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Research article

Elliptic and multiple-valued solutions of some higher order ordinary differential equations

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Abstract: In the present paper, by the complex method, the meromorphic solutions of the higher order ordinary differential equation $w^{(5)}+aw''+bw^2-cw+d=0$ are investigated, where a,b,c,d are constant complex numbers, and $b\neq 0$. Furthermore, by Theorem 1.1, we built elliptic and multiple-valued solutions for the higher order ordinary differential equations $u^{(6)}-u^{(5)}+u'^2-2u'u+u^2+2u'-2u+1=0$ and $u^{(6)}-u^{(5)}+au'''-au''+bu'^2-2bu'u+bu^2-cu'+cu+d=0$. At the end, we give some new meromorphic solutions for two higher-order KdV-like equations.

Keywords: the complex method; meromorphic solution; elliptic function; multiple-valued solution; KdV-like equation

1. Introduction

Non-linear differential equations are widely applied to represent complex phenomena in many natural sciences, and exact solutions contribute to well understanding of natural phenomena. Therefore, it is important to study the exact solutions of non-linear differential equations. There are many effective methods that are being used to find exact solutions of differential equations, such as the F-expansion method [1], the exponential function method [2,3], the tanh method [4], the inverse scattering transform method [5], the direct algebraic method [6], the sine-cosine method [7], the first integral method [8], the transformed rational function method [9], the Bäcklund transform method [10], the (G'/G)-expansion method [11] and the Lie group method [12].

We say $w(z)$ is a meromorphic function, which means that $w(z)$ is analytic in the complex plane $\mathbb{C}$ except for poles. In recent years, many researchers studied complex differential equations using the complex method [13] and Nevanlinna's theory, and build some new elliptic function solutions and simple periodic function solutions, for instance, see [14–17]. These results show that the complex method is an effective tool for constructing explicit meromorphic solutions for complex differential equations. In this paper, we consider the following partial differential equation

$$u_t + au_{xxx} + 2buu_x + u_{xxxxxx} = 0, \tag{1.1}$$

where $u(x,t)$ is a real-valued function, $a, b(\neq 0)$ are real constants. Equation (1.1) is a modified version of the Kuramoto-Sivashinsky equation in [18] which has aroused great interest in physical scientists in recent years. Define a meromorphic function f belongs to the class W (see [18]) if f is an elliptic function, a rational function of $e^{\alpha z}(\alpha \in \mathbb{C})$ or a rational function of z. The Kuramoto-Sivashinsky equation reads $\phi_t + \nu\phi_{xxxx} + b\phi_{xxx} + \mu\phi_{xx} + \phi\phi_x = 0, \quad \nu, b, \mu \in \mathbf{R}, \quad \nu \neq 0$. By the traveling wave transformation $\phi(x,t) = c + w(z), \quad z = x - ct$, it reduces to the ordinary differential equation

$$\nu w''' + bw'' + \mu w' + w^2/2 + A = 0, \quad \nu \neq 0. \tag{1.2}$$

Eremenko applied the Nevanlinna theory and found that all meromorphic solutions of Eq (1.2) belong to the class W, and if for some values of parameters such solution w exists, then all other meromorphic solutions form a one-parametric family $w(z - z_0)$. Further, elliptic solutions exist only if $b^2 = 16\,\mu\nu$, non-constant rational solutions exist if and only if $b = \mu = A = 0$, and all exponential solutions have the form of $P(\tan kz)$, where P is a polynomial [18]. In this direction, the motivation of this paper is, therefore, whether it is possible to study very high order differential equations, such as Eq (1.1), to study whether these equations have solutions in W, whether there are only solutions in W, and, further, find out the expressions for the solutions.

Take traveling wave transformation $u(x,t) = w(z), z = x - ct$ into Eq (1.1) and get the fifth-order algebraic ordinary differential equation (ODE)

$$w^{(5)} + aw'' + bw^2 - cw + d = 0, \tag{1.3}$$

where $a, b(\neq 0), c, d$ are constant complex numbers, and the superscript (k) denotes the kth derivative with respect to z.

Conte and Ng used the subequations method to obtain meromorphic solutions for the generalized third-order differential equation (see [19], pp. 2, Eq (3)). Demina and Kudryashov used the Laurent series method to study some non-linear partial differential equations, such as the Kawahara equation [20]. However, higher-order ODEs are rarely touched. Starting from this point, our aim is to prove that all meromorphic solutions for Eq (1.3) belong to the class W, and use the complex method and direct method of substitution to construct non-trivial elliptic and multiple-valued solutions of Eqs (1.3), (1.4) and (1.7). Further, we will prove the following results.

Theorem 1.1. *Equation (1.3) is integrable if and only if* $4bd - c^2 = 0$*, and all meromorphic solutions* $w(z)$ *belong to class W, with a movable quintuple pole at an arbitrary complex constant* z_0.

1) If $a \neq 0$*, the only elliptic solution is*

$$w_d(z) = -\frac{7560}{b}\wp(z - z_0)\wp'(z - z_0) - \frac{630a}{41b}\wp(z - z_0) + \frac{c}{2b},$$

where $z_0 \in \mathbb{C}, g_2 = 0, g_3 = \frac{2a^2}{226935}$*. Eq (1.3) is without rational and simply periodic function solution.*

2) If $a = 0$*, the only rational function solution is*

$$w_r(z) = \frac{15120}{b}\frac{1}{(z - z_0)^5} + \frac{c}{2b}.$$

Put $z = x - ct$ into former solutions, and the traveling wave solutions for Eq (1.1) will be obtained immediately.

For some values of the parameters, Theorem 1.1 shows that Eq (1.3) has only elliptic function solutions and rational function solutions. We can use the results of Theorem 1.1 to evaluate the existence of solutions to more complex sixth-order differential equations through a kind of functional transformation, for instance, Eqs (1.4) and (1.7). The results show that we obtain a class of innovative multiple-valued solutions for some complex ordinary differential equations. By Theorem 1.1, we prove the following theorems.

Theorem 1.2. *Consider the sixth-order ODE*

$$u^{(6)} - u^{(5)} + u'^2 - 2u'u + u^2 + 2u' - 2u + 1 = 0, u := u(z). \tag{1.4}$$

1) Equation (1.4) has the following form multiple-valued function solution

$$u_1(z) = 1 - \frac{3780}{(z-z_0)^4} + \frac{1260}{(z-z_0)^3} - \frac{630}{(z-z_0)^2} + \frac{630}{(z-z_0)} - 630e^{z-z_0}\int \frac{e^{-z+z_0}}{z-z_0}dz + \beta e^{z-z_0}, \tag{1.5}$$

where $\beta, z_0 \in \mathbb{C}$ are arbitrary.

2) Equation (1.4) bas the following meromorphic solution

$$u_2(z) = \beta e^{z-z_0} + 1, \tag{1.6}$$

where $\beta, z_0 \in \mathbb{C}$ are arbitrary.

Theorem 1.3. *Consider the sixth-order ODE*

$$u^{(6)} - u^{(5)} + au''' - au'' + bu'^2 - 2bu'u + bu^2 - cu' + cu + d = 0, u := u(z), \tag{1.7}$$

$a, b(\neq 0), c, d$ are constant complex numbers. If $4bd = c^2$, Eq (1.7) has the following elliptic and multi-valued solution

$$\begin{aligned} u_2(z) = &-\frac{630}{b}\left[\wp''(z-z_0) + \wp'(z-z_0) + \wp(z-z_0)\right] \\ &-\frac{25830+630a}{41b}e^{z-z_0}\int \wp(z)e^{-(z-z_0)}dz - \frac{c}{2b} + \beta e^{z-z_0}, \end{aligned} \tag{1.8}$$

where $\beta, z_0 \in \mathbb{C}$ are arbitrary, $g_2 = 0, g_3 = \frac{2a^2}{226935}$.

This paper is organized as follows. In Section 2, we will introduce some mathematical definitions, lemmas, and the complex method. In Section 3, we will prove the three theorems. In Section 4, we will give elliptic meromorphic solutions to the modified singularly perturbed generalized higher-order KdV equation and the special sixth-order KdV-like equation by virtue of Eq (1.3). In Section 5, we will give the conclusions and discussion and pose two unsolved conjectures for the readers.

2. Lemmas and the complex method

In this section, we introduce the related concepts, the lemmas, and the complex method [13].

Set $m \in \mathbf{N}, r_j \in \mathbf{N} \cup \{0\}$, $r = (r_0, r_1, ..., r_m)$, $j = 0, 1, ..., m$.

Define differential monomial

$$M_r[w](z) := [w(z)]^{r_0}[w'(z)]^{r_1}[w''(z)]^{r_2} \cdots [w^{(m)}(z)]^{r_m}.$$

$p(r) := r_0 + r_1 + \cdots + r_m$ is called the degree of $M_r[w]$.

Define differential polynomial

$$P(w, w', \cdots, w^{(m)}) := \sum_{r \in I} a_r M_r[w],$$

where a_r are constants, and I is a finite index set. The degree of $P(w, w', \cdots, w^{(m)})$ is defined by

$$\deg P(w, w', \cdots, w^{(m)}) := \max_{r \in I}\{p(r)\}.$$

Consider an autonomous algebraic ODE

$$P(w, w', \cdots, w^{(m)}) = bw^n + c \tag{2.1}$$

where P is a polynomial in $w(z)$ and its arguments with constant coefficients, $b(\neq 0)$ and c are complex constants. We investigate the solutions, which are in the form of the formal Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k (z - z_0)^k.$$

If there are exactly p distinct formal Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k \ (q > 0, c_{-q} \neq 0) \tag{2.2}$$

satisfy Eq (2.1), we say Eq (2.1) satisfies $\langle p, q\rangle$ condition. If we only determine p distinct principle parts $w(z) = \sum_{k=-q}^{-1} c_k z^k$ $(q > 0, c_{-q} \neq 0)$, we say Eq (2.1) satisfies weak $\langle p, q\rangle$ condition. If Eq (2.1) satisfies $\langle p, q\rangle$ condition, we say Eq (2.1) satisfies the finiteness property: has only finitely many formal Laurent series with finite principle part admitting the equation.

Let $\omega_1,\ \omega_2$ be two fixed complex numbers such that $\mathrm{Im}\frac{\omega_1}{\omega_2} > 0,\ L = L[2\omega_1,\ 2\omega_2]$ be discrete subset $L[2\omega_1,\ 2\omega_2] = \{\omega \mid \omega = 2m\omega_1 + 2n\omega_2,\ m, n \in \mathbf{Z}\}$, which is isomorphic to $\mathbf{Z} \times \mathbf{Z}$. The discriminant $\Delta = \Delta(c_1, c_2) := c_1^3 - 27c_2^2$.

Weierstrass $\wp(z) := \wp(z, g_2, g_3)$ function is a meromorphic function with two periods $2\omega_1,\ 2\omega_2$ and solves equation $(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, where g_2, g_3 are elliptic invariants defined by

$$g_2 = \sum_{(m,n)\neq(0,0)} \frac{60}{(2m\omega_1 + 2n\omega_2)^4}, g_3 = \sum_{(m,n)\neq(0,0)} \frac{140}{(2m\omega_1 + 2n\omega_2)^6},$$

and $g_2^3 - 27g_3^2 \neq 0$. The addition formula is

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2. \tag{2.3}$$

The Weierstrass $\wp$ function has the Laurent series expansion

$$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + O(|z|^6).$$

Furthermore, $\wp'(-z) = -\wp'(z)$, $2\wp''(z) = 12\wp^2(z) - g_2$, $\wp'''(z) = 12\wp(z)\wp'(z), \cdots$, any kth derivatives of $\wp$ can be deduced by the identities [21].

Lemma 2.1 ([13, 22, 23]). *Let $p, l, m, n \in \mathbf{N}$. If* $\deg P(w, w', ..., w^{(m)})) < n$ *and Eq (2.1) satisfies $\langle p, q\rangle$ condition, then all non-constant meromorphic solutions $w \in W$ and must be one of the following three forms:*

(i) Each elliptic solution with a pole at $z = 0$ can be written as

$$\begin{aligned} w(z) = & \sum_{i=1}^{l-1}\sum_{j=2}^{q} \frac{(-1)^j c_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z) + B_i}{\wp(z) - A_i}\right]^2 - \wp(z)\right) \\ & + \sum_{i=1}^{l-1}\frac{c_{-i1}}{2}\frac{\wp'(z) + B_i}{\wp(z) - A_i} + \sum_{j=2}^{q}\frac{(-1)^j c_{-lj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + c_0, \end{aligned} \tag{2.4}$$

where c_{-ij} are given by Eq (2.2), $B_i^2 = 4A_i^3 - g_2 A_i - g_3$ and $\sum_{i=1}^{l} c_{-i1} = 0, c_0 \in \mathbb{C}$.

(ii) Each rational function solution $w := R(z)$ is of the form

$$R(z) = \sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(z - z_i)^j} + c_0, \tag{2.5}$$

with $l(\leq p)$ distinct poles of multiplicity q.

(iii) Each simply periodic solution is a rational function $R(\xi)$ of $\xi = e^{\alpha z}(\alpha \in \mathbb{C})$. $R(\xi)$ has $l(\leq p)$ distinct poles of multiplicity q, and is of the form

$$R(\xi) = \sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(\xi - \xi_i)^j} + c_0. \tag{2.6}$$

Lemma 2.2 ([23]). *Let $p, l, m, n \in \mathbf{N}$, $degP(w, w', ..., w^{(m)})) < n$, m is an odd integer. If Eq (2.1) satisfies weak $\langle p, q\rangle$ condition and the dominant part $\widehat{E}(z, w) = w^{(m)} - aw^n$, then all meromorphic solutions $w(z)$ of Eq (2.1) belong to class W.*

Suppose that Eq (2.1) satisfies weak $\langle p, q\rangle$ condition, we can also construct meromorphic solutions by Eqs (2.4)–(2.6).

Apply the complex method [13], we will pose the following steps:

Step 1 Substituting the transform $T : u(x, y, t) \rightarrow w(z), \quad (x, y, t) \rightarrow z$ into a given PDE gives a non-linear ordinary differential Eq (2.1).

Step 2 Substitute (2.2) into Eq (2.1) to determine that weak $\langle p, q\rangle$ condition holds.

Step 3 By indeterminant relation Eqs (2.4)–(2.6) find the elliptic, rational and simply periodic solutions $w(z)$ of Eq (2.1) with pole at $z = 0$, respectively.

Step 4 By Lemma 2.1 obtain meromorphic solutions $w(z - z_0)$.

Step 5 Substituting the inverse transform T^{-1} into the meromorphic solutions $w(z - z_0)$, then get all exact solutions $u(x, y, t)$ of the original given PDE.

There is no unified method to handle all types of differential equations and obtain all types of solutions. One of the fundamental reasons we apply the complex method in the current paper is that by applying this method, we can obtain new meromoprhic solutions on the complex domains, e.g., W-class solutions.

3. The proof of the theorems

3.1. The Proof of Theorem 1.1

Proof. It is easy to check that Eq (1.3) has no nonconstant polynomial solution. Then, by Wiman-Valiron theory (see [24], Chapter 3), we have that the Eq (1.3) does not have transcendental entire solutions since there is only one top degree term in Eq (1.3). Suppose that $w(z)$ is a meromorphic solution of Eq (1.3), with a movable pole at $z = 0$, then in a neighborhood of $z = 0$, the Laurent series of $w(z)$ is the form of $\sum_{k=-q}^{\infty} c_k z^k$ ($q > 0, c_{-q} \neq 0$). The degree of the pole $z = 0$ and the coefficient c_{-q} can be uniquely determined by equating equation $w^{(5)} + bw^2 = 0$. Substituting $\sum_{k=-q}^{\infty} c_k z^k$ into Eq (1.3), we have $p = 1, q = 5$, $c_{-5} = \frac{15120}{b}, c_{-4} = 0, c_{-3} = 0, c_{-2} = -\frac{630a}{41b}, c_{-1} = 0, c_0 = \frac{c}{2b}$. Hence Eq (1.3) satisfies weak $\langle p, q\rangle = \langle 1, 5\rangle$ condition, and $w(z)$ has the following form Laurent series

$$w(z) = \frac{15120}{b}\frac{1}{z^5} - \frac{630a}{41b}\frac{1}{z^2} + \frac{c}{2b} + \dots$$

Furthermore, the dominant term is $\widehat{E}(z, w) = w''''' + bw^2$, therefore by Lemma 2.2, all meromorphic solutions $w \in W$. In the following, we are going to solve Eq (1.3).

1) $a \neq 0$.

By (2.4), we infer that the indeterminant of elliptic solution with pole $z = 0$ is

$$w_{d0}(z) = -\frac{c_{-5}}{24}\wp'''(z, g_2, g_3) + c_{-2}\wp(z, g_2, g_3) + c_0. \tag{3.1}$$

Substituting (3.1) into Eq (1.3), we have

$(967680a^2\wp^2(z) - 109800230400g_3\wp^2(z) - 12810026880g_2^2\wp(z) + 78109920ag_2\wp'(z) + 51660a^2g_2 + 6724db - 1681c^2 - 10065021120g_2g_3)/6724b = 0$.

Combining similar terms, we have

$(967680a^2 - 109800230400g_3)\wp(z)^2 - 12810026880g_2^2\wp(z) + 78109920ag_2\wp'(z) + 51660a^2g_2 + 6724db - 1681c^2 - 10065021120g_2g_3 = 0$.

Eliminating the coefficients for the above functional relation, we have $967680a^2 - 109800230400g_3 = 0, -12810026880g_2^2 = 0$, $78109920ag_2 = 0, 51660a^2g_2 + 6724db - 1681c^2 - 10065021120g_2g_3 = 0$, so $g_2 = 0, g_3 = 2a^2/226935$, and $6724bd - 1681c^2 = 0$.

Therefore, we yield that Eq (1.3) is integrable provided that $4bd - c^2 = 0$ (if $4bd - c^2 \neq 0$, the constant terms in the expansion of Eq (1.3) can not be vanished), then Eq (1.3) has the following elliptic solution

$$\begin{aligned} w_d(z) &= -\frac{630}{b}\wp'''(z-z_0) - \frac{630a}{41b}\wp(z-z_0) + \frac{c}{2b} \\ &= -\frac{7560}{b}\wp(z-z_0)\wp'(z-z_0) - \frac{630a}{41b}\wp(z-z_0) + \frac{c}{2b}, \end{aligned} \tag{3.2}$$

where $z_0 \in \mathbb{C}, g_2 = 0, g_3 = 2a^2/226935$. By additional formula and (2.4), we know that each elliptic function w can be written as $w = R_1(\wp) + R_2(\wp)\wp'$, where R_1, R_2 are uniquely determined rational functions.

By (2.5), we infer that the indeterminant of rational solution with pole $z = 0$ is

$$w_{r0}(z) = \frac{c_{-5}}{z^5} + \frac{c_{-2}}{z^2} + h, \tag{3.3}$$

h is constant. Substituting (3.3) into Eq (1.3), eliminating the coefficients, we have

$$\begin{cases} 30240h - \frac{15120c}{b} & = 0 \\ \frac{241920a^2}{1681b} & = 0 \\ -\frac{1260ah}{41} + \frac{630ca}{41b} & = 0 \\ bh^2 - ch + d & = 0. \end{cases}$$

It contradicts with $a \neq 0$, therefore, Eq (1.3) doesn't have any rational function solution.

By (2.6), we infer that the indeterminant of simply periodic solution with pole $z = 0$ is

$$R(z) = \frac{c_{-5}}{(\xi-1)^5} + \frac{c_{-2}}{(\xi-1)^2} + c_0, \tag{3.4}$$

setting $\xi = e^{\alpha z}$, substituting (3.4) into Eq (1.3), we have

$$(R'''''\xi^5 + 10R''''\xi^4 + 25R'''\xi^3 + 15R''\xi^2 + R'\xi)\alpha^5 + (aR''\xi^2 + aR\xi)\alpha^2 + bR^2 - cR + d = 0, \tag{3.5}$$

then eliminating the coefficients, letting the leading terms equal to zero, we have $\alpha^5 e^{5\alpha z} - 1 = 0$, hence $z = -\frac{1}{\alpha}\log\alpha$, but it is contradict with z is arbitrary. Therefore Eq (1.3) doesn't have any simple periodic solution.

2) $a = 0$.

By (3.2), letting $g_3 = 0$, it is obvious to see that

$$w_r(z) = -\frac{630}{b}\wp'''(z-z_0,0,0) + \frac{c}{2b} = \frac{15120}{b}\frac{1}{(z-z_0)^5} + \frac{c}{2b} \tag{3.6}$$

is the unique rational solution when $a = 0$, where $z_0 \in \mathbb{C}$.

Thus, we complete the proof of Theorem 1.1.

3.2. The proof of Theorem 1.2

Proof. We take a transformation $u(z) := (\int w(z)e^{-z}dz + \beta)e^{z}$ into Eq (1.4), where β is an arbitrary constant and $w(z)$ is a meromorphic function on the complex plane. Then, we reduce Eq (1.4) to

$$w^{(5)} + w^2 + 2w + 1 = 0. \tag{3.7}$$

1) By Theorem 1.1, all the non-constant meromorphic solution with pole $z = 0$ of Eq (3.7) is

$$w(z) = \frac{15120}{z^5} - 1, \tag{3.8}$$

so we get

$$\begin{aligned} u_0(z) &= (\int (\frac{15120}{z^5} - 1)e^{-z}dz + \beta)e^{z} \\ &= 1 - \frac{3780}{z^4} + \frac{1260}{z^3} - \frac{630}{z^2} + \frac{630}{z} + 630e^{z}\int \frac{e^{-z}}{z}dz + \beta e^{z}. \end{aligned} \tag{3.9}$$

Furthermore, the solutions with pole $z = z_0 \in \mathbb{C}$ of Eq (1.4) is

$$u_1(z) = 1 - \frac{3780}{(z-z_0)^4} + \frac{1260}{(z-z_0)^3} - \frac{630}{(z-z_0)^2} + \frac{630}{(z-z_0)} - 630e^{z-z_0}\int \frac{e^{-z+z_0}}{z-z_0}dz + \beta e^{z-z_0},$$

where $\beta, z_0 \in \mathbb{C}$ arbitrary. Clearly,

$$\int \frac{e^{-z}}{z}dz = \sum_{n=1}^{\infty} \frac{(-1)^n}{nn!}z^n + \log z + \gamma, \tag{3.10}$$

γ is a constant. According to the multiple-valued property of Logarithmic function $\log z$, solution (1.5) demonstrates that Eq (1.4) has a class of multiple-valued solutions that do not belong to W.

2) Since $w(z) = -1$ is the only constant meromorphic solution of (3.7), a trivial verification shows that only $u_2(z) = \beta e^{z-z_0} + 1$ satisfy Eq (1.4), where $\beta, z_0 \in \mathbb{C}$ arbitrary. It implies that all meromorphic solutions of Eq (1.4) be (1.6).

The proof of Theorem 1.2 is completed.

3.3. The Proof of Theorem 1.3

Proof. By the transformation $u(z) := (\int w(z)e^{-z}dz + \beta)e^{z}$, Eq (1.7) will be change into (1.3):

$$w^{(5)} + aw'' + bw^2 - cw + d = 0,$$

where β is arbitrary, $w(z)$ is meromorphic in the complex plane.

Letting $\wp(z) := \wp(z, g_2, g_3)$, here $g_2 = 0, g_3 = 2a^2/226935$. By Theorem 1.1, if $4bd = c^2$, we have,

$$\begin{aligned}
u_0(z) &= \left\{\iint \left[-\frac{630}{b}\wp'''(z) - \frac{630a}{41b}\wp(z) + \frac{c}{2b}\right] e^{-z}dz + \beta\right\} e^z \\
&= \left\{-\frac{630}{b}\int \wp'''(z)e^{-z}dz - \frac{630a}{41b}\int \wp(z)e^{-z}dz + \frac{c}{2b}\int e^{-z}dz + \beta\right\} e^z \\
&= \left\{-\frac{630}{b}\left[e^{-z}\wp''(z) + e^{-z}\wp'(z) + e^{-z}\wp(z) + \int \wp(z)e^{-z}dz\right] - \frac{630a}{41b}\int \wp(z)e^{-z}dz - \frac{c}{2b}e^{-z} + \beta\right\} e^z \\
&= -\frac{630}{b}\left[\wp''(z) + \wp'(z) + \wp(z) + e^z\int \wp(z)e^{-z}dz\right] - \frac{630a}{41b}e^z\int \wp(z)e^{-z}dz - \frac{c}{2b} + \beta e^z \\
&= -\frac{630}{b}\left[\wp''(z) + \wp'(z) + \wp(z)\right] - \frac{25830 + 630a}{41b}e^z\int \wp(z)e^{-z}dz - \frac{c}{2b} + \beta e^z.
\end{aligned} \tag{3.11}$$

Therefore, Eq (1.7) has the following elliptic multiple-valued solutions with pole $z = z_0 \in \mathbb{C}$:

$$u_2(z) = -\frac{630}{b}\left[\wp''(z - z_0) + \wp'(z - z_0) + \wp(z - z_0)\right] - \frac{25830 + 630a}{41b}e^{z-z_0}\int \wp(z)e^{-(z-z_0)}dz - \frac{c}{2b} + \beta e^{z-z_0},$$

where $\beta, z_0 \in \mathbb{C}$ is arbitrary, $4bd = c^2$, $g_2 = 0, g_3 = 2a^2/226935$.

The proof of Theorem 1.3 is completed.

4. Applications

The complex method has been applied in the process of many higher order differential equations, such as the six-order thin-film equation [15], the seventh-order KdV equation [16] with the assistance of Painlevé analysis and Nevanlinna theory. Many meromorphic solutions are constructed. In this section, the following sixth-order KdV-like equations are considered again, and the exact solutions are derived with the aid of Eq (1.3).

4.1. The modified singularly perturbed generalized higher-order KdV equation

The modified singularly perturbed generalized higher-order KdV equation [25] be

$$U_t + \alpha U^{n+1}U_x + \beta U_{xxx} + \epsilon U_{xxxxxx} = 0, \tag{4.1}$$

where α, β, ϵ are positive constants. Assume $n = 0$, substituting the traveling wave transformation

$$U(x,t) = w(z), \quad z = (\epsilon^{-1})^{1/6}x - ct$$

into Eq (4.1), then

$$-cw' + \alpha(\epsilon^{-1})^{1/6}ww' + \beta(\epsilon^{-1})^{1/2}w''' + w^{(6)} = 0, \tag{4.2}$$

integrating it yields

$$-cw + \alpha(\epsilon^{-1})^{1/6}w^2/2 + \beta(\epsilon^{-1})^{1/2}w'' + w^{(5)} + d = 0, \tag{4.3}$$

by Theorem 1.1, if and only if $2\alpha(\epsilon^{-1})^{1/6}d = c^2$, Eq (4.1) has elliptic meromorphic solutions:

$$w(z) = -15120\frac{\epsilon^{1/6}}{\alpha}\wp(z-z_0)\wp^{'}(z-z_0) - \frac{1260}{41}\frac{\beta}{\alpha}(\epsilon^{-1})^{1/3}\wp(z-z_0) + \frac{c\epsilon^{1/6}}{\alpha}, \tag{4.4}$$

where $z_0 \in \mathbb{C}, g_2 = 0, g_3 = 2\beta^2\epsilon^{-1}/226935$. Then substitute $z = (\epsilon^{-1})^{1/6}x - ct$ into (4.4), the traveling wave solutions for the modified singularly perturbed generalized higher-order KdV equation will be built. Suppose that $\beta = 0$ in Eq (4.1) and (4.4), the rational solutions will be derived instantly.

4.2. The special sixth-order KdV-like equation

We second give an example, modified from Kaya ([26], Example 2), consider a special sixth-order KdV equation as following

$$U_t + U_x + UU_x - U_{xxx} + U_{xxxxxx} = 0. \tag{4.5}$$

Substituting the traveling wave transformation

$$U(x,t) = w(z), \quad z = x - \lambda t$$

into Eq (4.5), then

$$(1-\lambda)w' + ww' - w''' + w^{(6)} = 0, \tag{4.6}$$

integrating it yields

$$(1-\lambda)w + w^2/2 - w'' + w^{(5)} + d = 0, \tag{4.7}$$

by Theorem 1.1, if and only if $2d = (\lambda-1)^2$, Eq (4.5) has elliptic meromorphic solutions:

$$w(z) = -15120\wp(z-z_0)\wp^{'}(z-z_0) + 1260/41\wp(z-z_0) + \lambda - 1, \tag{4.8}$$

where $z_0 \in \mathbb{C}, g_2 = 0, g_3 = 2/226935$. Then, substitute $z = x - \lambda t$ into (4.8), and the traveling wave solutions for Eq (4.5) will be built.

5. Conclusions and discussion

We prove that all meromorphic solutions for Eq (1.3) belong to the class W, and construct them by the complex method. Using a functional transformation $u(z) = (\int w(z)e^{-z}dz + \beta)e^z$, we obtain the elliptic and multiple-valued solutions for the high order nonlinear Eqs (1.4) and (1.7). At last, we give two applications on the KdV-like equations for Theorem 1.1. In conclusion, the complex method is an effective method for constructing explicit traveling wave solutions for some high-order nonlinear differential equations, such as elliptic solutions, simple periodic solutions and rational solutions. Most recently, the non-traveling wave rational solutions of a KdV-like equation [27] and a KP-like equation [28], the non-traveling wave soliton solutions of two types of nonlocal integrable nonlinear Schrödinger equation were investigated [29,30]. It is of great interest to investigate the traveling wave reduced KdV-like equation and the KP-like equation, and the nonlocal integrable nonlinear Schrödinger equations using the complex method to construct rational solutions and meromorphic solutions.

Furthermore, we would like to raise the unsolved conjectures for readers:

Conjecture 1 Equation (1.4) does not have any other multiple-valued function solutions except for the solution (1.5).
Conjecture 2 Equation (1.7) does not have any other multiple-valued function solutions except for the solution (1.8).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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