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Research article

Dynamic analysis of a stochastic epidemic model incorporating the double epidemic hypothesis and Crowley-Martin incidence term

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Abstract: The host population in epidemiology may actually be at risk of more than two infectious diseases with stochastic complicated interaction, e.g., HIV and HBV. In this paper, we propose a class of stochastic epidemic model that applies the double epidemic hypothesis and Crowley-Martin incidence rate in order to explore how stochastic disturbances affect the spread of diseases. While disregarding stochastic disturbances, we examine the dynamic features of the system in which the local stability of equilibria are totally determined by the basic reproduction numbers. We focus particularly on the threshold dynamics of the corresponding stochastic system, and we obtain the extinction and permanency conditions for a pair of infectious diseases. We find that the threshold dynamics of the deterministic and stochastic systems vary significantly: (i) disease outbreaks can be controlled by appropriate stochastic disturbances; (ii) diseases die out when the intensity of environmental perturbations is higher. The effects of certain important parameters on deterministic and stochastic disease transmission were obtained through numerical simulations. Our observations indicate that controlling epidemics should improve the effectiveness of prevention measures for susceptible individuals while improving the effectiveness of treatment for infected individuals.

Keywords: stochastic epidemic model; double epidemic hypothesis; basic reproduction number; Crowley-Martin incidence

1. Introduction

Smallpox, cholera, AIDS, COVID-19 and other infectious disease epidemics have wreaked immense havoc on the economy and way of life of the populace. Many mathematical models have been developed by researchers to explore the dynamical behavior of infectious diseases and thus control their transmission and gain a deeper understanding of these diseases [1–6]; among which, higher-order networks are widely used the spreading dynamics [7–9]. Compartmental models, which were originally established by Kermack and McKendrick [10], constitute a class of representative infectious

disease models that includes the SIR model [11], SIS model [12], $SIRS$ model [13], $SEIR$ model [14], $SI_1 \cdots I_kR$ [15] model and other variations [16–19].

Most epidemic models only concentrate on the transmission of a unique infectious disease; however, the host population may actually be at risk of more than two infectious diseases with complicated interaction, and they could occur as parallel, competitive or stimulative. A large percentage of people at risk for HIV infection is also at risk for HBV infection due to shared mechanisms of transmission [20]. Casalegno et al. [21] discovered that during the first half of fall 2009, in France, rhinovirus interference slowed the influenza pandemic and affected the transmission of the H1N1 virus. During the COVID-19 pandemic, it was discovered in [22] that the SARS-COV-2 Delta (B.1.617.2) variant had replaced the Alpha (B.1.1.7) variation on a significant scale, which is related to the Delta version's earlier invasion and superior transmissibility. In this paper, we only focus on two epidemics spreading parallelly, and we assume that an epidemic caused by one virus prevents the occurrence of the other. For related works, we recommend the references [23–25] and the references therein.

The rate at which new infections emerge, known as the disease incidence, is a crucial variable in mathematical models of infectious disease dynamics. The incidence rate has different forms which are commonly used as follows. It is assumed that the exposure rate is proportionate to the whole population and that the mass-action (bilinear) incidence is βSI [26]. The standard incidence is $\beta SI/N$, and it requires the assumption that the number of people exposed to a sick person per unit time is constant [27]. If the exposure rate is saturation of the susceptible S or infective I, the incidence will be the saturation incidence $\beta SI/(1 + aS)$ or $\beta SI/(1 + aI)$ [23]. Other incidence forms, such as the nonlinear incidence rate $\beta SI^p/(1 + \alpha S^q)$ and Beddington-DeAngelis incidence $\beta SI/(1 + aS + bI)$, have been discussed in [28, 29].

A particular Crowley-Martin functional response function was proposed in 1975 [30], and it is widely used in prey-predator models [31], eco-epidemic models [32] and epidemic models [33–35]. In infectious disease models, the Crowley-Martin incidence is represented by $\beta SI/(1 + aS)(1 + bI)$, which takes into account the interaction between susceptible and infected populations, where a measures the preventive effect of susceptible individuals and b measures the treatment effect with respect to infected individuals.

For these reasons, this paper presents a deterministic epidemic model with the double epidemic hypothesis and Crowley-Martin nonlinear incidence term. We divided the population into three compartments: the susceptible population S, the infected population I_1 infected with virus D_1 and the infected population I_2 infected with virus D_2. In addition, susceptible individuals enter at a rate of constant N, β_i is the rate of transmission from a susceptible person to an infected person, the natural and causal mortality rates of the population are m and δ_i respectively, and α_i is the rate of infected people transitioning to the susceptible class. The flowchart of disease transmission and progression is as shown in Figure 1; we formulate the following dynamical model:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= N - \frac{\beta_1 S I_1}{(1 + a_1 S)(1 + b_1 I_1)} - \frac{\beta_2 S I_2}{(1 + a_2 S)(1 + b_2 I_2)} + \alpha_1 I_1 + \alpha_2 I_2 - mS, \\
\frac{\mathrm{d}I_1}{\mathrm{d}t} &= \frac{\beta_1 S I_1}{(1 + a_1 S)(1 + b_1 I_1)} - (m + \alpha_1 + \delta_1) I_1, \\
\frac{\mathrm{d}I_2}{\mathrm{d}t} &= \frac{\beta_2 S I_2}{(1 + a_2 S)(1 + b_2 I_2)} - (m + \alpha_2 + \delta_2) I_2.
\end{aligned} \tag{1.1}$$

In fact, disease transmission is quite sensitive to disturbances caused by external environmental

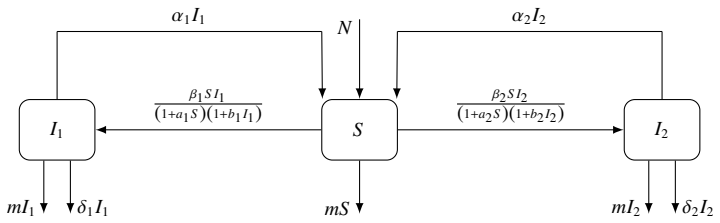


Figure 1. The disease transmission graph for model (1.1).

factors, such as temperature, light, rainstorms and human intervention. These stochastic factors could have a significant impact on almost all parameters of the model in multiple ways [36–40]. The transmission rate β oscillates around an average value as a result of the environment's ongoing oscillations brought on by the impact of white noise $\beta + \sigma\dot{B}(t)$, where $B(t)$ represents the standard Brownian motions and $\sigma > 0$ is the intensity of environmental fluctuations. Then, we obtain a stochastic epidemic model as follows:

$$\begin{cases} \mathrm{d}S = \left(N - \dfrac{\beta_1 S I_1}{(1+a_1 S)(1+b_1 I_1)} - \dfrac{\beta_2 S I_2}{(1+a_2 S)(1+b_2 I_2)} + \alpha_1 I_1 + \alpha_2 I_2 - mS\right)\mathrm{d}t \\ \qquad - \dfrac{\sigma_1 S I_1}{1 + a_1 S + b_1 I_1 + a_1 b_1 S I_1}\mathrm{d}B_1(t) - \dfrac{\sigma_2 S I_2}{1 + a_2 S + b_2 I_2 + a_2 b_2 S I_2}\mathrm{d}B_2(t), \\ \mathrm{d}I_1 = \left(\dfrac{\beta_1 S I_1}{(1+a_1 S)(1+b_1 I_1)} - (m + \alpha_1 + \delta_1) I_1\right)\mathrm{d}t + \dfrac{\sigma_1 S I_1}{(1+a_1 S)(1+b_1 I_1)}\mathrm{d}B_1(t), \\ \mathrm{d}I_2 = \left(\dfrac{\beta_2 S I_2}{(1+a_2 S)(1+b_2 I_2)} - (m + \alpha_2 + \delta_2) I_2\right)\mathrm{d}t + \dfrac{\sigma_2 S I_2}{(1+a_2 S)(1+b_2 I_2)}\mathrm{d}B_2(t). \end{cases} \tag{1.2}$$

The following describes how this manuscript is structured. In Section 2, we discuss the dynamics of deterministic systems, especially for the asymptotic stability of equilibria. In Section 3, we establish the extinction and persistence conditions of the corresponding stochastic system. In Section 4, through a number of numerical simulations, we explore the effects of the perturbation strength σ_i and parameters a_i and b_i on the dynamics of the system. The paper ends with a short discussion and conclusion.

2. Dynamics of deterministic system

Prior to investigating the stochastic system, it is also essential to ascertain the dynamical behaviors of the deterministic system.

For the deterministic system (1.1) or the stochastic system (1.2), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(S + I_1 + I_2) &= N - m(S + I_1 + I_2) - \delta_1 I_1 - \delta_2 I_2 \\ &\leq N - m(S + I_1 + I_2). \end{aligned}$$

This implies that

$$\limsup_{t\to\infty}(S + I_1 + I_2) \leq \frac{N}{m}.$$

We denote

$$\Gamma = \left\{(S(t), I_1(t), I_2(t)) \in \mathbb{R}^3_+ : S(t) + I_1(t) + I_2(t) \leq \frac{N}{m}, t \geq 0\right\};$$

then, regarding the solutions of system (1.1), Γ is a positively invariant set.

Utilizing the next generation matrix method [41, 42], we can obtain the basic reproduction number:

$$\mathcal{R}_i = \frac{\beta_i N}{(m + a_i N)(m + \alpha_i + \delta_i)}, \quad i = 1, 2.$$

The equilibrium equation is listed as follows:

$$\begin{aligned} &N - \frac{\beta_1 S I_1}{(1 + a_1 S)(1 + b_1 I_1)} - \frac{\beta_2 S I_2}{(1 + a_2 S)(1 + b_2 I_2)} + \alpha_1 I_1 + \alpha_2 I_2 - mS = 0, \\ &\frac{\beta_1 S I_1}{(1 + a_1 S)(1 + b_1 I_1)} - (m + \alpha_1 + \delta_1) I_1 = 0, \\ &\frac{\beta_2 S I_2}{(1 + a_2 S)(1 + b_2 I_2)} - (m + \alpha_2 + \delta_2) I_2 = 0. \end{aligned} \tag{2.1}$$

System (1.1) has four possible equilibria:

(i) disease-free equilibrium $E_0 = \left(\frac{N}{m}, 0, 0\right)$;

(ii) boundary equilibrium $E_1 = \left(\bar{S_1}, \bar{I_1}, 0\right)$, where $0 < \bar{S}_1 < \frac{N}{m}, \bar{I}_1 > 0$;

(iii) boundary equilibrium $E_2 = \left(\bar{S_2}, 0, \bar{I_2}\right)$, where $0 < \bar{S}_2 < \frac{N}{m}, \bar{I}_2 > 0$;

(iv) endemic equilibrium $E_3 = \left(S^*, I_1^*, I_2^*\right)$, where $S^*, I_1^*, I_2^* > 0$.

By (2.1), when $I_2 = 0$ and $I_1 \neq 0$, we obtain the boundary equilibrium

$$E_1 = \left(\frac{N - (m + \delta_1)\bar{I_1}}{m}, \bar{I_1}, 0\right),$$

and $\bar{I_1}$ is the positive root of

$$f\left(\bar{I_1}\right) = X\bar{I_1}^2 + Y\bar{I_1} + Z = 0,$$

where

$$X = a_1 b_1 (m + \delta_1) > 0;$$

when $\mathcal{R}_1 > 1$, we have

$$\begin{aligned} Y =& (m + a_1 N)\frac{m + \delta_1}{N}\left(-\frac{b_1 N}{m + \delta_1} + \frac{a_1 N}{m + a_1 N} - \mathcal{R}_1\right) < 0, \\ Z =& (m + a_1 N)(\mathcal{R}_1 - 1) > 0. \end{aligned}$$

If $\Delta = Y^2 - 4XZ = 0$, there exists a unique E_1 equilibrium with $\bar{I_1} = -\frac{Y}{2X}$; if $\Delta = Y^2 - 4XZ > 0$, then system (1.1) has two E_1 equilibria with $\bar{I_1} = \frac{-Y \pm \sqrt{Y^2 - 4XZ}}{2X}$.

Similarly, by (2.1), when $I_1 = 0$ and $I_2 \neq 0$, we obtain the boundary equilibrium

$$E_2 = \left(\frac{N - (m + \delta_2)\bar{I_2}}{m}, 0, \bar{I_2}\right);$$

if $\mathcal{R}_2 > 1$ then there may exist one or two-type equilibria E_2 as well.

By equilibrium equation (2.1), when $\mathcal{R}_i > 1, i = 1, 2$, the endemic equilibrium $E_3 = \left(S^*, I_1^*, I_2^*\right)$ exists and satisfies

$$E_3 = \left(\frac{N - (m + \delta_1)I_1^* - (m + \delta_2)I_2^*}{m}, I_1^*, I_2^*\right),$$

where the relationship between I_1^* and I_2^* satisfies

$$I_2^* = \frac{(m + \alpha_1 + \delta_1)\left(1 + b_1 I_1^*\right)\left[m + a_1(N - (m + \delta_1)I_1^*)\right] - \beta_1\left(N - \left(m + \delta_1 I_1^*\right)\right)}{(m + \delta_2)(a_1(m + \alpha_1 + \delta_1)(1 + b_1 I_1^* - \beta_1))}.$$

Theorem 1. *1). If $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 < 1$, then the disease-free equilibrium E_0 is locally asymptotically stable.*
2). If $\mathcal{R}_1 > 1$ and equilibrium E_1 exists, then the boundary equilibrium E_1 is locally asymptotically stable.
3). If $\mathcal{R}_2 > 1$ and equilibrium E_2 exists, then the boundary equilibrium E_2 is locally asymptotically stable.
4). If $\mathcal{R}_1 > 1$, $\mathcal{R}_2 > 1$ and equilibrium E_3 exists, then the endemic equilibrium E_3 is locally asymptotically stable.

The proof is given in Appendix A.

3. Dynamics of stochastic system

3.1. Existence and uniqueness of the positive solution

Lemma 1 ([43]). *For any initial value $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$, there exists a unique solution $(S(t), I_1(t), I_2(t))$ to system (1.2) on $t \geq 0$, and the solution will remain in $\mathbb{R}^3_+$ with probability 1, i.e., $(S(t), I_1(t), I_2(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ a.s.*

Proof. The proof of Lemma 1 is similar to that in Theorem 2.1 of [44]; we therefore omit it here.

Lemma 2 ([29]). *Γ is an almost positive invariant set of system (1.2), that is, if $(S(0), I_1(0), I_2(0)) \in \Gamma$, then $\mathbb{P}(S(t), I_1(t), I_2(t) \in \Gamma) = 1$ for all $t \geq 0$.*

Define the stochastic basic reproduction numbers

$$\begin{aligned}\mathcal{R}_i^s =& \frac{\beta_i N}{(m + a_i N)(m + \alpha_i + \delta_i)} - \frac{\sigma_i^2 N^2}{2(m + a_i N)^2(m + \alpha_i + \delta_i)} \\ =& \mathcal{R}_i - \frac{\sigma_i^2 N^2}{2(m + a_i N)^2(m + \alpha_i + \delta_i)}, \quad i = 1, 2.\end{aligned}$$

3.2. Extinction and persistence of stochastic system

We focus on disease extinction and persistence in this subsection since stochastic systems have distinct extinction and persistence conditions compared to deterministic systems. First, the following lemma is presented to demonstrate the extinction and persistence of diseases.

Lemma 3 ([42, 45]). *Let* $(S(t), I_1(t), I_2(t))$ *be a solution of system (1.2) with initial value* $(S(0), I_1(0), I_2(0)) \in \mathbb{R}^3_+$. *Then*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\sigma_i S(\tau)}{(1+a_iS(\tau))(1+b_iI_i(\tau))}\mathrm{d}B_i(\tau) = 0, \quad \lim_{t\to+\infty} \frac{1}{t}\int_0^t \sigma_i S(\tau)\mathrm{d}B_i(\tau) = 0, \quad i = 1, 2.$$

Theorem 2. *Suppose that one of the following two assumptions is satisfied:*
(H_1) $\sigma_i > \hat{\sigma}_i := \max\left\{\sqrt{\frac{\beta_i(Na_i+m)}{N}}, \frac{\beta_i}{\sqrt{2(m+\alpha_i+\delta_i)}}\right\}, \quad i = 1, 2;$
(H_2) $\sigma_i \le \sqrt{\frac{\beta_i(Na_i+m)}{N}}$ *and* $\mathcal{R}_i^s < 1, \quad i = 1, 2.$
Then the solution $(S(t), I_1(t), I_2(t))$ *of system (1.2) with any initial value* $(S(0), I_1(0), I_2(0)) \in \Gamma$ *satisfies that*

$$\lim_{t\to+\infty} S(t) = \frac{N}{m}, \quad \lim_{t\to+\infty} I_1(t) = \lim_{t\to+\infty} I_2(t) = 0.$$

Proof. By using Itô's formula, we have

$$\begin{aligned}\mathrm{d}\ln I_i(t) =& \left(\frac{\beta_i S}{(1+a_iS)(1+b_iI_i)} - \frac{\sigma_i^2 S^2}{2((1+a_iS)(1+b_iI_i))^2} - (m+\alpha_i+\delta_i)\right)\mathrm{d}t \\ &+ \frac{\sigma_i S}{(1+a_iS)(1+b_iI_i)}\mathrm{d}B_i(t), \quad i = 1, 2.\end{aligned} \tag{3.1}$$

Case 1: Under assumption (H_1), integrating both sides of (3.1), we have

$$\begin{aligned}\ln I_i(t) =& \int_0^t \left(\frac{\beta_i S(\tau)}{(1+a_iS(\tau))(1+b_iI_i(\tau))} - \frac{\sigma_i^2 S^2(\tau)}{2((1+a_iS(\tau))(1+b_iI_i(\tau)))^2}\right)\mathrm{d}\tau \\ &- (m+\alpha_i+\delta_i)t + Q_i(t) + \ln I_i(0) \\ =& -\frac{\sigma_i^2}{2}\int_0^t \left(\frac{S(\tau)}{(1+a_iS(\tau))(1+b_iI_i(\tau))} - \frac{\beta_i}{\sigma_i^2}\right)^2 \mathrm{d}\tau - (m+\alpha_i+\delta_i)t \\ &+ \frac{\beta_i^2}{2\sigma_i^2}t + Q_i(t) + \ln I_i(0) \\ \le& -(m+\alpha_i+\delta_i)t + \frac{\beta_i^2}{2\sigma_i^2}t + \ln I_i(0) + Q_i(t),\end{aligned} \tag{3.2}$$

where

$$Q_i(t) = \int_0^t \frac{\sigma_i S(\tau)}{(1+a_iS(\tau))(1+b_iI_i(\tau))}\mathrm{d}B_i(\tau); \tag{3.3}$$

dividing both sides of (3.2) by t, we have

$$\frac{\ln I_i(t)}{t} \le -\left(m+\alpha_i+\delta_i - \frac{\beta_i^2}{2\sigma_i^2}\right) + \frac{Q_i(t)}{t} + \frac{\ln I_i(0)}{t}; \tag{3.4}$$

by Lemma 3, we have

$$\lim_{t\to+\infty}\frac{Q_i(t)}{t}=0;$$

since $\sigma_i > \frac{\beta_i}{\sqrt{2(m+\alpha_i+\delta_i)}}$ for $i=1,2$, taking the limit superior on both sides of (3.4) leads to

$$\limsup_{t\to+\infty}\frac{\ln I_i(t)}{t}\le-\left(m+\alpha_i+\delta_i-\frac{\beta_i^2}{2\sigma_i^2}\right)<0.$$

Thus, $\lim_{t\to+\infty} I_i(t)=0$ a.s.

Case 2: Under assumption (H_2), similar to the calculation in Case 1, we have

$$\begin{aligned}\frac{\ln I_i(t)}{t}=&\frac{1}{t}\Bigg\{\int_0^t\Big(\frac{\beta_i S(\tau)}{(1+a_iS(\tau))(1+b_iI_i(\tau))}-\frac{\sigma_i^2S^2(\tau)}{2((1+a_iS(\tau))(1+b_iI_i(\tau)))^2}\\&-(m+\alpha_i+\delta_i)\Big)\mathrm{d}\tau+Q_i(t)+\ln I_i(0)\Bigg\}\\=&\frac{1}{t}\Bigg\{\int_0^t\Psi\left(\frac{S(\tau)}{(1+a_iS(\tau))(1+b_iI_i(\tau))}\right)\mathrm{d}\tau+Q_i(t)+\ln I_i(0)\Bigg\},\end{aligned}$$

where the function $\Upsilon(x)$ is defined as

$$\Upsilon: x\mapsto-\frac{1}{2}\sigma_i^2x^2+\beta_ix-(m+\alpha_i+\delta_i).$$

Take note that $\Upsilon(x)$ increases monotonically for $x\in\left[0,\frac{\beta_i}{\sigma_i^2}\right]$ and $x<\frac{N}{a_iN+m}$; thus, when $\sigma_i\le\sqrt{\frac{\beta_i(Na_i+m)}{N}}$, we have

$$\begin{aligned}\frac{\ln I_i(t)}{t}\le&\frac{\beta_iN}{m+a_iN}-\frac{\sigma_i^2N^2}{2(m+a_iN)^2}-(m+\alpha_i+\delta_i)+\frac{1}{t}(Q_i(t)+\ln I_i(0))\\=&(m+\alpha_i+\delta_i)\left(\mathcal{R}_i^s-1\right)+\frac{1}{t}(Q_i(t)+\ln I_i(0)).\end{aligned}\tag{3.5}$$

Taking the limit superior of both sides of (3.5) leads to

$$\limsup_{t\to+\infty}\frac{\ln I_i(t)}{t}\le(m+\alpha_i+\delta_i)(\mathcal{R}_i^s-1)<0;$$

which implies that $\lim_{t\to+\infty} I_i(t)=0, i=1,2$.

We suppose that $0<I_i(t)<\varepsilon_i (i=1,2)$ for all $t\ge0$; by the first equation of system (1.2), we have

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t}\ge N-(m+\beta_1\varepsilon_1+\beta_2\varepsilon_2+\sigma_1\varepsilon_1|\dot{B}_1(t)|+\sigma_2\varepsilon_2|\dot{B}_2(t)|)S(t).\tag{3.6}$$

Because $\varepsilon_1\to0$ and $\varepsilon_2\to0$, if we divide (3.6) by the limit inferior on both sides, we have

$$\liminf_{t\to+\infty}S(t)\ge\frac{N}{m}.\tag{3.7}$$

Combining this with $\limsup_{t\to+\infty} S(t) \le \frac{N}{m}$, it is easy to see that

$$\lim_{t\to+\infty} S(t) = \frac{N}{m} \ a.s.$$

We will present the persistence result of system (1.2) in the following theorem, whose proof is given in Appendix B.

Theorem 3. *If we assume that the solution to system (1.2) is* $(S(t), I_1(t), I_2(t))$ *and that* $(S(0), I_1(0), I_2(0)) \in \Gamma$ *is the initial value, then we get the following:*

(i) The disease I_2 *becomes extinct and the disease* I_1 *becomes permanent in the mean if* $\mathcal{R}_1^s > 1, \mathcal{R}_2^s < 1$ *and the disturbance intensity satisfies that* $\sigma_2 \le \sqrt{\beta_2(a_2 + \frac{m}{N})}$*. Additionally,* I_1 *satisfies*

$$\liminf_{t\to+\infty}\langle I_1(t)\rangle \ge \frac{(m + a_1N)(m + \alpha_1 + \delta_1)}{\beta_1(m + \delta_1) + b_1(m + a_1N)(m + \alpha_1 + \delta_1)} \left(\mathcal{R}_1^s - 1\right).$$

(ii) The disease I_1 *becomes extinct and the disease* I_2 *becomes permanent in the mean if* $\mathcal{R}_1^s < 1, \mathcal{R}_2^s > 1$ *and the disturbance intensity satisfies that* $\sigma_1 \le \sqrt{\beta_1(a_1 + \frac{m}{N})}$*. Additionally,* I_2 *satisfies*

$$\liminf_{t\to+\infty}\langle I_2(t)\rangle \ge \frac{(m + a_2N)(m + \alpha_2 + \delta_2)}{\beta_2(m + \delta_2) + b_2(m + a_2N)(m + \alpha_2 + \delta_2)}(\mathcal{R}_2^s - 1).$$

(iii) If $\mathcal{R}_i^s > 1$*, then the two infectious diseases* I_i *are permanent in mean; moreover,* I_i *satisfies*

$$\liminf_{t\to+\infty}\left\langle \sum_{i=1}^{2} I_i(t)\right\rangle \ge \frac{1}{\Delta_{\max}} \sum_{i=1}^{2} a_i(m + \alpha_i + \delta_i)(\mathcal{R}_i^s - 1),$$

where

$$\Delta_{\max} = \sum_{i=1}^{2}\left(\frac{\beta_1 + \beta_2}{m}(m + \delta_i) + b_i(m + \alpha_i + \delta_i)\right).$$

4. Simulations

In this section of this paper, we will continue with our investigation of the deterministic system and the stochastic system by using the numerical method. Before looking at how changes in the environment influence the spread of diseases and the effect of the parameters a_i and b_i on the dynamics of the disease, we first compare the extinction conditions for the same parameter values for the stochastic system and the deterministic system.

To simulate the behavior of the stochastic system (1.2), we made use of Milstein's method [45, 46]. Following [23, 47], with the exception of σ_i, the other parameter values of system (1.1) and system (1.2) were derived as given in Table 1. Then, we chose initial values as $(S(0), I_1(0), I_2(0)) = (15, 10, 5)$.

Table 1. Parameter values of system (1.1) and system (1.2) in numerical simulations.

Parameter	Description	Value	Sources
N	Influx of susceptible population	1	[23]
m	Rate of natural death	0.1	[23,47]
β_1	Transmission coefficient for virus D_1	1.2	[23,47]
β_2	Transmission coefficient for virus D_2	1.5	[23,47]
a_1	Preventive effect of virus D_1	1	[23]
a_2	Preventive effect of virus D_2	1.5	[23]
b_1	Treatment effect of virus D_1	2	[23]
b_2	Treatment effect of virus D_2	1	[23]
δ_1	Rate of virus D_1-related death	0.2	[23,47]
δ_2	Rate of virus D_2-related death	0.4	[23,47]
α_1	Recovery rate for virus D_1	0.9	[23,47]
α_2	Recovery rate for virus D_2	0.9	[23,47]

$$
\begin{cases}
S_{k+1} = S_k + \left(N - \dfrac{\beta_1 S_k I_{1k}}{(1+a_1S_k)(1+b_1I_{1k})} - \dfrac{\beta_2 S_k I_{2k}}{(1+a_2S_k)(1+b_2I_{2k})} + \alpha_1 I_{1k} + \alpha_2 I_{2k} - mS_k\right)\Delta t \\
\qquad - \left(\sigma_1 \dfrac{S_k I_{1k}}{(1+a_1S_k)(1+b_1I_{1k})} + \sigma_2 \dfrac{S_k I_{2k}}{(1+a_2S_k)(1+b_2I_{2k})}\right)\sqrt{\Delta t \xi_k} \\
\qquad - \left(\dfrac{\sigma_1^2}{2}\left(\dfrac{S_k I_{1k}}{(1+a_1S_k)(1+b_1I_{1k})}\right) + \dfrac{\sigma_2^2}{2}\left(\dfrac{S_k I_{2k}}{(1+a_2S_k)(1+b_2I_{2k})}\right)\right)\left(\xi_k^2 - 1\right)\Delta t, \\
I_{1k+1} = \dfrac{\sigma_1 S_k I_{1k}}{(1+a_1S_k)(1+b_1I_{1k})}\sqrt{\Delta t \xi_k} + \dfrac{\sigma_1^2}{2}\left(\dfrac{S_k I_{1k}}{(1+a_1S_k)(1+b_1I_{1k})}\right)\left(\xi_k^2 - 1\right)\Delta t + I_{1k} \\
\qquad + \left(\dfrac{\beta_1 S_k I_{1k}}{(1+a_1S_k)(1+b_1I_{1k})} - (m + \alpha_1 + \delta_1) I_{1k}\right)\Delta t, \\
I_{2k+1} = \dfrac{\sigma_2 S_k I_{2k}}{(1+a_2S_k)(1+b_2I_{2k})}\sqrt{\Delta t \xi_k} + \dfrac{\sigma_2^2}{2}\left(\dfrac{S_k I_{2k}}{(1+a_2S_k)(1+b_2I_{2k})}\right)\left(\xi_k^2 - 1\right)\Delta t + I_{2k} \\
\qquad + \left(\dfrac{\beta_2 S_k I_{2k}}{(1+a_2S_k)(1+b_2I_{2k})} - (m + \alpha_2 + \delta_2) I_{2k}\right)\Delta t.
\end{cases}
$$

4.1. Extinction conditions of two diseases in deterministic and stochastic systems

In order to investigate the dynamical differences that exist between systems (1.1) and (1.2), we give five examples for numerical simulations.

Example 1. When I_2 is facing extinction in a deterministic system, the stochastic perturbation could change I_1 from prevalence to extinction. When α_1 is changed to 0.7, $\sigma_1 = \sigma_2 = 1$, $\mathcal{R}_1 = 1.091 > 1, \mathcal{R}_1^s = 1.091 - 0.379 = 0.712 < 1, \mathcal{R}_2 = 0.9375 < 1$ and $\mathcal{R}_2^s = 0.9375 - 0.3125 = 0.625 < 1$. According to our previous analysis results, disease I_1 is prevalent and disease I_2 is subject to extinction in the deterministic system (1.1); however, disease I_1 and I_2 are both extinct in the stochastic system (1.2) (see Figure 2(a)).

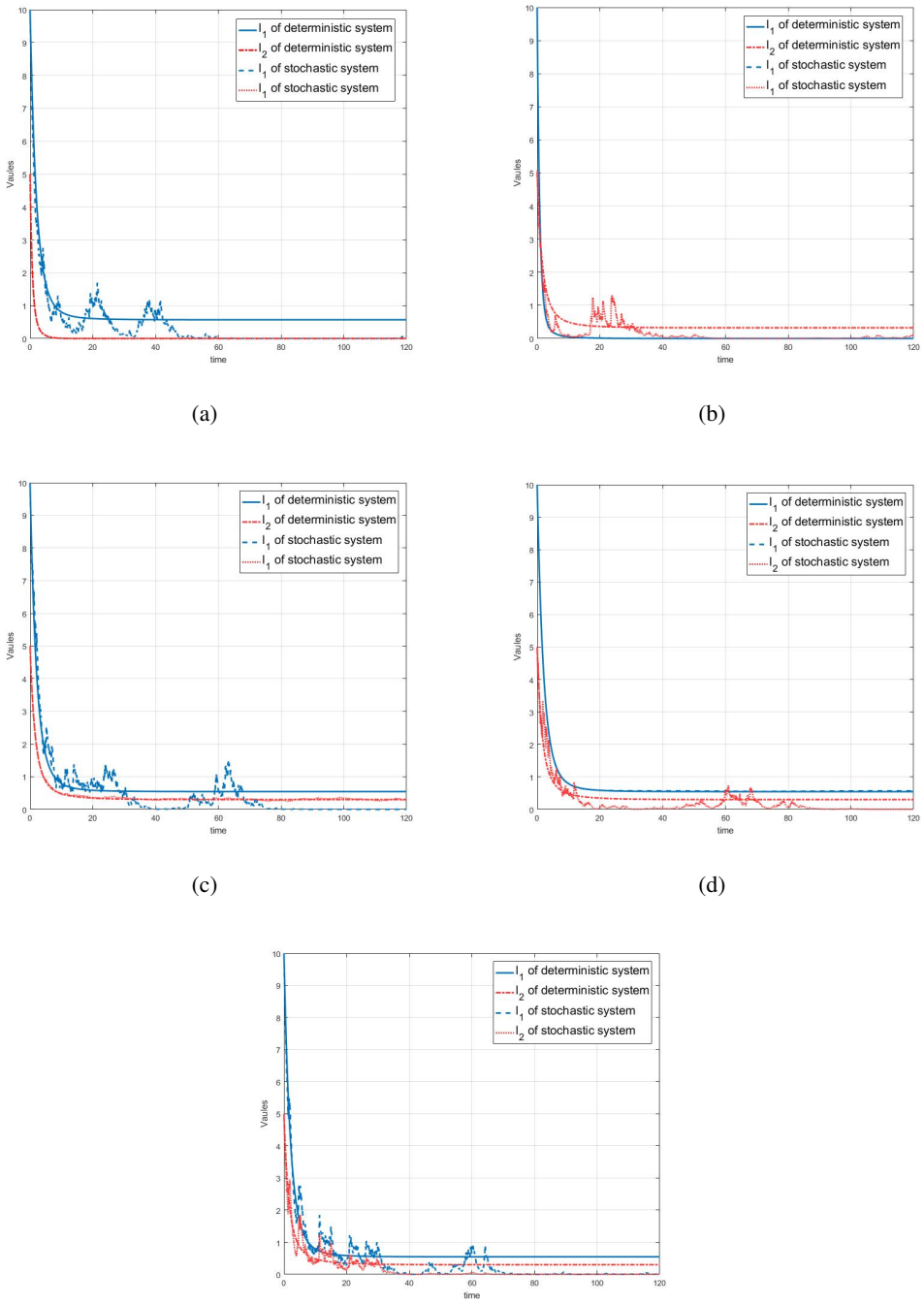


Figure 2. Dynamics and behavior comparisons between $I_1(t)$ and $I_2(t)$ for stochastic and deterministic systems. (a) $\mathcal{R}_1 > 1, \mathcal{R}_2 < 1, \mathcal{R}_1^s < 1, \mathcal{R}_2^s < 1$; (b) $\mathcal{R}_1 < 1, \mathcal{R}_2 > 1, \mathcal{R}_1^s < 1, \mathcal{R}_2^s < 1$; (c) $\mathcal{R}_1 > 1, \mathcal{R}_2 > 1, \mathcal{R}_1^s < 1, \mathcal{R}_2^s > 1$; (d) $\mathcal{R}_1 > 1, \mathcal{R}_2 > 1, \mathcal{R}_1^s > 1, \mathcal{R}_2^s < 1$; (e) $\mathcal{R}_1 > 1, \mathcal{R}_2 > 1, \mathcal{R}_1^s < 1, \mathcal{R}_2^s < 1$. The initial value of all solutions is (15,10,5). The time unit is day.

Example 2. When I_1 is subject to extinction in a deterministic system, the stochastic perturbation could change I_2 from a prevalence to extinction condition. When α_2 is changed to 0.7, δ_2 is 0.2, $a_2 = 1.3$, $\sigma_1 = \sigma_2 = 1$, $\mathcal{R}_1 = 0.909 < 1, \mathcal{R}_1^s = 0.909 - 0.416 = 0.496 < 1, \mathcal{R}_2 = 1.071 > 1$ and $\mathcal{R}_2^s = 1.071 - 0.255 = 0.816 < 1$. According to our previous analysis results, disease I_1 goes to extinction and disease I_2 is persistent in the deterministic system (1.1); also, disease I_1 and I_2 both go to extinction in the stochastic system (1.2) (see Figure 2(b)).

Example 3. When I_1 and I_2 are extinct in a deterministic system, the stochastic perturbation could change I_1 from a prevalence to extinction condition. When $\delta_2 = 0.1, \sigma_1 = 1$ and $\sigma_2 = 0.1$, it follows that $\mathcal{R}_1 = 1.091 > 1, \mathcal{R}_1^s = 1.091 - 0.413 = 0.678 < 1, \mathcal{R}_2 = 1.041 > 1$ and $\mathcal{R}_2^s = 1.041 - 0.001 = 1.04 > 1$. According to our previous analysis results, both disease I_1 and disease I_2 are persistent in the deterministic system (1.1); disease I_1 goes to extinction and I_2 is persistent in the stochastic system (1.2) (see Figure 2(c)).

Example 4. When I_1 and I_2 are extinct in a deterministic system, the stochastic perturbation could change I_2 from a prevalence to extinction condition. When $\delta_2 = 0.1, \sigma_1 = 0.1$ and $\sigma_2 = 1, \mathcal{R}_1 = 1.091 > 1, \mathcal{R}_1^s = 1.091 - 0.0041 = 1.0869 > 1, \mathcal{R}_2 = 1.041 > 1$ and $\mathcal{R}_2^s = 1.041 - 0.195 = 0.846 < 1$. According to our previous analysis results, both disease I_1 and disease I_2 are persistent in the deterministic system; also, disease I_1 is persistent and I_2 goes to extinction in the stochastic system (1.2) (see Figure 2(d)).

Example 5. When I_1 and I_2 are extinct in a deterministic system, the stochastic perturbation could change I_1 and I_2 from a prevalence to extinction condition. When $\delta_2 = 0.1, \sigma_1 = 1$ and $\sigma_2 = 1$, $\mathcal{R}_1 = 1.091 > 1$, $\mathcal{R}_1^s = 1.091 - 0.413 = 0.678 < 1$, $\mathcal{R}_2 = 1.041 > 1$ and $\mathcal{R}_2^s = 1.041 - 0.195 = 0.846 < 1$. According to our previous analysis results, both disease I_1 and disease I_2 are persistent in the deterministic system (1.1). In the stochastic system (1.2), both disease I_1 and disease I_2 go extinct (see Figure 2(e)).

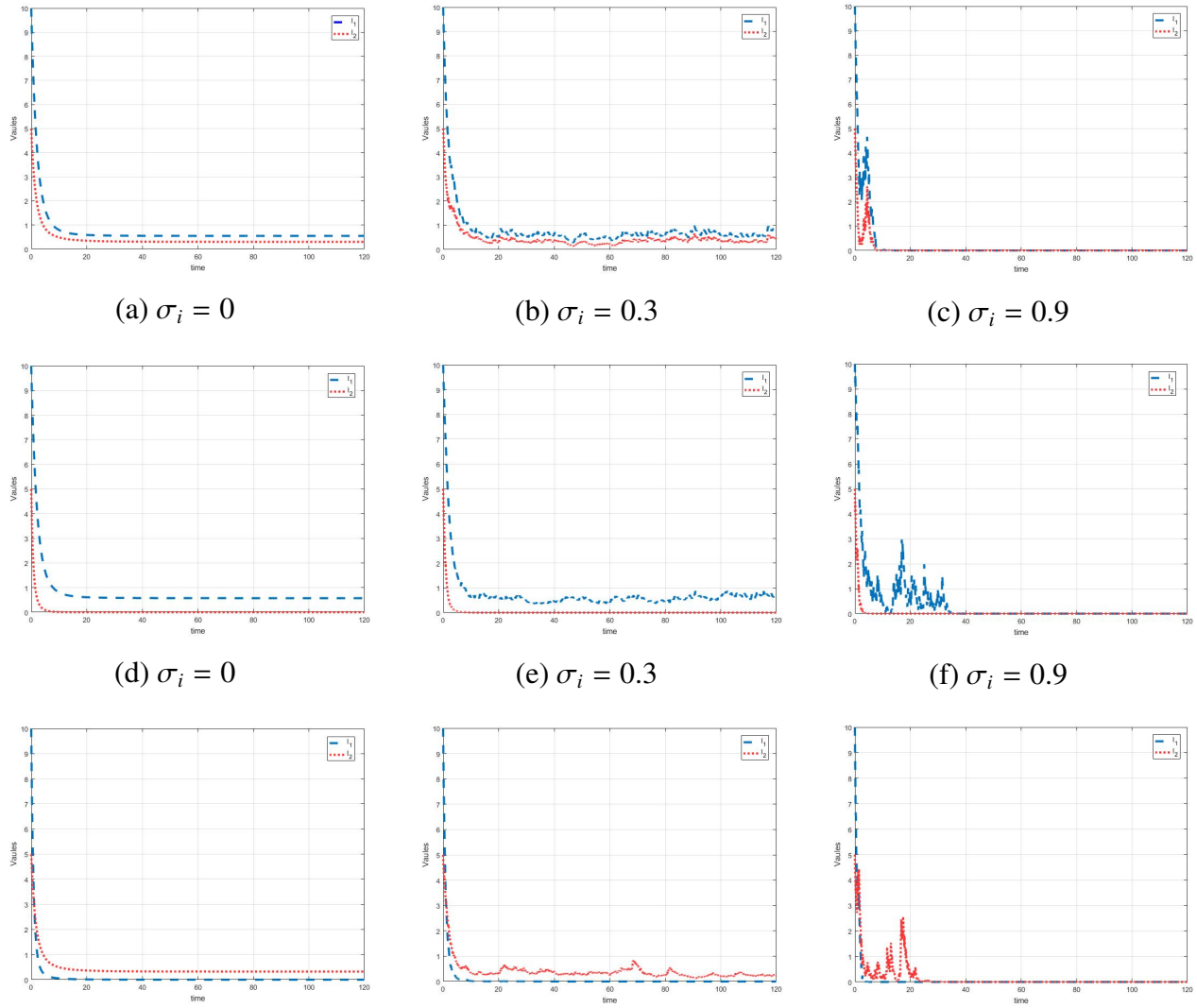
4.2. The impact of environmental noise

By (H_1) in Theorem 2, we can see that when the strengths of the perturbations are large, $\mathcal{R}_i^s$ loses its meaning and the diseases go to extinction. We chose different perturbation strengths for when $\sigma_i = 0, 0.3, 0.9$ to observe the trend of the disease. When σ_i is larger, the infectious disease I_i goes to extinction (see Figure 3). These simulations support our results for (H_1) in Theorem 2 well.

4.3. The impact of preventive effect a_i and treatment effect b_i

It should be noted that a_i and b_i of the Crowley-Martin incidence are key parameters. In this subsection, we discuss the effects of parameters a_i and b_i on the population and trend of infections by presenting some numerical simulations.

First, we study the influence of preventive effects a_i on the population of infective individuals in the deterministic system (1.1). For the case that the parameters in Table 1 are fixed in Table 1, we chose five different sets of values for a_i. For the deterministic system (1.1), it can be shown that the bigger the value of a_i, the quicker the extinction of disease I_i (see Figure 4). Second, we wanted to investigate the influence of the parameter a_i on the population of infective individuals in the stochastic system (1.2). We choose the perturbation intensity as $\sigma_i = 0.3, i = 1, 2$. At last, we observed the effect of a_i in the stochastic system (1.2) (see Figure 5). Similarly, we wanted to study the influence of treatment effects b_i on the population of infected individuals in deterministic and stochastic systems (see Figures 6 and 7).



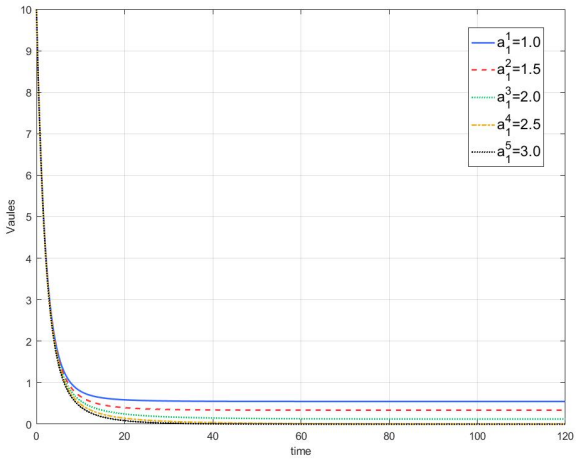
(a) $\sigma_i = 0$ (b) $\sigma_i = 0.3$ (c) $\sigma_i = 0.9$

(d) $\sigma_i = 0$ (e) $\sigma_i = 0.3$ (f) $\sigma_i = 0.9$

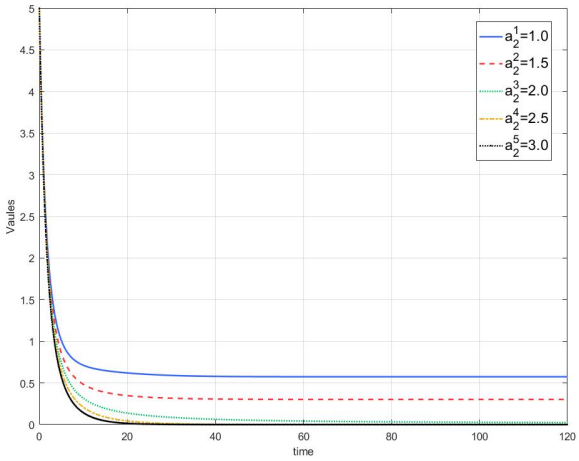
(g) $\sigma_i = 0$ (h) $\sigma_i = 0.3$ (i) $\sigma_i = 0.9$

Figure 3. The evolution of a single path of I_1 and I_2 for the stochastic system (1.2) when σ_i changes. And all other parameters are taken as in Table 1. (a)–(c) Influences of three different values of parameter σ_i influence the value of I_i when diseases I_1, I_2 are persistent initially; (d)–(f) influence of three different values of parameter σ_i on the value of I_i when I_1 is persistent and I_2 goes to extinction initially; (g)–(i) influences of three different values of parameter σ_i on the values of I_i when I_2 is persistent and I_1 goes to extinction initially. The initial value of all solutions is (15,10,5). The time unit is day.

From the above numerical simulations, we conclude that a larger a_i leads to a lower infected prevalence $I_i(t)$, and it may result in the extinction in deterministic and stochastic systems. This is because the parameter a_i affects $\mathcal{R}_i$ and $\mathcal{R}_i^s$. Additionally, we found that the infected population $I_i(t)$ also decreases when b_i increases, but b_i cannot lead to extinction.

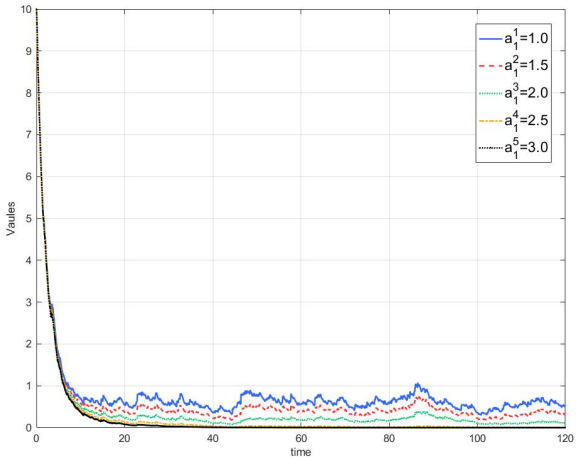


(a) $\alpha_i = 0.2$

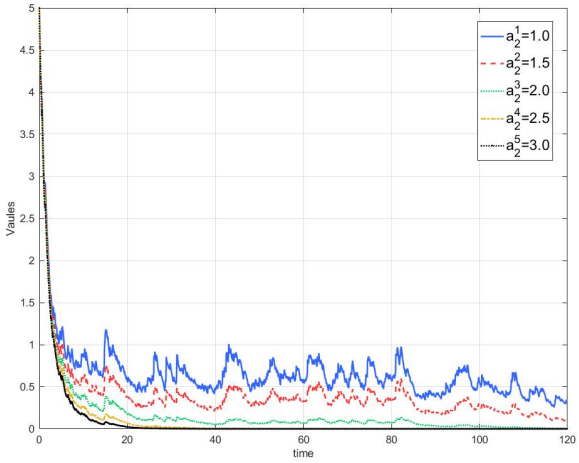


(b) $\alpha_i = 0.2$

Figure 4. The impact of the preventive effect a_i in the deterministic system (1.1). The preventive effect a_i reduces the number of infected individuals and keeps the susceptible individuals from becoming infected: (a) Influences of five different values of parameter a_1 on the values of I_1. (b) Influences of five different values of parameter a_2 on the values of I_2. The initial value of all solutions is (15,10,5). The time unit is day.

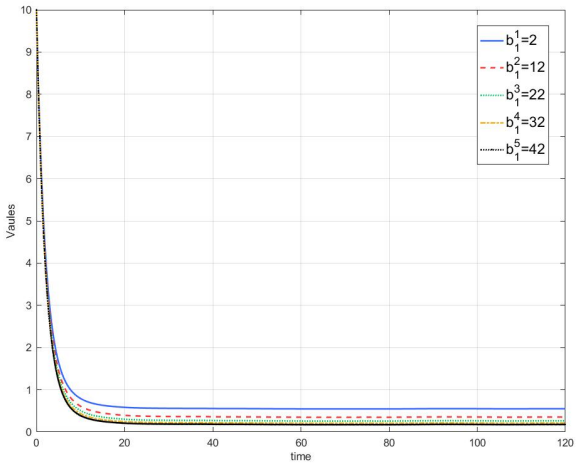


(c) $\sigma_i = 0.3, \alpha_i = 0.2$

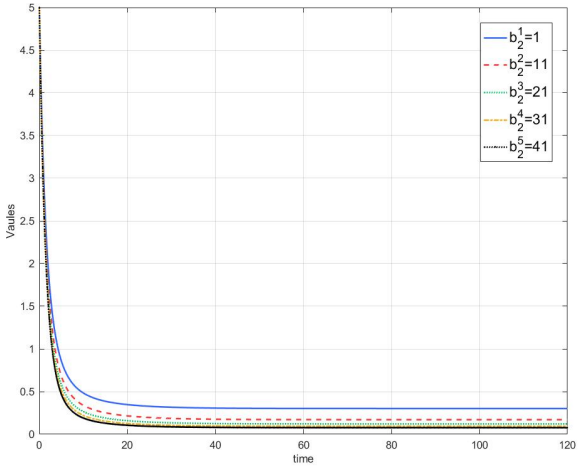


(d) $\sigma_i = 0.3, \alpha_i = 0.2$

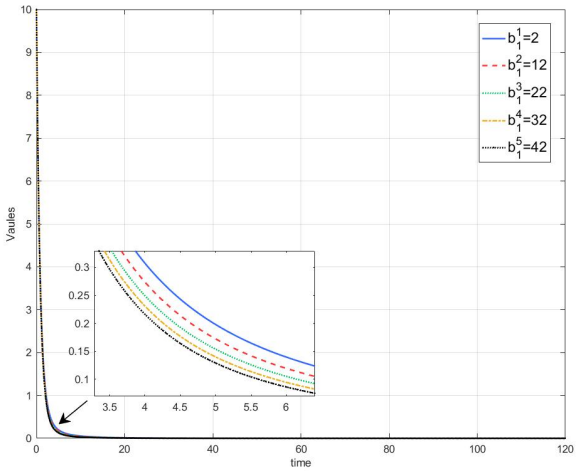
Figure 5. The impact of the preventive effect a_i in the stochastic system (1.2). The preventive effect a_i reduces the number of infected individuals and keeps the susceptible individuals from becoming infected: (c) Influences of five different values of parameter a_1 on the values of I_1. (d) Influences of five different values of parameter a_2 on the values of I_2. The initial value of all solutions is (15,10,5). The time unit is day.



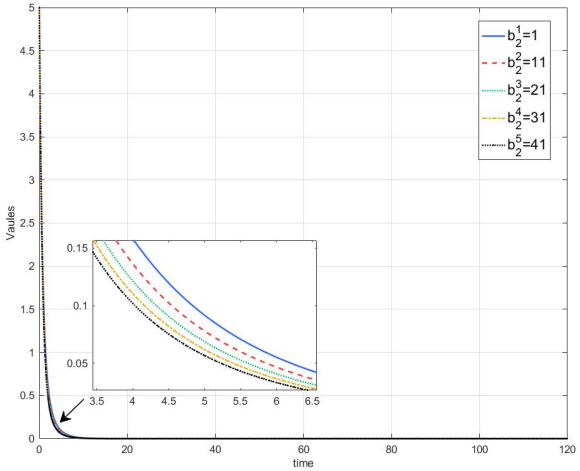
(a) $\alpha_i = 0.2$



(b) $\alpha_i = 0.2$

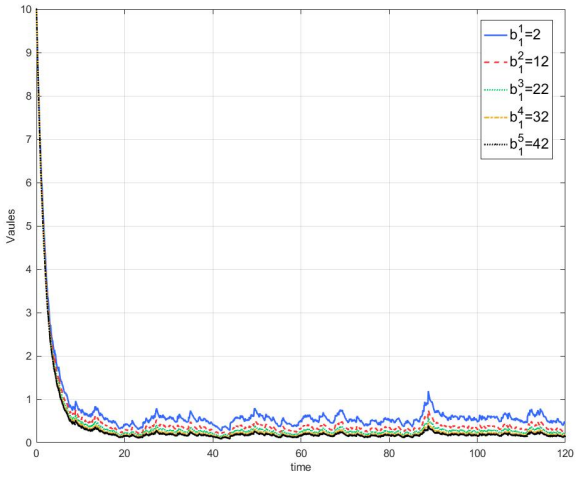


(c) $\alpha_i = 0.9$

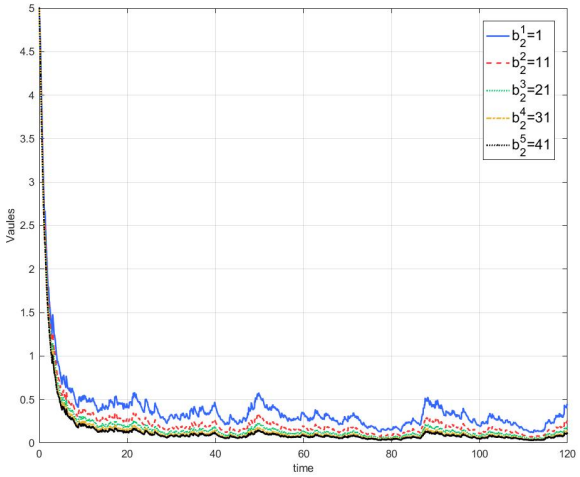


(d) $\alpha_i = 0.9$

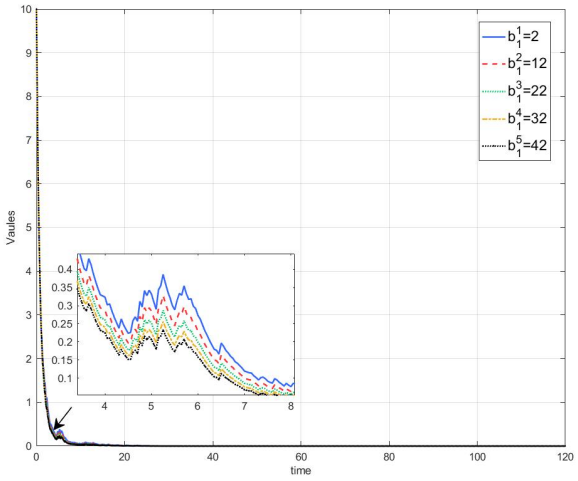
Figure 6. The impact of the treatment effect b_i in the deterministic system (1.1). The treatment effect b_i reduces the number of infected individuals and keeps the susceptible individuals from becoming infected: (a) Influences of five different values of parameter b_1 on the values of I_1. (b) Influences of five different values of parameter b_2 on the values of I_2. (c) Influences of five different values of parameter b_1 on the values of I_1. (d) Influences of five different values of parameter b_2 on the values of I_2. The initial value of all solutions is (15,10,5). The time unit is day.



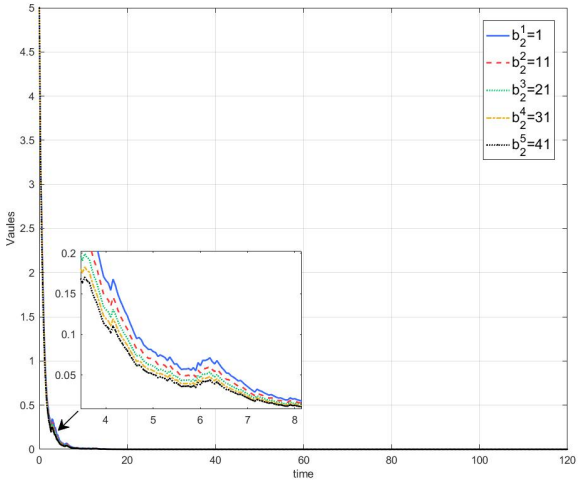
(a) $\alpha_i = 0.2$



(b) $\alpha_i = 0.2$



(c) $\alpha_i = 0.9$



(d) $\alpha_i = 0.9$

Figure 7. The impact of the treatment effect b_i in the stochastic system (1.2) when $\sigma_i = 0.3$. The treatment effect b_i reduces the number of infected individuals and keeps the susceptible individuals from becoming infected: (a) Influences of five different values of parameter b_1 on the values of I_1. (b) Influences of five different values of parameter b_2 on the values of I_2. (c) Influences of five different values of parameter b_1 on the values of I_1. (d) Influences of five different values of parameter b_2 on the values of I_2. The initial value of all solutions is (15,10,5). The time unit is day.

5. Discussion and conclusions

In this paper, we have proposed and studied a class of stochastic double disease models with Crowley-Martin incidence. We discussed the existence conditions and stability of the equilibrium points. E_0 is locally asymptotically stable when the basic reproduction number $\mathcal{R}_i < 1$; E_1 is locally asymptotically

stable when $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 < 1$; E_2 is locally asymptotically stable when $\mathcal{R}_2 > 1$ and $\mathcal{R}_1 < 1$; and E_3 is locally asymptotically stable when the basic reproduction number $\mathcal{R}_i > 1$. Subsequently, we have given the stochastic basic reproduction number $\mathcal{R}_i^*$ of the stochastic system and proven the stochastic extinction and persistence of the system. Finally, numerical simulations show that appropriate stochastic perturbations σ_i can control the spread of the disease, but larger stochastic perturbations can cause the disease to go extinct; the protection effect a_i can cause the disease to go extinct; the treatment effect b_i can reduce the number of infected individuals, but it cannot cause the disease to go extinct. Therefore, when treatment is given to infected individuals, protective measures for susceptible individuals are more necessary to completely eliminate the virus.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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Appendix A : Proof of Theorem 1

Proof. The system (1.1) has the following Jacobian matrix

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}, \tag{A1}$$

where

$$a_{11} = -\frac{\beta_1 I_1}{(1+a_1 S)^2(1+b_1 I_1)} - \frac{\beta_2 I_2}{(1+a_2 S)^2(1+b_2 I_2)} - m,$$

$$
\begin{aligned}
a_{12} &= -\frac{\beta_1 S}{(1+a_1S)(1+b_1I_1)^2} + \alpha_1,\\
a_{13} &= -\frac{\beta_2 S}{(1+a_2S)(1+b_2I_2)^2} + \alpha_2,\\
a_{21} &= \frac{\beta_1 I_1}{(1+a_1S)^2(1+b_1I_1)},\\
a_{22} &= \frac{\beta_1 S}{(1+a_1S)(1+b_1I_1)^2} - (m+\alpha_1+\delta_1),\\
a_{31} &= \frac{\beta_2 I_2}{(1+a_2S)^2(1+b_2I_2)},\\
a_{33} &= \frac{\beta_2 S}{(1+a_2S)^2(1+b_2I_2)} - (m+\alpha_2+\delta_2).
\end{aligned}
$$

Case 1. The evaluation of the Jacobian matrix at E_0 is represented by

$$
J(E_0) = \begin{pmatrix}
-m & -\frac{\beta_1 N}{m+a_1N} + \alpha_1 & -\frac{\beta_2 N}{m+a_2N} + \alpha_2\\
0 & \frac{\beta_1 N}{m+\alpha_1 N} - (m+\alpha_1+\delta_1) & 0\\
0 & 0 & \frac{\beta_2 N}{m+\alpha_2 N} - (m+\alpha_2+\delta_2)
\end{pmatrix},
$$

which has the following eigenvalues:

$$
\begin{aligned}
\lambda_1 &= -m,\\
\lambda_2 &= \frac{\beta_1 N}{m+\alpha_1 N} - (m+\alpha_1+\delta_1) = (m+\alpha_1+\delta_1)(\mathcal{R}_1 - 1),\\
\lambda_3 &= \frac{\beta_2 N}{m+\alpha_2 N} - (m+\alpha_2+\delta_2) = (m+\alpha_2+\delta_2)(\mathcal{R}_2 - 1).
\end{aligned}
$$

If $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 < 1$, then λ_1, λ_2 and $\lambda_3 < 0$. The disease-free equilibrium E_0 is locally asymptotically stable.

Case 2. The evaluation of the Jacobian matrix at E_1 is represented by

$$
J(E_1) = \begin{pmatrix}
a_{11} & a_{12} & a_{13}\\
a_{21} & a_{22} & 0\\
0 & 0 & a_{33}
\end{pmatrix},
$$

where

$$
\begin{aligned}
a_{11} &= -\frac{\beta_1 \bar{I}_1}{(1+a_1\bar{S}_1)^2(1+b_1\bar{I}_1)} - m,\\
a_{12} &= -\frac{\beta_1 \bar{S}_1}{(1+a_1\bar{S}_1)(1+b_1\bar{I}_1)^2} + \alpha_1,\\
a_{13} &= -\frac{\beta_2 \bar{S}_1}{1+a_2\bar{S}_1} + \alpha_2,\\
a_{21} &= \frac{\beta_1 \bar{I}_1}{(1+a_1\bar{S}_1)^2(1+b_1\bar{I}_1)},
\end{aligned}
$$

$$a_{22} = \frac{\beta_1 \bar{S_1}}{(1 + a_1\bar{S_1})(1 + b_1\bar{I_1})^2} - (m + \alpha_1 + \delta_1),$$
$$a_{33} = \frac{\beta_2 \bar{S_1}}{1 + a_2\bar{S_1}} - (m + \alpha_2 + \delta_2).$$

Using the equilibrium equation (2.1)

$$\frac{\beta_1 \bar{S_1}}{(1 + a_1\bar{S_1})(1 + b_1\bar{I_1})} = m + \alpha_1 + \delta_1;$$

we obtain

$$a_{22} = \frac{m + \alpha_1 + \delta_1}{1 + b_1\bar{I_1}} - (m + \alpha_1 + \delta_1) < 0;$$

one of the three eigenvalues of matrices $J(E_1)$'s is represented by

$$\lambda_1 = \frac{\beta_2 \bar{S_1}}{1 + a_2\bar{S_1}} - (m + \alpha_2 + \delta_2) < 0,$$

where $\mathcal{R}_2 < 1$ and $\bar{S_1} < \frac{N}{m}$ is used. What follows is the characteristic equation:

$$\lambda^2 + A_1\lambda + A_2 = 0,$$

where

$$\begin{aligned} A_1 =& -(a_{11} + a_{22}) \\ =& \frac{\beta_1\bar{I_1}}{\left(1 + a_1\bar{S_1}\right)^2\left(1 + b_1\bar{I_1}\right)} - \frac{\beta_1\bar{S_1}}{\left(1 + a_1\bar{S_1}\right)\left(1 + b_1\bar{I_1}\right)^2} + (m + \alpha_1 + \delta_1) + m \\ =& \frac{\beta_1\bar{I_1}}{\left(1 + a_1\bar{S_1}\right)^2\left(1 + b_1\bar{I_1}\right)} + \frac{\beta_1 b_1\bar{S_1}\bar{I_1}}{\left(1 + a_1\bar{S_1}\right)\left(1 + b_1\bar{I_1}\right)^2} + m \\ >& 0, \end{aligned}$$

$$\begin{aligned} A_2 =& a_{11}a_{22} - a_{21}a_{12} \\ =& \left(-\frac{\beta_1\bar{I_1}}{\left(1 + a_1\bar{S_1}\right)^2\left(1 + b_1\bar{I_1}\right)} - m\right)\left(\frac{\beta_1\bar{S_1}}{\left(1 + a_1\bar{S_1}\right)\left(1 + b_1\bar{I_1}\right)^2} - (m + \alpha_1 + \delta_1)\right) \\ =& (m + \delta_1)\frac{\beta_1\bar{I_1}(1 + b_1\bar{I_1}) - \beta_1 m\bar{S_1}(1 + a_1\bar{S_1})}{\left(1 + a_1\bar{S_1}\right)^2\left(1 + b_1\bar{I_1}\right)^2} + m(m + \alpha_1 + \delta_1) \\ =& \frac{\bar{I_1}(m + \delta_1)(1 + b_1\bar{I_1}) + \beta_1 m b_1\bar{S_1}\bar{I_1}(1 + a_1 S)}{\left(1 + a_1\bar{S_1}\right)^2\left(1 + b_1\bar{I_1}\right)^2} \\ >& 0. \end{aligned}$$

By the Routh-Hurwitz condition, if $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 < 1$, the boundary equilibrium E_1 is locally asymptotically stable. The proof process for Cases 2 and 3 are similar, so they are omitted.

Case 4. The evaluation of the Jacobian matrix at E_3 is represented by

$$J(E_3) = \begin{pmatrix} -a_{21} - a_{31} - m & -a_{22} - (m + \delta_1) & -a_{33} - (m + \delta_2) \\ \frac{\beta_1 I_1^*}{(1+a_1 S^*)^2(1+b_1 I_1^*)} & -\frac{\beta_1 b_1 S^* I_1^*}{(1+a_1 S^*)(1+b_1 I_1^*)^2} & 0 \\ \frac{\beta_2 I_2^*}{(1+a_2 S^*)^2(1+b_2 I_2^*)} & 0 & -\frac{\beta_2 b_2 S^* I_2^*}{(1+a_2 S^*)(1+b_2 I_2^*)^2} \end{pmatrix};$$

what follows is the characteristic equation:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where

$$\begin{aligned} \mathrm{A}_1 &= \frac{\beta_1 I_1^* \left(1 + b_1 I_1^* + b_1 S^* + a_1 b_1 S^{*2}\right)}{(1 + a_1 S^*)^2 \left(1 + b_1 I_1^*\right)^2} + \frac{\beta_2 I_2^* \left(1 + b_2 I_2^* + b_2 S^* + a_2 b_2 S^{*2}\right)}{(1 + a_2 S^*)^2 \left(1 + b_2 I_2^*\right)^2} \\ &:= C_1 + C_2 + m > 0, \end{aligned}$$

$$\begin{aligned} \mathrm{A}_2 =& (-a_{21} - a_{31} - m)a_{22} + (-a_{21} - a_{31} - m)a_{33} \\ &+ a_{22}a_{33} + a_{33}a_{31} + (m + \delta_2)a_{31} + a_{22}a_{21} + (m + \delta_1)a_{21} \\ =& a_{22}a_{33} - a_{22}a_{31} - a_{21}a_{33} + a_{21}(m + \delta_1) - ma_{22} + a_{31}(m + \delta_2) - ma_{33} \\ =& \frac{\beta_1\beta_2 S^* I_1^* I_2^* (b_1 b_2 S^* (1 + a_1 S^*)(1 + a_2 S^*) + b_1 (1 + a_1 S^*)(1 + b_2)^2 + b^2 (1 + a_2 S^* (1 + b_1)^2))}{\left((1 + a_1 S^*)\left(1 + b_1 I_1^*\right)(1 + a_2 S^*)\left(1 + b_2 I_2^*\right)\right)^2} \\ &+ \frac{\beta_1 I_1^* ((1 + b_1 I_1^*)(m + \delta_1) + m b_1 S^* (1 + a_1 S^*))}{(1 + a_1 S^*)^2 \left(1 + b_1 I_1^*\right)^2} + \frac{\beta_2 I_2^* ((1 + b_2 I_2^*)(m + \delta_2) + m b_2 S^* (1 + a_2 S^*))}{(1 + a_2 S^*)^2 \left(1 + b_2 I_2^*\right)^2} \\ :=& C_3 + C_4 + C_5 > 0, \end{aligned}$$

$$\begin{aligned} \mathrm{A}_3 =& -(m + \delta_1)a_{21}a_{33} - (m + \delta_2)a_{22}a_{31} + ma_{22}a_{33} \\ =& \frac{\beta_1\beta_2 b_2 S^* I_1^* I_2^* (m + \delta_1)}{(1 + a_1 S^*)^2 \left(1 + b_1 I_1^*\right)(1 + a_2 S^*)\left(1 + b_2 I_2^*\right)^2} + \frac{\beta_1\beta_2 b_1 S^* I_1^* I_2^* (m + \delta_2)}{(1 + a_1 S^*)\left(1 + b_1 I_1^*\right)^2 (1 + a_2 S^*)^2 \left(1 + b_2 I_2^*\right)} \\ &+ \frac{\beta_1\beta_2 m b_1 b_2 I_1^* I_2^* S^{*2}}{(1 + a_1 S^*)\left(1 + b_1 I_1^*\right)^2 (1 + a_2 S^*)\left(1 + b_2 I_2^*\right)^2} \\ >& 0. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{A}_1\mathrm{A}_2 - \mathrm{A}_3 &= (C_1 + C_2 + m)(C_3 + C_4 + C_5) - A_3 \\ &= (C_1 + C_{22})C_5 + C_1C_3 + C_2C_4 + m(C_3 + C_4 + C_5) + C_1C_4 + C_2C_3 - A_3 \end{aligned}$$

$$:= p + q,$$

where $p > 0$ when $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 > 1$, and we have

$$\begin{aligned} q =& C_1C_4 + C_2C_3 - A_3 \\ =& \frac{\beta_1\beta_2 I_1^* I_2^* \left(1 + b_2 I_2^*\right)\left((1 + b_1 I_1^*)(m + \delta_1) + mb_1 S(1 + a_1 S)\right) + b_1 S(1 + a_1 S)(mb_2 S(1 + a_2 S))}{\left((1 + a_1 S^*)\left(1 + b_1 I_1^*\right)(1 + a_2 S^*)\right)^2} \\ &+ \frac{\beta_1\beta_2 I_1^* I_2^* (1 + b_1 I_1)((1 + b_2 I_2)(m + \delta_2) + mb_2 S(1 + a_2 S))}{\left((1 + a_1 S^*)\left(1 + b_1 I_1^*\right)(1 + a_2 S^*)\right)^2} \\ >& 0. \end{aligned}$$

When both $\mathcal{R}_1$ and $\mathcal{R}_2$ are bigger than 1, the Routh-Hurwitz criteria show that the endemic equilibrium E_3 is locally asymptotically stable.

Appendix B : Proof of Theorem 3

Proof. Part (i). It is easy to see from the theorem that $\lim_{t\to+\infty} I_2(t) = 0$. Since $\mathcal{R}_1^s > 1$, there exists a ε small enough such that $0 < I_2(t) < \varepsilon$ for all t large enough; we have

$$\frac{\beta(N - (m + \delta_1)\varepsilon)}{(Na_1 + m)(m + \alpha_1 + \delta_1)} - \frac{\sigma_1^2 N^2}{2(Na_1 + m)(m + \alpha_1 + \delta_1)} > 1.$$

On the sides of the system (1.2), dividing by $t > 0$ and integrating from 0 to t gives

$$\begin{aligned} \Theta(t) &\triangleq \frac{S(t) - S(0)}{t} + \frac{I_1(t) - I_1(0)}{t} + \frac{I_2(t) - I_2(0)}{t} \\ &= N - m\langle S(t)\rangle - (m + \delta_1)\langle I_1(t)\rangle - (m + \delta_2)\langle I_2(t)\rangle \\ &\geq N - m\langle S(t)\rangle - (m + \delta_1)\langle I_1(t)\rangle - (m + \delta_2)\varepsilon; \end{aligned}$$

when $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\theta)d\theta$ is defined for an integrable function f on $[0, +\infty)$, one may get

$$\langle S(t)\rangle \geq \frac{N - (m + \delta_2)\varepsilon}{m} - \frac{m + \delta_1}{m}\langle I_1(t)\rangle - \frac{\Theta(t)}{m}.$$

By using Itô′s formula, it follows that

$$\begin{aligned} &\mathrm{d}\left(\left(1 + a_1\frac{N}{m}\right)\ln I_1 + b_1\left(1 + a_1\frac{N}{m}\right)I_1\right) \\ =& \left(\frac{\left(1 + a_1\frac{N}{m}\right)\beta_1 S}{(1 + a_1 S)(1 + b_1 I_1)} - \left(1 + a_1\frac{N}{m}\right)(m + \alpha_1 + \delta_1) - \frac{\left(1 + a_1\frac{N}{m}\right)\sigma_1^2 S^2}{2(1 + a_1 S)^2(1 + b_1 I_1)^2}\right)\mathrm{d}t \\ &+ b_1\left(1 + a_1\frac{N}{m}\right)\left(\frac{\beta_1 S I_1}{(1 + a_1 S)(1 + b_1 I_1)} - (m + \alpha_1 + \delta_1)I_1\right)\mathrm{d}t \\ &+ \frac{\left(1 + a_1\frac{N}{m}\right)\sigma_1 S}{(1 + a_1 S)(1 + b_1 I_1)}\mathrm{d}B_1(t) + \frac{b_1\left(1 + a_1\frac{N}{m}\right)\sigma_1 S I_1}{(1 + a_1 S)(1 + b_1 I_1)}\mathrm{d}B_1(t) \end{aligned}$$

$$\geq\left(\frac{\beta_1 S}{(1+b_1 I_1)}-\left(1+a_1\frac{N}{m}\right)(m+\alpha_1+\delta_1)-\frac{\sigma_1^2}{2}\left(1+a_1\frac{N}{m}\right)\left(\frac{N}{m+Na_1}\right)^2\right)\mathrm{d}t$$
$$+b_1\left(1+a_1\frac{N}{m}\right)\left(\frac{\beta_1 S I_1}{\left(1+a_1\frac{N}{m}\right)(1+b_1 I_1)}-(m+\alpha_1+\delta_1)I_1\right)\mathrm{d}t+\sigma_1 S(t)\mathrm{d}B_1(t)$$

$$\geq\left(\beta_1 S-\left(1+a_1\frac{N}{m}\right)(m+\alpha_1+\delta_1)-b_1\left(1+a_1\frac{N}{m}\right)(m+\alpha_1+\delta_1)I_1-\frac{\sigma^2}{2}\frac{\left(\frac{N}{m}\right)^2}{1+a_1\frac{N}{m}}\right)dt \tag{A2}$$
$$+\sigma_1 S\,\mathrm{d}B_1(t). \tag{A3}$$

On the sides of (A3), dividing by $t>0$ and integrating from 0 to t gives

$$\left(1+a_1\frac{N}{m}\right)\frac{\ln I_1(t)-\ln I_1(0)}{t}+b_1\left(1+a_1\frac{N}{m}\right)\frac{I_1(t)-I_1(0)}{t}$$
$$\geq\beta_1\langle S(t)\rangle-\frac{m+a_1N}{m}(m+\alpha_1+\delta_1)-b_1\frac{m+a_1N}{m}(m+\alpha_1+\delta_1)\langle I_1(t)\rangle-\frac{\sigma_1^2\left(\frac{N}{m}\right)^2}{\frac{2(m+a_1N)}{m}}+\frac{Q_2}{t}$$
$$\geq\beta_1\Big(\frac{N-(m+\delta_2)\varepsilon}{m}-\frac{m+\delta_2}{m}\langle I_1(t)\rangle-\frac{\Theta(t)}{m}\Big)-\frac{m+a_1N}{m}(m+\alpha_1+\delta_1)+\frac{Q_2(t)}{t}$$
$$-\frac{b_1(m+a_1N)}{m}(m+\alpha_1+\delta_1)\langle I_1(t)\rangle-\frac{\sigma_1^2\left(\frac{N}{m}\right)^2}{\frac{2(m+a_1N)}{m}}$$
$$=\frac{m+a_1N}{m}(m+\alpha_1+\delta_1)\Big(\frac{\beta_1(N-(N+\delta_2)\varepsilon)}{(m+a_1N)(m+\alpha_1+\delta_1)}-\frac{\sigma_1^2(\frac{N}{m})^2}{2(\frac{m+a_1N}{m})^2(m+\alpha_1+\delta_1)}-1\Big)$$
$$-\Big(\frac{\beta_1(m+\delta_1)}{m}+\frac{b_1(m+a_1N)}{m}(m+\alpha_1+\delta_1)\Big)\langle I_1(t)\rangle-\frac{\beta_1\Theta(t)}{m}+\frac{Q_2(t)}{t}, \tag{A4}$$

where $Q_2(t)=\int_0^t\sigma_1 S(\tau)\mathrm{d}B_1(\tau)$. It is possible to rewrite the inequality (A4) as

$$\langle I_1(t)\rangle\geq\frac{1}{\Delta}\Big\{\frac{m+a_1N}{m}(m+\alpha_1+\delta_1)(\frac{\beta_1(N-(N+\delta_2)\varepsilon)}{(m+a_1N)(m+\alpha_1+\delta_1)}-\frac{\sigma_1^2\left(\frac{N}{m}\right)^2}{2\left(\frac{m+a_1N}{m}\right)^2(m+\alpha_1+\delta_1)}-1)$$
$$-\frac{\beta_1\Theta(t)}{m}+\frac{Q_2(t)}{t}-\frac{1}{t}\frac{m+a_1N}{m}\Big[(\ln I_1(t)-\ln I_1(0))+b_1(I_1(t)-I_1(0))\Big]\Big\} \tag{A5}$$
$$\geq\begin{cases}\frac{1}{\Delta}\Big\{\frac{m+a_1N}{m}(m+\alpha_1+\delta_1)(\frac{\beta_1(N-(N+\delta_2)\varepsilon)}{(m+a_1N)(m+\alpha_1+\delta_1)}-\frac{\sigma_1^2\left(\frac{N}{m}\right)^2}{2\left(\frac{m+a_1N}{m}\right)^2(m+\alpha_1+\delta_1)}-1)\\ -\frac{\beta_1\Theta(t)}{m}+\frac{Q_2(t)}{t}+\frac{1}{t}\frac{m+a_1N}{m}\Big[\ln I_1(0)-b_1(I_1(t)-I_1(0))\Big]\Big\},0<I_1(t)<1,\\ \frac{1}{\Delta}\Big\{a_1(m+\alpha_1+\delta_1)(\frac{\beta_1(N-(N+\delta_2)\varepsilon)}{(m+a_1N)(m+\alpha_1+\delta_1)}-\frac{\sigma_1^2\left(\frac{N}{m}\right)^2}{2\left(\frac{m+a_1N}{m}\right)^2(m+\alpha_1+\delta_1)}-1)\\ -\frac{\beta_1\Theta(t)}{m}+\frac{Q_2(t)}{t}-\frac{1}{t}\frac{m+a_1N}{m}\Big[(\ln I_1(t)-\ln I_1(0))+b_1(I_1(t)-I_1(0))\Big]\Big\},1<I_1(t),\end{cases}$$

where $\Delta = \frac{\beta_1(m+\delta_1)}{m} + b_1(m + a_1N)(m + \alpha_1 + \delta_1)$.

By Lemma 3, we get that $\lim\limits_{t\to+\infty} \frac{Q_2(t)}{t} = 0$. We can observe that $I_1(t) \le \frac{N}{m}$; thus, we have that $\lim\limits_{t\to+\infty} \frac{I_1(t)}{t} = 0$ and $\lim\limits_{t\to+\infty} \frac{\ln I_1(t)}{t} = 0$ as $I_1(t) \ge 1$ and $\lim\limits_{t\to+\infty} \Theta(t) = 0$.

When the limit inferior of both sides of (A5) are taken into account, we have

$$\begin{aligned}\liminf_{t\to+\infty}\langle I_1(t)\rangle \ge& \frac{\left(1 + a_1\frac{N}{m}\right)(m + \alpha_1 + \delta_1)}{\Delta}\left(\frac{\beta_1(N - (N + \delta_2)\varepsilon)}{m\left(1 + a_1\frac{N}{m}\right)(m + \alpha_1 + \delta_1)} - \frac{\sigma_1^2\left(\frac{N}{m}\right)^2}{2\left(1 + a_1\frac{N}{m}\right)^2(m + \alpha_1 + \delta_1)} - 1\right)\\ \ge& 0.\end{aligned}$$

Allowing for $\varepsilon \to 0$, we have

$$\liminf_{t\to+\infty}\langle I_1(t)\rangle \ge \frac{(m + a_1N)(m + \alpha_1 + \delta_1)}{\beta_1(m + \delta_1) + b_1(m + a_1N)(m + \alpha_1 + \delta_1)}(\mathcal{R}_1^s - 1).$$

Due to the fact that the methods of proving parts (ii) and (i) are similar, this step will not be repeated. Part (iii). Take note that

$$\langle S(t)\rangle = \frac{N}{m} - \frac{m + \delta_1}{m}\langle I_1(t)\rangle - \frac{m + \delta_2}{m}\langle I_2(t)\rangle - \frac{\Theta(t)}{m}. \tag{A6}$$

Define

$$V(t) = \ln\left[I_1^{\left(1+a_1\frac{N}{m}\right)}(t)I_2^{\left(1+a_2\frac{N}{m}\right)}(t)\right] + \left[b_1(1 + a_1\frac{N}{m})I_1(t) + b_2(1 + a_2\frac{N}{m})I_2(t)\right]. \tag{A7}$$

Consequently, $V(t)$ is bounded. We have

$$\begin{aligned}D^+V(t) =& \sum_{i=1}^{2}\Big(\frac{\left(1 + a_i\frac{N}{m}\right)\beta_iS}{(1 + a_iS)(1 + b_iI_i)} - \left(1 + a_i\frac{N}{m}\right)(m + \alpha_i + \delta_i) - \frac{\left(1 + a_i\frac{N}{m}\right)\sigma_i^2S^2}{2((1 + a_iS)(1 + b_iI_i))^2}\Big)\mathrm{d}t\\ &+ \sum_{i=1}^{2}\frac{\left(1 + a_i\frac{N}{m}\right)\sigma_iS}{(1 + a_iS)(1 + b_iI_i)}\mathrm{d}B_i(t) + \sum_{i}^{2} b_i\left(1 + a_i\frac{N}{m}\right)\Big(\frac{\beta_iSI_i}{(1 + a_iS)(1 + b_iI_i)} - (m + \alpha_i + \delta_i)I_i\Big)\mathrm{d}t\\ &+ \sum_{i}^{2}\frac{b_i\left(1 + a_i\frac{N}{m}\right)\sigma_iSI_i}{(1 + a_iS)(1 + b_iI_i)}\mathrm{d}B_i(t)\\ \ge& \sum_{i=1}^{2}\Big(\frac{\left(1 + a_i\frac{N}{m}\right)\beta_iS}{1 + a_i\frac{N}{m} + b_iI_i + a_ib_i\frac{N}{m}I_i} - \left(1 + a_i\frac{N}{m}\right)(m + \alpha_i + \delta_i) - \frac{\left(1 + a_i\frac{N}{m}\right)\sigma_i^2S^2}{2((1 + a_iS)(1 + b_iI_i))^2}\Big)\mathrm{d}t\\ &+ \sum_{i=1}^{2}\frac{\left(1 + a_i\frac{N}{m}\right)\sigma_iS}{\left(1 + a_i\frac{N}{m}\right)(1 + b_iI_i)}\mathrm{d}B_i(t) + \sum_{i}^{2} b_i\left(1 + a_i\frac{N}{m}\right)\Big(\frac{\beta_iSI_i}{\left(1 + a_i\frac{N}{m}\right)(1 + b_iI_i)} - (m + \alpha_i + \delta_i)I_i\Big)\mathrm{d}t\\ &+ \sum_{i}^{2}\frac{b_i\left(1 + a_i\frac{N}{m}\right)\sigma_iSI_i}{1 + a_i\frac{N}{m} + b_iI_i + a_ib_i\frac{N}{m}I_i}\mathrm{d}B_i(t)\\ \ge& \left((\beta_1 + \beta_2)S - \sum_{i}^{2}(m + \alpha_i + \delta_i)(1 + b_iI_i)\left(1 + a_i\frac{N}{m}\right) + \sum_{i=1}^{2}\frac{\sigma_i^2\left(\frac{N}{m}\right)^2}{\frac{2(m+a_iN)}{m}}\right)\mathrm{d}t + \sum_{i}^{2}\sigma_iS\,\mathrm{d}B_i(t).\end{aligned} \tag{A8}$$

On the sides of (A8), dividing by $t > 0$ and integrating from 0 to t gives

$$\begin{aligned}\frac{V(t)-V(0)}{t} \geq & \sum_{i=1}^{2}\left(\beta_i\langle S(t)\rangle - (m+\alpha_i+\delta_i)\left(1+a_i\frac{N}{m}\right)(1+b_i\langle I_i(t)\rangle) - \frac{\sigma_i^2(\frac{N}{m})^2}{\frac{2(m+a_iN)}{m}} + \frac{Q_i}{t}\right)\\ =&(\beta_1+\beta_2)\frac{N}{m} - \sum_{i=1}^{2}(m+\alpha_i+\delta_i) - \sum_{i=1}^{2}\frac{\sigma_i^2\left(\frac{N}{m}\right)^2}{2\left(1+a_i\frac{N}{m}\right)} - \frac{\beta_1+\beta_2}{m}\Theta(t) + \sum_{i=1}^{2}\frac{Q_i}{t}\\ &- \sum_{i=1}^{2}\left(\frac{\beta_i}{m}(m+\delta_i) + b_i(m+\alpha_i+\delta_i)\left(1+a_i\frac{N}{m}\right)\right)\langle I_i(t)\rangle\\ \geq & \sum_{i=1}^{2}\left(1+a_i\frac{N}{m}\right)(m+\alpha_i+\delta_i)\left(\frac{\beta_i N}{(m+a_iN)(m+\alpha_i+\delta_i)} - \frac{\sigma_i^2N^2}{2\left(1+a_i\frac{N}{m}\right)(m+\alpha_i+\delta_i)} - 1\right)\\ &- \Delta_{\max}\left[\langle I_1(t)\rangle + \langle I_2(t)\rangle\right] - \sum_{i=1}^{2}\left(\frac{\beta_i}{m}\Theta(t) - \frac{Q_i}{t}\right), \end{aligned} \tag{A9}$$

where

$$Q_i(t) = \int_0^t \sigma_i S(\tau)dB_i(\tau), \quad \Delta_{\max} = \sum_{i=1}^{2}\left[\frac{\beta_1+\beta_2}{m}(m+\delta_i) + b_i(m+\alpha_i+\delta_i)\right].$$

It is possible to rewrite the inequality (A9) as

$$\begin{aligned}\left\langle\sum_{i=1}^{2} I_i(t)\right\rangle \geq & \frac{1}{\Delta_{\max}}\left(\sum_{i=1}^{2}\left(1+a_i\frac{N}{m}\right)(m+\alpha_i+\delta_i)\left(\frac{\beta_i N}{(m+a_iN)(m+\alpha_i+\delta_i)} - \frac{\sigma_i^2N^2}{2\left(1+a_i\frac{N}{m}\right)(m+\alpha_i+\delta_i)} - 1\right)\right.\\ &\left. + \frac{V(0)-V(t)}{t} - \sum_{i=1}^{2}\frac{\beta_i}{m}\Theta(t) + \sum_{i=1}^{2}\frac{Q_i}{t}\right). \end{aligned} \tag{A10}$$

By Lemma 3, we have that $\lim\limits_{t\to+\infty}\frac{Q_i(t)}{t} = 0$ for $i = 1, 2$. And we can see that $\lim\limits_{t\to+\infty}\Theta(t) = 0$ and $\lim\limits_{t\to+\infty}\frac{V(t)}{t} = 0$.

Taking the limit inferior of both sides of $(A10)$ yields

$$\liminf_{t\to+\infty}\left\langle\sum_{i=1}^{2} I_i(t)\right\rangle \geq \frac{1}{\Delta_{\max}}\sum_{i=1}^{2}\left(1+a_i\frac{N}{m}\right)(m+\alpha_i+\delta_i)(\mathcal{R}_i^s - 1) > 0.$$