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Schur's test, Bergman-type operators and Gleason's problem on radial-angular mixed spaces

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Abstract: Let $0 < p, q < \infty$, Φ be a generalized normal function and $L_{p,q}(\Phi)$ the radial-angular mixed space. In this paper, we first generalize the classical Schur's test to radial-angular mixed spaces setting and then find the sufficient and necessary condition for the boundedness of integral operators from $L_{p_1,p_2}(\Phi)$ to $L_{q_1,q_2}(\Phi)$ for $1 \le p_i, q_i \le \infty$ with $i \in \{1, 2\}$. Moreover, we also establish the boundedness of Bergman-type operators $P_{s,t}$, where $s \in \mathbb{R}$ and $t > 0$, on holomorphic radial-angular mixed space $H_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$. As an application, we finally solve Gleason's problem on $H_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$.

Keywords: Schur's test; Bergman-type operator; Gleason's problem

1. Introduction and main results

It is well known that Schur's tests give sufficient conditions for the boundedness of some integral operator T from one function space to another. Thus, they become one of the most useful tools for establishing boundedness of integral operators (see, for instance, [1]). Recall that, in 1911, Schur's test was first given by Schur [2] as a discrete form. After that, Schur's test has been generalized by many authors. For instance, in 1965, Schur's test for integral operators T on Lebesgue spaces $L^p(X, d\mu)$ was obtained in [3], where $1 \le p < \infty$ and the space $L^p(X, d\mu)$ always denotes the set of all measurable functions f on X such that

$$\|f\|_{L^p(X,d\mu)} := \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p} < \infty.$$

In 2015, Zhao [4] gave a generalization of Schur's test for the boundedness of T from one weighted Lebesgue spaces to another weighted Lebesgue spaces, and then applied to characterize boundedness of Forelli-Rudin operator. For the convenience of the reader, we present the following version of Schur's test from Zhao [4].

Theorem 1.1 ([4]). *Let μ and ν be two positive measures on the space X and K a nonnegative function on $X \times X$. Let T be an integral operator with kernel K defined by setting for any $x \in X$,*

$$Tf(x) := \int_X K(x,y) f(y)\, d\mu(y).$$

Suppose $1 < p \le q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let γ and δ be two real numbers such that $\gamma + \delta = 1$. If there exist two positive functions h_1 and h_2 with two positive constants C_1 and C_2 such that

$$\int_X [K(x,y)]^{\gamma p'} [h_1(y)]^{p'}\, d\mu(y) \le C_1 [h_2(x)]^{p'}$$

for almost all $x \in X$, and

$$\int_X [K(x,y)]^{\delta q} [h_2(x)]^{q}\, d\nu(x) \le C_2 [h_1(y)]^{q}$$

for almost all $y \in X$, then $T : L^p(X, d\mu) \to L^q(X, d\nu)$ is bounded with

$$\|T\|_{L^p(X,d\mu)\to L^q(X,d\nu)} \le C_1^{1/p'} C_2^{1/q}.$$

In this paper, we first extend Theorem 1.1 to the radial-angular mixed space $L_{p,q}(\Phi)$ for $1 \le p, q < \infty$, where Φ is a generalized normal function (see Definition 1.3 below for details). In addition, the similar Schur's tests for critical points are also addressed in this paper. Let $s \in \mathbb{R}$ and $t > 0$. We then establish the boundedness of Bergman-type operators $P_{s,t}$ from holomorphic radial-angular mixed space $H_{p,q}(\Phi)$ to $L_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$. At the end, we finally solve Gleason's problem on $H_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$ as an application.

To state our main results, we need some notations.

For $z := (z_1, \ldots, z_n)$, $\omega := (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n$, we denote the inner product of z and ω by

$$\langle z, \omega \rangle := z_1 \overline{\omega}_1 + \cdots + z_n \overline{\omega}_n$$

and $|z| := \sqrt{\langle z, z \rangle}$. Let

$$B := \{z \in \mathbb{C}^n : \ |z| < 1\}$$

denote the open unit ball in complex vector space $\mathbb{C}^n$ and ∂B its boundary. In addition, let $d\nu$ denote the Lebesgue measure, normalized such that $\int_B d\nu = 1$ and $d\sigma$ the surface measure on ∂B, again normalized such that $\int_{\partial B} d\sigma = 1$.

Recall that a positive continuous function φ defined on $[0, 1)$ is said to be *normal*, if there exist constants $0 < a < b$, $r_0 \in [0, 1)$ such that

(i) $\frac{\varphi(r)}{(1-r)^a}$ is nonincreasing in $r \in [r_0, 1)$ and $\lim_{r\to 1} \frac{\varphi(r)}{(1-r)^a} = 0$;
(ii) $\frac{\varphi(r)}{(1-r)^b}$ is nondecreasing in $r \in [r_0, 1)$ and $\lim_{r\to 1} \frac{\varphi(r)}{(1-r)^b} = \infty$.

Observe that the constants a and b are not uniquely for given normal function φ. Therefore, in what follows, let a_φ denote the superemum of all possible a satisfying (i) and b_φ the infimum of all possible b satisfying (ii). Then a_φ and b_φ are alwalys called *characteristic exponents* of φ.

We now present the notion of the generalized normal function Φ and related radial-angular mixed spaces $L_{p,q}(\Phi)$ from [5].

Definition 1.2. Let $\tau \in \mathbb{R}$ and φ be a normal function. A continuous function $\Phi : [0,1) \mapsto [0,\infty)$ is called *generalized normal function* if for any $r \in [0,1)$, $\Phi(r) := (1-r)^\tau \varphi(r)$.

Definition 1.3. Let $0 < p, q \leq \infty$ and Φ be a generalized normal function. The *radial-angular mixed space* $L_{p,q}(\Phi)$ is defined to be the set of all measurable complex functions f on B such that

$$\|f\|_{p,q,\Phi} := \left\{ \int_0^1 r^{2n-1}(1-r)^{-1}[\Phi(r)]^p [M_q(r,f)]^p \, dr \right\}^{1/p} < \infty$$

when $p \in (0,\infty)$ and

$$\|f\|_{\infty,q,\Phi} := \sup_{r\in(0,1)} \Phi(r) M_q(r,f) < \infty,$$

where $M_\infty(r,f) := \sup_{\zeta\in\partial B} |f(r\zeta)|$ and for any $q \in (0,\infty)$,

$$M_q(r,f) := \left[\int_{\partial B} |f(r\zeta)|^q \, d\sigma(\zeta) \right]^{1/q}.$$

We next state the main results about Schur's tests on radial-angular mixed spaces as follows. For any $\vec{p} := (p_1, p_2) \in (0,\infty)^2$, in what follows, we always denote $p_- := \min\{p_1, p_2\}$ and $p_+ := \max\{p_1, p_2\}$.

Theorem 1.4. *Let K a nonnegative function on $\partial B \times I \times \partial B \times I$. Let T be an integral operator with kernel K defined by setting for any $(x,y) \in \partial B \times I$,*

$$Tf(x,y) := \int_I \int_{\partial B} K(x,y,s,t) f(s,t) \, d\sigma(s) \, d\lambda(t).$$

Here and thereafter, $I := [0,1]$ and $d\lambda(r) := r^{2n-1}(1-r)^{-1}[\Phi(r)]^p \, dr$ for any $r \in I$. Suppose $\vec{p} := (p_1, p_2)$, $\vec{q} := (q_1, q_2) \in (1,\infty)^2$ satisfying $1 < p_- \leq p_+ \leq q_- < \infty$. Let γ and δ be two real numbers such that $\gamma + \delta = 1$. If there exist two positive functions h_1 and h_2 defined on $\partial B \times I$ with two positive constants C_1 and C_2 such that

$$\int_I \left\{ \int_{\partial B} [K(x,y,s,t)]^{\gamma p_1'} [h_1(s,t)]^{p_1'} \, d\sigma(s) \right\}^{p_2'/p_1'} d\lambda(t) \leq C_1 [h_2(x,y)]^{p_2'}$$

for almost all $(x,y) \in \partial B \times I$, and

$$\int_I \left\{ \int_{\partial B} [K(x,y,s,t)]^{\delta q_1} [h_2(s,t)]^{q_1} \, d\sigma(x) \right\}^{q_2/q_1} d\lambda(y) \leq C_2 [h_1(s,t)]^{q_2}$$

for almost all $(s,t) \in \partial B \times I$, then $T : L_{p_1,p_2}(\Phi) \to L_{q_1,q_2}(\Phi)$ is bounded with

$$\|T\|_{L_{p_1,p_2}(\Phi)\to L_{q_1,q_2}(\Phi)} \leq C_1^{1/p_2'} C_2^{1/q_2}.$$

We also obtain the Schur's tests for the endpoint cases.

Theorem 1.5. *Let γ, δ and T be an integral operator with kernel K defined as in Theorem 1.4. Suppose $\vec{p} := (1, p_2)$ and $\vec{q} := (q_1, q_2) \in (1, \infty)^2$ satisfying $1 < p_2 \le q_- < \infty$. If there exist two positive functions h_1 and h_2 defined on $\partial B \times I$ with two positive constants C_1 and C_2 such that*

$$\int_I \left\{ \operatorname*{ess\,sup}_{s \in \partial B} [K(x, y, s, t)]^\gamma h_1(s, t) \right\}^{p_2'} d\lambda(t) \le C_1 [h_2(x, y)]^{p_2'}$$

for almost all $(x, y) \in \partial B \times I$, and

$$\int_I \left\{ \int_{\partial B} [K(x, y, s, t)]^{\delta q_1} [h_2(x, y)]^{q_1} \, d\sigma(x) \right\}^{q_2/q_1} d\lambda(y) \le C_2 [h_1(s, t)]^{q_2},$$

for almost all $(s, t) \in \partial B \times I$, then $T : L_{p_1, p_2}(\Phi) \to L_{q_1, q_2}(\Phi)$ is bounded with

$$\|T\|_{L_{p_1,p_2}(\Phi) \to L_{q_1,q_2}(\Phi)} \le C_1^{1/p_2'} C_2^{1/q_2}.$$

Theorem 1.6. *Let γ, δ and T be an integral operator with kernel K defined as in Theorem 1.4. Suppose $\vec{p} := (p_1, 1)$ and $\vec{q} := (q_1, q_2) \in (1, \infty)^2$ satisfying $1 < p_1 \le q_- < \infty$. If there exist two positive functions h_1 and h_2 defined on $\partial B \times I$ with two positive constants C_1 and C_2 such that*

$$\operatorname*{ess\,sup}_{t \in I} \int_{\partial B} [K(x, y, s, t)]^{\gamma p_1'} [h_1(s, t)]^{p_1'} \, d\sigma(s) \le C_1 [h_2(x, y)]^{p_1'}$$

for almost all $(x, y) \in \partial B \times I$, and

$$\int_I \left\{ \int_{\partial B} [K(x, y, s, t)]^{\delta q_1} [h_2(x, y)]^{q_1} \, d\sigma(x) \right\}^{q_2/q_1} d\lambda(y) \le C_2 [h_1(s, t)]^{q_2},$$

for almost all $(s, t) \in \partial B \times I$, then $T : L_{p_1, p_2}(\Phi) \to L_{q_1, q_2}(\Phi)$ is bounded with

$$\|T\|_{L_{p_1,p_2}(\Phi) \to L_{q_1,q_2}(\Phi)} \le C_1^{1/p_1'} C_2^{1/q_2}.$$

Theorem 1.7. *Let γ, δ and T be an integral operator with kernel K defined as in Theorem 1.4. Suppose $\vec{q} := (q_1, q_2) \in [1, \infty)^2$. Let γ and δ be two real numbers such that $\gamma + \delta = 1$. If there exist two positive functions h_1 and h_2 defined on $\partial B \times I$ with two positive constants C_1 and C_2 such that*

$$\operatorname*{ess\,sup}_{(s,t) \in \partial B \times I} [K(x, y, s, t)]^\gamma h_1(s, t) \le C_1 h_2(x, y)$$

for almost all $(x, y) \in \partial B \times I$, and

$$\int_I \left\{ \int_{\partial B} [K(x, y, s, t)]^{\delta q_1} [h_2(s, t)]^{q_1} \, d\sigma(x) \right\}^{q_2/q_1} d\lambda(y) \le C_2 [h_1(s, t)]^{q_2}$$

for almost all $(s, t) \in \partial B \times I$, then $T : L_{1,1}(\Phi) \to L_{q_1, q_2}(\Phi)$ is bounded with

$$\|T\|_{L_{1,1}(\Phi) \to L_{q_1,q_2}(\Phi)} \le C_1 C_2^{1/q_2}.$$

Moreover, we find the following sufficient and necessary condition for boundedness of the integral operators T from $L_{p_1,p_2}(\Phi)$ to $L_{q_1,q_2}(\Phi)$.

Theorem 1.8. *Let T be an integral operator with kernel K defined as in Theorem 1.4. Suppose $\vec{p} := (p_1, p_2)$, $\vec{q} := (q_1, q_2) \in [1, \infty]^2$. Then T is a bounded operator from $L_{p_1,p_2}(\Phi)$ into $L_{q_1,q_2}(\Phi)$ with bound M if and only if for any $u \in L_{p_1,p_2}(\Phi)$ and $\upsilon \in L_{q_1',q_2'}(\Phi)$,*

$$\left|\int_I \int_{\partial B} \int_I \int_{\partial B} K(x, y, s, t)u(x, y)\upsilon(s, t)\, d\sigma(x)\, d\lambda(y)\, d\sigma(s)\, d\lambda(t)\right| \leq M\|u\|_{L_{p_1,p_2}(\Phi)}\|\upsilon\|_{L_{q_1',q_2'}(\Phi)}.$$

On the other hand, for $s \in \mathbb{R}$ and $t > 0$, the Bergman-type operator $P_{s,t}$ on $L_{p,q}(\Phi)$ is defined by

$$P_{s,t}f(z) := (1 - |z|^2)^s \int_B \frac{(1 - |w|^2)^{t-1} f(w)}{(1 - \langle z, w\rangle)^{n+t+s}}\, dv(w).$$

The boundedness of Bergman-type operators $P_{s,t}$ has been studied extensively (see, for instance, [4, 6–8]). Indeed, let φ be a normal function and $H(B)$ denote all the holomorphic functions in B. The boundedness of operator $P_{s,t}$ on $L_{p,q}(\varphi)$ for $1 \leq p, q < \infty$ was studied in [9]. Later on, the boundedness of $P_{s,t}$ from $H_{p,q}(\varphi) := H(B) \cap L_{p,q}(\varphi)$ to $L_{p,q}(\varphi)$ for $0 < p < 1$ and $1 \leq q < \infty$ was obtained in [10]. The only left case $0 < q < 1$ and $0 < p < \infty$ was sloved by Lou [11] in 2007. In this paper, we extend all these known results to radial-angular mixed spaces $L_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$.

To be exact, we establish the following boundedness of Bergman-type operators $P_{s,t}$.

Theorem 1.9. *Let $0 < p, q < \infty$ and $\Phi(\cdot) := (1 - \cdot)^\tau \varphi(\cdot)$ be a generalized normal function, where $\tau \in \mathbb{R}$ and φ is a normal function with characteristic exponents a_φ and b_φ. If*

$$t - \tau + \min\{n(1 - 1/q), 0\} > b_\varphi > a_\varphi > -s - \tau,$$

then $P_{s,t} : H_{p,q}(\Phi) \to L_{p,q}(\Phi)$ is a bounded operator.

Remark 1.10. For all $0 < p, q < \infty$, Theorem 1.9 gives the sufficient conditions such that the Bergman-type operator $P_{s,t}$ is bounded from $H_{p,q}(\Phi)$ to $L_{p,q}(\Phi)$. The conditions are different in the two cases: $1 \leq q < \infty$ and $0 < q < 1$.

(i) When $1 \leq q < \infty$ and $0 < p < \infty$, the sufficient condition is $t - \tau > b_\varphi > a_\varphi > -s - \tau$. We point out that when $\tau = 0$, then $\Phi \equiv \varphi$ and the radial-angular mixed space $L_{p,q}(\Phi)$ goes back to the space $L_{p,q}(\varphi)$. Thus, in this case, Theorem 1.9 holds true for $t > b_\varphi > a_\varphi > -s$. In this sence, Theorem 1.9 extends the main theorem established in [9, Theorem A(ii)] and [10].

(ii) When $0 < q < 1$ and $0 < p < \infty$, the sufficient condition is $t - \tau + n(1 - 1/q) > b_\varphi > a_\varphi > -s - \tau$. Similarly, we point out that, in this case, Theorem 1.9 with $\tau = 0$ goes back to [11, Theorem 1.1].

As an application, we prove that Gleason's problem in radial-angular mixed spaces $H_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$ is solvable. Let X be a space of holomorphic functions on a domain Ω in $\mathbb{C}^n$. Then Gleason's problem for X and any given point $a \in \Omega$, denoted by (Ω, a, X), is the following: Given $a \in \Omega$ and $f \in X$, do there exist functions $g_1, \ldots, g_n \in X$ such that

$$f(z) - f(a) = \sum_{k=1}^{n} (z_k - a_k) g_k(z)$$

for all $z \in \Omega$?

Gleason [12] originally asked the question for $(B, 0, A(B))$, where $A(B)$ is the ball algebra on the unit ball $B \subset \mathbb{C}^n$, consisting of holomorphic functions in B which are countinuous on $\bar{B}$. This problem $(B, 0, A(B))$ was solved by Leibenson. The difficulty of Gleason's problem depends on the domain Ω and the space X. Gleason's problem for different domains and function spaces have been studied extensively. For instance, with the help of the boundedness of Bergman-type operators, Zhu [13] and Choe [14] solved Gleason's problem for the Bergman space $L_{p,p}(\Phi)$ with $\Phi(r) := (1 - r^2)^{1/p}$ and $\Phi(r) := (1 - r^2)^{(1+\alpha)/p}$ respectively. Here, $1 \le p < \infty$ and $\alpha > -1$. Hu [15] studied Gleason's problem for harmonic mixed norm and Bloch spaces in convex domains. In particular, for given normal function φ, Gleason's problem on mixed spaces $L_{p,q}(\varphi)$ was addressed by [9] for the case $1 \le p, q < \infty$ and by [11] for the case $0 < p < \infty$, $0 < q < 1$. In this paper, we extend these conclusions and solve Gleason's problem for all $0 < p$, $q < \infty$ on $H_{p,q}(\Phi)$.

As an application of Theorem 1.9, we have the following result.

Theorem 1.11. *Let $0 < p$, $q < \infty$ and Φ be a generalized normal function. Gleason's problem can be solved on $H_{p,q}(\Phi)$. More precisely, for any intrger $m \ge 1$, there exist bounded linear operators A_α on $H_{p,q}(\Phi)$ such that if $f \in H_{p,q}(\Phi)$ and f and its partial derivatives of order $\le m - 1$ are zero at 0, then*

$$f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z)$$

for all $z \in B$. Here $\alpha := (\alpha_1, \dots, \alpha_n)$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and each α_i is a nonnegative integer.

The rest of this paper is organized as follows. In Section 2 we will prove main results about Schur's tests, namely, Theorems 1.4, 1.5, 1.6, 1.7 and 1.8. The proofs of Theorems 1.9 and 1.11 will be given in Section 3. Section 4 is the conclusions of this paper.

Finally, we make some conventions on notation. We always denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The notation $f \lesssim g$ means $f \le Cg$ and, if $f \lesssim g \lesssim f$, then we write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \le h$.

2. Proofs of Schur's tests

This section is devoted to proving Theorems 1.4, 1.5, 1.6, 1.7 and 1.8. To do this, we first recall the following definition of mixed Lebesgue spaces introduced by Benedek and Panzone [16] in 1961.

Definition 2.1. Let $\vec{p} := (p_1, p_2) \in (0, \infty)^2$ and (X_i, μ_i) be two totally non-trivial σ-finite measure spaces for $i \in \{1, 2\}$. The *mixed Lebesgue space* $L^{\vec{p}}(\prod_{i=1}^2 X_i, \prod_{i=1}^2 \mu_i)$ is defined to be the set of all measurable functions f on $X_1 \times X_2$ such that

$$\|f\|_{\vec{p}} := \left\| \|f\|_{L^{p_1}(X_1,\mu_1)} \right\|_{L^{p_2}(X_2,\mu_2)} := \left\{ \int_{X_2} \left[\int_{X_1} |f(x,y)|^{p_1} \, d\mu_1(x) \right]^{p_2/p_1} d\mu_2(y) \right\}^{1/p_2} < \infty.$$

We now write the radial-angular mixed space $L_{p,q}(\Phi)$ with the notion of mixed Lebesgue spaces in Definition 2.1. Indeed, if $p = q$, the space $L_{p,q}(\Phi)$ is the weighted Lebesgue space; if $p \ne q$, the space $L_{p,q}(\Phi)$ becomes the weighted mixed Lebesgue space. Precisely, Let $\alpha > -1$ and $d\nu_\alpha :=$

$(1-|z|^2)^\alpha\, d\nu(z)$. Recall that the weighted Lebesgue space $L^p_\alpha(B) := L^p(B, d\nu_\alpha)$ with $p > 0$ denotes the set of all measurable complex functions f on B such that

$$\|f\|_{L^p_\alpha(B)} := \left[\int_B |f(z)|^p\, d\nu_\alpha\right]^{1/p} = \left[\int_B |f(z)|^p (1-|z|^2)^\alpha\, d\nu(z)\right]^{1/p} < \infty.$$

Applying the integral formula in polar coordinates (see, for instance, [17]), we find that

$$\int_B |f(z)|^p (1-|z|^2)^\alpha\, d\nu(z) = 2n \int_0^1 r^{2n-1}(1-r^2)^\alpha M_p^p(r,f)\, dr.$$

Thus, $L^p_\alpha(B) = L_{p,p}((1-r^2)^{\frac{\alpha+1}{p}})$ in the sense of equivalent quasi-norms.

Moreover, with the notation of Definition 2.1, we can write

$$\|f\|_{p,q,\Phi} = \left\{\int_0^1 r^{2n-1}(1-r)^{-1}[\Phi(r)]^p [M_q(r,f)]^p\, dr\right\}^{1/p} = \left\|\|f_r\|_{L^q(\partial B,\, d\sigma)}\right\|_{L^p(I,\, d\lambda)}, \tag{2.1}$$

here and thereafter, $f_r(\cdot) := f(r\cdot)$, $I := [0,1]$, $d\lambda := r^{2n-1}(1-r)^{-1}[\Phi(r)]^p\, dr$. This implies that $L_{p,q}(\Phi) = L^{(p,q)}(\partial B \times I, d\sigma \times d\lambda)$. Therefore, to show Theorems 1.4, 1.5, 1.6, 1.7 and 1.8, it suffices to prove Schur's tests for mixed Lebesgue spaces.

In what follows, let $L^{\vec{p}}_\mu := L^{\vec{p}}(X \times X, \mu_1 \times \mu_2)$ and $L^{\vec{q}}_\nu := L^{\vec{q}}(X \times X, \nu_1 \times \nu_2)$ for $\vec{p}, \vec{q} \in (0,\infty)^2$, where (X, μ_i) and (X, ν_i) are measure spaces. Then we obtain the following propositions, which extend Schur's tests on weighted Lebesgue spaces given in Zhao [4].

Proposition 2.2. *Let μ_i and ν_i be positive measures on the space X for $i \in \{1,2\}$ and T an integral operator with nonnegative kernel K defined by setting for any $(x,y) \in X \times X$,*

$$Tf(x,y) := \int_X \int_X K(x,y,s,t) f(s,t)\, d\mu_1(s)\, d\mu_2(t).$$

Suppose $\vec{p} := (p_1, p_2)$, $\vec{q} := (q_1, q_2) \in (1,\infty)^2$ satisfying $1 < p_- \le p_+ \le q_- < \infty$. Let γ and δ be real numbers such that $\gamma + \delta = 1$. If there exist two positive functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that for almost all $(x,y) \in X \times X$,

$$\int_X \left\{\int_X [K(x,y,s,t)]^{\gamma p_1'} [h_1(s,t)]^{p_1'}\, d\mu_1(s)\right\}^{p_2'/p_1'} d\mu_2(t) \le C_1 [h_2(x,y)]^{p_2'} \tag{2.2}$$

and for almost all $(s,t) \in X \times X$,

$$\int_X \left\{\int_X [K(x,y,s,t)]^{\delta q_1} [h_2(s,t)]^{q_1}\, d\nu_1(x)\right\}^{q_2/q_1} d\nu_2(y) \le C_2 [h_1(s,t)]^{q_2} \tag{2.3}$$

then $T : L^{\vec{p}}_\mu \to L^{\vec{q}}_\nu$ is bounded with $\|T\|_{L^{\vec{p}}_\mu \to L^{\vec{q}}_\nu} \le C_1^{1/p_2'} C_2^{1/q_2}$.

Proof. If $f \in L^{\vec{p}}_\mu$, then for almost every $(x,y) \in X \times X$, we have

$$|Tf(x,y)| \le \int_X \int_X K(x,y,s,t)[h_1(s,t)]^{-1} h_1(s,t) |f(s,t)|\, d\mu_1(s)\, d\mu_2(t).$$

Using Hölder's inequality, we have

$$
\begin{aligned}
|Tf(x,y)| \le & \int_X \left\{ \int_X [K(x,y,s,t)]^{\gamma p_1'} [h_1(s,t)]^{p_1'} \, d\mu_1(s) \right\}^{1/p_1'} \\
& \times \left\{ \int_X [K(x,y,s,t)]^{\delta p_1} [h_1(s,t)]^{-p_1} |f(s,t)|^{p_1} \, d\mu_1(s) \right\}^{1/p_1} \, d\mu_2(t) \\
\le & \left\{ \int_X \left(\int_X [K(x,y,s,t)]^{\gamma p_1'} [h_1(s,t)]^{p_1'} \, d\mu_1(s) \right)^{p_2'/p_1'} d\mu_2(t) \right\}^{1/p_2'} \\
& \times \left\{ \int_X \left(\int_X [K(x,y,s,t)]^{\delta p_1} [h_1(s,t)]^{-p_1} |f(s,t)|^{p_1} \, d\mu_1(s) \right)^{p_2/p_1} d\mu_2(t) \right\}^{1/p_2},
\end{aligned}
$$

which, together with (2.2), further implies that

$$
\begin{aligned}
& |Tf(x,y)| \\
& \le C_1^{1/p_2'} h_2(x,y) \left\{ \int_{X_2} \left(\int_{X_1} [K(x,y,s,t)]^{\delta p_1} [h_1(s,t)]^{-p_1} |f(s,t)|^{p_1} \, d\mu_1(s) \right)^{p_2/p_1} d\mu_2(t) \right\}^{1/p_2}.
\end{aligned}
$$

In addition, from the assumption $1 < p_- \le p_+ \le q_- < \infty$, it follows that $q_1 \ge p_2$, $q_1 \ge p_1$, $q_2 \ge p_2$ and $q_2 \ge p_1$. Thus, combining the above inequality and Minkowski's inequality, we conclude that

$$
\begin{aligned}
& \|Tf\|_{L_\nu^{\vec{q}}} \\
& \le C_1^{1/p_2'} \left\{ \int_X \left[\int_X \left(\int_X \left[\int_X [K(x,y,s,t)]^{\delta p_1} [h_2(x,y)]^{q_1} |f(s,t)|^{q_1} [h_1(s,t)]^{-q_1} \right.\right.\right.\right. \\
& \quad \left.\left.\left.\left. \times d\nu_1(x) \right]^{\frac{q_2}{q_1}} d\nu_2(y) \right)^{\frac{p_1}{q_2}} d\mu_1(s) \right]^{\frac{p_2}{p_1}} d\mu_2(t) \right\}^{\frac{1}{p_2}} \\
& = C_1^{1/p_2'} \left\{ \int_X \left[\int_X |f(s,t)|^{p_1} [h_1(s,t)]^{-p_1} \left(\int_X \left[\int_X [K(x,y,s,t)]^{\delta p_1} [h_2(x,y)]^{q_1} \right.\right.\right.\right. \\
& \quad \left.\left.\left.\left. \times d\nu_1(x) \right]^{\frac{q_2}{q_1}} d\nu_2(y) \right)^{\frac{p_1}{q_2}} d\mu_1(s) \right]^{\frac{p_2}{p_1}} d\mu_2(t) \right\}^{\frac{1}{p_2}}.
\end{aligned}
$$

Now applying (2.3), we obtain

$$
\begin{aligned}
\|Tf\|_{L_\nu^{\vec{q}}} & \le C_1^{1/p_2'} C_2^{1/q_2} \left\{ \int_X \left[\int_X |f(s,t)|^{p_1} \, d\mu_1(s) \right]^{\frac{p_2}{p_1}} d\mu_2(t) \right\}^{\frac{1}{p_2}} \\
& = C_1^{1/p_2'} C_2^{1/q_2} \|f\|_{L_\mu^{\vec{p}}}.
\end{aligned}
$$

This finishes the proof of Proposition 2.2. □

Proposition 2.3. *Let γ, δ and T be an integral operator with kernel K defined as in Proposition 2.2. Suppose $\vec{p} := (1, p_2)$ and $\vec{q} := (q_1, q_2) \in (1, \infty)^2$ satisfying $1 < p_2 \le q_- < \infty$. If there exist two positive functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that for almost all $(x, y) \in X \times X$,*

$$\int_X \left\{ \operatorname*{ess\,sup}_{s\in X} [K(x, y, s, t)]^\gamma h_1(s, t) \right\}^{p_2'} d\mu_2(t) \le C_1 [h_2(x, y)]^{p_2'} \tag{2.4}$$

and, for almost all $(s, t) \in X \times X$,

$$\int_X \left\{ \int_X [K(x, y, s, t)]^{\delta q_1} [h_2(x, y)]^{q_1}\, d\nu_1(x) \right\}^{q_2/q_1} d\nu_2(y) \le C_2 [h_1(s, t)]^{q_2}, \tag{2.5}$$

then $T : L^{\vec{p}}_\mu \to L^{\vec{q}}_\nu$ is bounded with $\|T\|_{L^{\vec{p}}_\mu \to L^{\vec{q}}_\nu} \le C_1^{1/p_2'} C_2^{1/q_2}$.

Proof. Let $f \in L^{\vec{p}}_\mu$. From (2.4) and Hölder's inequality, we infer that, for almost every $(x, y) \in X \times X$,

$$\begin{aligned}
|Tf(x, y)| &\le \int_X \int_X K(x, y, s, t)[h_1(s, t)]^{-1} h_1(s, t)|f(s, t)|\, d\mu_1(s)\, d\mu_2(t) \\
&\le \int_X \operatorname*{ess\,sup}_{s\in X} \{[K(x, y, s, t)]^\gamma h_1(s, t)\} \\
&\quad \times \int_X [K(x, y, s, t)]^\delta [h_1(s, t)]^{-1} |f(s, t)|\, d\mu_1(s)\, d\mu_2(t) \\
&\le \left\{ \int_X \left(\operatorname*{ess\,sup}_{s\in X} \{[K(x, y, s, t)]^\gamma h_1(s, t)\} \right)^{p_2'} d\mu_2(t) \right\}^{1/p_2'} \\
&\quad \times \left\{ \int_X \left(\int_X [K(x, y, s, t)]^\delta [h_1(s, t)]^{-1} |f(s, t)|\, d\mu_1(s) \right)^{p_2} d\mu_2(t) \right\}^{1/p_2} \\
&\le C_1^{1/p_2'} h_2(x, y) \left\{ \int_X \left(\int_X [K(x, y, s, t)]^\delta [h_1(s, t)]^{-1} |f(s, t)|\, d\mu_1(s) \right)^{p_2} d\mu_2(t) \right\}^{1/p_2}.
\end{aligned}$$

Then, by the assumptions $q_1 \ge p_2$ and $q_2 \ge p_2$, and Minkowski's inequality, we conclude that

$$\begin{aligned}
&\|Tf\|_{L^{\vec{q}}_\nu} \\
&\le C_1^{1/p_2'} \left\{ \int_X \left[\int_X \left(\int_X \left[\int_X [K(x, y, s, t)]^{\delta q_1} [h_2(x, y)]^{q_1} |f(s, t)|^{q_1} [h_1(s, t)]^{-q_1} \right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. \times\, d\nu_1(x) \right]^{\frac{q_2}{q_1}} d\nu_2(y) \right)^{\frac{1}{q_2}} d\mu_1(s) \right]^{p_2} d\mu_2(t) \right\}^{\frac{1}{p_2}} \\
&= C_1^{1/p_2'} \left\{ \int_X \left[\int_X |f(s, t)| [h_1(s, t)]^{-1} \left(\int_X \left[\int_X [K(x, y, s, t)]^{\delta q_1} [h_2(x, y)]^{q_1} \right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. \times\, d\nu_1(x) \right]^{\frac{q_2}{q_1}} d\nu_2(y) \right)^{\frac{1}{q_2}} d\mu_1(s) \right]^{p_2} d\mu_2(t) \right\}^{\frac{1}{p_2}},
\end{aligned}$$

which, together with (2.5), further implies that

$$\|Tf\|_{L^{\vec{q}}_{\nu}} \le C_1^{1/p_2'} C_2^{1/q_2} \left\{ \int_X \left[\int_X |f(s,t)|\, d\mu_1(s) \right]^{p_2} d\mu_2(t) \right\}^{\frac{1}{p_2}} = C_1^{1/p_2'} C_2^{1/q_2} \|f\|_{L^{\vec{p}}_{\mu}}.$$

This finishes the proof of Proposition 1.5. □

Proposition 2.4. *Let γ, δ and T be an integral operator with kernel K defined as in Proposition 2.2. Suppose $\vec{p} := (p_1, 1)$ and $\vec{q} := (q_1, q_2) \in (1, \infty)^2$ satisfying $1 < p_1 \le q_- < \infty$. If there exist two positive functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that for almost all $(x, y) \in X \times X$,*

$$\operatorname*{ess\,sup}_{t \in X} \int_X [K(x,y,s,t)]^{\gamma p_1'} [h_1(s,t)]^{p_1'}\, d\mu_1(s) \le C_1 [h_2(x,y)]^{p_1'}$$

and, for almost all $(s,t) \in X \times X$,

$$\int_X \left(\int_X [K(x,y,s,t)]^{\delta q_1} [h_2(x,y)]^{q_1}\, d\nu_1(x) \right)^{q_2/q_1} d\nu_2(y) \le C_2 [h_1(s,t)]^{q_2},$$

then $T : L^{\vec{p}}_{\mu} \to L^{\vec{q}}_{\nu}$ is bounded with $\|T\|_{L^{\vec{p}}_{\mu} \to L^{\vec{q}}_{\nu}} \le C_1^{1/p_1'} C_2^{1/q_2}$.

Proof. This proposition is a symmetric case of Proposition 2.3. Therefore, the proof is similar and hence we omit it here. □

Proposition 2.5. *Let γ, δ and T be an integral operator with kernel K defined as in Proposition 2.2. Suppose $\vec{q} := (q_1, q_2) \in [1, \infty)^2$. If there exist two positive functions h_1 and h_2 defined on $X \times X$ with two positive constants C_1 and C_2 such that*

$$\operatorname*{ess\,sup}_{(s,t) \in X \times X} [K(x,y,s,t)]^{\gamma} h_1(s,t) \le C_1 h_2(x,y)$$

for almost all $(x, y) \in X \times X$, and

$$\int_X \left(\int_X [K(x,y,s,t)]^{\delta q_1} [h_2(s,t)]^{q_1}\, d\nu_1(x) \right)^{q_2/q_1} d\nu_2(y) \le C_2 [h_1(s,t)]^{q_2} \tag{2.6}$$

for almost all $(s,t) \in X \times X$, then $T : L^{\vec{1}}_{\mu} \to L^{\vec{q}}_{\nu}$ is bounded with $\|T\|_{L^{\vec{1}}_{\mu} \to L^{\vec{q}}_{\nu}} \le C_1 C_2^{1/q_2}$.

Proof. Let $f \in L^{\vec{1}}_{\mu}$. Then, by (2.6), we find that, for almost every $(x, y) \in X \times X$,

$$\begin{aligned}
|Tf(x,y)| &\le \int_X \int_X K(x,y,s,t)[h_1(s,t)]^{-1} h_1(s,t) |f(s,t)|\, d\mu_1(s)\, d\mu_2(t) \\
&\le \operatorname*{ess\,sup}_{(s,t) \in X \times X} [K(x,s,y,t)]^{\gamma} h_1(s,t) \\
&\quad \times \int_X \int_X [K(x,y,s,t)]^{\delta} [h_1(s,t)]^{-1} |f(s,t)|\, d\mu_1(s)\, d\mu_2(t) \\
&\le C_1 h_2(x,y) \int_X \int_X [K(x,y,s,t)]^{\delta} [h_1(s,t)]^{-1} |f(s,t)|\, d\mu_1(s)\, d\mu_2(t).
\end{aligned}$$

Now we first show the present proposition for the case $\vec{q} \in (1,\infty)^2$. In this case, applying the Minkowski's inequality, we deduce that

$$\begin{aligned}
&\|Tf\|_{L_\nu^{\vec{q}}} \\
&\le C_1 \int_X \int_X \left(\int_X \left[\int_X [K(x,y,s,t)]^{\delta q_1} [h_2(x,y)]^{q_1} [h_1(s,t)]^{-q_1} |f(s,t)|^{q_1} \right.\right. \\
&\quad \left.\left. \times \, d\nu_1(x) \right]^{\frac{q_2}{q_1}} d\nu_2(y) \right)^{\frac{1}{q_2}} d\mu_1(s)\, d\mu_2(t) \\
&= C_1 \int_X \int_X |f(s,t)| [h_1(s,t)]^{-1} \\
&\quad \times \left(\int_X \left[\int_X [K(x,y,s,t)]^{\delta q_1} [h_2(x,y)]^{q_1}\, d\nu_1(x) \right]^{\frac{q_2}{q_1}} d\nu_2(y) \right)^{\frac{1}{q_2}} d\mu_1(s)\, d\mu_2(t),
\end{aligned}$$

which, combined with (2.6), further implies that

$$\begin{aligned}
\|Tf\|_{L_\nu^{\vec{q}}} &\le C_1 C_2^{1/q_2} \int_X \int_X |f(s,t)|\, d\mu_1(s)\, d\mu_2(t) \\
&= C_1 C_2^{1/q_2} \|f\|_{L_\mu^1}.
\end{aligned}$$

This completes the proof in this case. For the case $q_i = 1$ for some $i \in \{1,2\}$, instead of using Minkowski's inequality, applying Fubini's theorem will also do the job. We omit the details in this case and hence finish the proof. □

The following result shows a necessary and sufficient condition of integral operators T associated with kernel K as in Proposition 2.2.

Proposition 2.6. *Let T be a integral operator with kernel K defined as in Proposition 2.2. Suppose $\vec{p} := (p_1, p_2)$, $\vec{q} := (q_1, q_2) \in [1,\infty]^2$. Then T is a bounded operator from $L^{\vec{p}}(\prod_{i=1}^2 X_i, \prod_{i=1}^2 \mu_i)$ into $L^{\vec{q}}(\prod_{i=1}^2 X_i, \prod_{i=1}^2 \nu_i)$ with bound M if and only if for any $u \in L^{\vec{p}}(\prod_{i=1}^2 X_i, \prod_{i=1}^2 \mu_i)$ and $\upsilon \in L^{\vec{q}'}(\prod_{i=1}^2 X_i, \prod_{i=1}^2 \nu_i)$,*

$$\left| \int_{X_2} \int_{X_1} \int_{X_2} \int_{X_1} K(x,y,s,t) u(x,y) \upsilon(s,t)\, d\nu_1(x)\, d\nu_2(y)\, d\mu_1(s)\, d\mu_2(t) \right| \le M \|u\|_{\vec{p}} \|\upsilon\|_{\vec{q}'}. \tag{2.7}$$

Proof. From [16, p. 304, Theorem 2], it follows that

$$\begin{aligned}
\|T\|_{L^{\vec{p}} \to L^{\vec{q}}} &:= \|T\|_{L^{\vec{p}}(\prod_{i=1}^2 X_i, \prod_{i=1}^2 \mu_i) \to L^{\vec{q}}(\prod_{i=1}^2 X_i, \prod_{i=1}^2 \nu_i)} = \sup_{\|u\|_{\vec{p}}=1} \|Tu\|_{\vec{q}} \\
&= \sup_{\|u\|_{\vec{p}}=1} \sup_{\|\upsilon\|_{\vec{q}'}=1} \left| \int_{X_2} \int_{X_1} Tu(s,t) \upsilon(s,t)\, d\mu_1(s)\, d\mu_2(t) \right|.
\end{aligned}$$

Thus, if (2.7) holds true, then

$$\|T\|_{L^{\vec{p}} \to L^{\vec{q}}} = \sup_{\|u\|_{\vec{p}}=1} \sup_{\|\upsilon\|_{\vec{q}'}=1} \left| \int_{X_2} \int_{X_1} \int_{X_2} \int_{X_1} K(x,y,s,t) u(x,y) \upsilon(s,t)\, d\nu_1(x)\, d\nu_2(y)\, d\mu_1(s)\, d\mu_2(t) \right|$$

$$\leq M.$$

Conversely, when $\|T\|_{L^{\vec{p}}\to L^{\vec{q}}} \leq M$, using Hölder's inequality, we find that

$$\begin{aligned}\left|\int_{X_2}\int_{X_1}\int_{X_2}\int_{X_1} K(x,y,s,t)u(x,y)\upsilon(s,t)\,d\nu_1(x)\,d\nu_2(y)\,d\mu_1(s)\,d\mu_2(t)\right| \\ \leq \int_{X_2}\int_{X_1} |Tu(s,t)\upsilon(s,t)|\,d\mu_1(s)\,d\mu_2(t) \\ \leq \|Tu\|_{\vec{q}}\|\upsilon\|_{\vec{q}'} \leq M\|u\|_{\vec{p}}\|\upsilon\|_{\vec{q}'}.\end{aligned}$$

This implies that (2.7) holds true and hence finishes the proof of Proposition 2.6. □

Finally, applying (2.1), we find that Theorems 1.4, 1.5, 1.6, 1.7 and 1.8 are just corollaries of Propositions 2.2, 2.3, 2.4, 2.5 and 2.6 respectively, and hence their proofs are finished.

3. Proofs of Theorems 1.9 and 1.11

In this section, we first give the proofs of boundedness of Bergman-type operators and then, as an application, solve Gleason' problem on $H_{p,q}(\Phi)$. To achieve this, we need the following lemmas.

Lemma 3.1. *([10, Lemma 4]) Let* $0 < p \leq 1$, $1 \leq q < \infty$, $t+s > 0$ *and* $f \in H(B)$. *Then there exists a positive constant C such that, for any* $0 \leq r < 1$,

$$[M_q(r, P_{s,t}f)]^p \leq C(1-r^2)^{ps}\int_0^1 \frac{\rho^{p(2n-1)}(1-\rho)^{pt-1}}{(1-r\rho)^{p(t+s)}}[M_q(\rho, f)]^p\,d\rho.$$

Lemma 3.2. *([9, Lemma 2.1]) Let* $0 < q < 1$ *and* $t + s > n(1 - 1/q)$. *Then, for any measurable function f on B, there exists a positive constant C such that, for any* $0 \leq r < 1$,

$$M_q(r, P_{s,t}f) \leq C(1-r^2)^s\left[\int_0^1 \frac{\rho^{q(2n-1)}(1-\rho)^{q(t+1)-2}}{(1-r\rho)^{q(n+t+s)-n}}[M_q(\rho, f)]^q\,d\rho\right]^{1/q}.$$

Lemma 3.3. *([19, Lemma 6]) Let* $s_1 > s_2 > 0$ *and* $0 \leq r < 1$. *Then there exists a positive constant C such that*

$$\int_0^1 \frac{(1-\rho)^{s_2-1}}{(1-r\rho)^{s_1}}\,d\rho \leq \frac{C}{(1-r)^{s_1-s_2}}.$$

Lemma 3.4. *([11, Lemma 2.1]) Let* $0 < p \leq 1$, $\alpha,\beta > 0$, $0 \leq r < 1$ *and* $f : [0,1) \to [0,\infty)$ *be increasing. Then there exists a positive constant C such that*

$$\left[\int_0^1 \frac{(1-\rho)^{\alpha-1}}{(1-r\rho)^{\beta}} f(\rho)\,d\rho\right]^p \leq C\int_0^1 \frac{(1-\rho)^{p\alpha-1}}{(1-r\rho)^{p\beta}}[f(\rho)]^p\,d\rho.$$

Lemma 3.5. *([9, Lemma 2.3]) Let* $0 < p < \infty$, $0 \leq r < 1$ *and* $\Phi(\cdot) := (1-\cdot)^\tau\varphi(\cdot)$ *be a generalized normal function, where* $\tau \in \mathbb{R}$ *and* φ *is a normal function with characteristic exponents* a_φ *and* b_φ. *If* $s - \tau + t > b_\varphi > a_\varphi > s - \tau$, *then there exists a positive constant C such that*

$$\int_0^1 \frac{[\Phi(\rho)]^p}{(1-\rho)^{ps+1}(1-r\rho)^{pt}}\,d\rho \leq C\frac{[\Phi(r)]^p}{(1-r)^{p(s+t)}}.$$

We now show Theorem 1.9.

Proof of Theorem 1.9. We prove Theorem 1.9 by separately considering $1 \le q < \infty$ in Step 1 and $0 < q < 1$ in Step 2.

Step 1) In this step we show that if $1 \le q < \infty$, $0 < p < \infty$ and $t - \tau > b_\varphi > a_\varphi > -s - \tau$, then $P_{s,t}$ is bounded from $H_{p,q}(\Phi)$ to $L_{p,q}(\Phi)$. To achieve this, We further deal with the two cases $0 < p \le 1$ and $1 < p < \infty$ separately.

Case i) $0 < p \le 1$ and $1 \le q < \infty$. In this case, let $f \in H_{p,q}(\Phi)$. From Lemma 3.1, Tonelli's theorem, Lemma 3.5 and the fact that $t - \tau > b_\varphi > a_\varphi > -s - \tau$, we infer that

$$\begin{aligned}
\|P_{s,t}f\|_{p,q,\Phi}^p &\le \int_0^1 (1-r)^{-1}[\Phi(r)]^p[M_q(r, P_{s,t}f)]^p\,dr \\
&\lesssim \int_0^1 (1-r)^{ps-1}[\Phi(r)]^p \int_0^1 \frac{\rho^{p(2n-1)}(1-\rho)^{pt-1}}{(1-r\rho)^{p(t+s)}}[M_q(\rho,f)]^p\,d\rho\,dr \\
&\sim \int_0^1 \rho^{p(2n-1)}(1-\rho)^{pt-1}[M_q(\rho,f)]^p \int_0^1 \frac{[\Phi(r)]^p}{(1-r)^{1-ps}(1-r\rho)^{p(t+s)}}\,dr\,d\rho \\
&\lesssim \int_0^1 \rho^{p(2n-1)}(1-\rho)^{-1}[\Phi(\rho)]^p[M_q(\rho,f)]^p\,d\rho.
\end{aligned}$$

Letting $\rho = r^{\frac{1}{p}}$, by the facts that $0 < p < 1$ and the monotonicity of $M_q(\rho, f)$ on ρ for holomorphic function f, we know that $M_q(r^{\frac{1}{p}}, f) \le M_q(r, f)$ and $(1 - r^{\frac{1}{p}})^{-1} \le (1-r)^{-1}$. In addition, since Φ is the generalized normal function, we have

$$[\Phi(r^{\frac{1}{p}})]^p := (1-r^{\frac{1}{p}})^{p\tau}[\varphi(r^{\frac{1}{p}})]^p \le (1-r^{\frac{1}{p}})^{p\tau}\frac{(1-r^{\frac{1}{p}})^{pb}}{(1-r)^{pb}}[\varphi(r)]^p \le \frac{(1-r^{\frac{1}{p}})^{p\tau}}{(1-r)^{p\tau}}\frac{(1-r^{\frac{1}{p}})^{pb}}{(1-r)^{pb}}[\Phi(r)]^p.$$

Thus, $[\Phi(r^{\frac{1}{p}})]^p \lesssim [\Phi(r)]^p$ and hence

$$\begin{aligned}
\|P_{s,t}f\|_{p,q,\Phi}^p &\lesssim \int_0^1 r^{2n-1}(1-r^{\frac{1}{p}})^{-1}[\Phi(r^{\frac{1}{p}})]^p[M_q(r^{\frac{1}{p}}, f)]^p\,dr \\
&\lesssim \int_0^1 r^{2n-1}(1-r)^{-1}[\Phi(r)]^p[M_q(r,f)]^p\,dr \sim \|f\|_{p,q,\Phi}^p.
\end{aligned}$$

This finishes the proof of Theorem 1.9 in the case $0 < p \le 1$ and $1 \le q < \infty$.

Case ii) $1 < p < \infty$ and $1 \le q < \infty$. In this case, let $f \in L_{p,q}(\Phi)$ and $t = t_1 + t_2 = t_3 + t_4$ such that $t_1 > 0$, $a_\varphi + t_1 + \tau > t_3$, $t_3 + s > t_1$ and $t_2 - \tau > b_\varphi$. We point out that these assumptions on t_i are reasonable. Indeed, taking a sufficiently small $\epsilon > 0$, let

$$t_1 := t - \tau - (1+\epsilon)b_\varphi, \;\; t_2 := \tau + (1+\epsilon)b_\varphi, \;\; t_3 := t - (1+\epsilon)b_\varphi + (1-\epsilon)a_\varphi$$

and $t_4 := (1+\epsilon)b_\varphi - (1-\epsilon)a_\varphi$. Then one can verify t_1, t_2, t_3 and t_4 satisfy all the above conditions from $t - \tau > b_\varphi > a_\varphi > -s - \tau$. With these assumptions, applying Lemma 3.1, Hölder's inequality and Lemma 3.3, we find that

$$M_q(r, P_{s,t}f) \lesssim (1-r^2)^s \int_0^1 \frac{\rho^{2n-1}(1-\rho^2)^{t-1}}{(1-r\rho)^{t+s}} M_q(\rho, f)\,d\rho$$

$$\lesssim (1-r^2)^s \left\{ \int_0^1 \frac{\rho^{p(2n-1)}(1-\rho^2)^{pt_2-1}}{(1-r\rho)^{pt_4}} [M_q(\rho, f)]^p \, d\rho \right\}^{\frac{1}{p}} \left[\int_0^1 \frac{(1-\rho^2)^{p't_1-1}}{(1-r\rho)^{p'(t_3+s)}} \, d\rho \right]^{\frac{1}{p'}}$$
$$\lesssim (1-r)^{t_1-t_3} \left\{ \int_0^1 \frac{\rho^{p(2n-1)}(1-\rho^2)^{pt_2-1}}{(1-r\rho)^{pt_4}} [M_q(\rho, f)]^p \, d\rho \right\}^{\frac{1}{p}}.$$

This, together with Tonelli's theorem and Lemma 3.5, further implies that

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\Phi}^p &\lesssim \int_0^1 (1-r)^{p(t_1-t_3)-1} [\Phi(r)]^p \int_0^1 \frac{\rho^{p(2n-1)}(1-\rho^2)^{pt_2-1}}{(1-r\rho)^{pt_4}} [M_q(\rho, f)]^p \, d\rho \, dr \\ &\sim \int_0^1 \rho^{p(2n-1)}(1-\rho^2)^{pt_2-1} [M_q(\rho, f)]^p \int_0^1 \frac{[\Phi(r)]^p}{(1-r)^{p(t_3-t_1)+1}(1-r\rho)^{pt_4}} \, dr \, d\rho \\ &\lesssim \int_0^1 \rho^{p(2n-1)}(1-\rho^2)^{-1} [M_q(\rho, f)]^p [\Phi(\rho)]^p \, d\rho \sim \|f\|_{p,q,\Phi}^p, \end{aligned}$$

which completes the proof of Theorem 1.9 in the case $1 < p < \infty$ and $1 \le q < \infty$. Thus, we finish the proof of Step 1.

Step 2) In this step we show that if $0 < q < 1$, $0 < p < \infty$ and $t - \tau + n(1 - 1/q) > b_\varphi > a_\varphi > -s - \tau$, then $P_{s,t}$ is bounded from $H_{p,q}(\Phi)$ to $L_{p,q}(\Phi)$. For this purpose, We considering the following two cases $p \le q$ and $p > q$ separately.

Case iii) $0 < q < 1$ and $p \le q$. In this case, let $f \in H_{p,q}(\Phi)$ and $g(z) := z^{2n-1} f(z)$. Applying Lemmas 3.2 and 3.4, Tonelli's theorem, Lemma 3.5 and the fact that $t > n(1/q - 1) + b_\varphi + \tau$, we infer that

$$\begin{aligned} \|P_{s,t}f\|_{p,q,\Phi}^p &\le \int_0^1 (1-r)^{-1} [\Phi(r)]^p [M_q(r, P_{s,t}f)]^p \, dr \\ &\lesssim \int_0^1 (1-r)^{ps-1} [\Phi(r)]^p \left\{ \int_0^1 \frac{\rho^{q(2n-1)}(1-\rho)^{q(t+1)-2}}{(1-r\rho)^{q(n+t+s)-n}} [M_q(\rho, f)]^q \, d\rho \right\}^{p/q} dr \\ &\lesssim \int_0^1 (1-r)^{ps-1} [\Phi(r)]^p \int_0^1 \frac{(1-\rho)^{p(t+1)-p/q-1}}{(1-r\rho)^{p(n+t+s)-np/q}} [M_q(\rho, g)]^p \, d\rho \, dr \\ &\sim \int_0^1 \rho^{p(2n-1)}(1-\rho)^{p(t+1)-p/q-1} [M_q(\rho, f)]^p \int_0^1 \frac{(1-r)^{ps-1}[\Phi(r)]^p}{(1-r\rho)^{p(n+t+s)-np/q}} \, dr \, d\rho \\ &\lesssim \int_0^1 \rho^{p(2n-1)}(1-\rho)^{p(1-n)(1-p/q)-1} [M_q(\rho, f)]^p [\Phi(\rho)]^p \, d\rho. \end{aligned}$$

Letting $\rho = r^{\frac{1}{p}}$, then by $0 < p \le q < 1$ and an argument similar to that used in the proof of Case i), we conclude that

$$\|P_{s,t}f\|_{p,q,\Phi}^p \lesssim \int_0^1 r^{2n-1}(1-r)^{-1} [M_q(r, f)]^p [\Phi(r)]^p \, dr \sim \|f\|_{p,q,\Phi}^p.$$

This finishes the proof of Theorem 1.9 in the present case.

Case iv) $0 < q < 1$ and $q < p < \infty$. In this case, let $f \in L_{p,q}(\Phi)$, $g(z) := z^{2n-1} f(z)$, $\eta := p/q > 1$ and $q(t+1) - 1 = L_1 + L_2 = L_3 + L_4$, where $L_1 > 0$, $L_3 > L_1$, $L_2/q + (n-1)(1 - 1/q) - \tau > b_\varphi$ and $a_\varphi > (L_3 - L_1)/q - s - \tau$. Observe that these conditions on L_i are satisfied by taking a sufficiently small $\epsilon > 0$ and setting

$$L_1 := q(t+1) - 1 - q(1+\epsilon)[b_\varphi + (1-n)(1 - 1/q)] - q\tau,$$

$$L_2 := q(1+\epsilon)[b_\varphi + (1-n)(1-1/q)] + q\tau,$$

$$L_3 := q(t+1) - 1 - q(1+\epsilon)[b_\varphi + (1-n)(1-1/q)] + q(\epsilon - \tau)$$

and

$$L_4 := q(1+\epsilon)[b_\varphi + (1-n)(1-1/q)] + q(\tau - \epsilon).$$

With these assumptions, applying Lemma 3.2, Hölder's inequality, Lemma 3.3, Tonelli's theorem and Lemma 3.5, we find that

$$\begin{aligned}
\|P_{s,t}f\|_{p,q,\Phi}^p &\lesssim \int_0^1 (1-r)^{ps-1}[\Phi(r)]^p \left\{\int_0^1 \frac{(1-\rho)^{q(t+1)-2}}{(1-r\rho)^{q(n+t+s)-n}}[M_q(\rho,g)]^q\,d\rho\right\}^{p/q} dr \\
&\lesssim \int_0^1 (1-r)^{ps-1}[\Phi(r)]^p \left[\int_0^1 \frac{(1-\rho)^{\eta' L_1-1}}{(1-r\rho)^{\eta' L_3}}\,d\rho\right]^{\eta/\eta'} \\
&\quad\times \int_0^1 \frac{(1-\rho)^{\eta L_2-1}}{(1-r\rho)^{(L_4-n+1+q(n+s-1))\eta}}[M_q(\rho,g)]^p\,d\rho\,dr \\
&\lesssim \int_0^1 (1-r)^{ps+\eta(L_1-L_3)-1}[\Phi(r)]^p \int_0^1 \frac{(1-\rho)^{\eta L_2-1}}{(1-r\rho)^{(L_4-n+1+q(n+s-1))\eta}}[M_q(\rho,g)]^p\,d\rho\,dr \\
&\sim \int_0^1 \rho^{p(2n-1)}(1-\rho)^{\eta L_2-1}[M_q(\rho,f)]^p \int_0^1 \frac{(1-r)^{p(s+(L_1-L_3)/q)-1}[\Phi(r)]^p}{(1-r\rho)^{p((L_4-n+1)/q+n+s-1)}}\,dr\,d\rho \\
&\lesssim \int_0^1 \rho^{p(2n-1)}(1-\rho)^{p(1-n)(1-p/q)-1}[M_q(\rho,f)]^p[\Phi(\rho)]^p\,d\rho \sim \|f\|_{p,q,\Phi}^p,
\end{aligned}$$

which completes the proof in this case and hence of Step 2. Combining this and Step 1, we obtain Theorem 1.9. □

Remark 3.6. Let $s \in \mathbb{R}$ and $t > 0$. Define $\widetilde{P}_{s,t}$ on $L_{p,q}(\Phi)$ by

$$\widetilde{P}_{s,t}f(z) := (1-|z|^2)^s \int_B \frac{(1-|w|^2)^{t-1}|f(w)|}{|1-\langle z,w\rangle|^{n+t+s}}\,dv(w).$$

Then the proof of Theorem 1.9 above actually show that, for any $f \in H_{p,q}(\Phi)$,

$$\left\|\widetilde{P}_{s,t}f\right\|_{p,q,\Phi} \lesssim \|f\|_{p,q,\Phi}.$$

This is important in the proof of Theorem 1.11.

Applying Theorem 1.9, we next prove Theorem 1.11.

Proof of Theorem 1.11. We may assume that $m = 1$. The general case follows from induction. In this case, note that $f(0) = 0$. By Leibenson's formula, we find that, for any $f \in H_{p,q}(\Phi)$ and $z \in B$,

$$f(z) = f(z) - f(0) = \int_0^1 \frac{d}{dr} f(rz)\,dr = \sum_{k=1}^n z_k \int_0^1 \frac{\partial f}{\partial z_k}(rz)\,dr.$$

For any $z \in B$, let $A_k f(z) := \int_0^1 \frac{\partial f}{\partial z_k}(rz)\,dr$. Then A_k is obviously linear. Therefore, to finish the proof, it remains to show that A_k is bounded on $H_{p,q}(\Phi)$. To this end, let $f_r(z) := f(rz)$ for any $0 < r < 1$ and

$z \in B$. Then we have $P_{0,t}f_r = f_r$ (see [9]). Letting $r \to 1^-$, by Theorem 1.9, we conclude that $P_{0,t}f = f$, namely, for any $z \in B$,

$$f(z) = \int_B \frac{(1-|w|^2)^{t-1} f(w)}{(1-\langle z,w\rangle)^{n+t}}\, dv(w).$$

By differential under the integral, we have

$$\begin{aligned} A_k f(z) &\sim \int_0^1 \int_B \frac{\overline{w_k}(1-|w|^2)^{t-1} f(w)}{(1-r\langle z,w\rangle)^{n+t+1}}\, dv(w)\, dr \\ &\sim \int_B \overline{w_k}(1-|w|^2)^{t-1} f(w) \int_0^1 \frac{1}{(1-r\langle z,w\rangle)^{n+t+1}}\, dr\, dv(w) \\ &\sim \int_B \frac{\overline{w_k}(1-|w|^2)^{t-1} f(w)}{(1-\langle z,w\rangle)^{n+t}} \frac{1-(1-\langle z,w\rangle)^{n+t}}{\langle z,w\rangle}\, dv(w). \end{aligned}$$

Observe that

$$\frac{1-(1-\langle z,w\rangle)^{n+t}}{\langle z,w\rangle} = \sum_{k=1}^{n+t-1} (1-\langle z,w\rangle)^k$$

is a polynomial in z and $\overline{w}$. Therefore, $|A_k f(z)| \lesssim |\widetilde{P}_{0,t} f(z)|$. From this and Remark 3.6, we infer that A_k is bounded on $H_{p,q}(\Phi)$ and hence finish the proof of Theorem 1.11. □

4. Conclusions

Let $0 < p, q < \infty$, Φ be a generalized normal function and $L_{p,q}(\Phi)$ the radial-angular mixed space. In this paper, we have generalized the classical Schur's test to radial-angular mixed spaces setting and then found the sufficient and necessary condition for the boundedness of integral operators from $L_{p_1,p_2}(\Phi)$ to $L_{q_1,q_2}(\Phi)$ for $1 \le p_i, q_i \le \infty$ with $i \in \{1,2\}$. Furthermore, we have also established the boundedness of Bergman-type operators $P_{s,t}$ on holomorphic radial-angular mixed space $H_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$. As an application, we finally solved Gleason's problem on $H_{p,q}(\Phi)$ for all possible $0 < p, q < \infty$.

Use of AI tools declaration

The authors have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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