Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(10): 6089–6119.
DOI: 10.3934/era.2023310
Received: 31 May 2023
Revised: 30 July 2023
Accepted: 01 September 2023
Published: 12 September 2023

Research article

On the regularization and matrix Lyapunov functions for fuzzy differential systems with uncertain parameters

Anatoliy Martynyuk[1], **Gani Stamov**[2,*], **Ivanka Stamova**[2] **and Yulya Martynyuk–Chernienko**[1]

[1] S. P. Timoshenko Institute of Mechanics, NAS of Ukraine, 3 Nesterov str., Kiev-57, Ukraine

[2] Department of Mathematics, University of Texas at San Antonio, One UTSA Circle, San Antonio, TX 78249, USA

* **Correspondence:** Email: gani.stamov@utsa.edu; Tel: +12104586103.

Abstract: In this paper, for a regularized fuzzy system, a generalization of the direct Lyapunov method is adapted on the base of matrix-valued Lyapunov-like functions. First, the new concept of a regularization scheme for fuzzy systems is discussed and the matrix-valued Lyapunov function technique is introduced. Then, sufficient conditions are established for the boundedness and stability of the equilibrium set of solutions of the regularized fuzzy system of differential equations. Scalar and vector Lyapunov-type functions are used based on an auxiliary matrix-valued function. Finally, a discussion is offered for the future directions of the proposed approach. Since the strategies for the analysis of the stability of fuzzy models are very important in numerous aspects, we expect that our results will inspire researchers to develop the introduced concept.

Keywords: fuzzy systems; regularization scheme; boundedness; stability; matrix Lyapunov functions

1. Introduction

One of the main assumptions in the classical stability of motion theory [1, 2] is the assumption of the invariance of the parameters of a system in the course of its movement.

Further development of the stability theory is associated with the investigation of dynamical systems with uncertain parameters. See, for example, [3] and some of the references therein, as well as some recent results on models with uncertain parameters [4–7]. The active research on the stability analysis for systems with parameter uncertainties shows the importance of the topic for practice. In fact, many real-world problems exhibit different types of uncertainties. That is why there are numerous studies that have addressed the elaboration of power techniques to investigate of the effect of parameter uncertainties [8–12].

One of the main tools in the study of the stability behavior of different classes of systems with uncertain parameters are the scalar and vector Lyapunov-type of functions [13–17]. The method of matrix-valued Lyapunov functions for the stability analysis of the solutions of continuous systems of differential equations has been developed and outlined in the monographs [18, 19].

Due to the benefits for theory and applications, the Lyapunov function strategy has attracted much attention from researchers of fuzzy systems of differential equations, and it is intensively applied in their stability analysis. For example, the book [20] offers an excellent overview of the state-of-the-art research on the theory of fuzzy differential equations and inclusions and provides a systematic account of the developments in their stability analysis. The stability theory for fuzzy differential equations based on Lyapunov functions has also been developed in numerous articles [21–25]. However, the method of matrix Lyapunov functions has not been elucidated for such systems, which is one of the main goals of this research.

In addition, for fuzzy systems with uncertain parameters, the development of new methods for the qualitative analysis of stability and boundedness remains an open problem.

In our earlier paper [26] we introduced a regularization scheme for systems of fuzzy differential equations with uncertain parameters as a new approach in the study of the properties of such systems. The proposed strategy reduces a collection of fuzzy differential equations to a simple form that allows for analysis of the properties of solutions of both the original fuzzy system of differential equations, as well as the reduced collection of differential equations. The idea is to use a family of mappings and regularize the fuzzy systems with respect to uncertain parameters.

In this paper, using the proposed approach in [26], sufficient conditions for the stability and boundedness of the equilibria of the regularized fuzzy system are proposed through the application of the Lyapunov function method on the basis of matrix-valued functions and non-linear integral inequalities. The established results contribute to the development of new methods for the study of fuzzy differential systems with uncertain parameters. This research also adopts the matrix-valued Lyapunov function strategy to their qualitative analysis.

The contributions of our paper are as follows:

(i) using a new regularization scheme for uncertain fuzzy differential equations with uncertain parameter we establish stability criteria for the regularized system via the matrix-valued Lyapunov function technique;

(ii) stability analysis for autonomous comparison problems is proposed and new stability results for autonomous systems are established;

(iii) boundedness and Lagrange stability criteria for the regularized system are proved;

(iv) the results offered show that the regularization scheme is a very advantageous technique which allows for analysis of the qualitative properties of solutions of both the original fuzzy system of differential equations, as well as the intermediate families of differential equations in a simple way.

The investigation is organized according to the following plan. In Section 2 some important notes on fuzzy sets and functions are provided. Section 3 is devoted to the new regularization approach developed in [26]. Matrix-valued Lyapunov functions and some of their properties are considered in Section 4. In Section 5 stability, uniform stability and asymptotic stability results for the regularized system are established. In Section 6, a stability analysis for autonomous comparison problems is conducted. Section 7 includes the boundedness results. Finally, Section 8 offers some comments and future directions for research.

2. Notes on fuzzy sets and functions

We assume that X is a basic set, and that, for any $x \in X$, $\varphi(x)$ is a membership function that takes its values from the interval $[0, 1]$. Following [27–29] for a fuzzy set with a membership function φ on X its κ-level sets $[\varphi]^\kappa$ are defined as

$$[\varphi]^\kappa = \{x \in X : \ \varphi(x) \geq \kappa\}, \ \kappa \in (0, 1]$$

and its support is given by

$$[\varphi]^0 = \overline{\bigcup_{\kappa \in (0,1]} [\varphi]^\kappa}.$$

We will use the following notation for the Hausdorff distance between two sets $\Xi, \Pi \in \mathbb{R}^n$ and $\Xi, \Pi \neq \emptyset$:

$$d_H(\Xi, \Pi) = \min\{H \geq 0 : \Xi \subseteq \{\Pi \cup \Pi_H(0)\}\,, \ \Pi \subseteq \{\Xi \cup \Pi_H(0)\}\},$$

where $\Pi_H(0) = \{x \in \mathbb{R}^n : \|x\| < H\}, \ H \geq 0$.

The defined Hausdorff distance $d_H(\Xi, \Pi)$ is a metric for any nonempty closed sets in $\mathbb{R}^n$. In addition, the pair (C^n, d_H) is a metric space, where C^n is the set of all nonempty closed sets in $\mathbb{R}^n$.

We will denote by E^n the space of all functions $\varphi : \mathbb{R}^n \to [0, 1]$ such that

1) φ is upper semicontinuous in the sense of Baire;
2) there exists an $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) = 1$;
3) u is fuzzy convex, i. e., $\varphi(\lambda x + (1 - \lambda)y) \geq \min\{\varphi(x), \varphi(y)\}$, $\lambda \in [0, 1]$;
4) the closure of the set $\{x \in \mathbb{R}^n : \varphi(x) > 0\}$ is a compact subset of $\mathbb{R}^n$.

It is well known [20, 26, 29] that, if a fuzzy set with a membership function φ is a fuzzy convex set, then $[\varphi]^\kappa$ is convex in $\mathbb{R}^n$ for any $\kappa \in [0, 1]$.

The distance between two sets $\varphi, \varpi \in E^n$ will be given as $d(\varphi, \varpi) = \sup\{|\varphi(x)-\varpi(x)| : \ x \in \mathbb{R}^n\}$; also, the least upper bound of the metric d on the space E^n is defined by $d[\varphi, \varpi] = \sup\{d_H([\varphi]^\kappa, [\varpi]^\kappa) : \ \kappa \in [0, 1]\}$ for $\varphi, \ \varpi \in E^n$, and it is a metric in E^n.

For any two sets $\varphi, \varpi \in E^n$ the element $\iota \in E^n$ such that $\varphi = \varpi + \iota$, which, if it exists, is the Hukuhara difference of φ and ϖ and is denoted by $\varphi - \varpi$.

The family of all nonempty compact convex subsets of $\mathbb{R}^n$ will be denoted by $P_k(\mathbb{R}^n)$.

The integral of a mapping F on a compact interval $T = [t_1, t_2], \ t_2 > t_1 > 0$ is denoted by $\int_a^b F(t)\, dt$; for any $0 < \kappa \leq 1$, it is given as

$$\int_T F(t)\, dt = \left\{ \int_T \bar{f}(t)\, dt \mid \ \bar{f} : \ T \to \mathbb{R}^n \text{ is a measurable selection for } F_\kappa \right\}.$$

The mapping $F : \ T \to E^n$ is said to be differentiable at $t_0 \in T$ if the value $F'(t_0)$ exists and $F'(t_0) \in E^n$ is such that both limits $\lim\{[F(t_0 + h) - F(t_0)]h^{-1} : h \to 0^+\}$ and $\lim\{[F(t_0) - F(t_0 - h)]h^{-1} : h \to 0^+\}$ exist and are equal to $F'(t_0)$. The above limits are considered in the metric space (E^n, d).

If F_κ is differentiable, the mapping F_κ is differentiable in the sense of Hukuhara for all $\kappa \in [0, 1]$ and $D_H F_\kappa(t) = [\, F'(t)\,]^\kappa$, where $D_H F_\kappa$ is the Hukuhara-type derivative of F_κ.

The family $\{D_H F_\kappa(t) : \kappa \in [0,1]\}$ determines an element $F'(t) \in E^n$. Also, if $F : T \to E^n$ is differentiable at $t \in T$, then the element $F'(t)$ is called the fuzzy derivative of $F(t)$ at the point t.

More on the concepts of fuzzy sets and functions is available in [30–32] and some of the references therein. For important results related to fuzzy differential equations, see [33–37].

3. Regularization procedure

Throughout the entire paper we consider the following fuzzy system with an uncertain parameter

$$\frac{du}{dt} = f(t,u,\alpha), \quad u(t_0) = u_0, \tag{3.1}$$

where $u \in E^{n^2}$; $f \in C(\mathbb{R}_+ \times E^{n^2} \times \mathcal{S}, E^{n^2})$; $E^{n^2} = E^n \times E^n$; $\alpha \in \mathcal{S}$ is an uncertain parameter; $\mathcal{S}$ is a compact set in $\mathbb{R}^d$.

As it is stated in [26], the parameter vector α represents the uncertainty in system (3.1). It can have a different nature and may represent different characteristics. For example, the uncertainty parameter α

(a) may describe an uncertain value of a certain physical parameter;
(b) may represent an estimate of an external disturbance;
(c) may describe an inaccurate measured value of the input effect of one of the subsystems on the other one;
(d) may represent some nonlinear elements of the considered mechanical system that are too complicated to be measured accurately;
(e) may be an indicator of the existence of some inaccuracies in the system (3.1);
(f) may be a union of the characteristics (a)–(e).

To regularize the system (3.1) with respect to the uncertain parameter α, in [26] we consider a family of mappings $f_\kappa(t,v)$ defined by

$$f_\kappa(t,v) = f_M(t,v)\kappa + (1-\kappa)f_m(t,v), \quad 0 \le \kappa \le 1, \tag{3.2}$$

where

$$f_m(t,v) = \overline{\text{co}} \bigcap_{\alpha \in \mathcal{S}} f(t,v,\alpha), \quad \mathcal{S} \subseteq \mathbb{R}^d, \tag{3.3}$$

$$f_M(t,v) = \overline{\text{co}} \bigcup_{\alpha \in \mathcal{S}} f(t,v,\alpha), \quad \mathcal{S} \subseteq \mathbb{R}^d. \tag{3.4}$$

Now, we propose to consider the following regularized with respect to $\alpha \in \mathcal{S}$ system of fuzzy differential equations of the type

$$\frac{dv}{dt} = f_\kappa(t,v), \quad v(t_0) = v_0, \tag{3.5}$$

where $f_\kappa \in C(I \times E^{n^2}, E^{n^2})$, $I = [t_0, t_0 + \tau]$, $t_0 \ge 0$, $\tau > 0$, $\kappa \in [0,1]$.

For the regularized system (3.5) we assume that $f_m(t,v)$, $f_M(t,v) \in C[\mathbb{R}_+ \times E^{n^2}, E^{n^2}]$. The solutions of the initial value problem (IVP) (3.5) are weakly continuous mappings $v : I \to E^{n^2}$ which satisfy the integral equation

$$v(t) = v_0 + \int_{t_0}^{t} f_\kappa(s, v(s))\, ds, \; t \in I, \; \kappa \in [0,1].$$

The proposed regularization scheme is a method that can help to reduce the analysis of the properties of the fuzzy system (3.1) with the uncertain parameter $\alpha \in \mathcal{S}$ to those of the regularized system (3.5).

4. Matrix-valued Lyapunov functions

We define a matrix-valued function

$$U(t,\cdot) = [u_{ij}(t,\cdot)], \quad i,j = 1,2, \tag{4.1}$$

with entries $u_{ij} \in \mathbb{R}$ that correspond to the family of regularized equations (3.5) as follows:

1) If $\kappa = 0$, then the entry $u_{11}(t,v_1) \in C(\mathbb{R}_+ \times E^{n^2},\ \mathbb{R}_+)$ corresponds to the fuzzy equation

$$\frac{dv_1}{dt} = f_m(t,v_1); \tag{4.2}$$

2) If $\kappa = 1$, then $u_{22}(t,v_2) \in C(\mathbb{R}_+ \times E^{n^2},\ \mathbb{R}_+)$ corresponds to the fuzzy equation

$$\frac{dv_2}{dt} = f_M(t,v_2); \tag{4.3}$$

3) If $0 < \kappa < 1$, then the entries $u_{12}(t,v) = u_{21}(t,v) \in C(\mathbb{R}_+ \times E^{n^2},\ \mathbb{R})$ correspond to the fuzzy regularized system

$$\frac{dv}{dt} = f_\kappa(t,v), \tag{4.4}$$

where $f_\kappa \in C(\mathbb{R}_+ \times E^{n^2},\ E^{n^2})$.

This way the fuzzy equations (4.2)–(4.4) become the base in the construction of the matrix-valued Lyapunov-type function (4.1).

Example 4.1. As a very simple example of a matrix-valued Lyapunov function, consider a function $U(t,\cdot)$ with the following entries:

$$u_{11}(t,v_1) = d[v_1,\theta_0];\ \ u_{22}(t,v_2) = d[v_2,\theta_0];\ u_{12}(t,v_1,v_2) = u_{21}(t,v_1,v_2) = d[v_1,v_2],$$

where $v_1, v_2 \in E^{n^2}$ and the state $\theta_0 \in E^{n^2}$ is defined as

$$\theta_0(x) = \begin{cases} 1, & \text{for } x = 0, \\ 0, & \text{for } x \in \mathbb{R}^n \setminus \{0\}. \end{cases}$$

Now, on the basis of the matrix function (4.1), and by means of a vector $\eta \in \mathbb{R}^2_+$, we construct a scalar Lyapunov-type function as follows

$$V(t,v,\eta) = \eta^T U(t,v)\eta, \tag{4.5}$$

where $V \in C(\mathbb{R}_+ \times E^{n^2} \times \mathbb{R}^2_+,\ \mathbb{R}_+)$.

Note that, in general, the vector η can be defined in one of the following ways:

(a) $\eta = y \in \mathbb{R}^2,\ y \neq 0$;

(b) $\eta = \xi \in C(\mathbb{R}^2,\ \mathbb{R}_+^2)$, $\xi(0) = 0$;
(c) $\eta = \psi \in C(\mathbb{R}_+ \times \mathbb{R}^2,\ \mathbb{R}_+^2)$, $\psi(t, 0) = 0$;
(d) $\eta \in \mathbb{R}_+^2$, $\eta > 0$.

Together with the defined scalar function (4.5), we will also use the following vector function, given by

$$\mathcal{L}(t, v, \eta) = AU(t, v)\eta, \tag{4.6}$$

where A is a constant (2×2) matrix, $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)^T$ and $\mathcal{L} \in C(\mathbb{R}_+ \times E^{n^2} \times \mathbb{R}_+^2,\ \mathbb{R}_+^2)$.

We will introduce the following concepts related to the function (4.1) based on the function (4.5).

Definition 4.2. The matrix-valued function $U : \mathbb{R}_+ \times E^{n^2} \to \mathbb{R}^2$ is said to be

1) positive semi-definite if there exist a neighborhood $D(\rho)$ of the state θ_0, $D(\rho) = \{v \in E^{n^2} : d[v, \theta_0] < \rho\}$, $0 < \rho < +\infty$ and a vector $\eta \in \mathbb{R}_+^2$ such that
   (a) the function $V(t, v, \eta)$ is continuous on $\mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2$;
   (b) the function $V(t, v, \eta)$ is nonnegative for all $(t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2$;
   (c) $V(t, 0, \eta) = 0$ for all $t \in \mathbb{R}_+$ and $\eta \in \mathbb{R}_+^2$;
2) positive definite if it is positive semi-definite and there exists a positive definite function $w(v) : D(\rho) \to \mathbb{R}_+$ such that $w(v) \leq V(t, v, \eta)$ for all $(t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2$;
3) decreasing if there exists a positive definite function $z(v) : D(\rho) \to \mathbb{R}_+$, such that $V(t, v, \eta) \leq z(v)$ for all $(t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2$;
4) radially unbounded if $d[v, \theta_0] \to \infty$ implies that $V(t, v, \eta) \to +\infty$ for all $t \in \mathbb{R}_+$ and $\eta \in \mathbb{R}_+^2$.

The notions of negative semi-definite and negative definite matrix-valued functions can be defined analogously.

Together with the consideration of Lyapunov-type functions (4.5) and (4.6), we will also introduce their total derivatives with respect to the regularized system (3.5), as follows:

$$D^+V(t, v, \eta) = \eta^T D^+U(t, v)\eta;$$

$$D^+\mathcal{L}(t, v, \eta) = AD^+U(t, v)\eta,$$

where $D^+U(t, v) = \limsup\{[U(t + h,\ v + hf_\kappa(t, v)) - U(t, v)]h^{-1} : h \to 0^+\}$.

Example 4.3. For the function considered in Example 4.1,

$$U(t, v) = \begin{pmatrix} d[v_1, \theta_0] & d[v_1, v_2] \\ d[v_1, v_2] & d[v_2, \theta_0] \end{pmatrix}$$

we have

$$D^+U(t, v) = \begin{pmatrix} D^+d[v_1, \theta_0] & D^+d[v_1, v_2] \\ D^+d[v_1, v_2] & D^+d[v_2, \theta_0] \end{pmatrix},$$

where

$$D^+d[v_1, \theta_0] = \lim_{h \to 0^+} \sup \frac{1}{h} \{d[v_1 + hf_m(t, v_1), \theta_0] - d[v_1, \theta_0]\};$$

$$D^+d[v_2, \theta_0] = \lim_{h \to 0^+} \sup \frac{1}{h} \{d[v_2 + hf_M(t, v_2), \theta_0] - d[v_2, \theta_0]\};$$

$$D^+d[v_1, v_2] = \lim_{h \to 0^+} \sup \frac{1}{h} \{d[v_1 + hf_m(t, v_1),\ v_2 + hf_M(t, v_2)] - d[v_1, v_2]\}.$$

The application of the function $U(t, v)$ and its total derivative allows for the investigation of the dynamical properties of the fuzzy system with uncertain parameters, as given by (3.1) on the basis of the simple fuzzy equations (4.2)–(4.4).

4.1. Scalar comparison principle

In this subsection, we will consider the family of regularized equations (3.5) for $(t, v) \in \mathbb{R}_+ \times D(\rho)$. Let $f_\kappa \in C(\mathbb{R}_+ \times D(\rho),\ E^{n^2})$ for any $\kappa \in [0, 1]$ and the solution $v(t)$ of the IVP (3.5) be defined on $[t_0, \infty)$. For the regularized system (3.5), we will present a scalar comparison principle by using the function (4.5).

Theorem 4.4. *Suppose that for the regularized system (3.5) there exist a matrix-valued function $U(t, \cdot)$ and a vector $\eta \in \mathbb{R}^2_+$ such that*

1) The function $V(t, v, \eta) \in C(\mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+,\ \mathbb{R}_+)$ and there exists a constant $L > 0$ such that

$$|V(t, v_1, \eta) - V(t, v_2, \eta)| \leq Ld[v_1, v_2],\ v_1, v_2 \in D(\rho),\ t \in \mathbb{R}_+,\ \eta \in \mathbb{R}^2_+;$$

2) There exists a function $g(t, \omega)$, $g \in C(\mathbb{R}^2_+,\ \mathbb{R})$ such that, for any $\kappa \in [0, 1]$

$$D^+V(t, v, \eta) \leq g(t, V(t, v, \eta)),\ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+;$$

3) The maximal solution $r_M(t; t_0, \omega_0)$ of the scalar comparison equation

$$\frac{d\omega}{dt} = g(t, \omega), \quad \omega(t_0) = \omega_0 \tag{4.7}$$

exists on $[t_0, \infty)$.

Then, $V(t_0, v_0, \eta) \leq \omega_0$ implies that

$$V(t, v(t), \eta) \leq r_M(t; t_0, \omega_0),\ t \in [t_0, \infty).$$

Proof. Since the solution $v(t)$ of the IVP (3.5) is defined on $[t_0, \infty)$, for $t \in [t_0, \infty)$, we set $m(t) = V(t, v(t), \eta)$ so that $m(t_0) \leq \omega_0$ and evaluate the difference $m(t + h) - m(t)$ for a sufficiently small $h > 0$. We have that

$$\begin{gathered} m(t+h) - m(t) = V(t+h, v(t+h), \eta) - V(t, v(t), \eta) = V(t+h, v(t+h), \eta) \\ -V(t+h, v(t) + hf_\kappa(t, v(t)), \eta) + V(t+h, v(t) \\ +hf_\kappa(t, v(t)), \eta) - V(t, v(t), \eta) \leq Ld[v(t+h), v(t) + hf_\kappa(t, v(t))] \\ +V(t+h, v(t) + hf_\kappa(t, v(t)), \eta) - V(t, v(t), \eta),\ \kappa \in [0, 1]. \end{gathered}$$

From the above estimate, we obtain

$$\begin{gathered} D^+m(t) = \lim_{h\to 0^+} \sup \frac{1}{h}[m(t+h) - m(t)] \\ \leq D^+V(t, v(t), \eta) + L \lim_{h\to 0^+} \sup \{d[v(t+h), v(t) + hf_\kappa(t, v(t))]\}\,. \end{gathered} \tag{4.8}$$

Let $v(t+h) = v(t) + z(t)$, where $z(t)$ is the Hukuhara difference for a sufficiently small $h > 0$. According to the properties of the metric $d[u, v]$, we get

$$d[v(t+h), v(t) + hf_\kappa(t, v(t))] = d\left[v(t) + z(t), v(t) + hf_\kappa(t, v(t))\right]$$

$$= d\left[z(t), hf_\kappa(t, v(t))\right] = d\left[v(t+h) - v(t), hf_\kappa(t, v(t))\right],$$

and, hence

$$\frac{1}{h}\, d[v(t+h), v(t) + hf_\kappa(t, v(t))] = d\left[\frac{v(t+h) - v(t)}{h},\ f_\kappa(t, v(t))\right]. \tag{4.9}$$

Taking the limit in (4.9), we obtain

$$\lim_{h\to 0^+} \sup \frac{1}{h}\{d[v(t+h), v(t) + hf_\kappa(t, v(t))]\}$$

$$= \lim_{h\to 0^+} \sup \frac{1}{h}\left\{d\left[\frac{v(t+h) - v(t)}{h},\ f_\kappa(t, v(t))\right]\right\} = d\left[\frac{dv(t)}{dt},\ f_\kappa(t, v(t))\right] \tag{4.10}$$

along any solution $v(t)$ of (3.5).

Taking into account (4.10), the estimate (4.8) takes the following form:

$$D^+ m(t) \le g(t, m(t)), \quad m(t_0) \le \omega_0, \tag{4.11}$$

to which we apply Theorem 1.5.1 [14] and get

$$m(t) \le r_M(t; t_0, \omega_0)$$

for all $t \ge t_0$. This proves Theorem 4.4.

We will now state some corollaries of Theorem 4.4. The Hahn class K of continuous and strictly increasing functions of the corresponding dimension that are zero at zero will be used [14].

Corollary 4.5. *Suppose that, in Theorem 4.4, instead of condition 2, we have the following:*

2′) $D^+ V(t, v, \eta) \le 0,\ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+$.

Then,

$$V(t, v(t), \eta) \le V(t_0, v_0, \eta),\ t \ge t_0.$$

Corollary 4.6. *Suppose that, in Theorem 4.4, instead of condition 2, we have the following:*

2″) $D^+ V(t, v, \eta) \le -a(\omega(t, v)) + g(t, V(t, v, \eta)),\ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+,$

where $\omega \in C(\mathbb{R}_+ \times D(\rho), \mathbb{R}_+)$*,* $a \in K$ *and* $g(t, \omega)$ *is a function decreasing on* ω *for any* $t \in \mathbb{R}_+$*.*

Then, $V(t_0, v_0, \eta) \le \omega_0$ *implies that*

$$V(t, v(t), \eta) + \int_{t_0}^{t} a[\omega(s, v(s))]\, ds \le r_M(t; t_0, \omega_0),\ t \ge t_0,\ \kappa \in [0, 1].$$

4.2. Quasilinear fuzzy systems

We represent the regularized system (3.5) in the following form:

$$\frac{dv}{dt} = A_\kappa(t)v + g_\kappa(t, v), \tag{4.12}$$

$$v(t_0) = v_0, \tag{4.13}$$

where $g_\kappa \in C(\mathbb{R}_+ \times E^{n^2})$ and $A_\kappa(t) : [t_0, \infty) \to E^{n^2}$ for any value of $\kappa \in [0, 1]$ is a semi-linear operator such that

(a) $A_\kappa(t)(u + v) = A_\kappa(t)u + A_\kappa(t)v,\ u, v \in E^{n^2}$;
(b) $A_\kappa(t)(\nu u) = \nu A_\kappa(t)u,\ \nu \in \mathbb{R}_+,\ u \in E^{n^2}$.

Assume that the solution of the problem described by (4.12) and (4.13) is well defined for $t \geq t_0$ and the operator $A_\kappa(t)$ is contracting, i.e., there exists $0 < \gamma < 1$ such that

$$d[A_\kappa(t)u,\ A_\kappa(t)v] \leq \gamma d[u, v],\ u, v \in E^{n^2}. \tag{4.14}$$

In addition, for a sufficiently small $h > 0$ the operator

$$Q(h, A_\kappa(t)) = I + hA_\kappa(t) + h^2A_\kappa^2(t) + \ldots + h^nA_\kappa^n(t) + \ldots$$

exists for $t \in \mathbb{R}_+$, $\kappa \in [0, 1]$ and $u \in E^{n^2}$, and it satisfies

$$\lim_{h\to 0} Q(h, A_\kappa(t))u = u. \tag{4.15}$$

The comparison principle for the quasilinear regularized problem described by (4.12) and (4.13) is represented as follows.

Theorem 4.7. *Suppose that for the regularized system (4.12), there exists a scalar Lyapunov-type function $V(t, v, \eta)$ that satisfies the following conditions:*

1) $V(t, v, \eta) \in C(\mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2, \mathbb{R}_+)$ and there exists a continuous on $\mathbb{R}_+$ function $L(t) \geq 0$ such that

$$|V(t, v_1, \eta) - V(t, v_2, \eta)| \leq L(t)d[v_1, v_2],\ v_1, v_2 \in D(\rho),\ t \in \mathbb{R}_+,\ \eta \in \mathbb{R}_+^2;$$

2) There exists a function $G(t, \omega)$, $G \in C(\mathbb{R}_+^2, \mathbb{R})$, such that for any $\kappa \in [0, 1]$

$$D^+V(t, v, \eta) \leq G(t, V(t, v, \eta)),\ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2;$$

3) The maximal solution $r_M(t; t_0, \omega_0)$ of the comparison equation

$$\frac{d\omega}{dt} = G(t, \omega), \quad \omega(t_0) = \omega_0 \tag{4.16}$$

exists on $[t_0, \infty)$.

Then, $V(t_0, v_0, \eta) \leq \omega_0$ implies that

$$V(t, v(t), \eta) \leq r_M(t; t_0, \omega_0),\ t \in [t_0, \infty). \tag{4.17}$$

The proof of Theorem 4.7 is obtained in a similar way as the proof of Theorem 4.4 taking into account the fact that

$$Q(h, A_\kappa(t))v + hg_\kappa(t, v) = v + hg_\kappa(t, v) + h(Q(h, A_\kappa(t))v. \tag{4.18}$$

From (4.18), we have the following for $m(t) = V(t, v(t), \eta)$:

$$m(t + h) - m(t) \le L(t + h)d[v(t + h),\ v(t) + h(A_\kappa(t)v(t) + g_\kappa(t, v(t)))]$$
$$+L(t + h)hd[Q(h, A_\kappa(t))A_\kappa(t)v(t),\ A_\kappa(t)v(t)] + V(t + h,\ Q(h, A_\kappa(t))v(t)$$
$$+hg_\kappa(t, v(t)), \eta) - V(t, v(t), \eta),\ \kappa \in [0, 1].$$

The above estimate, (4.14), (4.15) and condition 2 of Theorem 4.7 yield

$$D^+m(t) \le G(t, m(t)). \tag{4.19}$$

Next, we apply Theorem 3.1.1 [14] to obtain (4.17).

Corollary 4.8. *Suppose that, in Theorem 4.7, the functions* $V(t, v, \eta) = d[v, \theta_0]$ *and* $G = G(t, d[v, \theta_0])$, $t \in \mathbb{R}_+$, $v \in E^{n^2}$, $\eta \in \mathbb{R}^2_+$, *where* $\theta_0 \in E^{n^2}$ *is the state defined in Example 4.1.*

Then, $d[v_0, \theta_0] \le \omega_0$ *implies that*

$$d[v(t), \theta_0] \le r_M(t; t_0, \omega_0),\ t \ge t_0,\ \kappa \in [0, 1].$$

4.3. Vector comparison principle

Recall that a function $G(t, \omega)$, $G : \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{R}^2$ is quasimonotonic with respect to its second variable ω if $\omega_1 \le \omega_2$ and $\omega_{i1} = \omega_{i2}$ for $1 \le i \le 2$ imply that $G(t, \omega_1) \le G(t, \omega_2)$ for any two $\omega_1, \omega_2 \in \mathbb{R}^2_+$ and $t \in \mathbb{R}_+$.

In the case that $G(t, \omega) = A\omega$, where A is a (2×2) matrix with entries a_{ij}, the function $G(t, \omega)$ is quasimonotonic if $a_{ij} \ge 0$ for $i \ne j$.

Theorem 4.9. *Suppose that, for the regularized system (3.5), there exist a matrix-valued function* $U(t, \cdot)$ *and a vector* $\eta \in \mathbb{R}^2_+$ *such that the function (4.6) satisfies the following conditions:*

1) $\mathcal{L}(t, v, \eta) \in C(\mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+, \mathbb{R}^2_+)$ *and there exists a constant* $L > 0$ *such that*
$$\|\mathcal{L}(t, v_1, \eta) - \mathcal{L}(t, v_2, \eta)\| \le L\|D[v_1, v_2]\|,\ v_1, v_2 \in D(\rho),\ t \in \mathbb{R}_+,\ \eta \in \mathbb{R}^2_+,$$
where $D[v_1, v_2] = (d[v_1, u_1], d[v_2, u_2])^T$ *and* $\|.\|$ *is the vector norm in* $\mathbb{R}^2$*;*
2) *There exists a quasimonotonic with respect to* ω *function* $G(t, \omega) \in C(\mathbb{R}_+ \times \mathbb{R}^2_+, \mathbb{R}^2)$, $G(t, 0) = 0$, *such that, for any* $\kappa \in [0, 1]$,
$$D^+\mathcal{L}(t, v, \eta) \le G(t, \mathcal{L}(t, v, \eta)),\ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+; \tag{4.20}$$
3) *The maximal solution* $r_M(t; t_0, \omega_0)$ *of the vector comparison equation*
$$\frac{d\omega}{dt} = G(t, \omega), \quad \omega(t_0) = \omega_0 \tag{4.21}$$
exists on $[t_0, \infty)$.

Then, $\mathcal{L}(t_0, v_0, \eta) \leq \omega_0$ *implies that*

$$\mathcal{L}(t, v(t), \eta) \leq r_M(t; t_0, \omega_0),\ t \in [t_0, \infty),\ \kappa \in [0, 1].$$

The proof of Theorem 4.9 is similar to that of Theorem 4.4, so we omit it here.

Corollary 4.10. *Suppose that, in Theorem 4.9, the function* $G = G(t, \omega) = A\omega$, $t \in \mathbb{R}_+$, *where* A *is a* (2×2) *matrix with entries* $a_{ij} \geq 0$ *for* $i \neq j$.

Then,

$$\mathcal{L}(t, v(t), \eta) \leq \mathcal{L}(t_0, v_0, \eta)e^{A(t-t_0)},\ t \geq t_0,\ \kappa \in [0, 1].$$

5. Main stability analysis

Recall that the function diam $[u(t)]^\kappa$ is nondecreasing as $t \to \infty$. Hence, the direct application of the metric $\|u(t)\|$ in the stability analysis of the regularized system (3.5) is not suitable for its dynamical properties. For this reason, in our research we will introduce the following assumptions:

H_1. For any value of the uncertain parameter $\alpha \in \mathcal{S}$ the system (3.5) has a steady state θ_0 such that $f_\kappa(t, \theta_0) = \theta_0$ for all $t \in [t_0, \infty)$.
H_2. For the initial value $v_0 \in E^{n^2}$ and any $y_0 \in E^{n^2}$ there exists a Hukuhara difference $v_0 - y_0 = w_0$.
H_3. The solution $v(t) = v(t; t_0, v_0)$ of (3.5) exists for all $t \geq t_0$ and is unique for any $\kappa \in [0, 1]$.

Next, we introduce the following stability notions for the steady state θ_0.

Definition 5.1. The state θ_0 of (3.5) is:

$\mathcal{S}_1$ *equi-stable* if, for any $t_0 \in \mathbb{R}_+$ and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0)$ such that for any initial data $v_0 \in E^{n^2}$ with $d[v_0, \theta_0] < \delta$ we have that $d[v(t; t_0, v_0), \theta_0] < \varepsilon$ for all $t \geq t_0$ and all $\kappa \in [0, 1]$;
$\mathcal{S}_2$ *uniformly stable* if δ in $\mathcal{S}_1$ does not depend on t_0;
$\mathcal{S}_3$ *quasi-equi-asymptotically stable* if for any $t_0 \in \mathbb{R}_+$ and any $\xi > 0$ there exist $\delta_0(t_0, \xi) > 0$ and $\tau(t_0, \xi) \in \mathbb{R}_+$ such that $d[v_0, \theta_0] < \delta_0(t_0, \xi)$ implies that $d[v(t; t_0, v_0), \theta_0] < \xi$ for all $t \geq t_0 + \tau(t_0, \xi)$ and $\kappa \in [0, 1]$;
$\mathcal{S}_4$ *uniformly quasi-asymptotically stable* if δ_0 and τ in $\mathcal{S}_3$ are independent on t_0;
$\mathcal{S}_5$ *equi-asymptotically stable* if $\mathcal{S}_1$ and $\mathcal{S}_3$ hold simultaneously;
$\mathcal{S}_6$ *uniformly asymptotically stable* if $\mathcal{S}_2$ and $\mathcal{S}_4$ hold simultaneously;
$\mathcal{S}_7$ *uniformly exponentially stable* if for an arbitrary solution $v(t; t_0, v_0)$, we have

$$d[v(t), \theta_0] \leq \beta(d[v_0, \theta_0]) \exp[-\lambda(t - t_0)],$$

where $\beta(d) : [0, R] \to \mathbb{R}_+$ is nondecreasing on d for some $R > 0$ and $\lambda > 0$ is a constant.

Example 5.2. Consider on E^2 the following fuzzy equation

$$\frac{dv}{dt} = \mu v,\ v(0) = v_0 \in E^2, \tag{5.1}$$

where $(\mu \neq 0) \in [-1, 1]$, $\mu = \mu(\kappa)$, which we represent as follows:

$$\begin{cases} \dfrac{dv_1}{dt} = \mu v_2,\ v_2 = v_{20}, \\ \\ \dfrac{dv_2}{dt} = \mu v_1,\ v_1 = v_{10} \end{cases} \tag{5.2}$$

for $\kappa \in [0,1]$. At the same time, for an initial value $v_0 \in E^2$ on the level $[v_0]^\kappa = [v_{10}, v_{20}]^\kappa$ for $\kappa \in [0,1]$ the general solution of (5.2) has the form

$$\begin{cases} [v_1(t)]^\kappa = \frac{1}{2}(v_{10} + v_{20})e^{\mu t} + \frac{1}{2}(v_{10} - v_{20})e^{-\mu t}, \\ [v_2(t)]^\kappa = \frac{1}{2}(v_{10} + v_{20})e^{\mu t} - \frac{1}{2}(v_{10} - v_{20})e^{-\mu t} \end{cases} \tag{5.3}$$

for all $0 \le \kappa \le 1$ and $t \ge 0$.

It follows from (5.3) that at all κ-levels, the zero solution of (5.1) is unstable, even for sufficiently small values of v_{10}, v_{20} for any $\kappa \in [0,1]$. At the same time, the zero solution is stable under some additional conditions for the initial data $[v_0]^\kappa = [v_{10}, v_{20}]^\kappa$. Particularly, if $v_{10} + v_{20} = 0$ for $0 < \mu < 1$ or if $v_{10} - v_{20} = 0$ for $-1 < \mu < 0$, and any $\kappa \in [0,1]$, the relations given by (5.3) imply that $[v_1(t)]^\kappa$ and $[v_2(t)]^\kappa$ are decreasing for $t \to \infty$. These conditions are equivalent to the existence of the Hukuhara difference for $[v_{10}, v_{20}]^\kappa$, $\kappa \in [0,1]$.

We will now establish some stability criteria for the stationary state θ_0 of (3.5) based on a scalar auxiliary function of the type (4.5). Functions of the class $CK = \{a \in C[\mathbb{R}_+^2, \mathbb{R}_+] :\ a(t,u) \in K$ for each $t \in \mathbb{R}_+$ and $a(t,\wp) \to \infty$ as $\wp \to \infty\}$ will also be used.

Theorem 5.3. *Suppose that, for the regularized system (3.5), conditions H_1–H_3 hold, and that there exist a matrix-valued function $U(t,\cdot)$ and a vector $\eta \in \mathbb{R}_+^2$ such that the function (4.5) satisfies the following conditions:*

1) *$V(t,v,\eta)$ satisfies condition 1 of Theorem 4.4;*
2) *There exist functions $a \in K$ and $b \in CK$ and constant positive definite (2×2) matrices A_1 and A_2 such that*
$$a^T(d[v,\theta_0])A_1 a(d[v,\theta_0]) \le V(t,v,\eta) \le b^T(t,d[v,\theta_0])A_2 b(t,d[v,\theta_0]) \tag{5.4}$$
for $(t,v,\eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2$;
3) $$D^+V(t,v,\eta) \le 0,\ (t,v,\eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2.$$

Then the state θ_0 of (3.5) is equi-stable.

Proof. Under the condition 2) of Theorem 5.3, the estimate (5.4) can be represented as

$$\lambda_m(A_1)\overline{a}(d[v,\theta_0]) \le V(t,v,\eta) \le \lambda_M(A_2)\overline{b}(t,d[v,\theta_0]) \tag{5.5}$$

for $(t,v,\eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2$, where $\lambda_m(A_1) > 0$ and $\lambda_M(A_2) > 0$ are the minimal and maximal eigenvalues of the matrices A_1 and A_2, respectively, and the comparison functions $\overline{a} \in K$, $\overline{b} \in CK$ exist such that

$$\overline{a}(d[v,\theta_0]) \le a^T(d[v,\theta_0])a(d[v,\theta_0])$$

and

$$\overline{b}(t,d[v,\theta_0]) \ge b^T(t,d[v,\theta_0])b(t,d[v,\theta_0]).$$

Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. We choose $\delta = \delta(t_0,\varepsilon)$ so that

$$\lambda_M(A_2)\overline{b}(t_0,\delta) < \lambda_m(A_1)\overline{a}(\varepsilon). \tag{5.6}$$

We will show that, for such a choice of δ, the stationary state θ_0 of (3.5) is equi-stable. If this is not true, then there exist a solution $v^*(t; t_0, v_0)$ and a $t_1 > t_0$ such that, for $\kappa \in [0, 1]$,

$$d[v^*(t_1), \theta_0] = \varepsilon \quad \text{and} \quad d[v^*(t), \theta_0] \le \varepsilon < \rho, \;\; t_0 \le t < t_1. \tag{5.7}$$

According to condition 3 of Theorem 5.3 and Corollary 4.5, we have

$$V(t, v^*(t), \eta) \le V(t_0, v_0, \eta) \quad \text{for all} \quad t_0 \le t \le t_1 \text{ and } \kappa \in [0, 1]. \tag{5.8}$$

Hence, taking into account the estimates (5.5) and (5.6), we get

$$\lambda_m(A_1)\overline{a}(\varepsilon) = \lambda_m(A_1)\overline{a}(d[v^*(t_1), \theta_0]) \le V(t_1, v^*(t_1), \eta) \le V(t_0, v_0, \eta)$$

$$\le \lambda_M(A_2)\overline{b}(t_0, d[v_0, \theta_0]) < \lambda_m(A_1)\overline{a}(\varepsilon).$$

The obtained contradiction shows that $d[v_0, \theta_0] < \delta$ implies that $d[\, v(t; t_0, v_0), \theta_0 \,] < \varepsilon$ for all $t \ge t_0$ and all $\kappa \in [0, 1]$, which proves the theorem.

Theorem 5.4. *Suppose that, for the regularized system (3.5), conditions H_1–H_3 hold, there exists a function of the type (4.5) for which conditions 1 and 2 of Theorem 5.3 are satisfied and, instead of 3 we have the following:*

$3'$*) There exists a constant $\beta > 0$ such that*

$$D^+V(t, v, \eta) \le -\beta V(t, v, \eta), \;\; (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+.$$

Then, the state θ_0 of (3.5) is equi-asymptotically stable.

Proof. Since the conditions of Theorem 5.4 follow from the conditions of Theorem 5.3, the steady state θ_0 is equi-stable.

Let $\varepsilon = \rho$ and $\delta_0 = \delta_0(t_0, \rho)$. From Theorem 5.3, we have that $d[v_0, \theta_0] < \delta_0$ implies that $d[v(t), \theta_0] < \rho$ for all $t \ge t_0$ and $\kappa \in [0, 1]$.

From condition $3'$ of Theorem 5.4, we obtain

$$V(t, v(t), \eta) \le V(t_0, v_0, \eta) \exp[-\beta(t - t_0)], \;\; t \ge t_0, \;\; \kappa \in [0, 1].$$

For the given $\varepsilon > 0$ we choose

$$\tau(t_0, \varepsilon) = \frac{1}{\beta} \ln \frac{\lambda_M(A_2)\overline{b}(t_0, \delta_0)}{\lambda_m(A_1)\overline{a}(\varepsilon)} + 1.$$

Hence, for any $\kappa \in [0, 1]$

$$\lambda_m(A_1)\overline{a}(d[v(t), \theta_0]) \le V(t, v(t), \eta)$$

$$\le \lambda_M(A_2)\overline{b}(t_0, \delta) \exp[-\beta(t - t_0)] < \lambda_m(A_1)\overline{a}(\varepsilon), \;\; t \ge t_0 + \tau(t_0, \varepsilon).$$

From the above inequalities, for any initial data $v_0 \in E^{n^2}$ with $d[v_0, \theta_0] < \delta_0$, we have that $d[\, v(t; t_0, v_0), \theta_0 \,] < \varepsilon$ for $t \ge t_0 + \tau(t_0, \varepsilon)$ and any $\kappa \in [0, 1]$, which proves Theorem 5.4.

Theorem 5.5. *Suppose that, for the regularized system (3.5), conditions H_1–H_3 hold, and that there exist a matrix-valued function $U(t,\cdot)$ and a vector $\eta \in \mathbb{R}^2_+$ such that the function (4.5) satisfies the following conditions:*

1) *$V(t, v, \eta)$ satisfies condition 1 of Theorem 4.4 on $\mathbb{R}_+ \times (D(\rho) \cap D^c(\sigma)) \times \mathbb{R}^2_+$, where $D^c(\sigma)$ is the complement of $D(\sigma)$ for $0 < \sigma < \rho$;*
2) *$V(t, v, \eta)$ satisfies condition 2 of Theorem 5.3 on $\mathbb{R}_+ \times (D(\rho) \cap D^c(\sigma)) \times \mathbb{R}^2_+$ for the functions $a, b \in K$;*
3) *$V(t, v, \eta)$ satisfies condition 3 of Theorem 5.3 on $\mathbb{R}_+ \times (D(\rho) \cap D^c(\sigma)) \times \mathbb{R}^2_+$.*

Then the state θ_0 of (3.5) is uniformly stable.

Proof. Condition 2) of Theorem 5.5 leads to

$$\lambda_m(A_1)\overline{a}(d[v, \theta_0]) \leq V(t, v, \eta) \leq \lambda_M(A_2)\overline{b}(d[v, \theta_0]),\ (t, v, \eta) \in \mathbb{R}_+ \times (D(\rho) \cap D^c(\sigma)) \times \mathbb{R}^2_+.$$

Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. We can choose $\delta = \delta(\varepsilon) > 0$ so that

$$\lambda_M(A_2)\overline{b}(\delta) < \lambda_m(A_1)\overline{a}(\varepsilon). \tag{5.9}$$

We will show that for the above choice of $\delta > 0$ the stationary state θ_0 of the regularized system (3.5) is uniformly stable. If this is not true, then there exist a solution $v(t)$ of (3.5) and t_1, t_2, where $t_2 > t_1 > t_0$, such that $d[v(t_1), \theta_0] = \delta$, $d[v(t_2), \theta_0] = \varepsilon$ and $\delta \leq d[v(t), \theta_0] \leq \varepsilon < \rho$ for $t \in [t_1, t_2]$ and $\kappa \in [0, 1]$.

Set $\sigma = \delta$, and, according to the condition 3 of Theorem 5.5, we have

$$V(t_2, v(t_2), \eta) \leq V(t_1, v(t_1), \eta).$$

From the above inequality, we get

$$\lambda_m(A_1)\overline{a}(\varepsilon) = \lambda_m(A_1)\overline{a}(d[v(t_2), \theta_0]) \leq V(t_2, v(t_2), \eta) \leq V(t_1, v(t_1), \eta)$$
$$\leq \lambda_M(A_2)\overline{b}(d[v(t_1), \theta_0]) = \lambda_M(A_2)\overline{b}(\delta) < \lambda_m(A_1)\overline{a}(\varepsilon).$$

The obtained contradiction proves Theorem 5.5.

Theorem 5.6. *Suppose that, for the regularized system (3.5), conditions H_1–H_3 hold, there exists a function of the type (4.5) for which conditions 1 and 2 of Theorem 5.5 are satisfied and, instead of 3 we have the following:*

3′′) *There exists a function $c \in K$ such that*

$$D^+V(t, v, \eta) \leq -c(d[v, \theta_0]),\ (t, v, \eta) \in \mathbb{R}_+ \times (D(\rho) \cap D^c(\sigma)) \times \mathbb{R}^2_+.$$

Then, the state θ_0 of (3.5) is uniformly asymptotically stable.

Proof. Since all conditions of Theorem 5.5 are satisfied, the state θ_0 of (3.5) is uniformly stable. Then, for $\varepsilon = \rho$ and $\delta_0 = \delta_0(\rho)$, $d[v_0, \theta_0] < \delta_0$ implies that $d[v(t), \theta_0] < \rho$ for all $t \geq t_0$ and $\kappa \in [0, 1]$.

To prove Theorem 5.6, we have to show that the state θ_0 is attractive, i.e., that there exists a $t^* \geq t_0$ such that $d[v(t^*), \theta_0] < \delta$ for $t_0 \leq t^* \leq t_0 + \tau$, where $\tau = 1 + \frac{\lambda_M(A_2)\overline{b}(\delta_0)}{\lambda_m(A_1)\overline{a}(\delta)}$.

Suppose that the above is not true and $\delta \leq d[v(t), \theta_0]$ for $t_0 \leq t \leq t_0 + \tau$. Then, it follows from $3''$ that

$$V(t, v(t), \eta) \leq V(t_0, v_0, \eta) - \int_{t_0}^{t} c(d[v(s), \theta_0])\, ds,\ t_0 \leq t \leq t_0 + \tau.$$

From the last inequality for the given choice of τ, we have

$$0 \leq V(t_0 + \tau, v(t_0 + \tau), \eta) \leq \lambda_M(A_2)\overline{b}(\delta_0) - c(\delta)\tau < 0.$$

The obtained contradiction proves that $d[v_0, \theta_0] < \delta$ implies that $d[v(t; t_0, v_0), \theta_0] < \varepsilon$ for $t \geq t_0 + \tau$, i.e., the state θ_0 is attractive. Theorem 5.6 is proved.

In the next result, we will need the following concept.

Definition 5.7. Two functions $a, b \in K$ are said to be of the same order of magnitude if there exist positive constants k_1 and k_2 such that $k_1 a(r) \leq b(r) \leq k_2 a(r)$ for all $r \in \mathbb{R}_+$.

Theorem 5.8. *Suppose that, for the regularized system (3.5), conditions H_1–H_3 hold, and that there exist a matrix-valued function $U(t, \cdot)$ and a vector $\eta \in \mathbb{R}^2_+$ such that the function (4.5) satisfies the following conditions:*

1) *$V(t, v, \eta)$ satisfies condition 1 of Theorem 4.4;*
2) *There exist a comparison function $\sigma_1 \in K$, a positive constant Δ_1 and a (2×2) matrix function $F_1(\theta)$, $\theta \in \mathbb{R}$ such that*

$$\Delta_1 d^{\rho}[v, \theta_0] \leq V(t, v, \eta) \leq \sigma_1^T(d[v, \theta_0])F_1(\theta)\sigma_1(d[v, \theta_0])$$

   for $(t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+$ and $\rho > 1$;
3) *There exists a comparison function $\sigma_2 \in K$ and a (2×2) matrix function $F_2(\theta)$, $\theta \in \mathbb{R}$ such that*

$$D^+V(t, v, \eta) \leq \sigma_2^T(d[v, \theta_0])F_2(\theta)\sigma_2(d[v, \theta_0]),\ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+.$$

Then, if the matrix $F_1(\theta)(\theta \neq 0) \in \mathbb{R}^2_+$ is positive definite, the matrix $F_2(\theta)(\theta \neq 0) \in \mathbb{R}^2_+$ is negative definite and the comparison functions σ_1, σ_2 are of the same order of magnitude, the state θ_0 of (3.5) is uniformly exponentially stable.

Proof. We can represent the upper estimate of $V(t, v, \eta)$ in condition 2) of Theorem 5.8 as follows

$$V(t, v, \eta) \leq \lambda_M(F_1)\sigma_1^T(d[v, \theta_0])\sigma_1(d[v, \theta_0]), \tag{5.10}$$

where $\lambda_M(F_1) > 0$ is the maximal eigenvalue of the matrix F_1.

Since the function $\sigma_1 \in K$, there exists a function $\gamma \in K$ such that

$$\gamma(d[v, \theta_0]) \geq \sigma_1^T(d[v, \theta_0])\sigma_1(d[v, \theta_0]). \tag{5.11}$$

From (5.10) and (5.11), it follows that condition 2 of Theorem 5.8 takes the following form:

$$\Delta_1 d^{\rho}[v, \theta_0] \leq V(t, v, \eta) \leq \lambda_M(F_1)\gamma(d[v, \theta_0]),\ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}^2_+. \tag{5.12}$$

We represent condition 3 of Theorem 5.8 in the following form:

$$D^+V(t,v,\eta) \le \lambda_M(F_2)\pi(d[v,\theta_0]), \tag{5.13}$$

where the function $\pi \in K$ exists such that

$$\pi(d[v,\theta_0]) \ge \sigma_2^T(d[v,\theta_0])\sigma_2(d[v,\theta_0]).$$

Since the functions γ and π are of the same order of magnitude, there exist constants $K_1 > 0$ and $K_2 > 0$ such that

$$K_1\gamma(d[v,\theta_0]) \le \pi(d[v,\theta_0]) \le K_2\gamma(d[v,\theta_0]). \tag{5.14}$$

From (5.12) and (5.13) for any $\kappa \in [0,1]$, we obtain

$$V(t,v(t),\eta) \le V(t_0,v_0,\eta)\exp[\chi(t-t_0)] \tag{5.15}$$

where $\chi = \lambda_M(F_2)\lambda_M^{-1}(F_1),\ \chi < 0$.

Next, the estimate (5.15) and condition 2 of Theorem 5.8 imply that

$$d[v(t),\theta_0] \le \Delta_1^{-\frac{1}{\rho}}\lambda_M^{\frac{1}{\rho}}(F_1)\gamma^{\frac{1}{\rho}}(d[v_0,\theta_0])\exp[\frac{\chi}{\rho}(t-t_0)],\ t \ge t_0,\ \kappa \in [0,1]. \tag{5.16}$$

Hence, for

$$\beta(\cdot) = \Delta_1^{-\frac{1}{\rho}}\lambda_M^{\frac{1}{\rho}}(F_1)\gamma^{\frac{1}{\rho}}(d[v_0,\theta_0]) \text{ and } \lambda = -\frac{\chi}{\rho},$$

we have an exponential type convergence of an arbitrary solution $v_\kappa(t)$ of (3.5) to the state θ_0, which proves the theorem.

We will next establish stability criteria by means of the vector comparison function of type (4.6).

Theorem 5.9. *Suppose that, for the regularized system (3.5), conditions H_1–H_3 hold, and that there exist a matrix-valued function $U(t,\cdot)$ and a vector $\eta \in \mathbb{R}_+^2$ such that the function (4.6) satisfies the following conditions:*

1) *$\mathcal{L}(t,v,\eta)$ satisfies all conditions of Theorem 4.9;*
2) *There exist vector functions $a,b \in K$ and constant positive definite (2×2) matrices A_1 and A_2 such that for the function*

$$V_0(t,v,\eta) = \sum_{i=1}^{2}\mathcal{L}_i(t,v,\eta) \tag{5.17}$$

the inequalities

$$a^T(d[v,\theta_0])A_1a(d[v,\theta_0]) \le V_0(t,v,\eta) \le b^T(d[v,\theta_0])A_2b(d[v,\theta_0]) \tag{5.18}$$

hold for $(t,v,\eta) \in \mathbb{R}_+\times D(\rho)\times\mathbb{R}_+^2$.

Then, the stability properties of the zero solution of the vector comparison equation (4.21) imply the corresponding stability properties of the state θ_0 of (3.5).

Proof. We will study the equi-asymptotic stability of the steady state θ_0 of the regularized system (3.5).

Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Suppose that the zero solution of (4.21) is equi-asymptotically stable. Hence, it is stable, and for given $\lambda_m(A_1)\overline{a}(\varepsilon) > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that

$$\sum_{i=1}^{2} \omega_{i0} < \delta_1 \tag{5.19}$$

implies that

$$\sum_{i=1}^{2} r_{Mi}(t; t_0, \omega_0) < \lambda_m(A_1)\overline{a}(\varepsilon),\ t \geq t_0,$$

where $r_M(t; t_0, \omega_0) = (r_{M1}(t; t_0, \omega_0), r_{M2}(t; t_0, \omega_0))^T$ and $\overline{a} \in K$ exists such that $a^T(d[v, \theta_0])a(d[v, \theta_0]) \geq \overline{a}(d[v, \theta_0])$ on $D(\rho)$.

We set $\omega_0 = V_0(t_0, v_0, \eta)$ and choose $\delta = \delta(t_0, \varepsilon) > 0$ so that

$$\lambda_M(A_2)\overline{b}(\delta) < \lambda_m(A_1)\overline{a}(\varepsilon). \tag{5.20}$$

We will prove that if $d[\omega_0, \theta_0] < \delta$, then $d[v(t; t_0, v_0), \theta_0] < \varepsilon$ for all $t \geq t_0$ and all $\kappa \in [0, 1]$, where $v(t; t_0, v_0)$ is an arbitrary solution of (3.5).

If the above assertion is not true, then there exists $t_1 > t_0$ such that

$$d[v(t_1; t_0, v_0), \theta_0] = \varepsilon \quad \text{and} \quad d[v(t; t_0, v_0), \theta_0] \leq \varepsilon < \rho,\ 0 \leq t \leq t_1.$$

Since $\mathcal{L}(t, v, \eta)$ satisfies all conditions of Theorem 4.9, we have

$$\mathcal{L}(t, v(t), \eta) \leq r_M(t; t_0, \omega_0),\ t \in [t_0, t_1],\ \kappa \in [0, 1]. \tag{5.21}$$

From (5.18), we have

$$V_0(t_0, v_0, \eta) \leq \lambda_M(A_2)\overline{b}(d[v_0, \theta_0]) < \lambda_M(A_2)\overline{b}(\delta) < \delta_1$$

and, hence

$$\lambda_m(A_1)\overline{a}(\varepsilon) \leq V_0(t_1, v(t_1), \eta) \leq r_0(t_1; t_0, \omega_0) < \lambda_M(A_2)\overline{b}(\varepsilon) < \lambda_m(A_1)\overline{a}(\varepsilon), \tag{5.22}$$

where $r_0(t_1; t_0, \omega_0) = \sum_{i=1}^{2} r_{Mi}(t; t_0, \omega_0)$.

The contradiction (5.22) proves that the state θ_0 of the regularized system (3.5) is equi-stable.

We will next prove that it is attractive. Let $\varepsilon = \rho$ and $\hat{\delta}_0 = \delta(t_0, \rho) > 0$. We choose $0 < \sigma < \rho$ and for given $\lambda_m(A_1)\overline{a}(\sigma)$ and $t_0 \in \mathbb{R}_+$ choose $\delta_1^* = \delta_1^*(t_0, \sigma) > 0$ and $\tau = \tau(t_0, \sigma) > 0$ so that

$$\sum_{i=1}^{2} \omega_{i0} < \delta_1^* \tag{5.23}$$

implies that

$$\sum_{i=1}^{2} r_{M_i}(t; t_0, \omega_0) < \lambda_m(A_1)\overline{a}(\sigma),\ t \geq t_0 + \tau.$$

Let $\omega_0 = V_0(t_0, v_0, \eta)$. Determine $\delta_0^* = \delta_0^*(t_0, \sigma) > 0$ so that $\lambda_M(A_2)\overline{b}(\delta_0^*) < \delta_1^*$. Choose $\delta_0 = \min(\delta_1^*, \delta_0^*)$ and assume that $d[\omega_0, \theta_0] < \delta_0$. Hence, $d[v(t; t_0, v_0), \theta_0] < \rho$ for all $t \geq t_0$ and $\kappa \in [0, 1]$, and therefore, the estimate (5.21) is satisfied for all $t \geq t_0$, $\kappa \in [0, 1]$. Suppose that there exists a sequence $\{t_k\}$, $t_k \geq t_0 + \tau$, $t_k \to +\infty$ as $k \to +\infty$ and $\sigma \leq d[v(t_k), \theta_0]$, where $v(t)$ is an arbitrary solution of (3.5) with initial data $v_0 \in E^n$ such that $d[v_0, \theta_0] < \delta_0$, and such that the Hukuhara difference $u_0 - w_0 = v_0$ exists.

Finally, from (5.18) and (5.23), we have

$$\lambda_m(A_1)\overline{a}(\sigma) \leq V_0(t_k, v_\kappa(t_k), \eta) \leq r_0(t_k, t_0, \omega_0) < \lambda_m(A_1)\overline{a}(\sigma). \tag{5.24}$$

The obtained above contradiction (5.24) proves that the stationary state θ_0 of (3.5) is attractive and, therefore, quasi-asymptotically stable.

The other stability properties can be proved similarly by linking the dynamic properties of the regularized fuzzy system (3.5) and these of the comparison system (4.21).

Remark 5.10. The established stability criteria show that the idea of using a family of mappings and regularized fuzzy systems of type (3.1) with respect to uncertain parameters greatly benefits their stability analysis. In fact, due to some complications in the study of fuzzy differential systems with uncertain parameters, the proposed results in this direction are very few [20]. Hence, the proposed regularization procedure complements such published accomplishments and, due to the offered advantages is more appropriate for applications. In addition, the modifications of the Lyapunov theory-named matrix-valued Lyapunov functions further extend the advantages of the proposed strategy over the classical Lyapunov function strategies.

Example 5.11. We consider a fuzzy Cohen–Grossberg neural network of Lotka-Volterra type with two interacting species

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{r_1}{K_1} u_1(t)(K_1 - u_1(t) - e_{12}\alpha_{12}u_2(t)), \\ \dfrac{du_2}{dt} = \dfrac{r_2}{K_2} u_2(t)(K_2 - u_2(t) - e_{21}\alpha_{21}u_1(t)), \end{cases} \tag{5.25}$$

where $t \geq 0$; $u_1(t)$ and $u_2(t)$ are the populations of the two species at time t, respectively, r_1 and r_2 are intrinsic growth rates; K_1 and K_2 are the carrying capacities of the environment; e_{12} and e_{21} are inter-specific coefficients. All parameters r_1, r_2, K_1, K_2 and e_{12} and e_{21} are positive numbers. The uncertain parameters are α_{12} and α_{21}, which can take values from the interval [0, 1] and represent the interaction strength between the species.

Introduce the notations

$$f_1(t, u, \alpha) = \frac{r_1}{K_1} u_1(t)(K_1 - u_1(t) - e_{12}\alpha_{12}u_2(t)),$$

$$f_2(t, u, \alpha) = \frac{r_2}{K_2} u_2(t)(K_2 - u_2(t) - e_{21}\alpha_{21}u_1(t)),$$

$$f(t, u, \alpha) = (f_1, f_2)^T,\ \alpha = (\alpha_{12}, \alpha_{21})^T,\ u(t) = (u_1(t), u_2(t))^T.$$

Then, the model (5.25) has the following form:

$$\frac{du}{dt} = f(t, u, \alpha), \tag{5.26}$$

where $u \in E^2$ and $f \in C(\mathbb{R}_+ \times E^2 \times \mathcal{S}, E^2)$, $\mathcal{S} = [0,1] \times [0,1]$.

It is easy to show that for (5.25) there exists an equilibrium u^ε at

$$\begin{cases} u_1^\varepsilon = \dfrac{K_1 - K_2 e_{12}\alpha_{12}}{1 - e_{12}e_{21}\alpha_{12}\alpha_{21}}, \\ u_2^\varepsilon = \dfrac{K_2 - K_1 e_{21}\alpha_{21}}{1 - e_{12}e_{21}\alpha_{12}\alpha_{21}}, \end{cases} \tag{5.27}$$

which is positive for all permissible values of α_{12} and α_{21} whenever the carrying capacity ratio K_1/K_2 satisfies the condition

$$e_{12}\alpha_{12} < \frac{K_1}{K_2} < \frac{1}{e_{21}\alpha_{21}}.$$

Now, we consider a regularized system that corresponds to (5.26) and is given by

$$\frac{dv}{dt} = f_\kappa(t, v), \quad v(t_0) = v_0, \tag{5.28}$$

where $f_\kappa \in C(\mathbb{R}_+ \times E^2,\ E^2)$ and $f_m(t, v) = \emptyset$ and $f_\kappa(t, v) = f_M(t, v)\kappa$, $\kappa \in [0, 1]$.

Denote the steady state of (5.28) by θ_0. Suppose that there exists a function $\tilde{c}(t)$ such that

(i) $d\left[f_\kappa(t, v), \theta_0\right] \le \tilde{c}(t) d[v, \theta_0]$ for all $\kappa \in [0, 1]$ and $(t, v) \in \mathbb{R}_+ \times E^2$;

(ii) $\int\limits_0^\infty \tilde{c}(s) ds \le +\infty$.

If for the corresponding regularized equation (5.8) all conditions H_1–H_3 are satisfied, then (i) and (ii) guarantee the uniform stability of its stationary solution $\theta_0 \in E^2$.

In fact, the Lyapunov-type function $V(t, v, \eta) = d[v, \theta_0]$ satisfies all conditions of Theorem 5.5. More precisely, $V(t, v, \eta) = \eta^T U(t, v)\eta$, where the matrix $U(t, v)$ has entries $u_{ij}(t, v)$, $i, j = 1, 2$ defined as $u_{11}(t, v) = u_{22}(t, v) = \frac{1}{2} d[v, \theta_0]$, $u_{12}(t, v) = u_{21}(t, v) = 0$ and $\eta = (1, 1)^T$.

In addition, for any sufficiently small $h > 0$ we have

$$V\left(t, v + hf_\kappa(t, v), \eta\right) = d\left[v + hf_\kappa(t, v), \theta_0\right]$$

$$\le d[v, \theta_0] + hd\left[f_\kappa(t, v), \theta_0\right] \le d[v, \theta_0] + h\tilde{c}(t) d[v, \theta_0].$$

From the definition of the derivative $D^+V(t, v, \eta)$ and (i), we get

$$D^+V(t, v(t), \eta) \le \tilde{c}(t) d[v, \theta_0] \text{ for any } v \in E^2.$$

Since, condition (ii) is sufficient [18, 19] to guarantee the uniform stability of the zero solution of

$$\frac{dm(t)}{dt} = \tilde{c}(t) m(t), \ m(t_0) = m_0 \ge 0, \tag{5.29}$$

then according to the comparison principle, the state θ_0 of the regularized system (5.28) is uniformly stable, too.

6. Stability analysis for autonomous comparison problems

In this section, first, under the conditions of Theorem 5.9, we will analyze the stability of the zero solution of the comparison system (4.21) for the case that the vector function $G(t,\omega)$ is autonomous. In this case, the comparison system is in the following form:

$$\frac{d\omega}{dt} = \bar{G}(\omega), \quad \omega(t_0) = \omega_0 \geq 0, \tag{6.1}$$

where $\bar{G} \in C(\mathbb{R}_+^2, \mathbb{R}^2)$, $\bar{G} = (\bar{G}_1, \bar{G}_2)^T$, $\omega = (\omega_1, \omega_2)^T$ and the stability problems of the zero solution have effective resolutions under the following assumptions:

(i) The function $\bar{G}$ is quasimonotonic and nondecreasing with respect to ω on

$$\tilde{K} = \{\omega \in \mathbb{R}^2 : \ \omega_i \geq 0, \ \ i = 1, 2\};$$

(ii) There exists a local solution $\omega(t)$ of the IVP (6.1) which is uniquely determined by the given initial data;

(iii) There exists a neighborhood D^* of the state $\omega = 0$ such that, for any $\omega \in \overline{D}^*$, we have that $\bar{G}(\omega) \neq 0$ for $\omega \neq 0$ and $\bar{G}(0) = 0$.

In what follows, uniform asymptotic stability criteria will be established.

Theorem 6.1. *Suppose that, for the regularized system (3.5), conditions H_1–H_3 hold, and that there exist a matrix-valued function $U(t,\cdot)$ and a vector $\eta \in \mathbb{R}_+^2$ such that the function (4.6) satisfies the following conditions:*

1) $\mathcal{L}(t, v, \eta)$ satisfies condition 1 of Theorem 4.9 and condition 2 of Theorem 5.9;

2) $\mathcal{L}(t, v, \eta)$ is such that, for any $\kappa \in [0, 1]$, we have

$$D^+\mathcal{L}(t, v, \eta) \leq \bar{G}(\mathcal{L}(t, v, \eta)), \ (t, v, \eta) \in \mathbb{R}_+ \times D(\rho) \times \mathbb{R}_+^2. \tag{6.2}$$

3) For any $\delta > 0$, the system of inequalities

$$\bar{G}_i(\omega_1, \omega_2) < 0, \quad i = 1, 2$$

has a unique solution $\overline{\omega}_1$, $\overline{\omega}_2$ such that $0 < \overline{\omega}_i < \delta$ for $i = 1, 2$.

Then the state θ_0 of (3.5) is uniformly asymptotically stable.

Proof. Under condition 3 of Theorem 6.1, the isolated zero solution of the comparison system (6.1) is uniformly asymptotically stable [18, 19]. Further, applying the reasoning from the proof of Theorem 5.9, we complete the proof of Theorem 6.1.

Theorem 6.1 and Corollary 4.10 imply the validity of the following result.

Corollary 6.2. *Suppose that, in Theorem 6.1, the function $\bar{G} = \bar{G}(\omega) = A\omega$ and $t \in \mathbb{R}_+$, where A is a (2×2) matrix with entries $a_{ij} \geq 0$ for $i \neq j$.*

Then, if the system of inequalities

$$\sum_{j=1}^{2} a_{ij}\overline{\omega}_j < 0, \quad i = 1, 2$$

has a unique solution $\overline{\omega}_1$, $\overline{\omega}_2$ such that $0 < \overline{\omega}_j$ for all $j = 1, 2$, then the state θ_0 of the regularized fuzzy system (3.5) is uniformly asymptotically stable

Next, we suppose that

$$E^{n^m} = E^{n_1} \times \ldots \times E^{n_m} \quad \text{and} \quad E^{n_i} \cap E^{n_j} = \emptyset$$

for $n_i \neq n_j$, $i, j \in [1, m]$. We will represent an autonomous regularized system of the type (3.5) on $E^{n_1} \times \ldots \times E^{n_m}$ in the following form:

$$dv_i/dt = f^i_\kappa(v_i) + g^i_\kappa(v_1, \ldots, v_m), \tag{6.3}$$

where $f^i_\kappa \in C(E^{n_i}, E^{n_i})$ and $g^i_\kappa \in C(E^{n_1} \times \ldots \times E^{n_m}, E^{n_i})$ for $\kappa \in [0, 1]$, $i = 1, 2, \ldots, m$.

We will apply the following two metrics

$$d_0[u, v] = \sum_{i=1}^{m} d[u_i, v_i], \quad \text{where} \quad u_i, v_i \in E^{n_i}, \tag{6.4}$$

and

$$D[u, v] = (d[u_1, v_1], d[u_2, v_2], \ldots, d[u_m, v_m])^T, \tag{6.5}$$

where $D \in \mathbb{R}^m_+$ and $u, v \in E^{n^m}$. Note that the use of the measure (6.4) is related to the following condition

$$d_0[f_\kappa(u), f_\kappa(v)] = \sum_{i=1}^{m} d[f^i_\kappa(u), f^i_\kappa(v)] \leq g(d_0[u, v]) \quad \text{for all} \quad \kappa \in [0, 1], \tag{6.6}$$

where $g \in C(\mathbb{R}_+, \mathbb{R}_+)$.

The comparison principle for the autonomous regularized system (6.3) is represented as follows.

Theorem 6.3. *Suppose that, for the regularized system (6.3), there exists a Lyapunov-type function $\mathcal{L}(v, \eta)$ that satisfies the following conditions:*

1) *$\mathcal{L}(v, \eta) \in C(E^{n^m} \times \mathbb{R}^2_+, \mathbb{R}^m_+)$ and there exists a constant $(m \times m)$−matrix P with real entries such that*

$$\|\mathcal{L}(v_1, \eta) - \mathcal{L}(v_2, \eta)\| \leq |P| \|D[v_1, v_2]\|$$

for $v_1, v_2 \in E^{n^m}$, $\eta \in \mathbb{R}^2_+$, where $\|.\|$ is the norm in R^m and $|.|$ is the corresponding matrix norm;

2) *There exists a family $G(\omega)$, $G \in C(\mathbb{R}^m_+, \mathbb{R}^m)$, such that, for any $\kappa \in [0, 1]$*

$$D^+\mathcal{L}(v, \eta) \leq G(\mathcal{L}(v, \eta)), \ v \in E^{n^m}, \ \eta \in \mathbb{R}^2_+;$$

3) *The maximal solution $r_M(t, t_0, \omega_0)$ of the comparison equation*

$$\frac{d\omega}{dt} = G(\omega), \quad \omega(t_0) = \omega_0 \tag{6.7}$$

exists on $[t_0, \infty)$.

Then, $\mathcal{L}(v_0, \eta) \leq \omega_0$ implies that

$$\mathcal{L}(v(t), \eta) \leq r_M(t; t_0, \omega_0), \ t \in [t_0, \infty). \tag{6.8}$$

The steps of the proof of Theorem 6.3 are identical to those of the proof of Theorem 2.17 [19].

For the family of equations (6.3), the following result, whose proof is similar to the proof of Theorem 5.9, is valid.

Theorem 6.4. *Suppose that for the system (6.3) there exists a vector Lyapunov-type function $\mathcal{L}(v,\eta)$ such that the following holds:*

1) $\mathcal{L}(v,\eta)$ satisfies all conditions of Theorem 6.4 for $v \in \tilde{D}(\rho)$, where $\tilde{D}(\rho) = \{v \in E^{n^m} : d_0[v,\theta_0] < \rho\}$ and $\eta \in \mathbb{R}^2_+$;

2) There exist vector functions $a, b \in K$ and constant positive definite $(m \times m)$ matrices A_1 and A_2 such that, for the function

$$\mathcal{L}_0(v,\eta) = \sum_{i=1}^{m} \mathcal{L}_i(v,\eta) \tag{6.9}$$

for $v \in \tilde{D}\rho)$, $\eta \in \mathbb{R}^2_+$, we have

$$a^T(d_0[v,\theta_0])A_1 a(d_0[v,\theta_0]) \le \mathcal{L}_0(v,\eta) \le b^T(d_0[v,\theta_0])A_2 b(d_0[v,\theta_0]). \tag{6.10}$$

Then the stability properties of the zero solution of the vector comparison equation (6.7) with $G_\kappa(0) = 0$ imply the corresponding stability properties of the state θ_0 of (6.3).

Example 6.5. In this example, we will use functions $a \in K$ such that $a \to \infty$ as their variable approaches ∞. Such a class of functions will be denoted as KR. Consider Eq (6.3) under the following assumptions:

1) There exist functions $\mathcal{L}_i(v,\eta) \in C(E^{n^m} \times \mathbb{R}^2_+, \mathbb{R}_+)$ and $\psi_{i1}, \psi_{i2} \in KR$, $i = 1,2,\dots,m$, such that

$$\psi_{i1}(d_0[v_i,\theta_0]) \le \mathcal{L}_i(v,\eta) \le \psi_{i2}(d_0[v_i,\theta_0]);$$

2) For all $i, j = 1,2,\dots,m$, there exist constants $\sigma_i(\kappa) \in \mathbb{R}$ and $a_{ij}(\kappa)$ such that

   (a) $D^+\mathcal{L}_i(v,\eta) \le \sigma_i(\kappa)\psi_{i3}(d_0[v_i,\theta_0])$ with respect to the solutions of the family of equations

$$dv_i/dt = f_\kappa(v_i), \quad i = 1,2,\dots,m,\ \kappa \in [0,1]; \tag{6.11}$$

   (b) $D^+\mathcal{L}_i(v,\eta) \le \psi^{\frac{1}{2}}_{i3}(d_0[v_i,\theta_0]) \sum\limits_{j=1}^{m} a_{ij}(\kappa)\psi^{\frac{1}{2}}_{j}(d_0[v_i,\theta_0])$
   on the connection function $g^i_\kappa(v_1,\dots,v_m)$ between subsystems of (6.11) for any $\kappa \in [0,1]$, $i, j = 1,2,\dots,m$.

From 1) and 2), for the function $V(v,\eta) = b^T\mathcal{L}(v,\eta)$, $b \in \mathbb{R}^m_+$, we have

$$D^+V(v,\eta) \le \psi_3^T(d_0[v,\theta_0])\widetilde{S}\psi_3(d_0[v,\theta_0]),$$

where $\psi_3 = \left(\psi^{\frac{1}{2}}_{13}(d_0[v_1,\theta_0]),\dots,\psi^{\frac{1}{2}}_{m3}(d_0[v_m,\theta_0])\right)^T$, $\widetilde{S} = [s_{ij}(\kappa)]$, $i, j = 1,2,\dots,m$, $\kappa \in [0,1]$,

$$s_{ij}(\kappa) = \begin{cases} b_i(\sigma_i(\kappa) + a_{ij}(\kappa)) & \text{for } i = j, \\ \frac{1}{2}(b_i a_{ij}(\kappa) + b_j a_{ji}(\kappa)) & \text{for } i \ne j. \end{cases}$$

If the matrix $\overline{M} = \frac{1}{2}(\widetilde{S}^T(\kappa) + \widetilde{S}(\kappa))$ is positive definite for all $\kappa \in [0, 1]$, then

$$D^+V(\nu, \eta) \leq \lambda_M(\overline{M})\psi(d_0[\nu, \hat{\theta}_0]), \tag{6.12}$$

where $\lambda_M(\overline{M})$ is the maximal eigenvalue of $\overline{M}$ for any $\kappa \in [0, 1]$ and the function $\psi \in K$ exists such that $\psi_3^T(d_0[\nu, \theta_0])\psi_3(d_0[\nu, \theta_0]) \leq \psi(d_0[\nu, \theta_0])$. Condition 1) and (6.12) imply that the stationary state θ_0 of (6.3) is uniformly asymptotically stable.

7. Boundedness results

Consider the fuzzy system (3.1) and by means of the regularization process transform it to a regularized system of the following type:

$$\frac{d\nu}{dt} = f_\kappa(t, \nu) = g_\kappa(t, \nu) + h_\kappa(t, \nu), \tag{7.1}$$

$$\nu(t_0) = \nu_0, \ \nu_0 \in E^{n^2}, \tag{7.2}$$

where $f_\kappa(t, \nu)$ is defined by (3.2) for all $\kappa \in [0, 1]$, $g_\kappa, \ h_\kappa \in C(\mathbb{R}_+ \times E^{n^2}, \ E^{n^2})$, $h_\kappa(t, \nu) \neq 0$ for $\nu = 0$ and $t \in \mathbb{R}_+$.

For the above fuzzy system (7.1) ,assume that there exist a unique steady state θ_0 and functions $\tilde{f}(t)$ and $\tilde{m}(t)$ that are positive and integrable on any finite interval in $\mathbb{R}_+$ such that the following holds:

H_4. For any $\kappa \in [0, 1]$ and $(t, \nu) \in \mathbb{R}_+ \times E^{n^2}$,

$$d[g_\kappa(t, \nu), \theta_0] \leq \tilde{f}(t)d[\nu, \theta_0];$$

H_5. For any $\kappa \in [0, 1]$ and $(t, \nu) \in \mathbb{R}_+ \times E^{n^2}$ and $k > 1$

$$d[h_\kappa(t, \nu), \theta_0] \leq \tilde{m}(t)d^k[\nu, \theta_0];$$

H_6. For all $t \geq t_0$, $t_0 \in \mathbb{R}_+$

$$\Phi(t_0, t) = (k-1)d^{k-1}[\nu_0, \theta_0] \int_{t_0}^{t} \tilde{m}(s) \exp\left[(k-1) \int_{t_0}^{s} \tilde{f}(\tau)d\tau\right] ds < 1.$$

Now, the goal of our investigations is to obtain boundedness criteria for the family of solutions $\nu(t)$ of the set of fuzzy differential equations (7.1) for any $\kappa \in [0, 1]$.

Lemma 7.1. *Suppose that, for the system (7.1), conditions H_4–H_6 hold for all $t \in [t_0, a)$. Then, for the function $V(t, \nu, \eta) = d[\nu, \theta_0]$, the estimate*

$$d[\nu(t), \theta_0] \leq d[\nu_0, \theta_0] \exp\left(\int_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi(t_0, t))^{-\frac{1}{k-1}} \tag{7.3}$$

is satisfied for all $t \in [t_0, a)$.

Proof. Without loss of generality, we can consider the function $V(t, v, \eta) = d[v, \theta_0]$ for $d[v, \theta_0] \geq K^*$, where K^* is sufficiently large. We have that

$$K_1 d[v, \theta_0] \leq V(t, v, \eta) \leq K_2 d[v, \theta_0] \tag{7.4}$$

for any $0 < K_1 < K_2$ and all $v \in E^{n^2}$.

For the total derivative of the function $V(t, v, \eta)$ with respect to system (7.1) from H_4 and H_5, we obtain

$$D^+ V(t, v, \eta) \leq \tilde{f}(t) d[v, \theta_0] + \tilde{m}(t) d^k[v, \theta_0] \tag{7.5}$$

for any $(t, v) \in \mathbb{R}_+ \times E^{n^2}$, $\eta = (1, 1)^T$ and $k > 1$.

From the above estimate, we get

$$d[v(t), \theta_0] \leq d[v_0, \theta_0] + \int_{t_0}^{t} (\tilde{f}(s) d[v(s), \theta_0] + \tilde{m}(s) d^k[v(s), \theta_0]) ds, \ t \in [t_0, a). \tag{7.6}$$

Set $z(t) = V(t, v(t), \eta) = d[v(t), \theta_0]$. Then, $z(t_0) = d[v_0, \theta_0]$ and the estimate (7.6) has the following form:

$$z(t) \leq z(t_0) + \int_{t_0}^{t} \Big(\tilde{f}(s) z(s) + \tilde{m}(s) z^k(s)\Big) ds$$

$$= z(t_0) + \int_{t_0}^{t} \Big(\tilde{f}(s) + \tilde{m}(s) z^{k-1}(s)\Big) z(s) ds, \ t \in [t_0, a). \tag{7.7}$$

We apply the Gronwall-Bellman inequality to the above estimate and get

$$z(t) \leq z(t_0) \exp\bigg(\int_{t_0}^{t} \Big(\tilde{f}(s) + \tilde{m}(s) z^{k-1}(s)\Big) ds\bigg). \tag{7.8}$$

The estimate (7.8) leads to

$$z^{k-1}(t) \leq z^{k-1}(t_0) \exp\bigg((k-1) \int_{t_0}^{t} \Big(\tilde{f}(s) + \tilde{m}(s) z^{k-1}(s)\Big) ds\bigg). \tag{7.9}$$

We multiply both sides of (7.9) by the negative expression

$$-(k-1)\tilde{m}(t) \exp\bigg(-(k-1) \int_{t_0}^{t} \tilde{m}(s) z^{k-1}(s)\, ds\bigg),$$

and we obtain

$$-(k-1)\tilde{m}(t) z^{k-1}(t) \exp\bigg(-(k-1) \int_{t_0}^{t} \tilde{m}(s) z^{k-1}(s)\, ds\bigg)$$

$$\geq -(k-1)z^{k-1}(t_0)\tilde{m}(t)\exp\left((k-1)\int_{t_0}^{t}\tilde{f}(s)\,ds\right).$$

Hence,

$$\frac{d}{dt}\left(\exp\left(-(k-1)\int_{t_0}^{t}\tilde{m}(s)z^{k-1}(s)\,ds\right)\right)$$
$$\geq -(k-1)z^{k-1}(t_0)\tilde{m}(t)\exp\left((k-1)\int_{t_0}^{t}\tilde{f}(s)\,ds\right). \tag{7.10}$$

From (7.10), we obtain

$$\exp\left(-(k-1)\int_{t_0}^{t}\tilde{m}(s)z^{k-1}(s)\,ds\right)$$
$$\geq 1-(k-1)z^{k-1}(t_0)\int_{t_0}^{t}\tilde{m}(s)\exp\left((k-1)\int_{t_0}^{s}\tilde{f}(\tau)\,d\tau\right)ds. \tag{7.11}$$

The assumption H_6 and (7.11) lead to

$$\exp\left((k-1)\int_{t_0}^{t}\tilde{m}(s)z^{k-1}(s)\,ds\right)\leq \Phi^{-1}(t,t_0) \quad \text{for all} \quad t\in[t_0,a).$$

Considering (7.9) and (7.10), we find that

$$z^{k-1}(t)\leq \frac{z^{k-1}(t_0)\exp\left((k-1)\int_{t_0}^{t}\tilde{f}(s)ds\right)}{1-(k-1)z^{k-1}(t_0)\int_{t_0}^{t}\tilde{m}(s)\exp\left((k-1)\int_{t_0}^{s}\tilde{f}(\tau)d\tau\right)ds}. \tag{7.12}$$

Finally, (7.8) and (7.12) imply (7.3).

We introduce the following boundedness definitions [14,20].

Definition 7.2. The family of solutions $v(t)=v(t;t_0,v_0)$ of the initial value problem (7.1) and (7.2) is

$\mathcal{B}_1$ *bounded* if there exists a constant $r>0$ such that, for any $t_0\in\mathbb{R}_+$ and $v_0\in E^{n^2}$, we have that $d[v(t;t_0,v_0),\theta_0]<r$ for all $t\geq t_0$ and all $\kappa\in[0,1]$, where r may depend on any solution of the set of equations given by (7.1);

$\mathcal{B}_2$ *equi-bounded* if, for any $t_0\in\mathbb{R}_+$ and $\delta>0$, there exists $r=r(t_0,\delta)$ such that, for any initial data $v_0\in E^{n^2}$, $d[v_0,\theta_0]<\delta$ implies that $d[v(t;t_0,v_0),\theta_0]<r$ for all $t\geq t_0$ and all $\kappa\in[0,1]$;

$\mathcal{B}_3$ *uniformly bounded* if r in $\mathcal{B}_2$ does not depend on t_0;

$\mathcal{B}_4$ *quasi-equi-ultimately bounded* with a bound $\bar{r}$ if there exists $\bar{r}>0$ and, for any $\delta_0>0$, there exists $\tau=\tau(t_0,\delta)>0$ such that $d[v_0,\theta_0]<\delta_0$ implies that $d[v(t;t_0,v_0),\theta_0]<\bar{r}$ for all $t\geq t_0+\tau$ and $\kappa\in[0,1]$;

$\mathcal{B}_5$ *quasi-uniformly ultimately bounded* with a bound $\bar{r}$ if τ in $\mathcal{B}_4$ is independent on t_0;
$\mathcal{B}_6$ *equi-ultimately bounded* with a bound $\bar{r}$ if $\mathcal{B}_2$ and $\mathcal{B}_4$ hold simultaneously;
$\mathcal{B}_7$ *uniformly ultimately bounded* with a bound $\bar{r}$ if $\mathcal{B}_3$ and $\mathcal{B}_5$ hold simultaneously;
$\mathcal{B}_8$ *equi-stable in the Lagrange sense* if $\mathcal{B}_2$ and $\mathcal{S}_3$ hold simultaneously;
$\mathcal{B}_9$ *uniformly stable in the Lagrange sense* if $\mathcal{B}_3$ and $\mathcal{S}_4$ hold simultaneously.

Theorem 7.3. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Lemma 7.1 are satisfied and*

$$\exp\left(\int_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi(t, t_0))^{-\frac{1}{k-1}} < \frac{r}{d[v_0, \theta_0]}, \; r > 0, \; t \in [t_0, a). \tag{7.13}$$

Then, the family of solutions $v(t)$ is bounded.

Proof. If we suppose that the family of solutions $v(t)$ of (7.1) is unbounded, then, for any $r > 0$ there exists a $t_1 \in [t_0, \infty)$ such that

$$d[v(t), \theta_0] < r \;\text{ for }\; t \in [t_0, t_1) \;\text{ and }\; d[v(t_1), \theta_0] = r.$$

For the function $V(t, v, \eta) = d[v, \theta_0]$, the estimate (7.3) and condition (7.13) imply that $d[v(t_1), \theta_0] < r$ for all $\kappa \in [0, 1]$. The obtained contradiction proves the assertion of Theorem 7.3.

Next, we will need the following assumption:

(H_6^*). For any $\delta > 0$, $k > 1$ and $d[v_0, \theta_0] < \delta$, we have

$$\Phi^*(t_0, t) = (k-1)\delta^{k-1} \int_{t_0}^{t} \tilde{m}(s) \exp\left[(k-1)\int_{t_0}^{s} \tilde{f}(\tau)d\tau\right] ds < 1, \; t \in [t_0, a).$$

Theorem 7.4. *Assume that, for the set of fuzzy equations given by (7.1), the conditions H_4–H_6^* hold for all $t \in [t_0, a)$, and that, for any $\delta > 0$ and $t_0 \in \mathbb{R}_+$ there exists $r = r(t_0, \delta) > 0$ such that*

$$\exp\left(\int_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi^*(t_0, t))^{-\frac{1}{k-1}} < \frac{r(t_0, \delta)}{\delta} \tag{7.14}$$

for all $t \in [t_0, a)$.

Then, the family of solutions $v(t)$ is equi-bounded.

Proof. Analogously to Lemma 7.1, using conditions H_4–H_6^*, we obtain

$$d[v(t), \theta_0] < \delta \exp\left(\int_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi^*(t_0, t))^{-\frac{1}{k-1}}, \; t \in [t_0, a). \tag{7.15}$$

From the estimate (7.15) and condition (7.14), we obtain that $d[v(t), \theta_0] < r(t_0, \delta)$ for any $t \in [t_0, a)$ and $\kappa \in [0, 1]$ whenever $d[v_0, \theta_0] < \delta$. This proves Theorem 7.4.

The proofs of the following boundedness results are similar to the proof of Theorem 7.4.

Theorem 7.5. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Theorem 7.4 hold, and that, instead of (7.14), for any* $\delta > 0$ *there exists* $r^* = r^*(\delta) > 0$ *such that*

$$\exp\left(\int\limits_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi^*(t_0, t))^{-\frac{1}{k-1}} < \frac{r^*(\delta)}{\delta}$$

for all $t \in [t_0, a)$ *uniformly on* $t_0 \in \mathbb{R}_+$.

Then, the family of solutions $v(t)$ *is uniformly bounded.*

Theorem 7.6. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Theorem 7.4 hold, and that, instead of (7.14), for given* $\bar{\beta} > 0$ *and* $\tau = \tau(t_0, \delta) \in \mathbb{R}_+$ *we have*

$$\exp\left(\int\limits_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi^*(t_0, t))^{-\frac{1}{k-1}} < \frac{\bar{\beta}}{\delta}, \; t \geq t_0 + \tau.$$

Then, the family of solutions $v(t)$ *is quasi-equi-ultimately bounded.*

Theorem 7.7. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Theorem 7.4 hold, and that, instead of (7.14), for given* $\bar{r} > 0$ *and* $\tau^* = \tau^*(\delta) \in \mathbb{R}_+$ *we have*

$$\exp\left(\int\limits_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi^*(t_0, t))^{-\frac{1}{k-1}} < \frac{\bar{r}}{\delta}$$

for all $t \geq t_0 + \tau^*$ *uniformly on* $t_0 \in \mathbb{R}_+$.

Then, the family of solutions $v(t)$ *is quasi-uniformly ultimately bounded with a bound* $\bar{r}$.

The next result follows directly from Theorems 7.4 and 7.6.

Theorem 7.8. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Theorems 7.4 and 7.6 hold simultaneously.*

Then, the family of solutions $v(t)$ *is equi-ultimately bounded with a bound* $\bar{\beta}$.

The proof of the next result follows from the proofs of Theorems 7.5 and 7.7.

Theorem 7.9. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Theorems 7.5 and 7.7 hold simultaneously.*

Then, the family of solutions $v(t)$ *is uniformly ultimately bounded with a bound* $\bar{r}$.

Finally, we will establish Lagrange stability results for the steady state θ_0 of (7.1).

Theorem 7.10. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Theorem 7.4 hold, and that, for any* $t_0 \in \mathbb{R}_+$ *and* $\xi > 0$, *there exist* $\delta_0(t_0, \xi) > 0$ *and* $\tau = \tau(t_0, \xi) \in \mathbb{R}_+$ *such that*

$$\exp\left(\int\limits_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi(t_0, t))^{-\frac{1}{k-1}} < \frac{\xi}{\delta_0(t_0, \xi)}, \; t \geq t_0 + \tau(t_0, \xi). \tag{7.16}$$

Then, the family of solutions $v(t)$ *is equi-stable in the Lagrange sense.*

Proof. Since all conditions of Theorem 7.4 are satisfied, the family of solutions $v(t)$ is equi-bounded.

In addition, (7.3) and (7.16) imply that $d[v(t), \theta_0] < \xi$ for any $t \geq t_0 + \tau(t_0, \xi)$ and $\kappa \in [0, 1]$, whenever $d[v_0, \theta_0] < \delta_0(t_0, \xi)$. This proves that the steady state θ_0 of (7.1) is quasi-equi-asymptotically stable, which proves Theorem 7.10.

The proof of the last result is similar to the proof of Theorem 7.10.

Theorem 7.11. *Assume that, for the set of fuzzy equations given by (7.1), the conditions of Theorem 7.5 hold, and that, for any $\xi > 0$, there exist $\delta^*(\xi) > 0$ and $\tau^* = \tau^*(\xi) \in \mathbb{R}_+$ such that, uniformly on $t_0 \in \mathbb{R}_+$,*

$$\exp\left(\int_{t_0}^{t} \tilde{f}(s)ds\right)(1 - \Phi(t_0, t))^{-\frac{1}{k-1}} < \frac{\xi}{\delta^*(\xi)}, \ t \geq t_0 + \tau^*(\xi).$$

Then, the family of solutions $v(t)$ is uniformly stable in the Lagrange sense.

Remark 7.12. The proposed boundedness results extend and generalize some existing boundedness and Lagrange stability results for fuzzy systems [38] to the uncertain case. Also, they complement some recently published results [39] in the fuzzy case because of the use of the proposed regularization method.

8. Conclusions

In this paper, two approaches are proposed for the study of the stability and boundedness of solutions of fuzzy systems of differential equations with uncertain parameters. These approaches are based on the method of matrix-valued Lyapunov-type functions and an integral method based on nonlinear integral inequalities. By applying a new scheme of regularization of fuzzy equations with respect to the inaccuracy parameter [26], numerous new criteria for the stability and boundedness of solutions for regularized equations have been established via the proposed methods. The results obtained can be applied to various fuzzy systems and real-world models in which the effects of some uncertain parameters cannot be neglected. In addition, the proposed methods for the qualitative analysis of fuzzy equations are of decisive importance for many applications, in particular in the theory of motion control. The proposed approaches have significant potential for further generalizations and applications to important classes of fuzzy systems, including systems with delays and impulsive perturbations. An interesting topic for future research is the application of the proposed methods to fuzzy systems with fractional-order dynamics and systems whose dynamics are modeled by fuzzy equations with conformable derivatives of the state vector.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. J. P. LaSalle, S. Lefschetz, *Stability by Lyapunov's Second Method with Applications*, Academic Press, New York, 1961. Available from: https://www.elsevier.com/books/stability-by-liapunovs-direct-method-with-applications-by-joseph-l-salle-and-solomon-lefschetz/la-salle/978-0-12-437056-2.

2. A. M. Lyapunov, *Stability of Motion*, Academic Press, New York & London, 1966. Available from: https://www.elsevier.com/books/stability-of-motion/liapunov/978-1-4832-3009-2.

3. G. Leitmann, Deterministic control of uncertain systems via a constructive use of Lyapunov stability theory, in *System Modelling and Optimization*, Springer, (1990), 38–55. https://doi.org/10.1007/BFb0008354

4. M. J. Lacerda, P. Seiler, Stability of uncertain systems using Lyapunov functions with non-monotonic terms, *Automatica*, **82** (2017), 187–193. https://doi.org/10.1016/j.automatica.2017.04.042

5. G. Stamov, I. M. Stamova, Uncertain impulsive differential systems of fractional order: almost periodic solutions, *Int. J. Syst. Sci.*, **49** (2018), 631–638. https://doi.org/10.1080/00207721.2017.1416428

6. G. T. Stamov, I. M. Stamova, J. Cao, Uncertain impulsive functional differential systems of fractional order and almost periodicity, *J. Franklin Inst.*, **355** (2018), 5310–5323. https://doi.org/10.1016/j.jfranklin.2018.05.021

7. F. Z. Taousser, M. Defoort, M. Djemai, Stability analysis of a class of uncertain switched systems on time scale using Lyapunov functions, *Nonlinear Anal. Hybrid Syst.*, **16** (2015), 13–23. https://doi.org/10.1016/j.nahs.2014.12.001

8. Z. Liu, L. Jia, Cross-validation for the uncertain Chapman-Richards growth model with imprecise observations, *Int. J. Uncertainty Fuzziness Knowledge Based Syst.*, **28** (2020), 769–783. https://doi.org/10.1142/S0218488520500336

9. P. S. P. Pessim, V. J. S. Leite, M. J. Lacerda, Robust performance for uncertain systems via Lyapunov functions with higher order terms, *J. Franklin Inst.*, **356** (2019), 3072–3089. https://doi.org/10.1016/j.jfranklin.2019.02.004

10. T. J. Sullivan, *Introduction to Uncertainty Quantification*, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-23395-6

11. X. Xu, C. Huang, C. Li, G. Zhao, X. Li, C. Ma, Uncertain design optimization of automobile structures: a survey, *Electron. Res. Arch.*, **31** (2023), 1212–1239. https://doi.org/10.3934/era.2023062

12. G. W. Wei, Uncertain linguistic hybrid geometric mean operator and its application to group decision making under uncertain linguistic environment, *Int. J. Uncertainty Fuzziness Knowledge Based Syst.*, **17** (2009), 251–267. https://doi.org/10.1142/S021848850900584X

13. A. A. Martynyuk, Y. A. Martynyuk-Chernienko, *Uncertain Dynamical Systems——Stability and Motion Control*, CRC Press, Boca Raton, 2012. https://doi.org/10.1201/b11314

14. V. Lakshmikantham, S. Leela, A. A. Martynyuk, *Stability Analysis of Nonlinear Systems*, Birkhauser, Cham, 2015. https://doi.org/10.1007/978-3-319-27200-9

15. V. Lakshmikantham, V. M. Matrosov, S. Sivasundaram, *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Springer Dordrecht, 1991. https://doi.org/10.1007/978-94-015-7939-1

16. V. M. Matrosov, Vector Lyapunov function method: theory and applications to complex industrial systems, *IFAC Proc. Vol.*, **30** (1997), 49–62. https://doi.org/10.1016/S1474-6670(17)43346-5

17. I. M. Stamova, Vector Lyapunov functions for practical stability of nonlinear impulsive functional differential equations, *J. Math. Anal. Appl.*, **325** (2007), 612–623. https://doi.org/10.1016/j.jmaa.2006.02.019

18. A. A. Martynyuk, *Stability by Liapunov's Matrix Function Method with Applications*, CRC Press, New York, 1998. Available from: https://www.amazon.com/Stability-Liapunovs-Function-Applications-Mathematics/dp/0824701917.

19. A. A. Martynyuk, *Qualitative Methods in Nonlinear Dynamics. Novel Approaches to Liapunov's Matrix Functions*, Marcel Dekker, Inc., New York, 2002. Available from: https://www.routledge.com/Qualitative-Methods-in-Nonlinear-Dynamics-Novel-Approaches-to-Liapunovs/Martynyuk/p/book/9780367396770.

20. V. Lakshmikantham, R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, CRC Press, London, 2003. https://doi.org/10.1201/9780203011386

21. X. Fan, Z. Wang, A fuzzy Lyapunov function method to stability analysis of fractional-order T–S fuzzy systems, *IEEE Trans. Fuzzy Syst.*, **30** (2022), 2769–2776. https://doi.org/10.1109/TFUZZ.2021.3078289

22. M. B. Gücen, C. Yakar, Strict stability of fuzzy differential equations by Lyapunov functions, *Int. J. Inf. Control Comput. Sci.*, **11** (2018), 1267–1277. https://doi.org/10.5281/zenodo.1316718

23. V. Lakshmikantham, S. Leela, Stability theory of fuzzy differential equations via differential inequalities, *Math. Inequal. Appl.*, **2** (1999), 551–559. https://doi.org/10.7153/mia-02-46

24. S. Song, C. Wu, E. S. Lee, Asymptotic equilibrium and stability of fuzzy differential equations, *Comput. Math. Appl.*, **49** (2005), 1267–1277. https://doi.org/10.1016/j.camwa.2004.03.016

25. Z. P. Yang, W. J. Ren, Existence and stability results for quaternion fuzzy fractional differential equations in the sense of Hilfer, *J. Intell. Fuzzy Syst.*, **34** (2018), 167–175. https://doi.org/10.3233/JIFS-171042

26. A. Martynyuk, G. Stamov, I. Stamova, Y. Martynyuk-Chernienko, Regularization scheme for uncertain fuzzy differential equations: analysis of solutions, *Electron. Res. Arch.*, **31** (2023), 3832–3847. https://doi.org/10.3934/era.2023195

27. P. Diamond, P. Kloeden, *Metric Spaces of Fuzzy Sets: Theory and Applications*, World Scientific, Singapore, 1994. Available from: https://www.worldscientific.com/worldscibooks/10.1142/2326#t=aboutBook.

28. C. V. Negoita, D. A. Ralescu, *Applications of Fuzzy Sets to System Analysis*, Springer, Basel, 1975. Available from: https://link.springer.com/book/10.1007/978-3-0348-5921-9.

29. H. J. Zimmermann, *Fuzzy Set Theory——and Its Applications*, Springer, New York, 2001. Available from: https://link.springer.com/book/10.1007/978-94-010-0646-0.

30. K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, New York, 1992. https://doi.org/10.1515/9783110874228

31. M. L. Puri, D. A. Ralescu, Differentials of fuzzy functions, *J. Math. Anal. Appl.*, **91** (1983), 552–558. https://doi.org/10.1016/0022-247X(83)90169-5

32. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

33. J. J. Buckley, T. Feuring, Fuzzy differential equations, *Fuzzy Sets Syst.*, **110** (2000), 43–54. https://doi.org/10.1016/S0165-0114(98)00141-9

34. O. Kaleva, Fuzzy differential equations, *Fuzzy Sets Syst.*, **24** (1987), 301–317. https://doi.org/10.1016/0165-0114(87)90029-7

35. O. Kaleva, The Peano theorem for fuzzy differential equations revisited, *Fuzzy Sets Syst.*, **98** (1998), 147–148. https://doi.org/10.1016/S0165-0114(97)00415-6

36. D. Vorobiev, S. Seikkala, Towards the theory of fuzzy differential equations, *Fuzzy Sets Syst.*, **125** (2002), 231–237. https://doi.org/10.1016/S0165-0114(00)00131-7

37. M. Mazandarani, L. Xiu, A review on fuzzy differential equations, *IEEE Access*, **9** (2021), 62195–62211. https://doi.org/10.1109/ACCESS.2021.3074245

38. C. Yakar, M. Çiçek, M. B. Gücen, Practical stability, boundedness criteria and Lagrange stability of fuzzy differential systems, *Comput. Math. Appl.*, **64** (2012), 2118–2127. https://doi.org/10.1016/j.camwa.2012.04.008

39. A. Martynyuk, I. Stamova, Y. A. Martynyuk-Chernienko, On the boundedness and Lagrange stability of fractional-like neural network-based quasilinear systems, *Eur. Phys. J. Spec. Top.*, **231** (2022), 1789–1799. https://doi.org/10.1140/epjs/s11734-022-00447-3