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Research article

Upper semi-continuity of pullback attractors for bipolar fluids with delay

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Abstract: We investigate bipolar fluids with delay in a $2D$ channel $\Sigma = \mathbb{R} \times (-K, K)$ for some $K > 0$. The channel Σ is divided into a sequence of simply connected, bounded, and smooth sub-domains $\Sigma_n(n = 1, 2, 3 \cdots)$, such that $\Sigma_n \to \Sigma$ as $n \to \infty$. The paper demonstrates that the pullback attractors in the sub-domains Σ_n converge to the pullback attractor in the entire domain Σ as $n \to \infty$.

Keywords: bipolar fluid; delay; pullback attractor; upper semi-continuity

1. Introduction

Fluid models with delay terms are crucial for understanding and predicting the behavior of fluids where devices or control mechanisms are inserted into fluid domains to manipulate certain properties, such as temperature or velocity. For example, in a wind tunnel experiment, various control mechanisms are used to manipulate the flow of air around a model. These control mechanisms can include flaps, fans, or other devices that alter the velocity or pressure distribution in specific regions of the wind tunnel, see [1, 2]. Over the years, extensive research has been conducted on specific fluid models that incorporate delay terms, such as the Navier-Stokes with delay [3], the micropolar fluid with delay [4], the Kelvin-Voigt fluid with delay [5] and the viscoelastic fluid with delay [6], and so on.

The bipolar fluid model is a well-known incompressible non-Newtonian fluid model that was introduced in [7, 8]. It is commonly used to describe the motion of various materials, including molten plastics, synthetic fibers, paints, greases, polymer solutions, suspensions, adhesives, dyes, varnishes, and more. Accounting for the effect of the history-dependent behavior of the fluid, the bipolar fluid model can be described as follows

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nabla \cdot (\mu(u)e(u) - 2\mu_1 \Delta e(u)) + \nabla p = f(t, x) + g(t, u_t), \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

where the velocity is denoted by $u = u(t, x)$, the pressure by $p = p(t, x)$, the non-delay external force by $f(t, x)$, and the delayed external force by $g(t, u_t)$ with

$$u_t(s) = u(t + s, x), \quad s \in (-h, 0), \quad t > \tau.$$

The rate of the deformation tensor, denoted as $e(u)$, is a 2×2 matrix defined by its components $e_{ij}(u)$. These components can be written as

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2.$$

The variable viscosity $\mu(u)$ is taken as

$$\mu(u) = 2\mu_0(\eta + |e|^2)^{-\alpha/2}, \quad |e|^2 = \sum_{i,j=1}^{2} |e_{ij}|^2, \quad i, j = 1, 2.$$

The constitutive parameters h, η, μ_0, μ_1 are constants that satisfy $h, \eta, \mu_0, \mu_1 > 0$ and $0 < \alpha < 1$.

From a physical point of view, Equations (1.1) and (1.2) are often subject to (see [9, 10])

$$u(\tau, x) = u^{\text{in}}, \quad \tau \in \mathbb{R}, \quad x \in \Sigma, \tag{1.3}$$

$$u(t, x) = \phi^{\text{in}}(t - \tau, x), \quad (t, x) \in (\tau - h, \tau) \times \Sigma, \tag{1.4}$$

$$u = 0, \quad \tau_{ijk} n_j n_k = 0, \quad x \in \partial\Sigma, \quad i, j, k = 1, 2, \tag{1.5}$$

where $\Sigma = \mathbb{R} \times (-K, K)$ for some $K > 0$, $\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}$, (n_1, n_2) denotes the exterior unit normal to the boundary $\partial\Sigma$, $u = 0$ represents the non-slip condition and $\tau_{ijk} n_j n_k = 0$ means the first moment of the traction vanishes on $\partial\Sigma$.

Concerning Eqs (1.1)–(1.5) without the delay term $g(t, u_t)$ in 2D domains, there have been numerous studies on the well-posedness, regularity, and long time behavior of solutions. For example, we can refer to [11–19] in 2D bounded domains and [9, 10, 20, 21] in 2D unbounded ones. However, as for Eqs (1.1)–(1.5) with the delay term $g(t, u_t)$ in 2D domains, most of the existing results related to Eqs (1.1)–(1.5) concentrate on 2D bounded domains. Zhao, Zhou and Li in [22] first established the existence, uniqueness of solutions and the existence of pullback attractors. Later on, different frameworks were used to study the existence and stability of stationary solutions and the existence of pullback attractors, as seen in [23–25]. Recently, the authors in [26] initially paid attention to the 2D unbounded domain. They showed the existence, uniqueness of solutions and the existence of pullback attractors. Therefore, it is natural to inquire about the relation between the pullback attractors in 2D bounded domains and the 2D unbounded domain as the bounded domains approximate the unbounded domain.

It is well known that the upper semi-continuity of attractors, as introduced in [27], is a powerful concept for describing the relationship between attractors in nonlinear evolution equations. Several researchers have also established the upper semi-continuity of attractors for various physical models, as seen in [27–30], among others. In particular, Zhao et al. [16, 21] have studied the upper semi-continuity of global attractors and cocycle attractors for the bipolar fluid without delay (i.e., $g(t, u_t) = 0$) with respect to the domains from Σ_n to Σ. The key idea is to unify the attractors by means of a natural extension. Specifically, for any $n \in \mathbb{Z}_+$, let $u_n \in X(\Sigma_n)$ (where X is the phase space), define

$$\overline{u}_n = \begin{cases} u_n, & x \in \Sigma_n, \\ 0, & x \in \Sigma \backslash \Sigma_n. \end{cases} \tag{1.6}$$

Then one can regard the function $\overline{u}_n$ defined on Σ as the natural extension of the function u_n. Moreover, one has

$$\|u_n\|_{X(\Sigma)} = \|\overline{u}_n\|_{X(\Sigma)} = \|\overline{u}_n\|_{X(\Sigma_n)} = \|u_n\|_{X(\Sigma_n)}. \tag{1.7}$$

Borrowing the previous idea, in the present paper, we aim to establish the upper semi-continuity of pullback attractors for Eqs (1.1)–(1.5). Differently from [16, 21], the presence of delay introduces new challenges in the new phase space, requiring us to perform more precise estimations. Actually, the result presented in this paper generalizes the results of [16, 21].

The paper is structured as follows. In Section 2, we introduce the necessary preliminaries, and in Section 3, proceed to establish the upper semi-continuity of pullback attractors. We conclude the main result in Section 4.

Notation. In this paper, we use the following notations: $\mathbb{R}$ for the set of real numbers, $\mathbb{Z}_+$ for the set of non-negative integers and c as a generic constant, which may vary depending on the context. If the dependence needs to be explicitly emphasized, some notations like c_1 and $c(\cdot)$ will be used. $\mathcal{O}$ represents either Σ or Σ_n. We denote the 2D vector Lebesgue space and 2D vector Sobolev space as $\mathbb{L}^p(\mathcal{O})$ and $\mathbb{H}^m(\mathcal{O})$, respectively. The norms for these spaces are $\|\cdot\|_{\mathbb{L}^p(\mathcal{O})}$ and $\|\cdot\|_{\mathbb{H}^m(\mathcal{O})}$. Additionally, we define the following spaces: $\mathcal{V}(\mathcal{O})$ as the set of $\phi \in C_0^\infty(\mathcal{O}) \times C_0^\infty(\mathcal{O})$ satisfying $\phi = (\phi_1, \phi_2)$ and $\nabla \cdot \phi = 0$, $H(\mathcal{O})$ as the closure of $\mathcal{V}(\mathcal{O})$ in $\mathbb{L}^2(\mathcal{O})$ with the norm $\|\cdot\|_{H(\mathcal{O})}$ and dual space $H'(\mathcal{O}) = H(\mathcal{O})$, and $V(\mathcal{O})$ as the closure of $\mathcal{V}(\mathcal{O})$ in $\mathbb{H}^2(\mathcal{O})$ with the norm $\|\cdot\|_{V(\mathcal{O})}$ and dual space $V'(\mathcal{O})$. The inner product in $H(\mathcal{O})$ or $\mathbb{L}^2(\mathcal{O})$ is denoted by $(\cdot, \cdot)$. The dual pairing between $V(\mathcal{O})$ and $V'(\mathcal{O})$ is denoted by $\langle \cdot, \cdot \rangle$. The Hausdorff semi-distance between $\Lambda_1 \subset \Lambda$ and $\Lambda_2 \subset \Lambda$ is denoted by $\mathrm{Dist}_\Lambda(\Lambda_1, \Lambda_2)$, which is defined as $\mathrm{Dist}_\Lambda(\Lambda_1, \Lambda_2) = \sup_{x\in\Lambda_1} \inf_{y\in\Lambda_2} \|x - y\|_\Lambda$.

2. Preliminaries

In this section, we begin by reformulating Eqs (1.1)–(1.5) in an abstract form. We then recall the global existence, uniqueness of solutions and the existence of pullback attractors in the channel Σ and in sub-domains Σ_n.

For the purpose of abstract representation, we introduce the following operators for Eqs (1.1)–(1.5):

$$\langle Au, v\rangle = \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \frac{\partial e_{ij}(u)}{\partial x_k} \frac{\partial e_{ij}(v)}{\partial x_k} \mathrm{d}x, \quad u, v \in V(\mathcal{O}),$$

$$\langle N(u), v\rangle = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \mu(u) e_{ij}(u) e_{ij}(v)\, \mathrm{d}x, \quad u, v \in V(\mathcal{O}),$$

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j\, \mathrm{d}x, \quad \langle B(u), w\rangle = b(u, u, w), \quad u, v, w \in V(\mathcal{O}).$$

With the help of above notation, the weak form of Eqs (1.1)–(1.5) can be expressed when $\mathcal{O} = \Sigma$ as follows (see [22, 23, 26])

$$\frac{\partial u}{\partial t} + 2\mu_1 A u + B(u) + N(u) = f(t) + g(t, u_t), \tag{2.1}$$

$$u(\tau, x) = u^{\text{in}}, \quad x \in \Sigma, \tag{2.2}$$

$$u(t, x) = \phi^{\text{in}}(t-\tau) = \phi^{\text{in}}, \quad t \in (\tau - h, \tau), \quad x \in \Sigma, \tag{2.3}$$

and in the case of $O = \Sigma_n$ as follows

$$\frac{\partial u^{(n)}}{\partial t} + 2\mu_1 A u^{(n)} + B(u^{(n)}) + N(u^{(n)}) = f(t) + g(t, u_t^{(n)}), \tag{2.4}$$

$$u^{(n)}(\tau, x) = u^{(n),\text{in}}, \quad x \in \Sigma_n, \tag{2.5}$$

$$u^{(n)}(t, x) = \phi^{(n),\text{in}}(t-\tau, x) = \phi^{(n),\text{in}}, \quad t \in (\tau - h, \tau), \quad x \in \Sigma_n. \tag{2.6}$$

According to the definition of the operator A, the following estimate has been established in [9, 10].

Lemma 2.1. *There exists a constant c_1, which is dependent only on O, such that*

$$c_1 \|u\|^2_{V(O)} \leqslant \langle Au, u \rangle \leqslant \|u\|^2_{V(O)}.$$

From now on, we should consider $X = H, V, V'$ and spaces

$$C_{H(O)} = C([-h, 0]; H(O)), \;\; L^2_{X(O)} = L^2(-h, 0; X(O)), \;\; E^2_{X(O)} = X(O) \times L^2_{X(O)}.$$

In order to ensure the global existence, uniqueness of solutions and existence of pullback attractors for Eqs (2.1)–(2.3) and (2.4)–(2.6), certain assumptions about the external forces need to be imposed. First, the time-delay function $g : \mathbb{R} \times C_{H(O)} \mapsto \mathbb{L}^2(O)$ is required to satisfy:

(H1) *for any $\xi \in C_{H(O)}$, the function $t \in \mathbb{R} \mapsto g(t, \xi) \in \mathbb{L}^2(O)$ is measurable.*

(H2) $g(t, 0) = 0$ *for all $t \in \mathbb{R}$.*

(H3) *there exists a constant $L_g > 0$ such that for any $t \in \mathbb{R}$, $\xi, \eta \in C_{H(O)}$,*

$$\|g(t, \xi) - g(t, \eta)\|_{\mathbb{L}^2(O)} \leqslant L_g \|\xi - \eta\|_{C_{H(O)}}.$$

(H4) *there exists a constant $C_g \in (0, 2c_1\mu_1)$ such that*

$$\int_\tau^t \|g(s, u_s) - g(s, v_s)\|^2_{\mathbb{L}^2(O)} \mathrm{d}s \leqslant C_g^2 \int_{\tau-h}^t \|u(s) - v(s)\|^2_{H(O)} \mathrm{d}s,$$

and there exists a value $\gamma \in (0, \min\{4c_1\mu_1 - 2C_g, 4c_1^2\mu_1\})$ such that

$$\int_\tau^t \mathrm{e}^{\gamma s} \|g(s, u_s)\|^2_{\mathbb{L}^2(O)} \mathrm{d}s \leqslant C_g^2 \int_{\tau-h}^t \mathrm{e}^{\gamma s} \|u(s)\|^2_{H(O)} \mathrm{d}s,$$

for all $t \geqslant \tau$, $u, v \in L^2(\tau - h, t; H(O))$.

Second, the non-delay function f satisfies:

(H5) *Assume that $f \in L^2_b(\mathbb{R}; H(O))$ which means $f \in L^2_{loc}(\mathbb{R}; H(O))$ with*

$$\|f\|^2_{L^2_b} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|^2_{H(O)} \mathrm{d}s < +\infty.$$

Remark 1. According to the definitions of the spaces $C_{H(O)}$ and $H(O)$, the inequality in (H3) is not directly related to the first inequality in (H4). These two inequalities are independent of each other and describe different aspects of the functions and their differences.

Based on above assumptions, we can conclude that

Theorem 2.2. *([26]) Let the conditions* (H1)–(H5) *hold for the case of* $O = \Sigma$.

1) *Given* $(u^{\text{in}}, \phi^{\text{in}}) \in E^2_{H(\Sigma)}$, *there is a unique weak solution* $u = u(\cdot; \tau, u^{\text{in}}, \phi^{\text{in}})$ *of Eqs* (2.1)–(2.3) *satisfying* $u \in C([\tau, T]; H(\Sigma)) \cap L^2(\tau - h, T; H(\Sigma)) \cap L^2(\tau, T; V(\Sigma))$ *and* $\frac{d}{dt}u \in L^2(\tau, T; V'(\Sigma))$.

2) *The solution operators* $\{U(t, \tau)\}_{t \geqslant \tau} : E^2_{H(\Sigma)} \mapsto E^2_{H(\Sigma)}$ *defined by* $U(t, \tau) : (u^{\text{in}}, \phi^{\text{in}}) \mapsto (u, u_t)$ *generate a continuous process in space* $E^2_{H(\Sigma)}$.

3) *The family* $\widehat{\mathcal{B}}^{H(\Sigma)} = \{\mathcal{B}^{H(\Sigma)}(t) |\ t \in \mathbb{R}\}$ *given by*

$$\mathcal{B}^{H(\Sigma)}(t) = \Big\{(u, \phi) \in H(\Sigma) \times L^2_{V(\Sigma)} \Big|\ \|(u, \phi)\|_{H(\Sigma) \times L^2_{V(\Sigma)}} \leqslant \mathcal{R}_1(t), \|\frac{d}{dt}\phi\|_{L_{V'(\Sigma)}} \leqslant \mathcal{R}_2(t)\Big\}$$

is pullback absorbing for the process $\{U(t, \tau)\}_{t \geqslant \tau}$, *where*

$$\begin{aligned}
\mathcal{R}_1^2(t) &= 1 + c(1 + e^{\gamma h})\varrho_\gamma(t), \\
\mathcal{R}_2^2(t) &= 1 + ce^{2\gamma h}\varrho_\gamma(t)(1 + \varrho_\gamma(t)) + c \int_{t-h}^{t} \|f(s)\|^2_{H(\Sigma)} \mathrm{d}s, \\
\varrho_\gamma(t) &= \mathrm{e}^{-\gamma t} \int_{-\infty}^{t} \mathrm{e}^{\gamma s} \|f(s)\|^2_{H(\Sigma)} \mathrm{d}s.
\end{aligned}$$

4) *The process* $\{U(t, \tau)\}_{t \geqslant \tau}$ *is pullback* $\widehat{\mathcal{B}}^{H(\Sigma)}$*-asymptotically compact and possesses a unique pullback attractor* $\mathcal{A} = \{\mathcal{A}(t) |\ t \in \mathbb{R}\}$ *in* $E^2_{H(\Sigma)}$ *defined by*

$$\mathcal{A}(t) = \bigcap_{s \leqslant t} \overline{\bigcup_{\tau \leqslant s} U(t, \tau)\mathcal{B}^{H(\Sigma)}(\tau)}^{E^2_{H(\Sigma)}}.$$

In order to unify the pullback attractors in the channel Σ and in the sub-domains Σ_n, we consider the natural extensions Eqs (1.6) and (1.7) mentioned in Section 1. Then similar to Theorem 2.2, we can make slight modifications in [22] to get

Theorem 2.3. *Let the conditions* (H1)–(H5) *hold for the case of* $O = \Sigma$.

1) *Given* $(u^{(n),\text{in}}, \phi^{(n),\text{in}}) \in E^2_{H(\Sigma_n)}$, *there is a unique weak solution* $u^{(n)} = u^{(n)}(\cdot; \tau, u^{(n),\text{in}}, \phi^{(n),\text{in}})$ *of Eqs* (2.4)–(2.6) *satisfying* $u^n \in C([\tau, T]; H(\Sigma_n)) \cap L^2(\tau - h, T; H(\Sigma_n)) \cap L^2(\tau, T; V(\Sigma_n))$ *and* $\frac{d}{dt}u^{(n)} \in L^2(\tau, T; V'(\Sigma_n))$.

2) *The solution operators* $\{U_n(t, \tau)\}_{t \geqslant \tau} : E^2_{H(\Sigma_n)} \mapsto E^2_{H(\Sigma_n)}$ *defined by* $U_n(t, \tau) : (u^{(n),\text{in}}, \phi^{(n),\text{in}}) \mapsto (u^{(n)}, u^{(n)}_t)$ *generate a continuous process in space* $E^2_{H(\Sigma_n)}$.

3) *The family* $\widehat{\mathcal{B}}_n^{H(\Sigma_n)} = \{\mathcal{B}_n^{H(\Sigma_n)}(t)|\ t \in \mathbb{R}\}$ *given by*

$$\mathcal{B}_n^{H(\Sigma_n)}(t) = \left\{(u,\phi) \in H(\Sigma_n) \times L^2_{V(\Sigma_n)} \Big|\ \|(u,\phi)\|_{H(\Sigma_n)\times L^2_{V(\Sigma_n)}} \leqslant \mathcal{R}_1(t), \|\frac{d}{dt}\phi\|_{L_{V'(\Sigma_n)}} \leqslant \mathcal{R}_2(t)\right\}$$

is pullback absorbing for the process $\{U_n(t,\tau)\}_{t\geqslant\tau}$*. Note that* $\mathcal{R}_1(t)$ *and* $\mathcal{R}_2(t)$ *are the same in* Theorem 2.2.

4) *The process* $\{U_n(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $\widehat{\mathcal{B}}^{H(\Sigma_n)}$*-asymptotically compact and possesses a unique pullback attractor* $\mathcal{A}_n = \{\mathcal{A}_n(t)|\ t \in \mathbb{R}\}$ *in* $E^2_{H(\Sigma_n)}$ *defined by*

$$\mathcal{A}_n(t) = \bigcap_{s\leqslant t} \overline{\bigcup_{\tau\leqslant s} U_n(t,\tau)\mathcal{B}^{H(\Sigma_n)}(\tau)}^{E^2_{H(\Sigma_n)}}.$$

Due to the definitions of $\mathcal{R}_1(t), \mathcal{R}_2(t)$ and the fact

$$\begin{aligned}
\varrho_\gamma(t) &= \mathrm{e}^{-\gamma t}\int_{-\infty}^{t} e^{\gamma s}\|f(s)\|^2_{H(\Sigma)}\mathrm{d}s \\
&= \mathrm{e}^{-\gamma t}\int_{t-1}^{t} \mathrm{e}^{\gamma s}\|f(s)\|^2_{H(\Sigma)}\mathrm{d}s + \mathrm{e}^{-\gamma t}\int_{t-2}^{t-1} \mathrm{e}^{\gamma s}\|f(s)\|^2_{H(\Sigma)}\mathrm{d}s + \cdots \\
&\leqslant \int_{t-1}^{t} \|f(s)\|^2_{H(\Sigma)}\mathrm{d}s + \mathrm{e}^{-\gamma}\int_{t-2}^{t-1} \|f(s)\|^2_{H(\Sigma)}\mathrm{d}s + \cdots \\
&= \frac{\|f\|^2_{L^2_b}}{1-e^{-\gamma}},
\end{aligned} \tag{2.7}$$

we have that $\mathcal{B}^{H(\Sigma)}(t)$ and $\mathcal{B}_n^{H(\Sigma_n)}(t)$ in $E^2_{H(\Sigma)}$ are bounded uniformly with respect to the time t. Moreover, there exists a $\tau(t, \widehat{\mathcal{B}}^{H(\Sigma)})$ (independent of n) such that for $\tau \leqslant \tau(t, \widehat{\mathcal{B}}^{H(\Sigma)})$,

$$U(t,\tau)\mathcal{B}^{H(\Sigma)}(\tau) \subset \mathcal{B}^{H(\Sigma)}(t), \tag{2.8}$$

$$U_n(t,\tau)\mathcal{B}_n^{H(\Sigma_n)}(\tau) \subset \mathcal{B}_n^{H(\Sigma_n)}(t) \subset \mathcal{B}^{H(\Sigma)}(t). \tag{2.9}$$

3. Upper semi-continuity of pullback attractors

In this section, our objective is to establish the upper semi-continuity of pullback attractors. Specifically, we aim to prove the following theorem:

Theorem 3.1. *Suppose the conditions* (H1)–(H5) *hold for the case of* $O = \Sigma$*. Consider the families* $\mathcal{A}_n = \{\mathcal{A}_n(t)|\ t \in \mathbb{R}\}$ *and* $\mathcal{A} = \{\mathcal{A}(t)|\ t \in \mathbb{R}\}$ *as the pullback attractors for Eqs* (1.1)–(1.5) *in the domains* Σ_n *and* Σ*, respectively. Then for any* $t \in \mathbb{R}$*, we have*

$$\lim_{n\to\infty} \mathrm{Dist}_{H(\Sigma)\times L^2(-h,0;H(\Sigma))}(\mathcal{A}_n(t), \mathcal{A}(t)) = 0.$$

3.1. The key convergence of solutions

To prove Theorem 3.1, it is crucial to establish the strong convergence in the space $E^2_{H(\Sigma)}$ of any sequence $\{(u^{(n)}, \phi^{(n)})\}_{n\geqslant1}$, where $(u^{(n)}, \phi^{(n)})$ belongs to $\mathcal{A}_n(t)$, to some (u,ϕ) belonging to $\mathcal{A}(t)$.

Firstly, we obtain two auxiliary lemmas.

Lemma 3.2. *Suppose the conditions* (H1)–(H5) *hold for the case of* $O = \Sigma$. *Let* $\{(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\}_{n\geqslant 1}$ *be a sequence in* $H(\Sigma_n) \times L^2_{V(\Sigma_n)}$ *and* $(u^{\mathrm{in}}, \phi^{\mathrm{in}}) \in H(\Sigma) \times L^2_{V(\Sigma)}$ *satisfying the weak convergences as* $n \to \infty$,

$$(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) \rightharpoonup (u^{\mathrm{in}}, \phi^{\mathrm{in}}) \ \ in \ \ H(\Sigma) \times L^2_{V(\Sigma)}, \tag{3.1}$$

$$\frac{d}{dt}\phi^{(n),\mathrm{in}} \rightharpoonup \frac{d}{dt}\phi^{\mathrm{in}} \ \ in \ \ L^2_{V'(\Sigma)}. \tag{3.2}$$

Then for any $t \geqslant \tau$, *we obtain the following weak convergences as* $n \to \infty$,

$$u^{(n)}(t; \tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) \rightharpoonup u(t; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}}) \ \ in \ \ H(\Sigma), \tag{3.3}$$

$$u^{(n)}(\cdot; \tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) \rightharpoonup u(\cdot; \tau, u^{\mathrm{in}}, \phi^{\mathrm{in}}) \ \ in \ \ L^2(\tau - h, t; V(\Sigma)). \tag{3.4}$$

Proof. The fundamental energy estimates for Eqs (2.4)–(2.6) can be derived using the same method as shown in [26, Equations (3.1), (3.9), (3.27)]. The following inequality holds for any $\tau \leqslant t$,

$$\begin{aligned} &\|u^{(n)}(t)\|^2_{H(\Sigma_n)} + \eta \mathrm{e}^{-\gamma t} \int_\tau^t \mathrm{e}^{\gamma s} \|u^{(n)}(s)\|^2_{V(\Sigma_n)} \mathrm{d}s \\ &\quad \leqslant c\mathrm{e}^{\gamma(\tau - t)} \|(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|^2_{E^2_{H(\Sigma_n)}} + \beta^{-1} \mathrm{e}^{-\gamma t} \int_\tau^t \mathrm{e}^{\gamma s} \|f(s)\|^2_{H(\Sigma_n)} \mathrm{d}s, \end{aligned} \tag{3.5}$$

where C_g and γ are the constants from (H4), $\beta \in (0, 4c_1\mu_1 - 2C_g - \gamma)$ and $\eta = 4c_1\mu_1 - 2C_g - \gamma - \beta > 0$. We also have for any $T > 0$ and $\tau \leqslant t - T$,

$$\int_{t-T}^t \|u^{(n)}(s)\|^2_{V(\Sigma_n)} \mathrm{d}s \leqslant c\mathrm{e}^{\gamma(T + \tau - t)} \|(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|^2_{E^2_{H(\Sigma_n)}} + c\mathrm{e}^{\gamma(T - t)} \int_\tau^t \mathrm{e}^{\gamma s} \|f(s)\|^2_{H(\Sigma_n)} \mathrm{d}s. \tag{3.6}$$

Furthermore, we obtain for any $\tau \leqslant t$,

$$\begin{aligned} \int_{t-h}^t \Big\|\frac{d}{ds} u^{(n)}\Big\|^2_{V'(\Sigma_n)} \mathrm{d}s \leqslant{}& c \int_{t-h}^t \|f(s)\|^2_{H(\Sigma_n)} \mathrm{d}s \\ &+ c\left(\mathrm{e}^{\gamma(\tau - t)} \|(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|_{E^2_{H(\Sigma_n)}} + \varrho_\gamma(t)\right) \\ &+ c\left(\mathrm{e}^{\gamma(\tau - t)} \|(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|_{E^2_{H(\Sigma_n)}} + \varrho_\gamma(t)\right)^2, \end{aligned} \tag{3.7}$$

with $\varrho_\gamma(t)$ given in Theorem 2.2.

It follows from Eqs (3.5)–(3.7) that

$$u^{(n)} \text{ is bounded in } L^\infty(\tau, t; H(\Sigma)) \cap L^2(\tau, t; V(\Sigma)), \tag{3.8}$$

$$\frac{d}{ds} u^{(n)} \text{ is bounded in } L^2(\tau - h, t; V'(\Sigma)). \tag{3.9}$$

By Eq (3.8), we have the following weak star and weak convergences (for a subsequence)

$$u^{(n)} \rightharpoonup u \text{ in } L^\infty(\tau, t; H(\Sigma)), \tag{3.10}$$

$$u^{(n)} \rightharpoonup u \text{ in } L^2(\tau, t; V(\Sigma)). \tag{3.11}$$

By the standard method, we can demonstrate that the solution to Eqs (2.1)–(2.3) with initial data $(u^{\rm in}, \phi^{\rm in})$ is u. In fact, in view of Eqs (3.8) and (3.9), a subsequence can be extracted from the sequence $u^{(n)}$ and denoted as $u^{(n)}$ as well such that the following strong convergence holds

$$u^{(n)} \to u \text{ in } L^2(\tau, t; H(\Sigma_r)), \tag{3.12}$$

where Σ_r is any bounded sub-domain of Σ. By virtue of Eq (3.12), we can apply the limit operation to Eqs (2.4)–(2.6) for $u^{(n)}$. This implies that u is the solution to Eqs (2.1)–(2.3) with the given initial conditions $(u^{\rm in}, \phi^{\rm in})$. By uniqueness we conclude that Eqs (3.10)–(3.12) hold for the whole sequence.

Proof of Eq (3.4)*:* It follows from Eqs (3.1) and (3.11) that Eq (3.4) holds.

Proof of Eq (3.3)*:* From Eq (3.10), we have for any $v \in \mathcal{V}(\Sigma)$ and a.e. $t \geqslant \tau$ that

$$(u^{(n)}, v) \to (u, v).$$

Due to Eq (3.9), we obtain for all $t \geqslant \tau$ and $v \in \mathcal{V}(\Sigma)$,

$$(u^{(n)}(t) - u^{(n)}(t-h), v) = \int_{t-h}^{t} \langle \frac{d}{ds} u^{(n)}(s), v \rangle ds \leqslant \|v\|_{V(\Sigma)} \Big\| \frac{d}{ds} u^{(n)} \Big\|_{L^2(\tau-h,t;V'(\Sigma))} h^{1/2}. \tag{3.13}$$

In view of Eqs (3.9) and (3.13), we see $(u^{(n)}, v)$ is uniformly bounded and equicontinuous on $[\tau, t]$. Hence, we have for any $v \in \mathcal{V}(\Sigma)$ and all $t \geqslant \tau$,

$$(u^{(n)}, v) \to (u, v).$$

By taking advantage of the density of $\mathcal{V}(\Sigma)$ in $H(\Sigma)$, we arrive at Eq (3.3).

Similar to [26, Lemma 3.4] (also see [16, 21]), we have the following tail estimate:

Lemma 3.3. *Let the conditions* (H1)–(H5) *hold for the case of* $O = \Sigma$ *and* $(u^{(n),\rm in}, \phi^{(n),\rm in}) \in \mathcal{B}_n^{H(\Sigma_n)}(\tau)$. *Then given* $\epsilon > 0$*, there are* $\tau_n(\epsilon) < t$ *and* $r_n(\epsilon) > 0$ *satisfying for any* $r \in [r_n, n]$ *and* $\tau \leqslant \tau_n$*,*

$$\|u^{(n)}(t; \tau, u^{(n),\rm in}, \phi^{(n),\rm in})\|^2_{\mathbb{L}^2(\Sigma_n \setminus \Sigma_r)} \leqslant \epsilon.$$

With the help of Lemmas 3.2 and 3.3, we will establish the key convergence of solutions via the energy method introduced by Ball [31] and developed by Moise, Rosa and Wang [32].

Lemma 3.4. *Let the conditions* (H1)–(H5) *hold for the case of* $O = \Sigma$. *Then for each* $t \in \mathbb{R}$ *and any sequence*

$$\{(u^{(n)}, \phi^{(n)})\}_{n \geqslant 1} \ \textit{with} \ (u^{(n)}, \phi^{(n)}) \in \mathcal{A}_n(t),$$

there is a subsequence of (using the same index) $\{(u^{(n)}, \phi^{(n)})\}_{n \geqslant 1}$ *and some* $(u, \phi) \in \mathcal{A}(t)$ *such that the following strong convergence holds as* $n \to \infty$*,*

$$(u^{(n)}, \phi^{(n)}) \longrightarrow (u, \phi) \ \textit{in} \ E^2_{H(\Sigma)}. \tag{3.14}$$

Proof. In virtue of the invariance of the pullback attractor $\mathcal{A}_n$, there is a sequence $\{(u^{(n),\rm in}, \phi^{(n),\rm in})\}_{n \geqslant 1}$ with $(u^{(n),\rm in}, \phi^{(n),\rm in}) \in \mathcal{A}_n(\tau) \in \mathcal{A}_n$ such that

$$U_n(t, \tau)(u^{(n),\rm in}, \phi^{(n),\rm in}) = (u^{(n)}, \phi^{(n)}). \tag{3.15}$$

Due to the compactness of $\mathcal{A}_n(\tau)$, there is some subsequence (using the same index) of $\{(u^{(n),\text{in}}, \phi^{(n),\text{in}})\}_{n\geqslant 1}$ and some $(u^{\text{in}}, \phi^{\text{in}}) \in E^2_{H(\Sigma)}$ such that the following weak convergence holds as $n \to \infty$,

$$(u^{(n),\text{in}}, \phi^{(n),\text{in}}) \rightharpoonup (u^{\text{in}}, \phi^{\text{in}}) \text{ in } E^2_{H(\Sigma)}. \tag{3.16}$$

For the sequence $\{(u^{(n)}, \phi^{(n)})\}_{n\geqslant 1}$ with initial data $\{(u^{(n),\text{in}}, \phi^{(n),\text{in}})\}_{n\geqslant 1}$ in Eq (3.16), the compactness of $\mathcal{A}_n(t)$ deduces that there is some $(u, \phi) \in E^2_{H(\Sigma)}$ satisfying the following weak convergence as $n \to \infty$,

$$(u^{(n)}, \phi^{(n)}) \rightharpoonup (u, \phi) \text{ in } E^2_{H(\Sigma)}. \tag{3.17}$$

As an analogy of the proof of Lemma 3.2, u is the solution to Eqs (2.1)–(2.3) with initial data $(u^{\text{in}}, \phi^{\text{in}})$. On the other hand, it follows from Eqs (2.8) and (2.9) that for any $\tau \leqslant \tau(t, \widehat{\mathcal{B}}^{H(\Sigma)})$,

$$\mathcal{A}_n(t) = U_n(t, \tau)\mathcal{A}_n(\tau) \subset \mathcal{B}^{H(\Sigma)}(t).$$

Thanks to Eq (3.17), we have for any $\tau \leqslant \tau(t, \widehat{\mathcal{B}}^{H(\Sigma)})$ that $(u, \phi) \in \mathcal{B}^{H(\Sigma)}(t)$ and thus $(u, \phi) \in \mathcal{A}(t)$.

The proof of Lemma 3.4 will be concluded by demonstrating that the convergence of Eq (3.17) is strong. We finish the proof in two steps.

Step one: we show that as $n \to \infty$,

$$\phi^{(n)} \longrightarrow \phi \text{ strongly in } L^2_{H(\Sigma)}. \tag{3.18}$$

It follows from Eqs (2.8) and (2.9) that there is a $\tau(t-k, \widehat{\mathcal{B}}^{H(\Sigma)})$ satisfying for any $k \in \mathbb{Z}_+$ and $\tau \leqslant \tau(t-k, \widehat{\mathcal{B}}^{H(\Sigma)})$,

$$\mathcal{A}_n(t-k) = U_n(t-k, \tau)\mathcal{A}_n(\tau) \subset \mathcal{B}^{H(\Sigma)}(t-k).$$

Taking advantage of the definition of the pullback absorbing set $\widehat{\mathcal{B}}^{H(\Sigma)}$ and the diagonal argument, for each $k \in \mathbb{Z}_+$ and $\tau < t-k$, there is a $(u^{(k)}, \phi^{(k)}) \in \mathcal{A}(t-k)$ satisfying the following weak convergences (up to a subsequence) as $n \to \infty$,

$$U_n(t-k, \tau)(u^{(n),\text{in}}, \phi^{(n),\text{in}}) \rightharpoonup (u^{(k)}, \phi^{(k)}) \quad \text{in } H(\Sigma) \times L^2_{V(\Sigma)}, \tag{3.19}$$

$$\frac{d}{dt} u^{(n)}_{t-k}(\cdot; \tau, u^{(n),\text{in}}, \phi^{(n),\text{in}}) \rightharpoonup \frac{d}{dt}\phi^{(k)} \text{ in } L^2_{V'(\Sigma)}. \tag{3.20}$$

Thus, Equations (3.15), (3.17) and (3.19) together with the fact that the limit is unique, yield

$$(u, \phi) = (u^{(0)}, \phi^{(0)}).$$

Note that for any bounded set $\Sigma_r \subset \Sigma$, all the embeddings $V(\Sigma_r) \hookrightarrow H(\Sigma_r) \hookrightarrow V'(\Sigma_r)$ are compact. By Eqs (3.19) and (3.20) and Lemma 3.2, we obtain as $n \to \infty$,

$$u^{(n)}_{t-k}(\cdot; \tau, u^{(n),\text{in}}, \phi^{(n),\text{in}}) \longrightarrow \phi^{(k)} \text{ strongly in } L^2_{H(\Sigma_r)}.$$

Consequently, for any positive ϵ, there is some $n(\epsilon, r, k) > 0$ satisfying

$$\|u^{(n)}_{t-k}(\cdot; \tau, u^{(n),\text{in}}, \phi^{(n),\text{in}}) - \phi^{(k)}\|_{L^2_{H(\Sigma_r)}} < \frac{\epsilon}{3}, \quad \forall\, n \geqslant n(\epsilon, r, k). \tag{3.21}$$

Also, since for each $k \in \mathbb{Z}_+$, $\phi^{(k)} \in L^2_{V(\Sigma)} \hookrightarrow L^2_{H(\Sigma)}$ is a fixed element, there must exist some $r_1(k) > 0$ such that

$$\|\phi^{(k)}\|_{L^2_{H(\Sigma\setminus\Sigma_r)}} < \frac{\epsilon}{3}, \quad \forall\, r \geqslant r_1(k). \tag{3.22}$$

By Lemma 3.3, there exist $r(\epsilon, t-k, \widehat{\mathscr{B}}^{H(\Sigma)}) > 0$ and $\tau(\epsilon, t-k, \widehat{\mathscr{B}}^{H(\Sigma)}) < t-k$ satisfying for any $r > r(\epsilon, t-k, \widehat{\mathscr{B}}^{H(\Sigma)})$ and $\tau \leqslant \tau(\epsilon, t-k, \widehat{\mathscr{B}}^{H(\Sigma)})$,

$$\|u^{(n)}_{t-k}(\cdot;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|_{L^2_{H(\Sigma\setminus\Sigma_r)}} \leqslant h\frac{\epsilon}{3h} = \frac{\epsilon}{3}. \tag{3.23}$$

Therefore, by Eqs (3.21)–(3.23), we choose r and n large enough and τ small enough so that

$$\begin{aligned}
&\|u^{(n)}_{t-k}(\cdot;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) - \phi^{(k)}\|_{L^2_{H(\Sigma)}} \\
\leqslant\ & \|u^{(n)}_{t-k}(\cdot;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) - \phi^{(k)}\|_{L^2_{H(\Sigma\setminus\Sigma_r)}} + \|u^{(n)}_{t-k}(\cdot;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) - \phi^{(k)}\|_{L^2_{H(\Sigma_r)}} \\
\leqslant\ & \|u^{(n)}_{t-k}(\cdot;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|_{L^2_{H(\Sigma\setminus\Sigma_r)}} + \|\phi^{(k)}\|_{L^2_{H(\Sigma\setminus\Sigma_r)}} \\
&+ \|u^{(n)}_{t-k}(\cdot;\tau, u^{(n),\mathrm{in}}, \phi^{(m),\mathrm{in}}) - \phi^{(k)}\|_{L^2_{H(\Sigma_r)}} \leqslant \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

which implies that for each $k \in \mathbb{Z}_+$, it holds as $n \to \infty$,

$$u^{(n)}_{t-k}(\cdot;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) \longrightarrow \phi^{(k)} \ \text{ strongly in } \ L^2_{H(\Sigma)}. \tag{3.24}$$

Particularly, taking $k = 0$, we prove the claim of Eq (3.18).

Step two: we aim to prove that as $n \to \infty$, the following strong convergence holds:

$$u^{(n)}(t;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) \longrightarrow u \ \text{ in } \ H(\Sigma). \tag{3.25}$$

By Eqs (3.19) and (3.20) and Lemma 3.2, we have the weak convergence as $n \to \infty$,

$$U_n(t,\tau)(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) = U_n(t, t-k)U_n(t-k,\tau)(u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}}) \tag{3.26}$$

$$\rightharpoonup U(t, t-k)(u^{(k)}, \phi^{(k)}) \ \text{ in } \ E^2_{H(\Sigma)\times L^2_{V(\Sigma)}}. \tag{3.27}$$

As a result of Eqs (3.15), (3.17), (3.27) and the fact the limit is unique, we obtain

$$(u, \phi) = U(t, t-k)(u^{(k)}, \phi^{(k)}), \quad \forall\, k \in \mathbb{Z}_+. \tag{3.28}$$

Furthermore, considering Eqs (3.27), (3.28) and the lower semi-continuity of the norm, one can conclude

$$\|u\|^2_{H(\Sigma)} \leqslant \liminf_{m\to\infty} \|u^{(n)}(t;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|^2_{H(\Sigma)}. \tag{3.29}$$

Due to $H(\Sigma)$ being a Hilbert space, Equation (3.25) can be inferred from Eqs (3.27), (3.29) and the following remainder

$$\|u\|^2_{H(\Sigma)} \geqslant \limsup_{n\to\infty} \|u^{(n)}(t;\tau, u^{(n),\mathrm{in}}, \phi^{(n),\mathrm{in}})\|^2_{H(\Sigma)}. \tag{3.30}$$

Next we concentrate our attention on proving Eq (3.30). Defining a bilinear operator

$$[\![\cdot,\cdot]\!]_{V(\Sigma)} : V(\Sigma)\times V(\Sigma)\mapsto \mathbb{R}$$

by

$$[\![u,v]\!]_{V(\Sigma)} = 2\mu_1\langle Au,v\rangle - \frac{\gamma}{2}(u,v),\quad \forall\, u,v\in V(\Sigma). \tag{3.31}$$

Setting

$$[\![u]\!]^2_{V(\Sigma)} = [\![u,u]\!]_{V(\Sigma)}.$$

Thanks to Lemma 2.1, we obtain

$$(2c_1\mu_1 - \frac{\gamma}{2})\|u\|^2_{V(\Sigma)} \leqslant [\![u]\!]^2_{V(\Sigma)} \leqslant 2\mu_1\langle Au,u\rangle \leqslant 2\mu_1\|u\|^2_{V(\Sigma)}. \tag{3.32}$$

From Eq (3.32) and the condition $\gamma < 4c_1\mu_1$ in (H4), it is evident that $[\![\cdot]\!]_{V(\Sigma)}$ defines a norm in the space $V(\Sigma)$, establishing equivalence with the conventional norm $\|\cdot\|_{V(\Sigma)}$. From now on, we set

$$\|\cdot\| = \|\cdot\|_{H(\Sigma)}.$$

Multiplying Eq (2.4) by $u^{(n)}(t)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^{(n)}(t)\|^2 + \gamma\|u^{(n)}(t)\|^2 = 2\Gamma(f(t), g(t,u^{(n)}_t), u^{(n)}(t)), \tag{3.33}$$

where

$$\begin{aligned}\Gamma(f(t), g(t,u^{(n)}_t), u^{(n)}(t)) &= (f(t),u^{(n)}(t)) + (g(t,u^{(n)}_t),u^{(n)}(t))\\ &\quad - \langle N(u^{(n)}(t)),u^{(n)}(t)\rangle - [\![u^{(n)}(t)]\!]^2_{V(\Sigma)}.\end{aligned}$$

By employing the formula of constant variation, we can derive the energy equation as shown below:

$$\|u^{(n)}(t)\|^2 = \mathrm{e}^{-\gamma(t-\tau)}\|u^{(n),\mathrm{in}}\|^2 + 2\int_\tau^t \mathrm{e}^{-\gamma(t-\tau)}\Gamma(f(s), g(s,u^{(n)}_s), u^{(n)}(s))\mathrm{d}s. \tag{3.34}$$

Thus, for each $k\in\mathbb{Z}_+$, Equations (3.26) and (3.34) mean

$$\begin{aligned}\|u^{(n)}(t;\tau,u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})\|^2 &= \|u^{(n)}(t;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})\|^2\\ &= \mathrm{e}^{-\gamma k}\|u^{(n)}(t-k;\tau,u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})\|^2\\ &\quad + 2[G_1(t,n)+G_2(t,n)-G_3(t,n)-G_4(t,n)],\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}G_1(t,n) &= \int_{t-k}^t \mathrm{e}^{-\gamma(t-s)}(f(s), u^{(n)}(s;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})))\mathrm{d}s,\\ G_2(t,n) &= \int_{t-k}^t \mathrm{e}^{-\gamma(t-s)}(g(s,u^{(n)}_s(\cdot;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}}))),\\ &\qquad u^{(n)}(s;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})))\mathrm{d}s,\end{aligned}$$

$$G_3(t,n)=\int_{t-k}^{t}\mathrm{e}^{-\gamma(t-s)}\langle N(u^{(n)}(s;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}}))),\\ u^{(n)}(s;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}}))\rangle\mathrm{d}s,$$

$$G_4(t,n)=\int_{t-k}^{t}\mathrm{e}^{-\gamma(t-s)}[\![u^{(n)}(s;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}}))]\!]^2_{V(\Sigma)}\mathrm{d}s.$$

Limiting estimate of the first term in Eq (3.35)**:** By Eq (2.9), it holds for any $\tau\leqslant\tau(t-k,\widehat{\mathcal{B}}^{H(\Sigma)})$,

$$U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})\subset\mathcal{B}^{H(\Sigma)}(t-k),$$

which suggests that the following inequality holds:

$$\mathrm{e}^{-\gamma k}\|u^{(n)}(t-k;\tau,u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})\|^2\leqslant\mathrm{e}^{-\gamma k}\mathcal{R}_1^2(t-k).\tag{3.36}$$

Limiting estimate of the term G_1: We conclude from Eq (3.27) and Lemma 3.2 that the following weak convergence in $L^2(t-k,t;V(\Sigma))$ holds as $n\to\infty$,

$$u^{(n)}(\cdot;t-k,U_n(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}}))\rightharpoonup u(\cdot;t-k,u^{(k)},\phi^{(k)}).\tag{3.37}$$

Since $\mathrm{e}^{-\gamma(t-s)}f(s)\in L^2(t-k,t;H(\Sigma))$, Equation (3.37) indicates that

$$\lim_{n\to\infty}G_1(t,n)=\int_{t-k}^{t}\mathrm{e}^{-\gamma(t-s)}(f(s),u(s;t-k,u^{(k)},\phi^{(k)}))\mathrm{d}s.\tag{3.38}$$

Limiting estimate of the term G_4: It is evident that $\{\int_{t-k}^{t}\mathrm{e}^{-\gamma(t-s)}[\![u(s)]\!]^2_{V(\Sigma)}\mathrm{d}s\}^{1/2}$ operates as a norm in the space $L^2(t-k,t;V(\Sigma))$, establishing its equivalence to the usual norm $\|\cdot\|_{L^2(t-k,t;V(\Sigma))}$. Then Eq (3.37) indicates that

$$\int_{t-k}^{t}\mathrm{e}^{-\gamma(t-s)}[\![u(s,t-k,u^{(k)},\phi^{(k)})]\!]^2_{V(\Sigma)}\mathrm{d}s\leqslant\liminf_{m\to\infty}G_4(t,n).\tag{3.39}$$

Limiting estimate of the term G_2: To establish the limit of G_2, we employ the approach of inserting the term $g(s,u_s(\cdot;t-k,u^{(k)},\phi^{(k)})$. As an analogy of Eq (3.18), we obtain for each $s\in[t-k,t]$ as $n\to\infty$,

$$u_s^{(n)}(\cdot;t-k,U(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}}))\longrightarrow u_s(\cdot;t-k,u^{(k)},\phi^{(k)})\tag{3.40}$$

strongly in $L^2_{H(\Sigma)}$. It follows from the assumption (H2) and Eq (3.40) that as $n\to\infty$,

$$\begin{aligned}&\|g(s,u_s^{(n)}(\cdot;t-k,U(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})))-g(s,u_s(\cdot;t-k,u^{(k)},\phi^{(k)}))\|\\ &\leqslant L_g\|u_s^{(n)}(\cdot;t-k,U(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}}))-u_s(\cdot;t-k,u^{(k)},\phi^{(k)})\|_{L^2_{H(\Sigma)}}\\ &\longrightarrow 0,\quad\forall\, s\in[t-k,t],\end{aligned}$$

which yields that for any $s\in[t-k,t]$, the following strong convergence in $H(\Sigma)$ holds as $n\to\infty$,

$$g(s,u_s^{(n)}(\cdot;t-k,U(t-k,\tau)(u^{(n),\mathrm{in}},\phi^{(n),\mathrm{in}})))\longrightarrow g(s,u_s(\cdot;t-k,u^{(k)},\phi^{(k)})).$$

By considering the boundedness of

$$\int_{t-k}^{t} \|g(s, u_s^{(n)}(\cdot; t-k, U(t-k, \tau)(u^{(n),\text{in}}, \phi^{(n),\text{in}})))\|^2 \mathrm{d}s$$

and applying the Lebesgue dominated convergence theorem, the following strong convergence in $L^2(t-k, t; H(\Sigma))$ can be shown that as $n \to \infty$,

$$g(s, u_s^{(n)}(\cdot; t-k, U(t-k, \tau)(u^{(n),\text{in}}, \phi^{(n),\text{in}}))) \longrightarrow g(s, u_s(\cdot; t-k, u^{(k)}, \phi^{(k)})). \tag{3.41}$$

Then it can be inferred from Eqs (3.37) and (3.41) that

$$\lim_{n\to\infty} G_2(t, n) = \int_{t-k}^{t} \mathrm{e}^{-\gamma(t-s)}[g(s, u_s(\cdot; t-k, u^{(k)}, \phi^{(k)})), u(s; t-k, u^{(k)}, \phi^{(k)}))]\mathrm{d}s. \tag{3.42}$$

Limiting estimate of the term G_3: The limit of G_3 can be shown by using the technique of inserting the term $N(u(s; t-k, u^{(k)}, \phi^{(k)})$. The estimates are essentially the same as the estimates [16, Equation (3.50)] and we omit it, thus we obtain

$$\lim_{m\to\infty} G_3(t, n) = \int_{t-k}^{t} \mathrm{e}^{-\gamma(t-s)}\langle N(u(s; t-k, u^{(k)}, \phi^{(k)})), u(s; t-k, u^{(k)}, \phi^{(k)})\rangle \mathrm{d}s. \tag{3.43}$$

According to Eqs (3.33)–(3.36), (3.38) and (3.39) and (3.42)–(3.43), we conclude that

$$\limsup_{n\to\infty} \|u^{(n)}(t; \tau, u^{(n),\text{in}}, \phi^{(n),\text{in}})\|^2 \leqslant \mathrm{e}^{-\gamma k}\mathcal{R}_1^2(t-k) + 2\int_{t-k}^{t} \mathrm{e}^{-\gamma(t-s)}\Gamma(f(s), g(s, u_s^{(k)}), u^{(k)}(s))\mathrm{d}s. \tag{3.44}$$

Moreover, applying the energy equation to Eq (2.1), we obtain

$$\|u(t)\|^2 = \mathrm{e}^{-\gamma k}\|u^{(k)}\|^2 + 2\int_{t-k}^{t} \mathrm{e}^{-\gamma(t-s)}\Gamma(f(s), g(s, u_s^{(k)}), u^{(k)}(s))\mathrm{d}s. \tag{3.45}$$

Then Eqs (3.44) and (3.45) imply that

$$\limsup_{n\to\infty} \|u^{(n)}(t; \tau, u^{(n),\text{in}}, \phi^{(n),\text{in}})\|^2 \leqslant \mathrm{e}^{-\gamma k}\mathcal{R}_1^2(t-k) + \|u\|^2. \tag{3.46}$$

From Eq (2.9) and the definition of $\mathcal{R}_1^2(t)$ in Theorem 2.2, we know

$$\lim_{k\to\infty} \mathrm{e}^{-\gamma k}\mathcal{R}_1^2(t-k) = 0.$$

Hence, the statement of Eq (3.30) follows. We prove the assertion of this lemma.

3.2. Proof of Theorem 3.1

This subsection is dedicated to establishing the proof of Theorem 3.1.

The proof of Theorem 3.1: The approach taken to prove Theorem 1.1 involves a contradiction argument. Assume that the assertion is false, then for some $t_0 \in \mathbb{R}, \epsilon_0 > 0$, we can find a sequence $(u^{(n)}, \phi^{(n)}) \in \mathcal{A}_n(t_0)$, which satisfies

$$\mathrm{dist}_{E^2_{H(\Sigma)}}((u^{(n)}, \phi^{(n)}), \mathcal{A}(t_0)) \geqslant \epsilon_0. \tag{3.47}$$

However, according to Lemma 3.4, there is a subsequence (using the same index) of $\{(u^{(n)}, \phi^{(n)})\}_{n\geqslant 1}$ that can be found such that

$$\lim_{n\to\infty} \mathrm{dist}_{E^2_{H(\Sigma)}}((u^{(n)}, \phi^{(n)}), \mathcal{A}(t_0)) = 0,$$

which obviously contradicts with Eq (3.47). The proof is complete.

4. Conclusions

In this work, we consider the families $\mathcal{A}_n = \{\mathcal{A}_n(t)| \ t \in \mathbb{R}\}$ and $\mathcal{A} = \{\mathcal{A}(t)| \ t \in \mathbb{R}\}$ as the pullback attractors of the bipolar fluid with delay in the domains Σ_n and Σ, respectively. We demonstrate that the following equality holds for any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{Dist}_{E^2_{H(\Sigma)}}(\mathcal{A}_n(t), \mathcal{A}(t)) = 0. \tag{4.1}$$

According to the definition of the Hausdorff semi-distance, Equation (4.1) shows that the pullback attractors $\mathcal{A}_n = \{\mathcal{A}_n(t)| \ t \in \mathbb{R}\}$ in the sub-domains Σ_n converge to the pullback attractor $\mathcal{A} = \{\mathcal{A}(t)| \ t \in \mathbb{R}\}$ in the entire domain Σ as the sub-domains Σ_n approach the entire domain Σ, demonstrating the semi-continuity of the pullback attractors in the phase space $E^2_{H(\Sigma)}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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