m4_5, m5_6, m6_7, m7_8, m2_1,
m9_2, m10_9, m11_10, m6_11, m3_2, m4_3, m5_4, m6_5, m7_6, m8_7,t[1..11]],
lex;
```

The polynomials in the list $\mathcal{E}$ *are defined as described in Section 4.2.*

- $l_{1,2}l_{2,1}t_1 + m_{1,2}m_{2,1}t_1t_2$ *(because* $(1,2) \in E$*),*
- $l_{1,2}m_{2,1}t_1 + m_{1,2}m_{2,1}t_1t_2$ *(because* $(1,2) \in E$*),*
- ...

In CoCoA *syntax:*

```
E:=[l1_2*l2_1*t[1]+m1_2*m2_1*t[1]*t[2], l1_2*m2_1*t[1]+m1_2*m2_1*t[1]*t[2],
l2_9*l9_2*t[2]+m2_9*m9_2*t[2]*t[9], l2_9*m9_2*t[2]+m2_9*m9_2*t[2]*t[9],
l9_10*l10_9*t[9]+m9_10*m10_9*t[9]*t[10],
l9_10*m10_9*t[9]+m9_10*m10_9*t[9]*t[10],
l10_11*l11_10*t[10]+m10_11*m11_10*t[10]*t[11],
l10_11*m11_10*t[10]+m10_11*m11_10*t[10]*t[11],
l11_6*l6_11*t[11]+m11_6*m6_11*t[11]*t[6],
l11_6*m6_11*t[11]+m11_6*m6_11*t[11]*t[6],
l2_3*l3_2*t[2]+m2_3*m3_2*t[2]*t[3], l2_3*m3_2*t[2]+m2_3*m3_2*t[2]*t[3],
l3_4*l4_3*t[3]+m3_4*m4_3*t[3]*t[4], l3_4*m4_3*t[3]+m3_4*m4_3*t[3]*t[4],
l4_5*l5_4*t[4]+m4_5*m5_4*t[4]*t[5], l4_5*m5_4*t[4]+m4_5*m5_4*t[4]*t[5],
l5_6*l6_5*t[5]+m5_6*m6_5*t[5]*t[6], l5_6*m6_5*t[5]+m5_6*m6_5*t[5]*t[6],
l6_7*l7_6*t[6]+m6_7*m7_6*t[6]*t[7], l6_7*m7_6*t[6]+m6_7*m7_6*t[6]*t[7],
l7_8*l8_7*t[7]+m7_8*m8_7*t[7]*t[8], l7_8*m8_7*t[7]+m7_8*m8_7*t[7]*t[8],
l2_1*l1_2*t[2]+m2_1*m1_2*t[2]*t[1], l2_1*m1_2*t[2]+m2_1*m1_2*t[2]*t[1],
l9_2*l2_9*t[9]+m9_2*m2_9*t[9]*t[2], l9_2*m2_9*t[9]+m9_2*m2_9*t[9]*t[2],
l10_9*l9_10*t[10]+m10_9*m9_10*t[10]*t[9],
l10_9*m9_10*t[10]+m10_9*m9_10*t[10]*t[9],
l11_10*l10_11*t[11]+m11_10*m10_11*t[11]*t[10],
```

```
l11_10*m10_11*t[11]+m11_10*m10_11*t[11]*t[10],
l6_11*l11_6*t[6]+m6_11*m11_6*t[6]*t[11],
l6_11*m11_6*t[6]+m6_11*m11_6*t[6]*t[11],
l3_2*l2_3*t[3]+m3_2*m2_3*t[3]*t[2], l3_2*m2_3*t[3]+m3_2*m2_3*t[3]*t[2],
l4_3*l3_4*t[4]+m4_3*m3_4*t[4]*t[3], l4_3*m3_4*t[4]+m4_3*m3_4*t[4]*t[3],
l5_4*l4_5*t[5]+m5_4*m4_5*t[5]*t[4], l5_4*m4_5*t[5]+m5_4*m4_5*t[5]*t[4],
l6_5*l5_6*t[6]+m6_5*m5_6*t[6]*t[5], l6_5*m5_6*t[6]+m6_5*m5_6*t[6]*t[5],
l7_6*l6_7*t[7]+m7_6*m6_7*t[7]*t[6], l7_6*m6_7*t[7]+m7_6*m6_7*t[7]*t[6],
l8_7*l7_8*t[8]+m8_7*m7_8*t[8]*t[7], l8_7*m7_8*t[8]+m8_7*m7_8*t[8]*t[7],
t[1]^2, t[2]^2, t[3]^2, t[4]^2, t[5]^2, t[6]^2, t[7]^2, t[8]^2, t[9]^2,
t[10]^2, t[11]^2];
```

Step 2. Let us now consider the configuration of Example 3.1 (Figure 3.2). In this configuration:

$$P = \{(1,2),(9,10),(10,11),(2,3),(3,4),(11,6),(6,7),(7,8),(2,1),\\ (11,10),(5,4),(3,2),(6,11),(8,7)\}$$
$$E - P = \{(2,9),(4,5),(5,6),(9,2),(10,9),(4,3),(6,5),(7,6)\}$$

so we define the monomial p as follows:

$$p = l_{1,2}m_{2,9}l_{9,10}l_{10,11}l_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}l_{7,8}l_{2,1}m_{9,2}m_{10,9}l_{11,10}l_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}l_{8,7}$$

As may be seen, the variable m_{29} appears in the monomial p since it is not possible to pass from section S_2 to section S_9. Similarly, the variable l_{12} appears in p since it is possible to pass from section S_1 to section S_2.

In CoCoA *syntax:*

```
p:=l1_2*m2_9*l9_10*l10_11*l11_6*l2_3*l3_4*m4_5*m5_6*l6_7*l7_8*l2_1*m9_2*
m10_9*l11_10*l6_11*l3_2*m4_3*l5_4*m6_5*m7_6*l8_7;
```

Example 4.2. *Let us now consider the placement of trains of Example 3.2 (Figure 3.3): there are two trains located in sections* 1 *and* 10*, respectively.*

Step 3. In this case: $Q = \{1, 10\}$. Therefore, we define the monomial

$$q = t_1 t_{10}$$

In CoCoA *syntax:*

```
q:= t[1]*t[10];
```

Step 4. In order to solve the railway interlocking problem (P, Q) we need to check whether $\mathrm{NR}(pq, \mathcal{E}) = 0$ *or not. In* CoCoA *syntax:*

```
NR(p*q,E)=0;
```

As the output of CoCoA *is "false", the proposed situation is safe.*

Example 4.3. *Let us now consider the placement of trains of Example 3.3 (Figure 3.4) instead: a new train is placed in section S_8. We only need to perform Steps 3 and 4.*

Step 3. We need to recalculate the polynomial q. Now: $Q = \{1, 10, 8\}$*, and, therefore, we define the monomial*

$$q = t_1 t_{10} t_8$$

In CoCoA *syntax:*

```
q:= t[1]*t[10]*t[8];
```

Step 4. Again, we need to check whether $\mathrm{NR}(pq, \mathcal{E}) = 0$ *or not. In* CoCoA *syntax:*

```
NR(p*q,E)=0;
```

The output of CoCoA *is "true" in this case. Consequently, the proposed situation is dangerous.*

Example 4.4. *If the aspect of the colour light signal between section* S_{10} *and section* S_{11} *were changed, now indicating* stop *(see Figure 3.5 in Example 3.4), we would only need to recompute Steps 2 and 4.*

Step 2. We need to recalculate the polynomial p. Now we have:

$$P = \{(1,2), (9,10), (2,3), (3,4), (11,6), (6,7), (7,8), (2,1), (11,10), (5,4), (3,2), (6,11), (8,7)\}$$
$$E - P = \{(2,9), (10,11), (4,5), (5,6), (9,2), (10,9), (4,3), (6,5), (7,6)\}$$

and therefore:

$$p = l_{1,2}m_{2,9}l_{9,10}m_{10,11}l_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}l_{7,8}l_{2,1}m_{9,2}m_{10,9}l_{11,10}l_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}l_{8,7}$$

($l_{10,11}$ *has been substituted by* $m_{10,11}$*). In* CoCoA *syntax:*

```
p:=l1_2*m2_9*l9_10*m10_11*l11_6*l2_3*l3_4*m4_5*m5_6*l6_7*l7_8*l2_1*m9_2*
m10_9*l11_10*l6_11*l3_2*m4_3*l5_4*m6_5*m7_6*l8_7;
```

Step 4. We need to check if the situation is dangerous. In CoCoA *syntax:*

```
NR(p*q,E)=0;
```

returns "false". Consequently, the proposed situation is safe.

4.4. Intuition of our approach

In this section, we will provide the intuition behind the choice of polynomials and the reason why the division algorithm provides the solution for detecting whether an interlocking problem is in a dangerous situation or not.

As may be seen in the previous section, in order to detect if an interlocking problem (P, Q) is in a dangerous situation we need to check if $\mathrm{NR}(pq, \mathcal{E}) = 0$. Although the definition NR is not simple and requires many technical details, we provide the intuition behind the value $\mathrm{NR}(pq, \mathcal{E})$: the remainder of the polynomial, pq by a list of polynomials $\mathcal{E}$.

In the same way as ordinary division (for polynomials with one variable and natural numbers), the division of a polynomial p (with several variables) by a list of polynomials $G = [f_1 \ldots f_m]$ is $p = \alpha_1 f_1 + \ldots \alpha_m f_m + r$. The value $\mathrm{NR}(p, G)$ just denotes r, the remainder of the division.

The algorithm for calculating $\mathrm{NR}(p, G)$,in the same way as ordinary division, involves calculating intermediate dividends: it sets $r_0 = pq$ and for each step i, it choose a polynomial $f \in G$ and obtain a new intermediate dividend r_{i+1}, fulfilling

$$r_i = f \cdot z_{i+1} + r_{i+1}$$

where z_{i+1} is another polynomial. The new intermediate dividend r_{i+1} is a polynomial 'simpler'* than r_i. The last intermediate dividend r_n is one such that it not possible to obtain a polynomial r_{n+1} simpler than r_n. By definition, the value $\mathrm{NR}(pq, \mathcal{E})$ is the last intermediate dividend r_n. That is to say, $\mathrm{NR}(p, G) = r_n$. In Definition A.6 in Appendix A we give more details of this operator.

In our approach, we calculate $\mathrm{NR}(pq, \mathcal{E})$ where pq represents the interlocking problem (P, Q). As we will see in Appendix A, each intermediate dividend r_i is a polynomial with the form $p_i q_i$ such that r_i represents an interlocking problem (P_i, Q_i). This interlocking problem (P_i, Q_i) is strongly related to (P, Q). Indeed, (P_i, Q_i) is in a dangerous situation if and only if (P, Q) is in a dangerous situation. By means of this property, we can, consequently, detect if (P, Q) is in a dangerous situation by checking if (P_n, Q_n) (the interlocking problem associated to $r_n = \mathrm{NR}(pq, \mathcal{E})$) is in a dangerous situation. Besides, we will prove in Appendix A another important property: the interlocking problem (P_n, Q_n) is in a dangerous situation if and only if $r_n = 0$. By means of these two properties we conclude that (P, Q) is in a dangerous situation if and only if $\mathrm{NR}(pq, \mathcal{E}) = 0$. We will prove all these previous results in Appendix A.

We will illustrate all these ideas by means of an example. We will consider the intermediate dividends of $\mathrm{NR}(pq, \mathcal{E})$ for the interlocking problem in Figure 3.4 and the interlocking problems associated to them in Figure 4.1.

- The interlocking problem A in Figure 4.1 is the same as the interlocking problem in Figure 3.4. The monomial p_A and q_A are the following:
  - $p_A = l_{1,2}m_{2,9}l_{9,10}l_{10,11}l_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}l_{7,8}l_{2,1}m_{9,2}m_{10,9}l_{11,10}l_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}l_{8,7}$
  - $q_A = t_1 t_{10} t_8$
- In the interlocking problem A in Figure 4.1 there is a train located in section s_{10} that can pass from s_{10} to s_{11}. We can use the polynomial $g_{10,11} = l_{10,11}l_{11,10}t_{10} + m_{10,11}m_{11,10}t_{10}t_{11} \in \mathcal{E}$ in the algorithm of division and we have that:

$$p_A q_A = z_1 \cdot g_{10,11} + r_1$$

  where:
  - $z_1 = l_{1,2}m_{2,9}l_{9,10}l_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}l_{7,8}l_{2,1}m_{9,2}m_{10,9}l_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}l_{8,7}t_1t_8$. The monomial z_B is irrelevant for our purpose.
  - $r_1 = p_B q_B$ is the first intermediate dividend, where p_B and q_B are monomials describing the interlocking problem B in Figure 4.1:
    - $p_B = l_{1,2}m_{2,9}l_{9,10}m_{10,11}l_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}l_{7,8}l_{2,1}m_{9,2}m_{10,9}m_{11,10}l_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}l_{8,7}$
    - $q_B = t_1 t_{10} t_8 t_{11}$

*Obviously, in order to provide a formal definition, it is required to consider many mathematical issues: define when r_{i+1} is simpler than r_i (this is done by means of leading terms of the polynomial); analyze if the output takes into account the election of f when several choices are possible. All these mathematical questions are deeply considered in [34]. We will briefly describe them in Section A.

The interlocking problems A and B are closely related: A is in a dangerous situation if and only if B is in dangerous situation.

- In the interlocking problem B in Figure 4.1 there is a train located in section s_8 that can pass from s_8 to s_7. We can use the polynomial $g_{8,7} = l_{8,7}l_{7,8}t_8 + m_{8,7}m_{7,8}t_8t_7 \in \mathcal{E}$ in the algorithm of division and we have that:

$$p_Bq_B = z_2 \cdot g_{8,7} + r_2$$

where:

  - $z_2 = l_{1,2}m_{2,9}l_{9,10}m_{10,11}l_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}l_{2,1}m_{9,2}m_{10,9}m_{11,10}l_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}t_1t_{10}t_{11}$. The monomial z_C is irrelevant for our purpose.
  - $r_2 = p_Cq_C$ is the second intermediate dividend, where p_C and q_C are monomials describing the interlocking problem C in Figure 4.1:
    * $p_C = l_{1,2}m_{2,9}l_{9,10}m_{10,11}l_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}m_{7,8}l_{2,1}m_{9,2}m_{10,9}m_{11,10}l_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}m_{8,7}$
    * $q_C = t_1t_{10}t_8t_{11}t_7$

  In the same way, the interlocking problems B and C are closely related: B is in a dangerous situation if and only if C is in dangerous situation.

- In the interlocking problem C in Figure 4.1 there is a train located in section s_{11} that can pass from s_{11} to s_6. We can use the polynomial $g_{11,6} = l_{11,6}l_{6,11}t_{11} + m_{11,6}m_{6,11}t_{11}t_6 \in \mathcal{E}$ in the algorithm of division and we have that:

$$p_Cq_C = z_3 \cdot g_{11,6} + r_3$$

where:

  - $z_3 = l_{1,2}m_{2,9}l_{9,10}m_{10,11}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}m_{7,8}l_{2,1}m_{9,2}m_{10,9}m_{11,10}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}m_{8,7}t_1t_{10}t_1t_{10}t_8t_7$. The monomial z_D is irrelevant for our purpose.
  - $r_3 = p_Dq_D$ is the third intermediate dividend, where p_D and q_D are monomials describing the interlocking problem D in Figure 4.1:
    * $p_D = l_{1,2}m_{2,9}l_{9,10}m_{10,11}m_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}l_{6,7}m_{7,8}l_{2,1}m_{9,2}m_{10,9}m_{11,10}m_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}m_{8,7}$
    * $q_D = t_1t_{10}t_8t_{11}t_7t_6$

  In the same way, the interlocking problems C and D are closely related: C is in a dangerous situation if and only if D is in dangerous situation.

- In the interlocking problem D in Figure 4.1 there is a train located in section s_7 that can pass from s_7 to s_6. We can use the polynomial $g'_{6,7} = l_{6,7}m_{7,6}t_6 + m_{6,7}m_{7,6}t_7t_6 \in \mathcal{E}$ in the algorithm of division and we have that:

$$p_Dq_D = z_4 \cdot g'_{6,7} + r_4$$

where:

  - $z_4 = l_{1,2}m_{2,9}l_{9,10}m_{10,11}m_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}m_{7,8}l_{2,1}m_{9,2}m_{10,9}m_{11,10}m_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{8,7}t_1t_{10}t_8t_{11}t_7$. The monomial z_E is irrelevant for our purpose.
  - $r_4 = p_Eq_E$ is the fourth intermediate dividend, where p_E and q_E are monomials describing the interlocking problem E in Figure 4.1:
    * $p_E = l_{1,2}m_{2,9}l_{9,10}m_{10,11}m_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}m_{6,7}m_{7,8}l_{2,1}m_{9,2}m_{10,9}m_{11,10}m_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}m_{8,7}$
    * $q_E = t_1t_{10}t_8t_{11}t_7t_6^2$

- In the interlocking problem E in Figure 4.1 there are two trains located in section s_6 (we will say that it as a trivial interlocking problem), and, therefore, the interlocking problem E is in a dangerous situation. Consequently, our original interlocking problem A is also in a dangerous situation. We can use the polynomial $t_6^2 \in \mathcal{E}$ in the algorithm of division and we have that:

$$p_E q_E = z_5 \cdot t_6^2 + r_5$$

where

- $z_5 = l_{1,2}m_{2,9}l_{9,10}m_{10,11}m_{11,6}l_{2,3}l_{3,4}m_{4,5}m_{5,6}m_{6,7}m_{7,8}l_{2,1}m_{9,2}m_{10,9}m_{11,10}m_{6,11}l_{3,2}m_{4,3}l_{5,4}m_{6,5}m_{7,6}m_{8,7}t_1t_{10}t_8t_{11}t_7$. The monomial z_E is irrelevant for our purpose.
- $r_5 = 0$.

We have that r_5 is the last intermediate dividend. Therefore, $\mathrm{NR}(p_A q_A, \mathcal{E}) = r_5 = 0$.

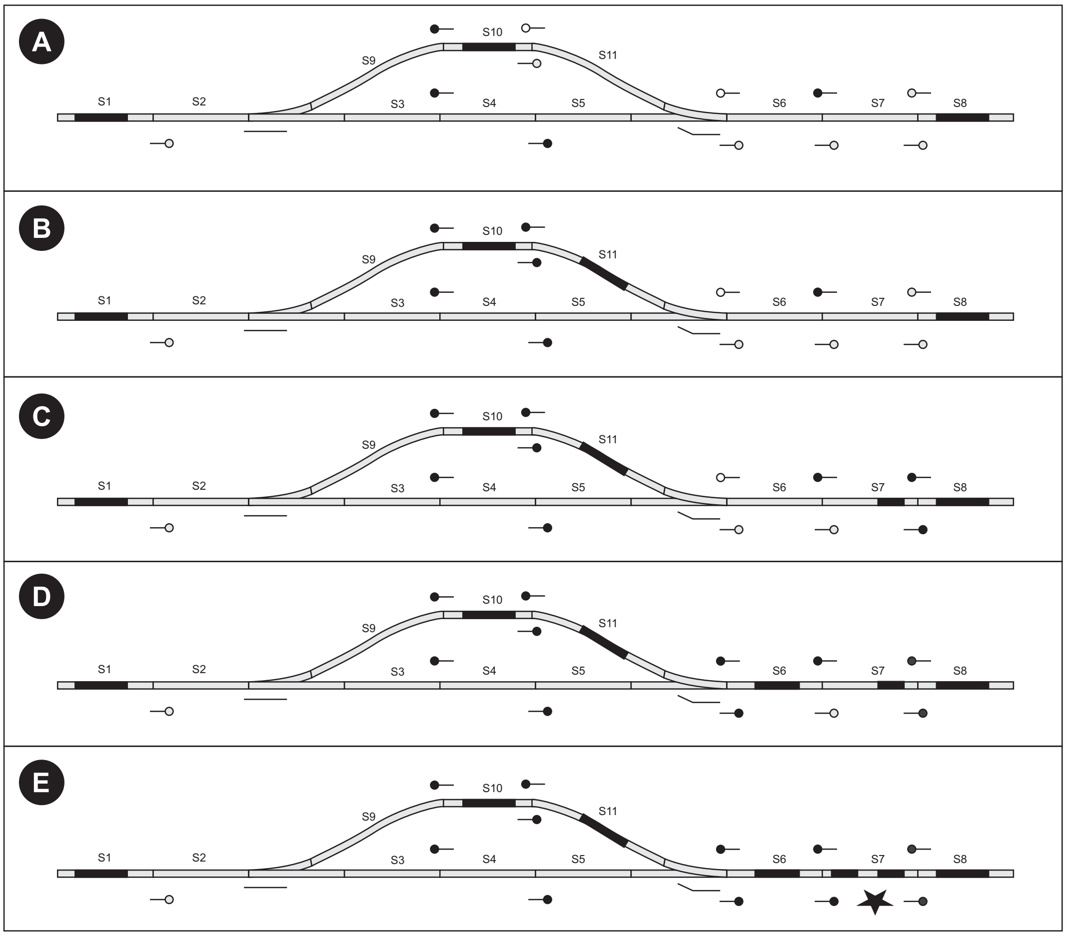

Figure 4.1. Successive interlocking problems associated to intermediate dividends obtained by calculating $\mathrm{NR}(pq, \mathcal{E})$.

5. Worst case computational complexity

In this section we will discuss that the complexity of every step of our approach is linear. We will consider that N is the number of sections in the railway station and K is the size of the set E (see Definition 3.1).

Step 1. Obtaining $\mathcal{E}$. The number of variables of the polynomial ring is $N + 3K$. Since K is $O(N)$, we have that the number of variables is $O(N)$. The number of polynomials in $\mathcal{E}'$ is $N + 2K$. Since K is $O(N)$ (see Remark 3.1), we have that the number of polynomials is $O(N)$. Each polynomials is the sum of two monomials at most. Consequently, the time complexity of this step is $O(N)$.

Step 2. Calculating the polynomial p. Since the number of variables in the polynomial ring is $O(N)$, we have that the time complexity of this step is $O(N)$.

Step 3. Calculating the polynomial q. Since the number of variables in the polynomial ring is $O(N)$, we have that the time complexity of this step is $O(N)$.

Step 4. Checking $\mathrm{NR}(pq, \mathcal{E}) = 0$. The complexity of generic algorithms for calculating the normal remainder of a polynomial pq with respect to a list of polynomials like $\mathcal{E}$ strongly depends on the structure used for representing the polynomials and the kind of polynomials in the list ($\mathcal{E}$ in our case). The polynomials in $\mathcal{E}$ are just sum of two monomials with a constant number of variables and pq is a monomial with a high number of variables. In Appendix B we describe an algorithm with complexity O(N) for calculating this task when pq and the polynomials in E fulfill these characteristics.

6. Model extension for trains occupying more than one section at the same time

Till now, we have analysed the case where trains are placed just in one section. Here we will see that the railway interlocking problems for trains placed in more than one section can be reduced to railway interlocking problems for trains placed in one section.

Let us consider a railway interlocking problem (P, Q). Lets us suppose that a train is placed in sections $\{S_{v_1}, \ldots, S_{v_m}\}$, where $(v_i, v_{i+1}) \in E$, $i \in \{1 \ldots m-1\}$. Clearly, the proposed railway interlocking problem is equivalent to consider that the train is just in S_{v_1} and that it is possible to pass from section S_{v_i} to $S_{v_{i+1}}$ and from section $S_{v_{i+1}}$ to S_{v_i} for every $i \in \{1 \ldots m-1\}$. In other words, if there are colour light signals along the train occupying more than one section, it is considered that the indication of them all is *proceed*, whichever their real indications are. This can be achieved by considering the railway interlocking problem (P', Q) where:

$$\begin{aligned} P' = P &\cup \{(v_i, v_{i+1}) \,|\, (v_i, v_{i+1}) \notin P,\ i \in \{1 \ldots m-1\}\} \cup \\ &\cup \{(v_{i+1}, v_i) \,|\, (v_{i+1}, v_i) \notin P,\ i \in \{1 \ldots m-1\}\} \end{aligned}$$

instead of (P, Q).

Consequently, we can reduce the general case to the specific one in which every train is placed in just one section.

Example 6.1. *We will illustrate this in the example of the railway station with the specific configuration depicted in Figure 3.2 (Example 3.1, Section 3). There we have:*

$$E = \{(1,2),(2,9),(9,10),(10,11),(11,6),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8)$$
$$(2,1),(9,2),(10,9),(11,10),(6,11),(3,2),(4,3),(5,4),(6,5),(7,6),(8,7)\}$$
$$P = \{(1,2),(9,10),(10,11),(2,3),(3,4),(11,6),(6,7),(7,8),(2,1),(11,10),$$
$$(5,4),(3,2),(6,11),(8,7)\}$$

In Figure 6.1, we have a situation in which some trains are placed in more than one section: there is a train placed in sections $[S_1, S_2]$ *and there is a train placed in sections* $[S_3, S_4, S_5]$.

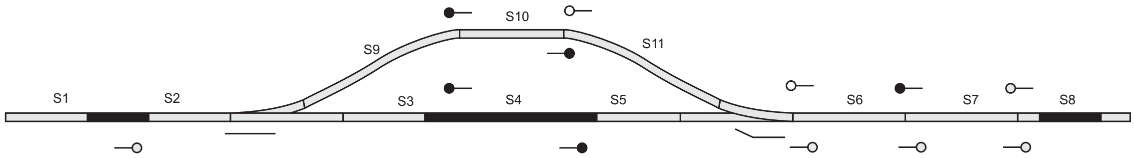

Figure 6.1. A configuration of the railway station.

The general railway interlocking problem of Figure 6.1 is equivalent to the railway interlocking problem (P', Q') *depicted in Figure 6.2, in which all trains are placed in just one section, and where:*

$$P' = P \cup \{(1,2),(2,1),(3,4),(4,3),(4,5),(5,4)\} =$$
$$= \{(1,2),(9,10),(10,11),(2,3),(3,4),(11,6),(6,7),(7,8),$$
$$(2,1),(11,10),(5,4),(3,2),(6,11),(8,7),(4,3),(4,5)\}$$
$$Q' = \{1,3,8\}$$

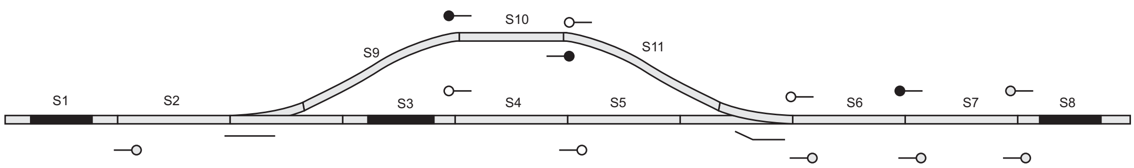

Figure 6.2. A configuration of the railway station.

7. Experimental evaluation

In this section we analyse the performance of the approach presented in this paper, comparing it with other previous algebraic approaches. Indeed, we have made a comparison between the times required to decide upon the safety of a proposed situation in very large stations, with different numbers of sections (N) and trains (M). As may be seen, the new method proposed is always much more efficient than the other approaches proposed by the authors. This is possible because the present approach has worst case linear complexity (see Section 5).

In Table 1, we show the times required to decide upon the safety of different proposed situations (we have not included model in [31] because its performance is very poor bad for large stations). These times refer to the average performance of ten different configurations (both safe and unsafe) of a railway station with N sections and M trains involved.

Indeed, our new approach takes always less than 1 second to decide upon the safety of the proposed situation, even when the number of sections is huge, while the other approaches are much slower. In this case the first column corresponds to the track layout of a former Spanish railway station. The next columns correspond to concatenating that station with itself several times (we have observed that there are no significant changes in the performance of our approach if we add these stations in parallel by means of turnouts).

Time of our approach include the four steps in Section 4.2: it includes the calculation of the polynomials of the polynomials for the topology, configuration and trains.

Table 1. Time comparative of different methods implemented for deciding upon the safety of a situation proposed to a railway interlocking system in certain railway stations with N sections and M trains involved.

	N = 52 M = 5	N = 156 M = 15	N = 260 M = 25	N = 520 M = 50	N = 780 M = 75	N = 1040 M = 100	N = 1560 M = 150
Model described in [35]	10.23 s	250.250 s	> 1 h	$> 1h$	> 1 h	> 1 h	> 1 h
Model described in [17]	15.356 s	380.368 s	> 1 h	$> 1h$	> 1 h	> 1 h	> 1 h
Model described in [18]	1.124 s	38.292 s	554.180 s	> 1 h	> 1 h	> 1 h	> 1 h
Model described in [19]	0.320 s	0.920 s	2.180 s	> 1 h	> 1 h	> 1 h	> 1 h
Model described in [37]	0.020 s	0.047 s	1.038 s	16.458 s	> 1 h	> 1 h	> 1 h
Our approach in *CoCoA*	$< 0.001s$	0.015 s	0.015 s	0.078 s	0.156 s	0.250 s	0.610 s

Madrid-Chamartín-Clara Campoamor is the biggest railway station in Spain. Now it has 21 passing tracks of two gauges (15 of the Iberian gauge, traditionally used in Spain and Portugal, and 6 of the standard gauge, used in the Spanish high-speed lines) [40]. The railway station is under renovation and it will have in the next future 25 passing tracks (13 of Iberian gauge and 12 of standard gauge) [41]. We have considered in Table 2 the track layout of this railway station when all its tracks were of Iberian gauge (in order to consider a real world example as big as possible). This stations contains 250 sections, 81 semaphores, 100 turnouts.

Table 2. Time comparative of different methods implemented for deciding upon the safety of a situation proposed to a railway interlocking system in Madrid Chamartin railway station when all its tracks were Iberian gauge (and there are M trains involved). The models [17, 18, 35] are not used because their timings are too long.

	M = 10	M = 20	M = 30	M = 50	M = 100
Model described in [37]	646 ms	799 ms	1045 ms	1124 ms	1522 ms
Model described in [19]	2340 ms	2340 ms	2340 ms	2340 ms	2340 ms
Our approach	0.14 ms	0.14 ms	0.14 ms	0.14 ms	0.14 ms

8. Conclusions

In this paper we have presented a new algebraic model for detecting dangerous situation in a railway station. According to this model, the position of trains in a railway station configuration is considered unsafe if and only if the remainder of a certain monomial (representing the configuration and the current position of the trains) divided by a list of polynomials vanishes. This connects two seemingly different fields: computer algebra and interlocking problems. Not only is this interesting from a theoretical perspective, but it also offers practical advantages. We have implemented this in CoCoA, resulting in very short program code. In this way, any improvement in the implementation of division algorithms immediately results in faster performance in our program. We have compared execution times with other fast models previously implemented by the authors and found that our approach eliminates the need to calculate a Groebner basis. Besides, our approach can be used as 'accelerated-time simulations', allowing for the analysis of different modifications in a railway network without the need to physically implement them with the enormous cost that it would require [42–45].

However, our approach has a significant limitation. While it is mathematically interesting, Computer Algebra Systems have not been certified for use in safety-critical implementations [46]. Obtaining ad hoc certification is much more costly than certifying a search engine over a table. As a result, our approach currently has no chance of being implemented in real-world systems. Nevertheless, we believe that our approach has value from a mathematical perspective and can be extended in several directions:

- Develop a library that allows for easy definition of the list E for any topology; that defines switch changes and semaphore colors through functions and train movements. Updates to the polynomials p and q through multiplication and division of variables.
- Develop a graphical environment that allows for the visual design of a station and obtains the polynomials in E, p, and q in a computer algebra system like CoCoA.
- Extend our model. Although usual stations do not contain cycles and for each pair of sections there is only one path, it would be interesting from a theoretical point of view to extend the model so that this restriction is not necessary.
- Study alternatives for general graphs, not just those derived from railway stations.
- Extend the interlocking problem to include other problems related to expert systems in railway stations, such as automatically detecting which semaphores and switches cannot be changed because they would imply a dangerous situation. Since our approach expresses the interlocking problem as an algebraic system similar to those used for implementing expert systems [16], we believe that our model can be easily integrated into these expert systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

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This paper is dedicated to the memory of Eugenio Roanes-Lozano. Eugenio passed away suddenly just days after submitting this paper. He was an exceptional researcher in various fields, one of which was his extraordinary passion for trains, with numerous publications on the subject. This paper represents the culmination of his research in this area. Eugenio was a significant figure to us in many ways: as a researcher, teacher, colleague, and most importantly, as a friend. We would like to high-

light his immense humanity. Eugenio, it was not yet time for you to take your last train. You will be deeply missed.

Conflict of interest

The authors declare there are no conflicts of interest.

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A. Proof of our approach

In this section, we will formally prove that the new algebraic approach works.

Distinguishing different kinds of railway interlocking problems

Given the details of a railway station through its potential connectivity, E (see Definition 3.1), we will study the relation between different possible railway interlocking problems for this railway station.

We will start defining when railway interlocking problem is *trivial*.

Definition A.1. *Given the potential connectivity of railway station, E, a configuration in it, P, and a placement of trains, Q, we say that the railway interlocking problem (P, Q) is trivial if and only if Q contains two repeated elements.*

Obviously, a trivial railway interlocking problem is always in a dangerous situation (because there is more than one train at a certain section).

We will define a transitive relation between railway interlocking problems (see Definition A.2). To illustrate the intuition behind this definition, consider Figure A.1. On the right side of the figure, there is a railway interlocking problem A where a train is in Section S_i (i.e., $i \in Q$), and can move from section S_i to section S_j. From this interlocking problem A, we can derive another railway interlocking problem, B, illustrated on the left side of the figure. In this interlocking problem, B, there is another train in section S_j but sections S_i and S_j are now separated: it is no longer possible to move from Section S_i to section S_j or vice versa. We say that the interlocking problem B is derived from A. Problems B and A are closely related. According to Proposition A.1 we can detect if A is in a dangerous situation by analyzing B. That is to say, A is in a dangerous situation if and only if B is in a dangerous situation. As may be seen, the first and the third items of the formal definition (see Definition A.2) extend this idea so that this relation is reflective and transitive. The relation "is derived from" records possible train movements (Q_2 contains both old and new train positions) while changing the configuration of the possible movements in the railway network (P_2 does not correspond to the configuration resulting from the train movement from i to j).

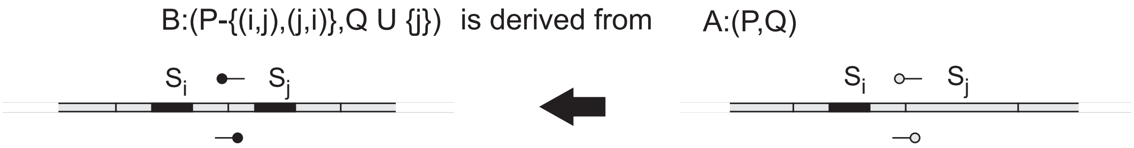

Figure A.1. Illustration of the relation "is derived from".

Definition A.2. *Let* (P_1, Q_1) *and* (P_2, Q_2) *be two possible railway interlocking problems in a railway station detailed by E. We will say that* (P_2, Q_2) *is immediately derived from* (P_1, Q_1) *if and only if* $\exists(i, j) \in P_1$ *such that:*

- $i \in Q_1$
  and
- $P_2 = P_1 - \{(i, j), (j, i)\}$
  and
- $Q_2 = Q_1 \cup \{j\}$

We define the relation "derived from" as the transitive and reflexive closure of the "immediately derived from" relation. That is to say: (P_2, Q_2) *is derived from* (P_1, Q_1) *if and only if*

- $P_1 = P_2$ *and* $Q_1 = Q_2$
- (P_1, Q_1) *is immediately derived from* (P_2, Q_2)
- *there is a railway interlocking problem* (P_3, Q_3) *of E such that* (P_2, Q_2) *is derived from* (P_3, Q_3) *and* (P_3, Q_3) *is derived from* (P_1, Q_1).

(The key idea underlying the second condition is that if there is a train in a section S_i and $(i, j) \in P$, we can consider instead that such train can reach S_j, adding that section to Q_2 and to erase (i, j) and (j, i) from $P2$ –if they belong to $P2$.)

In Figure 4.1 there are more examples of this relation: the interlocking problem B is derived from A, C is derived from B, D is derived from C, E is derived from D.

Proposition A.1. *Let* (P_1, Q_1) *and* (P_2, Q_2) *be two railway interlocking problems in a railway station detailed by E such that* (P_2, Q_2) *is derived from* (P_1, Q_1). *We have that:*

$$(P_1, Q_1) \text{ is in a dangerous situation} \Leftrightarrow (P_2, Q_2) \text{ is in a dangerous situation.}$$

Proof. As (P_2, Q_2) is derived from (P_1, Q_1), from Definition A.2 it follows that we can suppose that $(i, j) \in P_1$, $i \in Q_1$, $P_2 = P_1 - \{(i, j), (j, i)\}$ and $Q_2 = Q_1 \cup \{j\}$.

⇒) (P_1, Q_1) is in a dangerous situation. Let $[u_1, \ldots u_n]$ and $[v_1 \ldots v_m]$ be two paths fulfilling Definition 3.5 for (P_1, Q_1).

We will consider the following cases:

Case $\exists k < n$ **such that** $u_k = j$ **and** $u_{k+1} = i$**.** By (7) in Definition 3.5, we have that for every $1 \leq k' \leq k$ $u_{k'} \neq i$ and, therefore, $(u_{k'}, u_{k'+1}) \notin \{(i, j), (j, i)\}$. Therefore, $[u_1 \ldots u_k]$ and $[j]$ fulfil all conditions in Definition 3.5 for (P_2, Q_2).

Case $\exists k' < m$ **such that** $v_{k'} = j$ **and** $v_{k'+1} = i$**.** It is analogous to the previous case.

Case $\exists k < n$ such that $u_k = i$ and $u_{k+1} = j$. Since $i \in Q_1$, by (5) in Definition 3.5, we have that $u_1 = i$. We will consider the following subcases:

$v_1 = i$. Then $[u_1]$ and $[v_1]$ fulfil all conditions in Definition 3.5 for (P_2, Q_2).

$v_1 \neq i$. Since $u_1 = i$, by (6) in Definition 3.5, we have that for every $1 \leq k' < m$, $v_{k'} \neq i$. Besides, we have that $v_m \neq i$ since $v_m = u_n \neq u_1 = i$, by (4) and (7) in Definition 3.5 (since we are considering that $n > 1$ in this case). Therefore, $(v_{k'}, v_{k'+1}) \notin \{(i, j), (j, i)\}$.
Since $u_1 = i$, by (7) in Definition 3.5 we have that for every $2 \leq k \leq n$ $u_k \neq i$, and therefore, $(u_k, u_{k+1}) \notin \{(i, j), (j, i)\}$.
Consequently, $[u_2, \ldots u_n]$ and $[v_1 \ldots v_m]$ fulfil all conditions in Definition 3.5 for (P_2, Q_2).

Case $\exists k'$ such that $v_{k'} = i$ and $v_{k'+1} = j$. It is analogous to the previous case.

Case for every $1 \leq k < n$ $(u_k, u_{k+1}) \notin \{(i, j), (j, i)\}$ and for every $1 \leq k' < m$ $(v_{k'}, v_{k'+1}) \notin \{(i, j), (j, i)\}$. It is immediate to prove that $[u_1, \ldots u_n]$ and $[v_1 \ldots v_m]$ fulfil all conditions in Definition 3.5 for (P_2, Q_2).

⇐) Let $[u_1 \ldots u_n]$ and $[v_1 \ldots v_m]$ be paths fulfilling Definition 3.5 for the railway interlocking problem (P_2, Q_2). Since $P_2 \subset P_1$, we have that for every $1 \leq k < n$ $(u_k, u_{k+1}) \in P_1$ and that for every $1 \leq k' < m$ $(v_{k'}, v_{k'+1}) \in P_1$. We will consider the following cases:

Case 1. $u_1 = j = v_1$. We have that $[i, j]$ and $[j]$ are paths fulfilling Definition 3.5 for (P_1, Q_1).

Case 2. $u_1 \neq j \neq v_1$. We have that $[u_1 \ldots u_n]$ and $[v_1 \ldots v_m]$ are paths fulfilling Definition 3.5 for (P_1, Q_1).

Case 3. $u_1 = j \neq v_1$. We have that $v_1 \neq i$. Otherwise, we would have that $[v_1, u_1 \ldots, u_n]$ and $[v_1 \ldots v_m]$ would be two different paths from v_1 to $v_m = u_n$ for (P_1, Q_1) and this is excluded (see Remark 3.5). Consequently, $[i, u_1 \ldots u_n]$ and $[v_1 \ldots v_m]$ are paths fulfilling Definition 3.5 for (P_1, Q_1) because:

- $(i, u_1) = (i, j) \in P_1$.
- $\{v_1, i\} \subseteq Q_1$ because $i \in Q_1$, $v_1 \neq i$, and, since $v_1 \neq j$ $v_1 \in Q_1$.

Case 4. $v_1 = j \neq u_1$. It is analogous to the previous case.

Although every trivial railway interlocking problem is in a dangerous situation, not all railway interlocking problems in a dangerous situations are trivial ones. However, a trivial railway interlocking problem can be derived from any railway interlocking problem in a dangerous situation (Theorem A.1).

Theorem A.1. *A railway interlocking problem (P, Q) is in a dangerous situation if and only if a trivial railway interlocking problem is derived from (P, Q).*

Proof. ⇒) Let us suppose that (P, Q) is in a dangerous situation.
Let $[u_1, \ldots, u_n]$ and $[v_1, \ldots, v_m]$ be paths fulfilling Definition 3.5. That is to say, we have that $u_n = v_m$, $(u_i, u_{i+1}) \in P$, $(v_i, v_{i+1}) \in P$, $u_1 \in Q$ and $v_1 \in Q$.

We'll proceed by induction on the sum of the lengths of the two paths, $n + m$.

Case n = m = 1: The paths are $[u_1]$ and $[v_1]$. Since $u_1 = v_1$ we have that this is a trivial railway interlocking problem.

Case $n > 1$ **or** $m > 1$**.** Let us suppose that $n > 1$ (the case $m > 1$ is identical). Since $(u_1, u_2) \in P$ and $u_1 \in Q$, we have that $(P_2, Q_2) = (P - \{(u_1, u_2), (u_2, u_1)\}, Q \cup \{u_2\})$ is derived from (P, Q). Then (P_2, Q_2) is in a dangerous situation because the paths $[u_2, \ldots, u_n]$ and $[v_1, \ldots, v_m]$ fulfil the conditions in Definition 3.5. By induction hypothesis, a trivial railway interlocking problem is derived from (P_2, Q_2), and therefore, this trivial railway interlocking problem is derived from (P, Q).

$\Leftarrow$) Let us suppose that a trivial railway interlocking problem is derived from the railway interlocking problem (P, Q). As a consequence of Definition A.1, a trivial railway interlocking problem is always in a dangerous situation, and therefore, by Proposition A.1, we have that (P, Q) is in a dangerous situation too.

Algebraic Preliminaries. Some introductory notes about polynomial rings

In this section we will briefly describe some results about polynomial rings that we will need to prove the validity of our approach (see, for instance, [34], for a detailed introduction to the topic).

Definition A.3. *A monomial in $x_1, \ldots, x_N$ is a product of the form $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots x_N^{\alpha_N}$. An element in the polynomial ring $\mathbb{K}[x_1, \ldots, x_N]$ is a polynomial in the variables $x_1, \ldots, x_N$ whose coefficients lie in the field $\mathbb{K}$, that is, a finite linear combination of monomials with coefficients in $\mathbb{K}$:*

$$\sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_N)} a_{(\alpha_1,\alpha_2,\ldots,\alpha_N)} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$$

where $a_{(\alpha_1,\alpha_2,\ldots,\alpha_N)} \in \mathbb{K}$ and $\alpha_1, \alpha_2, ..., \alpha_N \in \mathbb{Z}_{\geq 0}$, written sometimes, for the sake of brevity

$$\sum_{\alpha} a_\alpha x^\alpha \ , \ a_\alpha \in \mathbb{K} \ , \ \alpha \in \mathbb{Z}_{\geq 0}^N$$

If $a_{(\alpha_1,\alpha_2,\ldots,\alpha_N)} \neq 0$ then $a_{(\alpha_1,\alpha_2,\ldots,\alpha_N)} x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_N}$ is called a term. If $\mathbb{K} = \mathbb{Q}$, an example of polynomial in $\mathbb{Q}[x_1, x_2, x_3]$ is $p = 2x_1^2 + 7x_1x_2x_3 - x_3 + \frac{1}{2}x_2x_3$.

Later in this paper we will focus on the case $\mathbb{K} = \mathbb{Z}_2$. An element in the polynomial ring $\mathcal{A} = \mathbb{Z}_2[x_1, \ldots, x_N]$ is a polynomial in the variables $x_1, \ldots, x_N$ whose coefficients lie in the field $\mathbb{Z}_2 = \{0, 1\}$. An example of this kind of polynomials is $p = x_1x_2 + x_3 + x_2x_3$. We must take into account that the coefficients lie in $\mathbb{Z}_2$, and therefore, we have that, for instance, (since 1+1=0 in $\mathbb{Z}_2$):

$$x_1x_2 + x_3x_4 + x_3x_4 = x_1x_2 + (1 + 1)x_3x_4 = x_1x_2$$

The algebraic structure *ideal* is key in our approach.

Definition A.4. *An ideal of a commutative ring is a subring that verifies that the product of any element of the ring by any element of the subring belongs to the subring.*

Example A.1. *For instance, the set of even numbers is an ideal of $\mathbb{Z}$. For example, the set of polynomials in $\mathbb{Z}[x]$ with a trailing coefficient equal to 0 (i.e., the multiples of x) form an ideal of $\mathbb{Z}[x]$.*

Proposition A.2. *Given a set of polynomials $G = \{f_1, \ldots, f_m\} \subseteq \mathbb{K}[x_1, \ldots, x_N]$, the smallest ideal containing G, denoted $\langle f_1, \ldots, f_m \rangle$, turns out to be:*

$$\langle f_1, \ldots, f_m \rangle = \{\alpha_1 f_1 + \ldots \alpha_m f_m \,|\, \alpha_1, \ldots, \alpha_m \in \mathbb{K}[x_1, \ldots, x_N]\}$$

Notation A.1. *For the sake of simplicity, if G is a list of polynomials, we will denote* $\langle G \rangle$ *the ideal generated by the polynomials in G.*

Once an order for the variables is given, for instance $x_1 > x_2 > \ldots > x_m$, a total order for the monomials can be defined (see [34], pp. 52–57). Although there are many monomial orderings, we will use the *pure lexicographical order* in this paper. Simplifying it, if we have, for instance, $x < y$ and the monomials $x^2y = xxy$ and $x^3 = xxx$, think of them as if they were words to be ordered in a dictionary: $xxx < xxy$.

Definition A.5. *The leading monomial of a polynomial* $p \in \mathbb{K}[x_1, \ldots, x_N]$, *denoted* $\mathrm{LM}(p)$, *is its greatest monomial with respect to the chosen monomial ordering. The corresponding term is its leading term, and is denoted* $\mathrm{LT}(p)$.

Example A.2. *For example, in* $\mathbb{R}[x_1, x_2, x_3]$ *with* $x_1 > x_2 > x_3$ *and the lexicographic monomial order:*

$$LM(2x_1^2 + 7x_1x_2x_3 - x_3 + \frac{1}{2}x_2x_3) = x_1^2$$

and

$$LT(2x_1^2 + 7x_1x_2x_3 - x_3 + \frac{1}{2}x_2x_3) = 2x_1^2$$

Remark A.1. *In this paper we will focus on the case* $\mathbb{K} = \mathbb{Z}_2$, *and, obviously, in these polynomial rings:* $LT(p) = LM(p)$.

Theorem A.2. *Once the monomial ordering is defined in* $\mathbb{K}[x_1, \ldots, x_N]$, *the division of a polynomial p by an ordered m-list of polynomials* $G = (f_1, \ldots, f_m)$ *can be defined. This division outputs:*

$$p = \alpha_1 f_1 + \ldots + \alpha_m f_m + r$$

where $\alpha_1, \ldots, \alpha_m, r \in \mathbb{K}[x_1, \ldots, x_N]$ *and either* $r = 0$ *or r is a linear combination of monomials, none of which is divisible by any of* $\mathrm{LT}(f_1) \ldots \mathrm{LT}(f_m)$.
A proof can be found in (see [34], pp. 62–64).

Notation A.2. *The polynomial r, remainder of p on division by G, will be denoted* $\mathrm{NR}(p, G)$ *or* $\overline{p}^G$.

The proof in [34] (pp. 62–64) provides an algorithm to find the polynomials $\alpha_1, \ldots \alpha_m, r \in \mathbb{K}[x_1, \ldots, x_N]$ in Theorem A.2. In each step of this algorithm we obtain an intermediate-dividend. We will require only some of the first intermediate-dividends of this algorithm.

Definition A.6. *Given* $p \in \mathbb{K}[x_1, \ldots, x_N]$ *and an ordered m-list of polynomials* $G = (f_1, \ldots, f_m) \subseteq \mathbb{K}[x_1, \ldots, x_N]$, *we recursively define the* first-intermediate-dividends *of p on division by G,* $r_0, \ldots, r_n \in \mathbb{K}[x_1, \ldots, x_N]$, *as follows:*

- $r_0 = p$,
- *if* $i > 0$ *then*
  - *if* $\exists j \in \{1, \ldots, m\}$ *such that* $\mathrm{LT}(f_j) | \mathrm{LT}(r_i)$ *then*

$$r_{i+1} = r_i - \frac{\mathrm{LT}(r_i)}{\mathrm{LT}(f_k)} f_k$$

where k is the minimum j such that $\mathrm{LT}(f_j) | \mathrm{LT}(r_i)$,

– *if* $\forall j \in \{1, \ldots, m\}, \mathrm{LT}(f_j) \nmid \mathrm{LT}(r_i)$ *then* $r_n = r_i$.

According to the proof [34] (pp. 62–64), we have:

Proposition A.3. *Let* $p \in \mathbb{K}[x_1, \ldots, x_N]$ *and let* $G = (f_1, \ldots, f_m) \subseteq \mathbb{K}[x_1, \ldots, x_N]$ *be an ordered m-list of polynomials. Let* $r_0, \ldots r_n \in \mathbb{K}[x_1, \ldots, x_N]$ *be the first-intermediate-dividends of p on division by G. We have that:*

$$r_n = 0 \text{ if and only if } \mathrm{NR}(p, G) = 0$$

Groebner bases are very special ideal bases: among other nice properties, they allow to decide whether a polynomial belongs to the ideal or not (see Theorem A.3). As we wil see, Grobener bases will play an important role in the proof in our main result, Theorem A.11.

Theorem A.3. *Let p be a polynomial in* $\mathbb{K}[x_1, \ldots, x_N]$ *and let G be a Groebner basis for an ideal* $I \subset \mathbb{K}[x_1, \ldots, x_N]$. *Then we have that:*

$$p \in I \Leftrightarrow \mathrm{NR}(p, G) = 0$$

A proof can be found in [34] (Proposition 1 and Corollary 2, pp. 79–80).

Notation A.3. *Let* $p_1, p_2 \in \mathbb{K}[x_1, \ldots, x_N]$. $\mathrm{LCM}(p_1, p_2)$ *denotes least common multiple of* p_1 *and* p_2.

Definition A.7. *Let* $p_1, p_2 \in \mathbb{K}[x_1, \ldots, x_N]$. *The S-polynomial of* p_1 *and* p_2 *is:*

$$S(p_1, p_2) = \frac{\mathrm{LCM}(\mathrm{LM}(p_1), \mathrm{LM}(p_2))}{\mathrm{LT}(p_1)} p_1 - \frac{\mathrm{LCM}(\mathrm{LM}(p_1), \mathrm{LM}(p_2))}{\mathrm{LT}(p_2)} p_2$$

Remark A.2. *Obviously, in case the polynomial ring considered is* $\mathcal{A} = \mathbb{Z}_2[x_1, \ldots, x_N]$, *the* $-$ *in Definition A.7 can be substituted by a* $+$.

Remark A.3. *In case the polynomial ring considered is* $\mathcal{A} = \mathbb{Z}_2[x_1, \ldots, x_N]$, *if* $p \in \mathcal{A}$, $p \neq 0$, *then* $\mathrm{LM}(p) = \mathrm{LT}(p)$.

The following theorem allows to verify if G is a Groebner basis:

Theorem A.4. *Let I be a polynomial ideal in* $\mathbb{K}[x_1, \ldots, x_N]$ *and let* $G = \{f_1, \ldots, f_m\}$ *be a basis of I. Then:*

$$G \text{ is a Groebner basis} \Leftrightarrow \overline{S(f_i, f_j)}^{G^*} = 0$$

(where G^* *is an an ordered m-list of the polynomials in G, ordered in some way).*

A proof can be found in [34] (Theorem 6, pp. 82–85).

The railway interlocking problem in algebraic terms

Given a railway station and the description, E, of its potential connectivity between sections (see Definition 3.1), we will define the list $\mathcal{E}'$ from the list $\mathcal{E}$ (see Section 4.1) as follows:

$$\mathcal{E}' = \mathcal{E} \cup \bigcup_{(i,j) \in E} m_{ij}^2$$

Lemma A.5. *Let E be the description of the potential connectivity between sections of a railway station. Then:*

$$\mathcal{E}' \text{ is a Groebner basis of } \langle \mathcal{E}' \rangle.$$

Proof. It is enough to prove that $\forall f_1, f_2 \in \mathcal{E}'$, $\overline{S(f_1, f_2)}^{\mathcal{E}'} = 0$. But the elements of $\mathcal{E}'$ are of the form g_{ij} or g'_{ij} or t_i^2 or m_{ij}^2, and, although tedious, it can be easily checked by hand or using a CAS, that in the 4^2 possible combinations, the S-polynomial vanishes.

Hereinafter, p and q are the polynomials representing the given configuration of the railway station, P, and the placement of the trains, Q, respectively (see Section 4.1).

Lemma A.6. *Let (P, Q) be a railway interlocking problem for a railway station whose connectivity is described by E. Let $r_0, r_1 \dots r_n$ be the first-intermediate-dividends of pq on division by any chosen ordering of the elements $\mathcal{E}'$. Then:*

i) $\forall i \in \{0, \dots, n-1\}$ *there is a railway interlocking problem (P_i, Q_i) derived from (P, Q) such that $r_i = p_i q_i$.*
ii) If $\mathrm{NR}(pq, \mathcal{E}') = 0$ *the railway interlocking problem (P, Q) is in a dangerous situation.*

Proof.

i) For $i = 0$, we have that $r_0 = pq$ and, therefore, $(P_0, Q_0) = (P, Q)$ is a railway interlocking problem derived from (P, Q).
Suppose that $0 < i < n$, $r_{i-1} = p_{i-1}q_{i-1}$ and (P_{i-1}, Q_{i-1}) is a railway interlocking problem derived from (P, Q). We will prove that r_i represents a railway interlocking problem, (P_i, Q_i), derived from (P, Q). According to Definition A.6, we have that:

$$r_i = r_{i-1} - \frac{\mathrm{LT}(r_{i-1})}{\mathrm{LT}(f_k)} f_k$$

where f_k is a polynomial in $\mathcal{E}'$ such that $\mathrm{LT}(f_k) | \mathrm{LT}(r_{i-1})$. We have also that $\mathrm{LT}(r_{i-1}) = \mathrm{LT}(p_{i-1}q_{i-1}) = p_{i-1}q_{i-1}$, where p_{i-1} is a monomial that contains only variables of the kind l_{xy} and m_{xy}, and q_{i-1} is a monomial that contains only variables of the kind t_x (see Section 4.1).
We have that $f_k \neq t_k^2$. Otherwise, we would have that $r_i = 0$, and therefore, $i = n$ (but we have supposed that $i < n$).
We have that $f_k \neq m_{xy}^2$ because pq is a monomial where the exponent of the m_{xy} variables is at most 1.
Consequently, there are two cases left:

Case $f_k = g_{xy}$. We have that g_{xy} is of the form:
$g_{xy} = l_{xy}l_{yx}t_x + m_{xy}m_{yx}t_xt_y$.
Since $\mathrm{LT}(f_k) | \mathrm{LT}(r_{i-1})$ we have that $\mathrm{LT}(g_{x,y}) = l_{xy}l_{yx}t_x | p_{i-1}q_{i-1}$. Thus, we have that $l_{xy}l_{yx} | p_i$ and $t_x | q_i$ (remember that p_{i-1} is a monomial that contains only variables of the kind l_{xy} and m_{xy}, and q_{i-1} is a monomial that contains only variables of the kind t_x). Consequently:

- p_{i-1} is of the form $p_{i-1} = p' \cdot l_{xy}l_{yx}$ where p' is a monomial.
- q_{i-1} is of the form $q_{i-1} = q' \cdot t_x$ where q' is a monomial.

Therefore, we have that

$$r_i = r_{i-1} - \frac{\mathrm{LT}(r_{i-1})}{\mathrm{LT}(f_k)} f_k = p' \cdot m_{xy} m_{yx} q' \cdot t_x t_y$$

We define:

- $p_i = p' \cdot m_{xy} m_{yx}$
- $q_i = q' \cdot t_x t_y$

Just observe that p_i is a monomial that contains the same variables as p_{i-1} except that $l_{x,y}$ and $l_{y,x}$ in p_{i-1} have respectively turned to be $m_{x,y}$ and $m_{y,x}$. In the same way, q_i is a monomial that contains all the variables in q_{i-1} and also the variable t_y. Consequently, we have that $r_i = p_i q_i$ are polynomials representing the railway interlocking problem (P_i, Q_i) where $P_i = P_{i-1} - \{(x, y), (y, x)\}$ and $Q_i = Q_{i-1} \cup \{y\}$.

Case $f_k = g'_{x,y}$**.** We have that g'_{xy} is of the form:
$g'_{xy} = l_{xy} m_{yx} t_x + m_{xy} m_{yx} t_x t_y$.
Since $\mathrm{LT}(f_k) | \mathrm{LT}(r_{i-1})$ we have that $\mathrm{LT}(g'_{x,y}) = l_{xy} m_{yx} t_x | p_{i-1} q_{i-1}$. Thus, we have that $l_{xy} m_{yx} | p_i$ and $t_x | q_i$ (remember that p_{i-1} is a monomial that contains only variables of the kind l_{xy} and m_{xy}, and q_{i-1} is a monomial that contains only variables of the kind t_x). Consequently:

- p_{i-1} is of the form $p_{i-1} = p' \cdot l_{xy} m_{yx}$ where p' is a monomial
- q_{i-1} is of the form $q_{i-1} = q' \cdot t_x$ where q' is a monomial

Therefore, we have that:

$$r_i = r_{i-1} - \frac{\mathrm{LT}(r_{i-1})}{\mathrm{LT}(f_k)} f_k = p' \cdot m_{xy} m_{yx} q' \cdot t_x t_y$$

We define:

- $p_i = p' \cdot m_{xy} m_{yx}$
- $q_i = q' \cdot t_x t_y$

Just observe that p_i is a monomial that contains the same variables as p_{i-1} except that $l_{x,y}$ in p_{i-1} has turned to be $m_{x,y}$. In the same way, q_i is a monomial that contains all the variables in q_{i-1} and also the variable t_y. Consequently, we have that $r_i = p_i q_i$ is a monomial representing the railway interlocking problem (P_i, Q_i) where $P_i = P_{i-1} - \{(x, y)\} = P_{i-1} - \{(x, y), (y, x)\}$ (just observe that $(y, x) \notin P_{i-1}$) and $Q_i = Q_{i-1} \cup \{y\}$.

Consequently, we have that $r_i = p_i q_i$ represents that the interlocking problem (P_i, Q_i) is derived from (P_{i-1}, Q_{i-1}). By the transitivity property of the derived relation between railway interlocking problems (see Definition A.2), we have that (P_i, Q_i) is also derived from (P, Q).

ii) According to Proposition A.3, we have that $r_n = \mathrm{NR}(pq, \mathcal{E}') = 0$. Consequently,

$$0 = r_{n-1} - \frac{\mathrm{LT}(r_{n-1})}{\mathrm{LT}(f_k)} f_k$$

That is to say, $p_{n-1} q_{n-1} = \frac{p_{n-1} q_{n-1}}{\mathrm{LT}(f_k)} f_k$
Since $p_{n-1} q_{n-1}$ is a monomial, we have that f_k must be also a monomial. Consequently, f_k cannot be either of the form g_{ij} or the form g'_{ij}. Besides, f_k cannot be either of the form m_i^2 since pq is a

monomial where the exponent of the m_{ij} variables is at most 1. Therefore, we have that $f_k = t_x^2$.
Since $\mathrm{LT}(f_k)|\mathrm{LT}(r_{n-1})$, we have that $t_x^2|p_{n-1}q_{n-1}$, and consequently, the interlocking problem (P_{n-1}, Q_{n-1}) is a trivial railway interlocking because Q_{n-1} contains a repeated element x. Therefore,(P_{n-1}, Q_{n-1}) is in a dangerous situation.
By i) we have that (P_{n-1}, Q_{n-1}) is derived from (P, Q). Since (P_{n-1}, Q_{n-1}) is in a dangerous situation, by Theorem A.1, we have that (P, Q) is also in a dangerous situation.

Lemma A.7. *Let (P, Q) be a railway interlocking problem for a railway station described by E. We have that:*

$$\mathrm{NR}(pq, \mathcal{E}) = \mathrm{NR}(pq, \mathcal{E}')$$

Proof. Since the algorithm for calculating $\mathrm{NR}(pq, \mathcal{E}')$ in Lemma A.6 discards the case $f_k = m_{ij}^2$, we have that $\mathrm{NR}(pq, \mathcal{E}) = \mathrm{NR}(pq, \mathcal{E}')$.

According to previous results (see ii in Lemmas A.6 and A.7), we have a sufficient condition to detect if an interlocking problem (P, Q) is in a dangerous situation: if $\mathrm{NR}(pq, \mathcal{E}) = 0$ then the interlocking problem (P, Q) is in a dangerous situation. Now, we will prove that this is also a necessary condition: if the interlocking problem (P, Q) is in a dangerous situation, then $\mathrm{NR}(pq, \mathcal{E}) = 0$. This proof is more complicated and requires translating the problem in terms of a ideal membership problem. An important result for the proof will be that $\mathcal{E}'$ is a Groebner basis.

Lemma A.8. *Let (P_0, Q_0) be a trivial railway interlocking problem for a railway station described by E. We have that:*

$$p_0q_0 \in \langle \mathcal{E}' \rangle$$

Proof. According to Definition A.1, i must appear more than once in Q_0. Let k be the number of times i appears in Q_0. Since $k \geq 2$, we have that $k = m + 2$ where $m \geq 0$. Therefore, we have that q_0 is of the form $q_0 = q' \cdot t_i^k = q' \cdot t_i^{m+2} = q' \cdot t_i^m \cdot t_i^2$ where q' is a monomial. Consequently, $p_0q_0 = p_0 \cdot q' \cdot t_i^m \cdot t_i^2 \in \langle \mathcal{E}' \rangle$ since $t_i^2 \in \mathcal{E}'$.

Lemma A.9. *Let (P_1, Q_1) and (P_2, Q_2) be two railway interlocking problems for a railway station described by E such that (P_2, Q_2) is derived from (P_1, Q_1). We have that:*

$$p_1q_1 \in \langle \mathcal{E}' \rangle \Leftrightarrow p_2q_2 \in \langle \mathcal{E}' \rangle$$

Proof. Let $(i, j) \in P_1$ be such that $i \in Q_1$
and $P_2 = P_1 - \{(i, j), (j, i)\}$
and $Q_2 = Q_1 \cup \{j\}$.
Since $i \in Q_1$ and $Q_2 = Q_1 \cup \{j\}$, we have that q_1 and q_2 are of the form (for a certain monomial q):
$q_1 = q \cdot t_i$
$q_2 = q \cdot t_it_j$
Since $(i, j) \in P_1$ we have that $l_i|p_1$.

We will consider two cases:

Case $(j, i) \in P_1$**.** We have that $l_{ji}|p_1$. Since $P_2 = P_1 \cup \{(i, j)(j, i)\}$, we have that p_1 and p_2 are of the form (for a certain monomial p):
$p_1 = p \cdot l_{ij}l_{ji}$
$p_2 = p \cdot m_{ij}m_{ji}$

Besides, we have that $g_{ij} = l_{ij}l_{ji}t_i + m_{ij}m_{ji}t_it_j \in \mathcal{E}'$. Therefore, we have that:
$pq \cdot g_{ij} = pq \cdot l_{ij}l_{ji}t_i + pq \cdot m_{ij}m_{ji}t_it_j = p_1q_1 + p_2q_2$
Consequently,

$$p_1q_1 \in \langle \mathcal{E}' \rangle \Leftrightarrow p_2q_2 \in \langle \mathcal{E}' \rangle$$

Case $(j, i) \notin P_1$**.** We have that $m_{ji}|p_1$. Since $P_2 = P_1 \cup \{(i, j), (j, i)\}$, we have that the monomials p_1, p_2 are of the form (where p is a certain monomials):
$p_1 = p \cdot l_{ij}m_{ji}$
$p_2 = p \cdot m_{ij}m_{ji}$
Besides, we have that $g'_{ij} = l_{ij}m_{ji}t_i + m_{ij}m_{ji}t_it_j \in \mathcal{E}'$. Therefore, we have that:
$pq \cdot g'_{ij} = pq \cdot l_{ij}m_{ji}t_i + pq \cdot m_{ij}m_{ji}t_it_j = p_1q_1 + p_2q_2$
Consequently,

$$p_1q_1 \in \langle \mathcal{E}' \rangle \Leftrightarrow p_2q_2 \in \langle \mathcal{E}' \rangle$$

We have the following theorem:

Theorem A.10. *Let (P, Q) be a railway interlocking problem for a railway station described by E.*

$$(P, Q) \textit{ is in a dangerous situation} \Leftrightarrow pq \in \langle \mathcal{E}' \rangle$$

Proof. **⇒)** Let (P, Q) be a railway interlocking problem in a dangerous situation. According to Theorem A.1, we have that a trivial railway interlocking problem is derived from (P, Q). According to Lemmas A.8 and A.9, we have that $pq \in \langle \mathcal{E}' \rangle$.
⇐) Suppose that $pq \in \langle \mathcal{E}' \rangle$. Since $\mathcal{E}'$ is a Groebner basis (see Lemma A.5), we have, by Theorem A.3, that $\mathrm{NR}(pq, \mathcal{E}') = 0$. Consequently, by Lemma A.6, we have that (P, Q) is in a dangerous situation.

Since the list $\mathcal{E}'$ is indeed a Groebner basis (see Lemma A.5), the ideal membership problem of the previous theorem can be easily solved (see Theorem A.11).

Theorem A.11. *Let (P, Q) be a railway interlocking problem for a railway station described by E. We have that:*

$$(P, Q) \textit{ is in a dangerous situation} \Leftrightarrow \mathrm{NR}(pq, \mathcal{E}) = 0$$

Proof. This is a immediate consequence of Theorem A.3, Lemma A.5, Theorem A.10 and Lemma A.7.

B. A linear algorithm for calculating *NR* for our algebraic approach

Here we will propose an efficient algorithm for calculating $\mathrm{NR}(p, \mathcal{E})$ for the specific case that:

- The coefficients of polynomials lie in $\mathbb{Z}_2$.
- p is a monomial.
- $\mathcal{E} = \{g_i\}$ is a list of polynomials such that each g_i is the sum of two monomials p_i and q_i. That is to say, $g_i = p_i + q_i$ where $p_i > q_i$. We will use the notation $\mathrm{LT}(g_i) = p_i$ and $M(g_i) = q_i$.

In this specific case, the general algorithm for calculating NR can be simplified since all intermediate dividends are monomials and the algorithm can run in $\mathrm{O}(N)$ if we use appropriate data structures:

- In order to represent $p = \prod_j {x_j}^{v_j}$ we use the array $[v_1 \dots v_N]$. An empty array [] will represent the monomial $p = 0$. We will use this data structure to represent the intermediate dividends of the algorithm. For example, we represent the monomial $p = x_1 x_2^3$ by means of the array $p : [1, 3, 0, \dots, 0]$.
- In order to represent the monomials p_i and q_i in $g_i = p_i + q_i$ we use a list with the form $\{(x_i, v_i)\}$. In this case a list $\{(x_j, v_j)\}$ will represent the monomial $q_i = \prod_j {x_j}^{v_j}$. For example, we represent the monomial $q_i = x_1 x_2^3$ by means of the list $\{(x_1, 1), (x_2, 3)\}$.
- In order to represent the polynomial $g_i = p_i + q_i$, we use two lists: one for representing p_i, another for representing q_i. For example, for $\mathcal{E} = \{g_1, g_2, g_3\}$ where $g_1 = x_1x_2 + x_2x_3$; $g_2 = x_2^2 + x_5x_6$; $g_3 = x_3 + x_5$ we represent each polynomial g_i in the following way:
  - $g_1 = x_1x_2 + x_2x_3$ is represented by these two lists: $\mathrm{LT}(g_1) = \{(1, 1), (2, 1)\}; M(g_1) = \{(2, 1), (3, 1)\}$.
  - $g_2 = x_2^2 + x_5x_6$ is represented by these two lists: $\mathrm{LT}(g_2) = \{(2, 2)\}; M(g_2) = \{(5, 1), (6, 1)\}$.
  - $g_3 = x_3 + x_5$ is represented by these two lists: $\mathrm{LT}(g_3) = \{(3, 1)\}; M(g_3) = \{(5, 1)\}$.
- For each variable x_j we consider F_{x_j}, the list of the polynomials $g_i \in \mathcal{E}$ such that the variables are in $\mathrm{LT}(g)$: that is to say, there is some natural e such that $(x_j, e) \in \mathrm{LT}(g)$. For example, in case that $\mathcal{E} = [g_1, g_2, g_3] = [x_1x_2 + x_2x_3, x_2^2 + x_5x_6, x_3 + x_5]$, we have that:
  - $F_{x_1} = \{g_1\}$ because x_1 appears in the monomial $\mathrm{LT}(g_1)$.
  - $F_{x_2} = \{g_1, g_2\}$ because x_2 appears in the monomials $\mathrm{LT}(g_1)$ and $\mathrm{LT}(g_2)$.
  - $F_{x_3} = \{g_3\}$ because x_2 appears in the monomial $\mathrm{LT}(g_3)$.
  - $F_{x_4} = F_{x_5} = F_{x_6} = \emptyset$ because x_4, x_5, x_6 do not appear in $\mathrm{LT}(g_i)$ for any g_i.
- For each polynomial g_i, we consider V_{g_i}, the set of the variables that are in $M(g_i)$. For example, in case that $\mathcal{E} = [g_1, g_2, g_3] = [x_1x_2 + x_2x_3, x_2^2 + x_5x_6, x_3 + x_5]$, we have that:
  - $V_{g_1} = \{2, 3\}$ because x_2 and x_3 appears in $M(g_1)$.
  - $V_{g_2} = \{5, 6\}$ because x_5 and x_6 appears in $M(g_2)$.
  - $V_{g_3} = \{5\}$ because x_5 appears in $M(g_3)$.

The algorithm for calculating the remainder of a monomial p with respect to a list of polynomials $\mathcal{E}$ is the following:

```
NR(int[] monomial, list of polynomials E)
  int[] p = Clone monomial
  Calculate F_i for every variable i
  Calculate V_g for every polynomial g in E
  Stack<Polynomial> check = E
  do
     g = check.Pop();
     if (CanBeUsedToReduce(g, p))
         Reduce(g,p);
         if (p==[]) return;
         for each Variable j in V_g)
           for each (Polynomial f in F_j)
              check.Push(f);
```

```
(13) while (check.Count > 0);
(14) return p;
```

The algorithm `CanBeUsedToReduce(g,p)` outputs `true` if $\mathrm{LT}(g)||p$ (that is to say, if the polynomial g can be used to reduce the monomial p). This algorithm is very simple and checks for every $(a,b) \in \mathrm{LT}(g)$ if `p[a]> b`. Since the size of $\mathrm{LT}(g)$ is constant in our algebraic model, this algorithm runs in O(1).

The algorithm `Reduce(g,p)` calculates a new intermediate dividend (that is to say, reduces the monomial p with respect to the polynomial g). This algorithm is also very simple due to the data structures used. For every $(a,b) \in \mathrm{LT}(g)$ and $(a,c) \in M(g)$, we update `p[a]` with the value `p[a]-b+c`. Since the size of the sets $\mathrm{LT}(g)$ and $M(g)$ are constant in our algebraic model, this algorithm runs in O(1).

The algorithm uses a stack, `check`, in which the algorithm stores the polynomials g_i that may be used to reduce the monomial p. The algorithm NR runs in $\mathrm{O}(N)$ since the step `Reduce` can only be applied N times at most and every time the monomial is reduced, the number of polynomials added to `check` is constant.