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Research article

On the global existence and blow-up for the double dispersion equation with exponential term

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Abstract: This paper deals with the initial boundary value problem for the double dispersion equation with nonlinear damped term and exponential growth nonlinearity in two space dimensions. We first establish the local well-posedness in the natural energy space by the standard Galërkin method and contraction mapping principle. Then, we prove the solution is global in time by taking the initial data inside the potential well and the solution blows up in finite time as the initial data in the unstable set. Moreover, finite time blow-up results are provided for negative initial energy and for arbitrary positive initial energy respectively.

Keywords: double dispersion equation; nonlinear damped; exponential nonlinearity; global existence; blow-up

1. Introduction

This paper is devoted to the following initial-boundary value problem for the double dispersion equation

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u - \Delta h(u_t) + \Delta f(u) = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega \\ u(x,t) = \Delta u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^2$ having smooth boundary $\partial\Omega$ and $u = u(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. The damped term $h(u_t)$ is given by $h(u_t) = |u_t|^{q-1} u_t$ with $q > 1$, and the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ admits

(H1) for each $\beta > 0$, there exists a positive constant $C_\beta > 0$ depending only on β such that

$$|f(t)| \le C_\beta e^{\beta t^2}, \quad |f'(t)| \le C_\beta e^{\beta t^2}, \quad \text{for all } t \in \mathbb{R}.$$

On account of the possibility of energy exchange through lateral surfaces of the waveguide in the physical study of nonlinear wave propagation in waveguide, the longitudinal displacement $u(x,t)$ of the rod satisfies the following double dispersion equation (DDE) [1,2]

$$u_{tt} - u_{xx} = \frac{1}{4}\left(6u^2 + au_{tt} - bu_{xx}\right)_{xx},$$

and the general cubic double dispersion equation (CDDE)

$$u_{tt} - u_{xx} = \frac{1}{4}\left(cu^3 + 6u^2 + au_{tt} - bu_{xx} + du_t\right)_{xx}. \tag{1.2}$$

Here $a, b, c > 0$ and $d \neq 0$ are some constants depending on the Young modulus, the shearing modulus, the density of the waveguide and the Poisson coefficient.

Due to the wide applications in the real world, the initial value problem and initial-boundary value problem of double dispersion equation have drawn much attention from mathematicians. In [3–5], the authors studied global solutions with $f'(s) \geq C$ (bounded below) for following the generalized DDE which includes Eq (1.2) as special cases,

$$u_{tt} - au_{xxtt} + bu_{xxxx} - u_{xx} - du_{xxt} = f(u)_{xx}. \tag{1.3}$$

Moreover, they also showed the nonexistence of the global solution under some other conditions. When $d = 0$, Liu [6] investigated the global existence and nonexistence of solutions for the initial-boundary value problem of (1.3) with $|f(u)| \leq C|u|^p$.

For the multidimensional generalized form of (1.3)

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u - d\Delta u_t = \Delta f(u), \tag{1.4}$$

Polat [7] researched the existence of global solutions also in the cases of $f'(s) \geq C$. Wang [8] considered the global existence and asymptotic behavior of the small amplitude solution in the time-weighted Sobolev space for the Cauchy problem of (1.4) with $|f(u)| \leq C|u|^{\alpha-j}$. Wang [9, 10] investigated the asymptotic profile of solutions for the Cauchy problem of (1.4) with $f(u) = O(u^2)$. Su [11] researched the existence and nonexistence of a global solution in the natural energy space for the initial-boundary value problem of (1.4) with $f(u) = \beta|u|^{p-1}u$. So, motivated by this fact and the so-called Moser-Trudinger type inequalities [12–15], it was natural to consider nonlinearities with exponential growth.

For the generalized double dispersion (1.4) with exponential nonlinearity, Zhang [16, 17] proved the existence and nonexistence of global weak solution for the Cauchy problem of (1.4) with $d = 0$. Guo [18] established the sufficient conditions of finite time blow-up of solutions in the cases of arbitrary positive initial energy for the Cauchy problem of (1.4) with $d = 0$. As far as we are concerned, there are no results on the global existence and finite time blow-up of solutions for the initial-boundary value problem of (1.4) with both nonlinear damped and exponential nonlinearity.

Furthermore, there are many works that focus on the wave equation with exponential nonlinearity. Global well-posedness in the defocusing case was established by Nakamura [19] for small data, Atallah [20] in the radial case, then by Ibrahim [21] and Struwe [22,23], see also [24–29] and their references.

It is the aim of this manuscript to obtain results about the global existence and finite time blow-up of solutions under sufficient conditions for the problem (1.1) in the case that f is a source term and admits an exponential growth and $h(u_t)$ is polynomial growth. This is the first attempt in the literature to take into account both nonlinear damped term and the exponential nonlinear source for the problem (1.1).

This paper is organized as follows. We first establish the local well-posedness by the standard Galërkin method and contraction mapping principle(see Theorem 3.1). By means of the potential well theory, we provide the sufficient conditions of global solutions with subcritical initial energy (see Theorem 4.1). In the case of the negative initial energy or the initial data in the potential well, we prove the local solutions will blow up in finite time (see Theorems 5.1 and 5.2). Moreover, we also constructed the sufficient conditions of finite time blow-up of solutions in the case of arbitrary positive initial energy (see Theorem 5.3).

It is worth mentioning here that the sufficient conditions of blow-up of solutions can be investigated based on the concave method [17,18,27] in the absence of the nonlinear damped term $h(u_t)$ or $h(u_t) = u_t$ in (1.1). However, when nonlinear damping and exponential source terms are both present in the equation, it seems that their method does not work on our problem directly. We research the blow-up of solutions in the cases of $E(0) < 0$ or subcritical initial energy mainly by exploiting an argument from the one devised in [30–32], which investigated the blow-up of solutions to the wave equations with polynomial-type nonlinearity of form $|s|^m s$, and the case of $E(0) > 0$ mainly by [33].

We conclude this section with several notations given. The notation $(\cdot,\cdot)$ stands for the L^2-inner product and $\langle\cdot,\cdot\rangle$ is used for the notation of duality pairing between dual space. For brevity, we use the same letter C to denote different positive constants, and $C(\cdots)$ to denote positive constants depending on the quantities appearing in the parenthesis.

2. Preliminaries

For brevity, we use the following abbreviations:

$$L^p = L^p(\Omega) \quad H_0^1 = H_0^1(\Omega), \quad \|\cdot\| = \|\cdot\|_{L^2}, \quad \|\cdot\|_{H_0^1}^2 = \|\cdot\|^2 + \|\nabla\cdot\|^2,$$

with $1 \le p \le \infty$. $H^{-1} = H^{-1}(\Omega)$ is the dual space of H_0^1. Let $A = -\Delta$. Then, $\langle Au, v\rangle = (\nabla u, \nabla v)$, for $u, v \in H_0^1$ and the domain of A is $D(A) = H^2 \cap H_0^1$; A is a positive, self-adjoint and invertible operator and the inverse operator A^{-1} is compact [34]. Consequently, the operator A possesses an infinitely countable positive eigenvalues:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \to +\infty,$$

and a corresponding sequence of eigenfunctions $\{e_j : j = 1, 2, \cdots\}$ that forms an orthogonal basis for L^2. Also, the sequence $\{e_j : j = 1, 2, \cdots\}$ is an orthogonal basis for H_0^1. In addition, the linear span of $\{e_j : j = 1, 2, \cdots\}$ is dense in L^p for any $1 \le p < \infty$. Since the domain Ω is smooth, then $e_j \in C^\infty(\Omega)$.

For any $u \in L^2$, there exists $u_j = u_j(t) = (u, e_j)$ such that

$$u = \sum_{j=1}^{\infty} u_j e_j, \quad \|u\|^2 = \sum_{j=1}^{\infty} |u_j|^2.$$

The powers of A are defined as follows [34]

$$A^s u = \sum_{j=1}^{\infty} \lambda_j^s u_j e_j, \quad s \in \mathbb{R}.$$

For $s \geq 0$, $A^s : D(A^s) \subset L^2 \to L^2$, and the domain of A^s is given by

$$D(A^s) = \{u \in L^2 : u = \sum_{j=1}^{\infty} u_j e_j,\ \sum_{j=1}^{\infty} \lambda_j^{2s} |u_j|^2 < \infty\},$$

which endowed with the graph norm

$$\|u\|_{D(A^s)} = \|A^s u\| = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} |u_j|^2 \right)^{\frac{1}{2}}$$

and the associated scalar product

$$(u, v)_{D(A^s)} = \sum_{j=1}^{\infty} \lambda_j^{2s} u_j v_j,$$

where $v_j = (v, e_j)$. Especially, $D(A^s) = H_0^{2s}$ for $\frac{1}{4} < s \leq \frac{1}{2}$ (see [35]).

It worth to be mentioned that we introduce the space H_s (with $s > 0$) as the domain $D(A^s)$ equipped with the graph norm $\|u\|_{D(A^s)} = \|A^s u\|$. If s is negative, we define H_s as the completion of L^2 with respect to the norm $\|u\|_{D(A^s)} = \|A^{-|s|} u\|$.

By the Poincaré inequality, one can easily see that if $u \in L^2$ then $(-\Delta)^{-\frac{1}{2}} u \in L^2$. So, we can define a Hilbert space as follows

$$\mathcal{H} = (L^2, (\cdot, \cdot)_{\mathcal{H}}),$$

where the scalar product $(\cdot, \cdot)_{\mathcal{H}}$ is defined by

$$(u, v)_{\mathcal{H}} = (u, v) + \left((-\Delta)^{-\frac{1}{2}} u, (-\Delta)^{-\frac{1}{2}} v \right)$$

and the norm $\| \cdot \|_{\mathcal{H}}$ is defined by

$$\|u\|_{\mathcal{H}}^2 = \|u\|^2 + \left\| (-\Delta)^{-\frac{1}{2}} u \right\|^2. \tag{2.1}$$

We note that the Poincaré inequality implies that the norms $\| \cdot \|$ and $\| \cdot \|_{\mathcal{H}}$ are equivalent norms on L^2.

Applying the operator $(-\Delta)^{-1}$ to (1.1). Then (1.1) becomes

$$(-\Delta)^{-1} u_{tt} + u_{tt} + u + (-\Delta)^{-1} \Delta^2 u + h(u_t) - f(u) = 0. \tag{2.2}$$

Since $(-\Delta)^{-1} \Delta^2 u = -\Delta u$, for any $u \in \left\{ u \in H^4 \cap H_0^1 : \Delta u \mid_{\partial\Omega} = 0 \right\}$ (see [34, Lemma 1.7]), then the Eq (2.2) can be written as

$$(-\Delta)^{-1} u_{tt} + u_{tt} + u - \Delta u + h(u_t) - f(u) = 0. \tag{2.3}$$

Then, the weak solution of (2.3) with the initial data of (1.1) and boundary value condition $u \mid_{\partial\Omega} = 0$ is said to be the weak solution of the problem (1.1). This leads to the following definition.

Definition 2.1 (weak solution)**.** *A function $u \in C\left([0, T]; H_0^1\right) \cap C^1\left([0, T]; L^2\right)$ with $u_t \in L^{q+1}((0, T) \times \Omega)$ is said to be a weak solution to the problem* (1.1) *over* $[0, T]$*, if and only if for any $t \in [0, T]$, it satisfies*

$$\left((-\Delta)^{-\frac{1}{2}} u_t, (-\Delta)^{-\frac{1}{2}} \varphi \right) + (u_t, \varphi) + \int_0^t \left[(u, \varphi) + (\nabla u, \nabla \varphi) + \langle h(u_t), \varphi \rangle - (f(u), \varphi) \right] \mathrm{d}\tau$$

$$= \left((-\Delta)^{-\frac{1}{2}}u_1, (-\Delta)^{-\frac{1}{2}}\varphi\right) + (u_1, \varphi),$$

for all test functions $\varphi \in H_0^1$, *and*

$$u(x,0) = u_0(x) \in H_0^1,\ u_t(x,0) = u_1(x) \in L^2.$$

Now, we give the Trudinger-Moser inequality [14, 15] which will be used repeatedly to estimate the exponential nonlinearity.

Lemma 2.1. *For all* $u \in W_0^{1,n}(n \geq 2)$

$$e^{\alpha|u|^{\frac{n}{n-1}}} \in L^1, \quad \textit{for all } \alpha > 0, \tag{2.4}$$

and there exist positive constants $C(n)$ *which depends on n only, such that*

$$\sup_{\|u\|_{W_0^{1,n}} \leq 1} \int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}} \mathrm{d}x \leq C(n)|\Omega|, \quad \textit{for all } \alpha \leq \alpha(n),$$

where $|\Omega| = \int_\Omega \mathrm{d}x$, $\alpha(n) = nw_{n-1}^{\frac{1}{n-1}}$, *and* w_{n-1} *is the* $(n-1)$*-dimensional measure of the* $(n-1)$*-sphere.*

Remark 2.1. *From Lemma 2.1, we know that* $f(u) \in L^1$ *(in fact, from* (2.4)*,* $f(u) \in L^p$ *for any* $p \geq 1$ *) for all* $u \in H_0^1$, *and let*

$$F(u) = \int_0^u f(\tau)\mathrm{d}\tau,$$

we also infer that $F(u) \in L^1$ *and*

$$|F(u)| \leq C_\beta |u| e^{\beta|u|^2}.$$

The following lemma is for the convergence of the approximate solutions.

Lemma 2.2 ([36])**.** *Let X and Y be Banach spaces, X′ and Y′ be the dual spaces of X and Y respectively, and X a dense subset of Y and the inclusion map of X into Y continuous. Assume that*

$$u_n \to u \quad \textit{weakly-star in} \quad L^\infty(0,T;X');$$
$$u_{nt} \to \chi \quad \textit{weakly-star in} \quad L^\infty(0,T;Y')$$

then $\chi = u_t$ *in* $L^\infty(0,T;Y')$.

The next lemma is straightforward and follows from the continuity and monotonicity of the function $h(u) = |u|^{q-1}u = |u|^q sgn(u)$.

Lemma 2.3. *Let* $h(u) = |u|^{q-1}u = |u|^q sgn(u)$. *Then h generates a monotone operator from* $L^{1+\frac{1}{q}}(\Omega)$ *into* $L^{1+\frac{1}{q}}(\Omega)$, *i.e.,*

$$\langle h(u) - h(v), u - v\rangle \geq 0 \quad \forall u, v \in L^{q+1}.$$

Moreover, the mapping $\lambda \mapsto \langle h(u + \lambda v), \eta\rangle$ *is continuous from* $\mathbb{R}$ *to* $\mathbb{R}$ *for every fixed* $u, v, \eta \in L^{q+1}$.

The continuity of the solutions needs the following result which can be found in [37].

Lemma 2.4. *Let V and Y be Banach spaces, V reflexive, V a dense subset of Y and the inclusion map of V into Y continuous. Then,*

$$L^\infty(0,T;V) \cap C_\omega([0,T];Y) = C_\omega([0,T];V),$$

where

$$C_w([0,T];Y) = \{u \in L^\infty(0,T;Y); \langle u(t), y'\rangle \textit{ is continuous on } [0,T], \textit{for all } y' \in Y'\}.$$

3. Local solutions

This section focus on the local well-posedness of (1.1) in the natural energy space. First, we establish the existence of a local weak solution to the corresponding linear problem of (1.1) by using a standard Galerkin approximation scheme based on the eigenfunctions $\{e_j\}_{j=1}^{\infty}$ of the operator $A = -\Delta$. The well-posedness of the nonlinear problem (1.1) shall be researched by the contraction mapping principle, and mainly by the Trudinger-Moser inequality to deal with the nonlinearity.

Theorem 3.1. *Let f satisfy (H1), $u_0 \in H_0^1$, and $u_1 \in L^2$. Then there exists a unique weak solution u of* (1.1) *in $\Omega \times (0, T_{\max})$, where $T_{\max}$ is the life time of solutions. In addition, u satisfies the energy identity*

$$E(t) + \int_0^t \|u_\tau\|_{q+1}^{q+1} \mathrm{d}\tau = E(0), \tag{3.1}$$

where

$$E(t) = \frac{1}{2}\left(\|u_t\|_{\mathcal{H}}^2 + \|u\|_{H_0^1}^2\right) - \int_\Omega F(u)\mathrm{d}x.$$

Here and after $\|\cdot\|_{\mathcal{H}}$ is denoted by (2.1). *Moreover, if*

$$\sup_{t\in[0,T_{\max})} \left(\|u_t\|_{\mathcal{H}}^2 + \|u\|_{H_0^1}^2\right) < \infty$$

then $T_{\max} = \infty$.

In order to prove Theorem 3.1, we firstly consider the following auxiliary result.

Lemma 3.1. *Let $T > 0$, $u_0 \in H_0^1$, $u_1 \in L^2$, and $u \in C\left([0, T]; H_0^1\right)$ with the norm $\max_{t\in[0,T]} \|u\|_{H_0^1} \le R$ for some constant $R > 0$. Assume that f satisfies (H1). Then, there exists a unique*

$$v \in C\left([0, T]; H_0^1\right) \cap C^1\left([0, T]; L^2\right), v_t \in L^{q+1}(\Omega \times (0, T))$$

which solves the linear problem

$$\begin{cases} (-\Delta)^{-1} v_{tt} + v_{tt} - \Delta v + v + h(v_t) - f(u) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ v(x, 0) = u_0(x), \quad v_t(x, 0) = u_1(x), & \text{in } \Omega, \\ v \mid_{\partial\Omega} = 0, & \text{on } \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{3.2}$$

Proof. Let $\{e_j\}_{j=1}^{\infty}$ be the orthonormal basis for L^2, as described in Section 2. For $m \in \mathbb{N}$ given, we seek m functions $g_{1,m}, g_{2,m}, ..., g_{m,m} \in C^2[0, T]$ such that

$$v_m(x, t) = \sum_{j=1}^{m} g_{j,m} e_j(x) \tag{3.3}$$

solves the problem

$$\left((-\Delta)^{-1} v_{mtt} + v_{mtt} + v_m - \Delta v_m + h(v_{mt}) - f(u), \eta\right) = 0, \tag{3.4}$$

$$v_m(x,0)=\sum_{j=1}^{m}\rho_j e_j \to u_0 \quad \text{in } H_0^1, \quad \text{as } m\to\infty, \tag{3.5}$$

$$v_{mt}(x,0)=\sum_{j=1}^{m}\xi_j e_j \to u_1 \quad \text{in } L^2, \quad \text{as } m\to\infty, \tag{3.6}$$

for every $\eta \in \text{Span}\{e_1, e_2, \dots, e_m\}$. Taking $\eta = e_j$ in (3.4) yields the following Cauchy problem for a linear ordinary differential equation with unknown $g_{j,m}$:

$$\begin{aligned}&(1+\lambda_j^{-1})g''_{j,m}+(1+\lambda_j)g_{j,m}+\left(h(v_{mt}),e_j\right)=(f(u),e_j),\\ &g_{j,m}(0)=\rho_j, \quad g'_{j,m}(0)=\xi_j \quad (j=1,2,...,m).\end{aligned}$$

Then, by the standard ordinary differential equations theory, for all j , the above Cauchy problem yields a unique global solution $g_{j,m} \in C^2(0,t_m)$. In turn, this gives a unique $v_m(t) \in H^2 \cap H_0^1$ for every $t \in (0,t_m)$ defined by (3.3) and satisfying (3.4)–(3.6). Taking $\eta = v_{mt}$ in (3.4), and integrating over $[0,t] \subset (0,t_m)$, we obtain

$$\frac{1}{2}\left(\|v_{mt}\|_{\mathcal{H}}^2+\|v_m\|_{H_0^1}^2\right)+\int_0^t \|v_{m\tau}\|_{L^{q+1}}^{q+1}\mathrm{d}\tau=\frac{1}{2}\left(\|v_{mt}(x,0)\|_{\mathcal{H}}^2+\|v_m(x,0)\|_{H_0^1}^2\right)+\int_0^t\int_\Omega f(u)v_{m\tau}\mathrm{d}x\mathrm{d}\tau. \tag{3.7}$$

We estimate the last term in the right-hand side of (3.7) thanks to the assumption (H1), the Hölder inequality. More precisely,

$$\int_0^t\int_\Omega f(u)v_{m\tau}\mathrm{d}x\mathrm{d}\tau \le C_\beta\int_0^t\int_\Omega e^{\beta u^2}|v_{m\tau}|\mathrm{d}x\mathrm{d}\tau \le C_\beta\int_0^t \|v_{m\tau}\|_{L^{q+1}}\left(\int_\Omega e^{\frac{q+1}{q}\beta u^2}\mathrm{d}x\right)^{\frac{q}{q+1}}\mathrm{d}\tau.$$

By means of the Young inequality, we achieve

$$\int_0^t\int_\Omega f(u)v_{m\tau}\mathrm{d}x\mathrm{d}\tau \le \varepsilon\int_0^t \|v_{m\tau}\|_{L^{q+1}}^{q+1}\mathrm{d}\tau + C_{\beta,\varepsilon}\int_0^t\int_\Omega e^{\frac{q+1}{q}\beta u^2}\mathrm{d}x\mathrm{d}\tau,$$

for a appropriate small constant $0<\varepsilon<1$. Since the assumption on u that $\max\limits_{t\in[0,T]}\|u\|_{H_0^1}\le R$, we find

$$\int_\Omega e^{\frac{q+1}{q}\beta u^2}\mathrm{d}x=\int_\Omega e^{\frac{q+1}{q}\beta\|u\|_{H_0^1}^2\left(\frac{|u|}{\|u\|_{H_0^1}}\right)^2}\mathrm{d}x \le \int_\Omega e^{\frac{q+1}{q}\beta R^2\left(\frac{|u|}{\|u\|_{H_0^1}}\right)^2}\mathrm{d}x.$$

With the Moser-Trudinger inequality, for each $\beta < \frac{4q\pi}{(q+1)R^2}$, we get

$$\int_\Omega e^{\frac{q+1}{q}\beta u^2}\mathrm{d}x \le C(\beta,\Omega).$$

Thus,

$$\int_0^t\int_\Omega f(u)v_{m\tau}\mathrm{d}x\mathrm{d}\tau \le \varepsilon\int_0^t \|v_{m\tau}\|_{L^{q+1}}^{q+1}\mathrm{d}\tau + C_T. \tag{3.8}$$

From the facts (3.5)–(3.8) we obtain

$$\|v_{mt}\|_{\mathcal{H}}^2 + \|v_m\|_{H_0^1}^2 + \int_0^t \|v_{m\tau}\|_{L^{q+1}}^{q+1} \mathrm{d}\tau \le C_T, \tag{3.9}$$

for every $m \ge 1$, where $C_T > 0$ is independent of m. So we can extend the approximated solutions v_m to the whole interval $[0, T]$ and the uniform estimate (3.9) also holds on $[0, T]$. Besides we get

$$\begin{aligned} &v_m \text{ is bounded in } L^\infty(0, T; H_0^1),\\ &v_{mt} \text{ is bounded in } L^\infty(0, T; L^2) \cap L^{q+1}((0, T) \times \Omega). \end{aligned}$$

By the definition of $h(v_{mt})$, we have

$$\int_0^t \int_\Omega |h(v_{m\tau})|^{\frac{q+1}{q}} \mathrm{d}x\mathrm{d}\tau = \int_0^t \int_\Omega |v_{m\tau}|^{q+1} \mathrm{d}\tau \le C_T.$$

That means

$$h(v_{mt}) \text{ is bounded in } L^{\frac{q+1}{q}}((0, T) \times \Omega).$$

Moreover, by the Eq (3.4) we know that

$$v_{mtt} \text{ is bounded in } L^{\frac{q+1}{q}}(0, T; H^{-1}).$$

Therefore, we can extract from the sequence $\{v_m\}$ a subsequence which we still denote by $\{v_m\}$ such that

$$\begin{aligned} &v_m \to v \quad \text{weakly-star in} \quad L^\infty(0, T; H_0^1), \\ &h(v_{mt}) \to \chi \quad \text{weakly in} \quad L^{\frac{q+1}{q}}((0, T) \times \Omega). \end{aligned} \tag{3.10}$$

Since $n = 2$ and Ω is bounded we deduce immediately that $u \in H_0^1 \hookrightarrow L^r$ for all $1 \le r < +\infty$. From Lemma 2.2, we have that

$$\begin{aligned} &v_{mt} \to v_t \quad \text{weakly-star in} \quad L^\infty(0, T; \mathcal{H}) \cap L^{q+1}((0, T) \times \Omega), \\ &v_{mtt} \to v_{tt} \quad \text{weakly-star in} \quad L^{\frac{q+1}{q}}(0, T; H^{-1}). \end{aligned} \tag{3.11}$$

Integrating (3.4) with respect to t from 0 to t and let $m \to \infty$ we obtain

$$\begin{aligned} &\left((-\Delta)^{-\frac{1}{2}} v_t, (-\Delta)^{-\frac{1}{2}} \eta\right) + (v_t, \eta) + \int_0^t \left[(v, \eta) + (\nabla v, \nabla \eta) + \langle \chi, \eta \rangle - (f(u), \eta)\right] \mathrm{d}\tau \\ =&\left((-\Delta)^{-\frac{1}{2}} u_1, (-\Delta)^{-\frac{1}{2}} \eta\right) + (u_1, \eta), \end{aligned}$$

for all test functions $\eta \in H_0^1$. So, we get a solution v to the following initial-boundary problem

$$\begin{cases} (-\Delta)^{-1} v_{tt} + v_{tt} - \Delta v + v + \chi - f(u) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ v(x, 0) = u_0(x), \quad v_t(x, 0) = u_1(x), & \text{in } \Omega, \\ v\,|_{\partial\Omega} = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases}$$

and

$$v \in L^\infty(0,T;H_0^1), \quad v_t \in L^\infty(0,T;L^2), \quad v_{tt} \in L^{\frac{q+1}{q}}(0,T;H^{-1}).$$

Consequently,

$$v \in H^1(0,T;L^2) \hookrightarrow C([0,T];L^2), v_t \in H^1(0,T;H^{-1}) \hookrightarrow C([0,T];H^{-1}).$$

By making use of Lemma 2.4, we deduce immediately that

$$v \in C_w([0,T];H_0^1), \quad v_t \in C_w([0,T];L^2). \tag{3.12}$$

Noting that

$$v_{tt} - \Delta v = (I-\Delta)^{-1}\Delta(f(u)+h(v_t)) \in L^\infty(0,T;L^2) + L^{1+\frac{1}{q}}((0,T)\times\Omega),$$

then form Corollary 4.1 of Starauss in [37], we conclude that $\|v_t\|^2 + \|\nabla v\|^2$ is a continuous function of t. Combining with (3.12), we have

$$v \in C([0,T];H_0^1), \quad v_t \in C([0,T];L^2).$$

So far, we shall have proved the existence of the solution of (3.2), if we show that $\chi = h(v_t)$. Observing the monotonicity and continuity of $h(v_t)$, we follow the idea of monotonicity argument used in [38].

On the one hand, noting that

$$(-\Delta)^{-1}v_{tt} + v_{tt} + v - \Delta v = -\chi + f(u) \in L^{\frac{q+1}{q}}((0,T)\times\Omega) + L^1(0,T;L^2),$$

and by considering ideas from [37, Theorem 4.1], v satisfies for all $t \in (0,T]$ the energy identity

$$\frac{1}{2}\left(\|v_t\|_{\mathcal{H}}^2 + \|v\|_{H_0^1}^2\right) + \int_0^t \langle \chi, v_\tau\rangle \mathrm{d}\tau = \frac{1}{2}\left(\|u_1\|_{\mathcal{H}}^2 + \|u_0\|_{H_0^1}^2\right) + \int_0^t\int_\Omega f(u)v_\tau \mathrm{d}x\mathrm{d}\tau. \tag{3.13}$$

On the other hand, it follows from (3.7), (3.10) and (3.11) that

$$\frac{1}{2}\left(\|v_t\|_{\mathcal{H}}^2 + \|v\|_{H_0^1}^2\right) + \liminf_{m\to\infty}\int_0^t \langle h(v_{m\tau}), v_{m\tau}\rangle \mathrm{d}\tau \le \frac{1}{2}\left(\|u_1\|_{\mathcal{H}}^2 + \|u_0\|_{H_0^1}^2\right) + \int_0^t\int_\Omega f(u)v_\tau \mathrm{d}x\mathrm{d}\tau. \tag{3.14}$$

Therefore, (3.13) and (3.14) yield

$$\liminf_{m\to\infty}\int_0^t \langle h(v_{m\tau}), v_{m\tau}\rangle \mathrm{d}\tau \le \int_0^t \langle \chi, v_\tau\rangle \mathrm{d}\tau. \tag{3.15}$$

Now, let $\varphi \in L^{1+\frac{1}{q}}(0,T\times\Omega)$ be arbitrary. Then it follows from Lemma 2.3 and (3.15) that

$$0 \le \liminf_{m\to\infty}\int_0^t \langle h(v_{m\tau}) - h(\varphi), v_{m\tau} - \varphi\rangle \mathrm{d}\tau \le \int_0^t \langle \chi - h(\varphi), v_\tau - \varphi\rangle \mathrm{d}\tau. \tag{3.16}$$

By choosing $\varphi(t) = v_t - \lambda\psi(t)$, where $\lambda \in \mathbb{R}$, (3.16) yields

$$\int_0^t \langle \chi - h\left(v_\tau - \lambda\psi(t)\right), \psi(t)\rangle \mathrm{d}\tau \ge 0,$$

for all $\lambda \geq 0$ and $\psi \in L^{q+1}(0, T \times \Omega)$. By letting $\lambda \to 0$ and recalling the continuity of h, we achieve

$$\int_0^t \langle \chi - h(v_\tau), \psi(t) \rangle \mathrm{d}\tau \geq 0, \quad \forall \psi \in L^{q+1}(0, T \times \Omega),$$

which implies $\chi = h(v_t)$.

The last work is the uniqueness of solutions, which follows arguing for contradiction: if v and w are two solutions of (3.2) which share the same initial data, by subtracting the equations and v is substituted by $v - w$, we would get

$$\frac{1}{2}\left(\|v_t - w_t\|_{\mathcal{H}}^2 + \|v - w\|_{H_0^1}^2 \right) + \int_0^t \int_\Omega (h(v_\tau) - h(w_\tau))(v_\tau - w_\tau) \mathrm{d}\tau = 0. \tag{3.17}$$

Observing the Mazur inequality

$$2^{-p} |x - y|^{p+1} \leq |x|x|^p - y|y|^p| \leq (p+1)|x - y| \left(|x|^p + |y|^p \right), \quad \forall p \geq 0, x, y \in \mathbb{R},$$

we get

$$\left(|s|^{q-1} s - |t|^{q-1} t \right)(s - t) \geq C|s - t|^{q+1}, \quad \text{for all } s, t \in \mathbb{R}. \tag{3.18}$$

So, (3.17) can be estimated as

$$\frac{1}{2}\left(\|v_t - w_t\|_{\mathcal{H}}^2 + \|v - w\|_{H_0^1}^2 \right) + C \int_0^t \|v_\tau - w_\tau\|_{L^{q+1}}^{q+1} \mathrm{d}\tau \leq 0,$$

which immediately yields $w \equiv v$. The proof of Lemma 3.1 is now completed.

Now we turn to the nonlinear problem (1.1) by using the contraction mapping principle.

Proof of Theorem 3.1 Taking $(u_0, u_1) \in H_0^1 \times L^2$, let $R^2 = 2(\|u_0\|_{H_0^1}^2 + \|u_1\|_{\mathcal{H}}^2)$ and for any $T > 0$ considering a closed convex set $\mathcal{M}_T$ to be a ball of radius R in the space $\mathcal{M} = C\left([0, T]; H_0^1\right) \cap C^1\left([0, T]; L^2\right)$,

$$\mathcal{M}_T = \{u \in \mathcal{M} : \|u\|_{\mathcal{M}_T} \leq R\}$$

with the norm $\|u\|_{\mathcal{M}_T}^2 = \max\limits_{t \in [0,T]} \left\{ \|u\|_{H_0^1}^2 + \|u_t\|_{\mathcal{H}}^2 \right\}$ and metric $d(u, v) = \|u - v\|_{\mathcal{M}_T}$. Obviously, $(\mathcal{M}_T, d)$ is a complete metric space. By Lemma 3.1, for any $u \in \mathcal{M}_T$ we may define $v = \Phi(u)$, being v the unique solution to problem (3.2). We claim that, for a suitable $T > 0$, Φ is a contractive map satisfying $\Phi(\mathcal{M}_T) \subseteq \mathcal{M}_T$.

It follows from the fact (3.13) that

$$\frac{1}{2}\left(\|v_t\|_{\mathcal{H}}^2 + \|v\|_{H_0^1}^2 \right) + \int_0^t \|v_\tau\|_{L^{q+1}}^{q+1} \mathrm{d}\tau = \frac{1}{2}\left(\|u_1\|_{\mathcal{H}}^2 + \|u_0\|_{H_0^1}^2 \right) + \int_0^t \int_\Omega f(u) v_\tau \mathrm{d}x \mathrm{d}\tau. \tag{3.19}$$

For the last term, we argue in the same spirit (although slightly differently) as for (3.8) and we get

$$\left| \int_0^t \int_\Omega f(u) v_\tau \mathrm{d}x \mathrm{d}\tau \right| \leq \varepsilon \int_0^t \|v_\tau\|_{L^{q+1}}^{q+1} \mathrm{d}\tau + C(\beta, \Omega) T. \tag{3.20}$$

Combining (3.19) with (3.20) and taking the maximum over $[0, T]$ gives

$$\|v\|^2_{\mathcal{M}_T} \le \frac{1}{2}R^2 + C(\beta, \Omega)T.$$

Choosing T sufficiently small, we get $\|v\|_{\mathcal{M}_T} \le R$, which shows that $\Phi(\mathcal{M}_T) \subseteq \mathcal{M}_T$. Now, taking w_1 and w_2 in $\mathcal{M}_T$, subtracting the two equations (3.2) for $v_1 = \Phi(u_1)$ and $v_2 = \Phi(u_2)$, and setting $z = v_1 - v_2$ then z is the unique solution of the problem

$$\begin{cases} (-\Delta)^{-\frac{1}{2}} z_{tt} + z_{tt} - \Delta z + z + h(v_{1t}) - h(v_{2t}) = f(u_1) - f(u_2), & \text{in } \Omega \times \mathbb{R}^+, \\ z(x, 0) = 0, \quad z_t(x, 0) = 0, & \text{in } \Omega, \\ z \mid_{\partial\Omega} = 0, & \text{on } \partial\Omega \times \mathbb{R}^+. \end{cases}$$

Taking the assumption (H1) into account, we have

$$|f(u_1) - f(u_2)| \le C_\beta \left(e^{\beta|u_1|^2} + e^{\beta|u_2|^2} \right) |u_1 - u_2|. \tag{3.21}$$

Using (3.18) again it holds

$$\frac{1}{2}\left(\|z_t\|^2_{\mathcal{H}} + \|z\|^2_{H_0^1} \right) + C \int_0^t \|z_\tau\|^{q+1}_{L^{q+1}} \mathrm{d}\tau \le \int_0^t \int_\Omega (f(u_1) - f(u_2)) z_\tau \mathrm{d}x\mathrm{d}\tau.$$

By (3.21), Lemma 2.1, the Hölder inequality and the Sobolev imbedding inequality, it deduces that

$$\begin{aligned} \int_0^t \int_\Omega (f(u_1) - f(u_2)) z_\tau \mathrm{d}x\mathrm{d}\tau \le & C_\beta \int_0^t \int_\Omega \left(e^{\beta|u_1|^2} + e^{\beta|u_2|^2} \right) |u_1 - u_2||z_\tau| \mathrm{d}x\mathrm{d}\tau \\ \le & C_\beta \int_0^t \|z_\tau\| \|u_1 - u_2\|_{L^{q+1}} \int_\Omega \left(e^{\frac{2(q+1)}{q-1}\beta|u_1|^2} + e^{\frac{2(q+1)}{q-1}\beta|u_2|^2} \right) \mathrm{d}x\mathrm{d}\tau \\ \le & C(\beta, \Omega) \int_0^t \|z_\tau\|^2 \|u_1 - u_2\|^2_{H_0^1} \mathrm{d}\tau, \end{aligned} \tag{3.22}$$

for each $\beta < \frac{2(q-1)\pi}{(q+1)R^2}$, where we have used the fact that $\frac{1}{2} + \frac{1}{q+1} + \frac{q-1}{2(q+1)} = 1$. Using the Young inequality, the estimate (3.22) becomes

$$\int_0^t \int_\Omega (f(u_1) - f(u_2)) z_\tau \mathrm{d}x\mathrm{d}\tau \le \frac{1}{2} C^2(\beta, \Omega) T \|u_1 - u_2\|^2_{\mathcal{M}_T} + \frac{1}{2} \int_0^t \|z_\tau\|^2 \mathrm{d}\tau.$$

Consequently,

$$\frac{1}{2}\left(\|z_t\|^2_{\mathcal{H}} + \|z\|^2_{H_0^1} \right) + C \int_0^t \|z_\tau\|^{q+1}_{L^{q+1}} \mathrm{d}\tau \le \frac{1}{2} C^2(\beta, \Omega) T \|u_1 - u_2\|^2_{\mathcal{M}_T} + \frac{1}{2} \int_0^t \|z_\tau\|^2 \mathrm{d}\tau.$$

Applying Gronwall inequality and taking the maximum over $[0, T]$, it follows that

$$\|z\|_{\mathcal{M}_T} = \|\Phi(u_1) - \Phi(u_2)\|_{\mathcal{M}_T} \le \delta \|u_1 - u_2\|_{\mathcal{M}_T},$$

for some $\delta < 1$ provided T is sufficiently small. By the contraction mapping principle, there exists a unique solution u to (1.1) defined on $[0, T]$, $u \in C\left([0, T]; H_0^1\right) \cap C^1\left([0, T]; L^2\right)$ and from (3.19) the energy equality (3.1) holds. Theorem 3.1 is proved.

4. Global solutions

The goal of this section is to prove that the local solution established in Theorem 3.1 can be extended globally in time when the initial data inside the potential well. So that we firstly introduce Nehari functional, Nehari manifold and the stable sets. Let

$$J(u) = \frac{1}{2}\|u\|^2_{H_0^1} - \int_\Omega F(u)\mathrm{d}x,$$

and

$$I(u) = \|u\|^2_{H_0^1} - \int_\Omega uf(u)\mathrm{d}x.$$

Related to the functional J, we have the well known Nehari manifold

$$\mathcal{N} = \left\{u \in H_0^1 \setminus \{0\}; I(u) = 0\right\}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function verifying both the assumption (H1) and the following properties:

(H2) The function f satisfies the following condition near the origin

$$\lim_{t\to 0} \frac{f(t)}{t} = 0.$$

(H3) There exists $\theta > 2$ such that

$$0 < \theta F(t) < f(t)t \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{4.1}$$

(H4) There are constants $R_0, M_0 > 0$ such that, for all $|t| \geq R_0$,

$$0 < F(t) \leq M_0|f(t)|.$$

Then, it can be checked that the mountain-pass level d can be characterized as

$$d = \inf\left\{\sup_{\lambda\geq 0} J(\lambda u) : u \in H_0^1, u \neq 0\right\} = \inf_{u\in\mathcal{N}} J(u) > 0.$$

The proof is refered to [17, 27].

Hereafter, we will denote by W and V the following sets:

$$W = \left\{u \in H_0^1; I(u) > 0, J(u) < d\right\} \cup \{0\},$$
$$V = \left\{u \in H_0^1; I(u) < 0, J(u) < d\right\}.$$

Under the assumptions imposed on f, it is can be proved that W and V are invariant sets related to (1.1) (Lemmas 4.1 and 5.1), which provide the possibilities for us to establish the global existence and nonexistence of the solutions.

Remark 4.1. *An interesting question is whether there is a function that meets the conditions (H1)–(H4). Now, we construct explicitly an example such that all assumptions (H1)–(H4) are satisfied.*

Example 1. *Let $f(t) = |t|^{p-1}te^{\frac{\beta}{2}t^2}$ with $p > 1$. Then $f(t)$ satisfies (H1)–(H4).*

Proof. It follows from the fundamental inequality

$$t^b \le \left(\frac{b}{ea}\right)^b e^{at}, \quad \text{for } a > 0, b > 0, t \in (0, +\infty)$$

that

$$|f(t)| = |t|^p e^{\frac{\beta}{2}t^2} \le \left(\frac{2p}{e\beta}\right)^p e^{\frac{\beta}{2}t} e^{\frac{\beta}{2}t^2} \le Ce^{\beta t^2}.$$

Which shows $f(t)$ satisfies (H1). Obviously, $f(t)$ satisfies (H2).

For (H3), by the definition of $F(t)$, we get

$$\begin{aligned} F(t) &= \int_0^t f(\tau)\mathrm{d}\tau = \int_0^t |\tau|^{p-1}\tau e^{\frac{\beta}{2}|\tau|^2}\mathrm{d}\tau = \int_0^t |\tau|^{p-1}\tau \sum_{n=0}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |\tau|^{2n}}{n!}\mathrm{d}\tau \\ &= \int_0^t \sum_{n=0}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |\tau|^{2n+p-1}\tau}{n!}\mathrm{d}\tau = \sum_{n=0}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |t|^{2n+p+1}}{n!(2n+p+1)}. \end{aligned} \tag{4.2}$$

For $p > 1$, we have

$$(p+1)F(t) < |t|^{p+1} \sum_{n=0}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |t|^{2n}}{n!} = |t|^{p+1} e^{\frac{\beta}{2}|t|^2} = tf(t). \tag{4.3}$$

Thus, $f(t)$ satisfies (H3).

At last, we show f verifies (H4). It follows from (4.2) that

$$\begin{aligned} F(t) =&|t|^p \sum_{n=0}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |t|^{2n+1}}{n!(2n+p+1)} = |t|^p \sum_{n=1}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^{n-1} |t|^{2n-1}}{(n-1)!(2(n-1)+p+1)} \\ =&|t|^p \sum_{n=1}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |t|^{2n}}{n!} \cdot \frac{2n}{\beta(2(n-1)+p+1)|t|}. \end{aligned}$$

Obviously, for any constant $R_0 > 0$, $|t|^{-1} < \frac{1}{R_0}$ for $|t| > R_0$, and $\frac{2n}{\beta(2(n-1)+p+1)} < \frac{1}{\beta}$. Thus,

$$F(t) \le M_0|t|^p \sum_{n=1}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |t|^{2n}}{n!} \le M_0|t|^p \sum_{n=0}^{+\infty} \frac{\left(\frac{\beta}{2}\right)^n |t|^{2n}}{n!} = M_0|f(t)|,$$

where $M_0 = \frac{1}{\beta R_0}$. f satisfies (H4).

The following result is about the invariance of the set W under the flow of (1.1).

Lemma 4.1 ([17,27])**.** *Let f satisfy (H1)–(H4) and u be the unique local solution to* (1.1)*. Assume that $E(0) < d$, then for all $t \in [0, T]$, u belongs to W provide that u_0 belongs to W, and*

$$\|u\|^2_{H^1_0} < \frac{2\theta d}{\theta - 2}.$$

Now we concerned with the global existence of the solution for the problem (1.1).

Theorem 4.1 (Global solutions for $E(0) < d$)**.** *Let $f(u)$ satisfy (H1)–(H4), and u be the unique local solution to* (1.1). *Assume that $E(0) < d$, and $I(u_0) > 0$ or $\|u_0\|_{H_0^1} = 0$. Then, u exists globally and $u \in W$, for all $t \in [0, \infty)$.*

Proof. From (4.1) we have

$$\begin{aligned}E(t) =&\frac{1}{2}\|u_t\|_{\mathcal{H}}^2 + \frac{1}{2}\|u\|_{H_0^1}^2 - \int_\Omega F(u)\mathrm{d}x\\ \geq&\frac{1}{2}\|u_t\|_{\mathcal{H}}^2 + \frac{\theta-2}{2\theta}\|u\|_{H_0^1}^2 + \frac{1}{\theta}\left(\|u\|_{H_0^1}^2 - \int_\Omega uf(u)\mathrm{d}x\right)\\ =&\frac{1}{2}\|u_t\|_{\mathcal{H}}^2 + \frac{\theta-2}{2\theta}\|u\|_{H_0^1}^2 + \frac{1}{\theta}I(u).\end{aligned}$$

Since $E(0) < d$ and $I(u_0) > 0$ or $u_0 = 0$, it follows from Lemma 4.1 that $I(u) > 0$ or $\|u\|_{H_0^1} = 0$ for all $t \in [0, T_{\max})$, which implies that

$$\frac{1}{2}\|u_t\|_{\mathcal{H}}^2 + \frac{\theta-2}{2\theta}\|u\|_{H_0^1}^2 + \int_0^t \|u_\tau\|_{L^{q+1}}^{q+1}\mathrm{d}\tau < E(0) < d,$$

for all $t \in [0, T_{\max})$ and $\theta > 2$. Therefore, by virtue of Theorem 3.1 it yields $T_{\max} = \infty$.

5. Blow-up of solutions

This section is concerned with results on finite time blow-up of the solutions for the problem (1.1) with $f(u) = |u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$, $p > 1$. We firstly prove that when the initial energy is negative, the local solution can not be extended globally in time. Secondly, in the case of $E(0) < d$, we prove that when the initial data $u_0 \in V$, the local solution blows up in finite time. Lastly, we construct the sufficient conditions of finite time blow-up of the solutions with arbitrary positive initial energy.

Now, we state the main results on the finite time blow-up of the solutions.

Theorem 5.1 (Blow-up for $E(0) \leq 0$)**.** *Let $f(u) = |u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$ for $p > 1$, and u be the unique local solution to* (1.1). *Assume that $E(0) < 0$ or $E(0) = 0$ and $(u_0, u_1)_{\mathcal{H}} > 0$. Then, the problem* (1.1) *does not admit any global weak solution.*

From Theorem 5.1 we can easily obtain the following result.

Corollary 5.1. *Let $f(u) = |u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$ with $p > 1$ and u be the unique local solution to* (1.1). *If there exists $t_0 \in [0, T]$ such that $E(t_0) < 0$, then the problem* (1.1) *does not admit any global weak solution.*

The result about finite time blow-up of the solutions with the subcritical initial energy is based on the following three lemmas.

Lemma 5.1 ([17,27])**.** *Let $f(u) = |u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$ with $p > 1$ and u be the unique local solution to* (1.1). *Assume that $E(0) < d$, then for all $t \in [0, T]$, u belongs to V provide that u_0 belongs to V, and*

$$\|u\|_{H_0^1}^2 > 2d. \tag{5.1}$$

Lemma 5.2. *Let* $f(u) = |u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$ *with* $p > 1$ *and*

$$\varphi(\lambda) = \frac{1}{2}\int_\Omega \lambda u f(\lambda u)\mathrm{d}x - \int_\Omega F(\lambda u)\mathrm{d}x.$$

Then $\varphi(\lambda)$ *is strictly increasing on* $0 < \lambda < \infty$.

Proof. By a simple calculation, we check

$$u\left(uf'(u) - f(u)\right) \geq 0, \tag{5.2}$$

and the equality holds only for $u = 0$. With (5.2) we get

$$\begin{aligned}\frac{\mathrm{d}}{d\lambda}\varphi(\lambda) =& \frac{1}{2}\int_\Omega uf(\lambda u)\mathrm{d}x + \frac{1}{2}\int_\Omega \lambda u^2 f'(\lambda u)\mathrm{d}x - \int_\Omega uf(\lambda u)\mathrm{d}x\\ =&\frac{1}{2}\left(\int_\Omega \lambda u^2 f'(\lambda u)\mathrm{d}x - \int_\Omega uf(\lambda u)\mathrm{d}x\right)\\ =&\frac{1}{2\lambda}\int_\Omega \lambda u\left(\lambda u f'(\lambda u) - f(\lambda u)\right)\mathrm{d}x \geq 0,\end{aligned}$$

the equality holds only for $u = 0$. Thus, the proof of Lemma 5.2 is completed.

Lemma 5.3. *Let* $f(u) = |u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$ *with* $p > 1$ *and* u *be the unique local solution to* (1.1). *Assume that* u *belongs to* V *then*

$$I(u) < -2(d - E(0))$$

Proof. We define the function $W(\lambda) = I(\lambda u)$ for $\lambda > 0$. Observe that $W(1) = I(u) < 0$ and

$$W(\lambda) = \lambda^2\|u\|^2_{H^1_0} - \int_\Omega \lambda u f(\lambda u)\mathrm{d}x > 0$$

for λ sufficiently small. Hence there exists some $\lambda_0 \in (0, 1)$ such that $W(\lambda_0) = I(\lambda_0 u) = 0$. That is $\lambda_0^2\|u\|^2_{H^1_0} = \int_\Omega \lambda_0 u f(\lambda_0 u)\mathrm{d}x$. By the definition of d, we obtain

$$\begin{aligned}d = \inf_{u\in\mathcal{N}} J(u) &\leq J(\lambda_0 u)\\ =&\frac{1}{2}\lambda_0^2\|u\|^2_{H^1_0} - \int_\Omega F(\lambda_0 u)\mathrm{d}x\\ =&\frac{1}{2}\int_\Omega \lambda_0 u f(\lambda_0 u)\mathrm{d}x - \int_\Omega F(\lambda_0 u)\mathrm{d}x.\end{aligned}$$

It follows from Lemma 5.2 that

$$\begin{aligned}&\frac{1}{2}\int_\Omega \lambda_0 u f(\lambda_0 u)\mathrm{d}x - \int_\Omega F(\lambda_0 u)\mathrm{d}x\\ <&\frac{1}{2}\int_\Omega uf(u)\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x\\ =&\frac{1}{2}\|u\|^2_{H^1_0} - \int_\Omega F(u)\mathrm{d}x - \frac{1}{2}\left(\|u\|^2_{H^1_0} - \int_\Omega uf(u)\mathrm{d}x\right)\end{aligned}$$

$$=J(u)-\frac{1}{2}I(u)\le E(t)-\frac{1}{2}I(u)\le E(0)-\frac{1}{2}I(u).$$

That is,

$$I(u)<2(E(0)-d).$$

This completes the proof of Lemma 5.3.

Theorem 5.2 (Blow-up for $E(0)<d$)**.** *Let $f(u)=|u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$ with $p>1$ and u be the unique local solution to* (1.1)*. Assume that $E(0)<\gamma d$ for some constant $0<\gamma\le\frac{1}{2}$, and $u_0\in V$. Then, the problem* (1.1) *does not admit any global weak solution.*

Theorem 5.3 (Blow-up for $E(0)>0$)**.** *Let $f(u)=|u|^{p-1}ue^{\frac{\beta}{2}|u|^2}$ with $p>1$ and u be the unique local solution to* (1.1)*. Assume that $E(0)>0$, and*

$$(u_0,u_1)_{\mathcal{H}}>\frac{q}{q+1}M^{\frac{1}{q}}E(0)>0, \tag{5.3}$$

then the problem (1.1) *does not admit any global weak solution, where M is the root of the equation*

$$\frac{K(M)}{\omega(M)}=\frac{qM^{\frac{1}{q}}}{q+1}$$

on $(M_0,+\infty)$*,*

$$M_0=\begin{cases}\frac{C_{\beta,\Omega}(p+1)}{(q+1)(p-1)}, & \text{if } p\le q,\\ \frac{(q-1)\lambda_1+p-q}{(p-1)^2\lambda_1}, & \text{if } p>q,\end{cases}\qquad K(M)=\begin{cases}(p+1)\left(1-\frac{C_{\beta,\Omega}}{(q+1)M}\right), & \text{if } p\le q,\\ p+1-\frac{q-1}{M(p-1)}, & \text{if } p>q,\end{cases}$$

$$\omega(M)=\begin{cases}\sqrt{\lambda_1(K(M)+2)(K(M)-2)}, & \text{if } p\le q,\\ \sqrt{(K(M)+2)\left[(K(M)-2)\lambda_1-\frac{p-q}{(p-1)M}\right]}, & \text{if } p>q,\end{cases}$$

$C_{\beta,\Omega}>0$ is a constant depending on β and Ω , and λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary data on the smooth bounded domain $\Omega\subset\mathbb{R}^2$.

Now, we concerned with our proofs of Theorems 5.1–5.3.

Proof of Theorem 5.1 Arguing by contradiction, we suppose that $T_{\max}=+\infty$. For any $0<T<\infty$, we define

$$G(t)=\|u\|_{\mathcal{H}}^2=\left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2+\|u\|^2,\quad \forall t\in[0,T]. \tag{5.4}$$

Then,

$$G'(t)=2\left((-\Delta)^{-\frac{1}{2}}u,(-\Delta)^{-\frac{1}{2}}u_t\right)+2(u,u_t).$$

Note that the standard approximation argument shows that $G''(t)$ exists and by the Eq (2.3), we have

$$\begin{aligned}G''(t)=&-2\|u\|_{H_0^1}^2+2\|u_t\|_{\mathcal{H}}^2+2\int_\Omega uf(u)\mathrm{d}x-2\int_\Omega uh(u_t)\mathrm{d}x\\ =&4\|u_t\|_{\mathcal{H}}^2-4E(t)+2\int_\Omega(uf(u)-2F(u))\,\mathrm{d}x-2\int_\Omega|u_t|^{q-1}u_tu\mathrm{d}x.\end{aligned}$$

Let

$$H(t) = -E(t).$$

It follows from the energy equality (3.1) that

$$H'(t) = \|u_t\|_{L^{q+1}}^{q+1} \geq 0.$$

So that $H(t)$ is increasing on $[0, T]$ and noting the fact (4.3), we have

$$0 \leq -E(0) = H(0) \leq H(t) \leq \int_\Omega F(u)\mathrm{d}x < \frac{1}{p+1}\int_\Omega uf(u)\mathrm{d}x. \tag{5.5}$$

Now considering the following function defined on $[0, T]$

$$L(t) = H^{1-\alpha}(t) + \epsilon G'(t),$$

where $0 < \alpha < 1$ and $\epsilon > 0$ are sufficiently small and given later. A simple computation entails

$$\begin{aligned} L'(t) =&(1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon G''(t) \\ =&(1-\alpha)H^{-\alpha}(t)H'(t) + 4\epsilon\|u_t\|_{\mathcal{H}}^2 - 4\epsilon E(t) + 2\epsilon\int_\Omega (uf(u) - 2F(u))\,\mathrm{d}x - 2\epsilon\int_\Omega |u_t|^{q-1}u_t u\mathrm{d}x. \end{aligned} \tag{5.6}$$

By the Young inequality we have

$$\left|\int_\Omega |u_t|^{q-1}u_t u\mathrm{d}x\right| \leq \frac{\delta^{q+1}}{q+1}\|u\|_{L^{q+1}}^{q+1} + \frac{q}{q+1}\delta^{-\frac{q+1}{q}}\|u_t\|_{L^{q+1}}^{q+1}, \tag{5.7}$$

for some $\delta > 0$ to be fixed later. Substituting (5.7) into (5.6), and noting $H(t) = -E(t)$ we obtain

$$\begin{aligned} L'(t) \geq& \left[(1-\alpha)H^{-\alpha}(t) - \frac{2\epsilon q}{q+1}\delta^{-\frac{q+1}{q}}\right]H'(t) + 4\epsilon\|u_t\|_{\mathcal{H}}^2 + 4\epsilon H(t) \\ &+ 2\epsilon\int_\Omega\left(f(u)u - 2F(u) - \frac{\delta^{q+1}}{q+1}\|u\|_{L^{q+1}}^{q+1}\right)\mathrm{d}x. \end{aligned} \tag{5.8}$$

Taking δ such that $\delta^{-\frac{q+1}{q}} = kH^{-\alpha}(t)$, the constant $k > 0$ will be chosen later, then from (5.5), we have

$$\delta^{q+1} = k^{-q}H^{\alpha q}(t) \leq k^{-q}\left(\int_\Omega F(u)\mathrm{d}x\right)^{\alpha q} \leq (p+1)^{-\alpha q}k^{-q}\left(\int_\Omega uf(u)\mathrm{d}x\right)^{\alpha q}.$$

For any $m > q$, by means of the Hölder inequality, we obtain

$$\|u\|_{L^{q+1}}^{q+1} \leq C_\Omega\|u\|_{L^{m+1}}^{q+1} = C_\Omega\left(\int_\Omega |u|^{m+1}\mathrm{d}x\right)^{\frac{q+1}{m+1}}.$$

Taking $m > p$, it holds

$$\int_\Omega |u|^{m+1}\mathrm{d}x = \int_\Omega |u|^{p+1}|u|^{m-p}\mathrm{d}x \leq C(\beta)\int_\Omega |u|^{p+1}e^{\frac{\beta}{2}u^2}\mathrm{d}x = C(\beta)\int_\Omega uf(u)\mathrm{d}x. \tag{5.9}$$

Then, taking $m > \max\{p, q\}$, we arrive at

$$\|u\|_{L^{q+1}}^{q+1} \le C_{\Omega,\beta}\left(\int_\Omega uf(u)\mathrm{d}x\right)^{\frac{q+1}{m+1}}.$$

Thus,

$$\delta^{q+1}\|u\|_{L^{q+1}}^{q+1} \le C_{\Omega,\beta}(p+1)^{-\alpha q}k^{-q}\left(\int_\Omega uf(u)\mathrm{d}x\right)^{\alpha q+\frac{q+1}{m+1}}.$$

Taking $0 < \alpha \le \frac{m-q}{q(m+1)}$ such that $\alpha q + \frac{q+1}{m+1} \le 1$ and $s = \alpha q + \frac{q+1}{m+1} - 1 \le 0$. By (5.5) again,

$$\delta^{q+1}\|u\|_{L^{q+1}}^{q+1} \le C_{\Omega,\beta}(p+1)^{-\alpha q+s}k^{-q}H(0)^s\int_\Omega uf(u)\mathrm{d}x.$$

Therefore, (5.8) can be estimated by

$$\begin{aligned}L'(t) \ge&\left[(1-\alpha) - \frac{2\epsilon kq}{q+1}\right]H^{-\alpha}(t)H'(t) + 4\epsilon\|u_t\|_{\mathcal{H}}^2 + 4\epsilon H(t)\\ &+ 2\epsilon\left(1 - \frac{2}{p+1} - C_{\Omega,\beta}(p+1)^{\frac{q+1}{m+1}-1}k^{-q}H(0)^s\right)\int_\Omega uf(u)\mathrm{d}x.\end{aligned}$$

Taking $k > 0$ sufficiently large such that $1 - \frac{2}{p+1} - C_{\Omega,\beta}(p+1)^{\frac{q+1}{m+1}-1}k^{-q}H(0)^s > 0$, and for the fixed k picking $\epsilon > 0$ appropriate small such that $(1-\alpha) - \frac{2\epsilon kq}{q+1} > 0$ and

$$L(0) = H^{1-\alpha}(0) + 2\epsilon(u_0, u_1)_{\mathcal{H}} > 0, \tag{5.10}$$

for $H(0) > 0$. Thus, we have

$$L'(t) \ge\gamma\epsilon\left(H(t) + \|u_t\|_{\mathcal{H}}^2 + \int_\Omega uf(u)\mathrm{d}x\right). \tag{5.11}$$

On the other hand, by the Hölder inequality and the Young inequality, we have

$$\begin{aligned}G'(t)^{\frac{1}{1-\alpha}} \le&2^{\frac{1}{1-\alpha}}|(u,u_t)_{\mathcal{H}}|^{\frac{1}{1-\alpha}}\\ \le&C\left\|(-\Delta)^{-\frac{1}{2}}u\right\|^{\frac{1}{1-\alpha}}\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^{\frac{1}{1-\alpha}} + C\|u\|^{\frac{1}{1-\alpha}}\|u_t\|^{\frac{1}{1-\alpha}}\\ \le&C\left(\|u\|_{\mathcal{H}}^{\frac{2}{1-2\alpha}} + \|u_t\|_{\mathcal{H}}^2\right) \le C\left(\|u\|^{\frac{2}{1-2\alpha}} + \|u_t\|_{\mathcal{H}}^2\right),\end{aligned}$$

for $0 < \alpha < \frac{1}{2}$.

Case 1. When $\|u\| \ge 1$, taking $\alpha \le \frac{p-1}{2(p+1)}$ such that $\frac{2}{1-2\alpha} \le p+1$, then by the Hölder inequality, we achieve

$$\|u\|^{\frac{2}{1-2\alpha}} = \|u\|^{-(p+1)+\frac{2}{1-2\alpha}}\|u\|^{p+1} \le C\|u\|_{L^{p+1}}^{p+1} \le C\int_\Omega uf(u)\mathrm{d}x$$

Case 2. When $\|u\| \le 1$, it follows from the energy equality (3.1), the assumption $E(0) < 0$ and the last inequality of (5.5) that

$$\frac{1}{2}\|u\|^2 \le \int_\Omega F(u)\mathrm{d}x \le \frac{1}{p+1}\int_\Omega uf(u)\mathrm{d}x.$$

Noting that $\frac{2}{1-2\alpha} > 2$, then

$$\|u\|^{\frac{2}{1-2\alpha}} \le \|u\|^2 \le \frac{2}{p+1}\int_\Omega uf(u)\mathrm{d}x.$$

Therefore, we obtain

$$G'(t)^{\frac{1}{1-\alpha}} \le C\left(\int_\Omega uf(u)\mathrm{d}x + \|u_t\|_{\mathcal{H}}^2\right).$$

Consequently, by taking $0 < \alpha \le \min\left\{\frac{m-q}{q(m+1)}, \frac{p-1}{2(p+1)}\right\}$ and from the above argument we have

$$L^{\frac{1}{1-\alpha}}(t) = \left(H^{1-\alpha}(t) + 2\epsilon G'(t)\right)^{\frac{1}{1-\alpha}} \le C\left(H(t) + \|u_t\|_{\mathcal{H}}^2 + \int_\Omega uf(u)\mathrm{d}x\right).$$

Combining with (5.11), it holds that

$$L'(t) \ge CL(t)^{\frac{1}{1-\alpha}}, \quad \forall t \in [0,T].$$

The fact (5.10) and the assumption $(u_0, u_1)_{\mathcal{H}} > 0$ if $E(0) = 0$ entail $L(0) > 0$ for $E(0) \le 0$. Then by a calculation we have

$$L(t) \ge \left[L(0)^{-\frac{\alpha}{1-\alpha}} - \frac{\alpha C}{1-\alpha}t\right]^{-\frac{1-\alpha}{\alpha}}, \quad \forall t \in [0,T],$$

which shows that $L(t)$ blows up in finite time

$$T_{\max} \le \frac{1-\alpha}{\alpha C}L(0)^{-\frac{\alpha}{1-\alpha}}.$$

This completes the proof of Theorem 5.1.

The proof of Theorem 5.2 Arguing by contradiction, we suppose that $T_{\max} = +\infty$. Taking $d_1 \in (E(0), \gamma d)$ for $0 < \gamma \le \frac{1}{2}$ and setting

$$\tilde{H}(t) = d_1 - E(t).$$

It follows from the energy equality (3.1) that

$$\tilde{H}'(t) = \|u_t\|_{L^{q+1}}^{q+1} \ge 0.$$

So that $\tilde{H}(t)$ is increasing on $[0,T]$. Noting that $u(t) \in V$ we conclude from (5.1) that

$$\frac{1}{2}\|u\|_{H_0^1}^2 - d_1 > d - d_1 > 0,$$

which deduce that

$$0 < \tilde{H}(0) \le \tilde{H}(t) \le \int_\Omega F(u)\mathrm{d}x < \frac{1}{p+1}\int_\Omega uf(u)\mathrm{d}x, \quad \forall t \in [0,T].$$

Considering the following function defined on $[0,T]$

$$\tilde{L}(t) = \tilde{H}^{1-\alpha}(t) + \epsilon G'(t),$$

where $G(t)$ is defined by (5.4), $0<\alpha<1$ and $\epsilon>0$ are sufficiently small and given later. By the direct calculation as (5.6), we deduce that

$$\begin{aligned}\tilde{L}'(t)=&(1-\alpha)\tilde{H}^{-\alpha}(t)\tilde{H}'(t)+4\epsilon\|u_t\|_{\mathcal{H}}^2-4\epsilon E(t)+2\epsilon\int_\Omega(uf(u)-2F(u))\,\mathrm{d}x-2\epsilon\int_\Omega|u_t|^{q-1}u_tu\mathrm{d}x\\=&(1-\alpha)\tilde{H}^{-\alpha}(t)\tilde{H}'(t)+3\epsilon\|u_t\|_{\mathcal{H}}^2-2\epsilon d_1-\epsilon\|u\|_{H_0^1}^2\\&+2\epsilon\tilde{H}(t)+2\epsilon\int_\Omega(f(u)u-F(u))\,\mathrm{d}x-2\epsilon\int_\Omega|u_t|^{q-1}u_tu\mathrm{d}x\\=&(1-\alpha)\tilde{H}^{-\alpha}(t)\tilde{H}'(t)+3\epsilon\|u_t\|_{\mathcal{H}}^2-2\epsilon d_1-\epsilon I(u)\\&+2\epsilon\tilde{H}(t)+\epsilon\int_\Omega(f(u)u-2F(u))\,\mathrm{d}x-2\epsilon\int_\Omega|u_t|^{q-1}u_tu\mathrm{d}x.\end{aligned}\tag{5.12}$$

It follows from Lemma 5.3 that

$$-2\epsilon d_1-\epsilon I(u)>-2\epsilon d_1+2\epsilon(d-E(0))>2\epsilon(1-2\gamma)d\geq 0$$

for $0<\gamma\leq\frac{1}{2}$. Substituting the above inequality into (5.12), it holds

$$\begin{aligned}\tilde{L}'(t)\geq&(1-\alpha)\tilde{H}^{-\alpha}(t)\tilde{H}'(t)+3\epsilon\|u_t\|_{\mathcal{H}}^2+2\epsilon\tilde{H}(t)\\&+\epsilon\int_\Omega(f(u)u-2F(u))\,\mathrm{d}x-2\epsilon\int_\Omega|u_t|^{q-1}u_tu\mathrm{d}x.\end{aligned}$$

The remainder of the argument is analogous to that in (5.6) and so omitted. The proof Theorem 5.2 is completed.

The proof of Theorem 5.3 Arguing by contradiction, we suppose that $T_{\max}=+\infty$. For $t\in[0,+\infty)$, considering the function

$$L_1(t)=G'(t)-\frac{2q}{q+1}M^{\frac{1}{q}}E(t),$$

where $G(t)$ is given by (5.4). Similar to the proof of Theorem 5.1, by the estimate (5.7) with $\delta^{-\frac{q+1}{q}}=M^{\frac{1}{q}}$, we have

$$\begin{aligned}L_1'(t)=&-2\|u\|_{H_0^1}^2+2\|u_t\|_{\mathcal{H}}^2+2\int_\Omega uf(u)\mathrm{d}x-2\int_\Omega|u_t|^{q-1}u_tu\mathrm{d}x+\frac{2q}{q+1}M^{\frac{1}{q}}\|u_t\|_{L^{q+1}}^{q+1}\\\geq&-2\|u\|_{H_0^1}^2+2\|u_t\|_{\mathcal{H}}^2+2\int_\Omega uf(u)\mathrm{d}x-\frac{2}{(q+1)M}\int_\Omega|u|^{q+1}\mathrm{d}x.\end{aligned}$$

Case 1. For $q\geq p$, by means of the fact (5.9), and recalling the assumption (4.3), we have

$$\begin{aligned}L_1'(t)\geq&-2\|u\|_{H_0^1}^2+2\|u_t\|_{\mathcal{H}}^2+2\left(1-\frac{C_{\beta,\Omega}}{(q+1)M}\right)\int_\Omega uf(u)\mathrm{d}x\\\geq&-2\|u\|_{H_0^1}^2+2\|u_t\|_{\mathcal{H}}^2+2(p+1)\left(1-\frac{C_{\beta,\Omega}}{(q+1)M}\right)\int_\Omega F(u)\mathrm{d}x.\end{aligned}$$

Taking $M>M_0=\frac{(p+1)C_{\beta,\Omega}}{(q+1)(p-1)}$ such that $(p+1)\left(1-\frac{C_{\beta,\Omega}}{(q+1)M}\right)-2>0$, and by using the Poincaré inequality, we deduce

$$L_1'(t)\geq\lambda_1\left[(p+1)\left(1-\frac{C_{\beta,\Omega}}{(q+1)M}\right)-2\right]\|u\|_{\mathcal{H}}^2+\left[(p+1)\left(1-\frac{C_{\beta,\Omega}}{(q+1)M}\right)+2\right]\|u_t\|_{\mathcal{H}}^2$$

$$-2(p+1)\left(1-\frac{C_{\beta,\Omega}}{(q+1)M}\right)E(t)$$
$$=\lambda_1\left(K(M)-2\right)\|u\|_{\mathcal{H}}^2+\left(K(M)+2\right)\|u_t\|_{\mathcal{H}}^2-2K(M)E(t).$$

By using the Cauchy inequality,

$$\begin{aligned}\lambda_1\left(K(M)-2\right)\|u\|_{\mathcal{H}}^2+\left(K(M)+2\right)\|u_t\|_{\mathcal{H}}^2 &\geq 2\sqrt{\lambda_1\left(K(M)-2\right)\left(K(M)+2\right)}(u,u_t)_{\mathcal{H}}\\ &=2\omega(M)(u,u_t)_{\mathcal{H}},\end{aligned}$$

we get

$$L_1'(t)\geq\omega(M)\left(2(u,u_t)_{\mathcal{H}}-\frac{2K(M)}{\omega(M)}E(t)\right) \tag{5.13}$$

By a simple calculation, we have

$$\lim_{M\to M_0}\frac{K(M)}{\omega(M)}=+\infty,\qquad \lim_{M\to M_0}\frac{q}{q+1}M^{\frac{1}{q}}=\frac{q}{q+1}M_0^{\frac{1}{q}}$$
$$\lim_{M\to+\infty}\frac{K(M)}{\omega(M)}=\frac{p+1}{\sqrt{\lambda_1(p-1)(p+3)}},\qquad \lim_{M\to+\infty}\frac{q}{q+1}M^{\frac{1}{q}}=+\infty.$$

Obviously, there exists $M>M_0$ such that

$$\frac{K(M)}{\omega(M)}=\frac{q}{q+1}M^{\frac{1}{q}},$$

and the estimate (5.13) becomes

$$L_1'(t)\geq\omega(M))L_1(t).$$

The condition (5.3) guarantees $L_1(0)>0$. Thus, it yields

$$L_1(t)\geq L_1(0)e^{\omega(M)t},\quad \forall t\geq 0.$$

By the assumption that u is the global solution, we have, from Corollary 5.1, we have $0\leq E(t)\leq E(0)$. Thus,

$$G'(t)\geq L_1(0)e^{\omega(M)t},\quad \forall t\geq 0.$$

Therefore,

$$G(t)=\|u\|_{\mathcal{H}}^2\geq\|u_0\|_{\mathcal{H}}^2+\frac{1}{\alpha(M)}L_1(0)\left(e^{\omega(M)t}-1\right),\quad \forall t\geq 0. \tag{5.14}$$

Case 2. For $q<p$, observe that the function

$$g(y)=\frac{a^y}{y},a\geq 0,a\neq 1,y>0$$

is convex. By the properties of convex functions, we have

$$\frac{1}{q+1}\int_\Omega|u|^{q+1}\mathrm{d}x\leq\frac{p-q}{2(p-1)}\int_\Omega|u|^2\mathrm{d}x+\frac{q-1}{(p+1)(p-1)}\int_\Omega|u|^{p+1}\mathrm{d}x$$

$$\leq \frac{p-q}{2(p-1)}\|u\|^2 + \frac{q-1}{(p+1)(p-1)}\int_\Omega uf(u)\mathrm{d}x.$$

Thus, by the the Poincaré inequality we have

$$\begin{aligned} L_1'(t) \geq & -2\|u\|_{H_0^1}^2 - \frac{p-q}{M(p-1)}\|u\|^2 + 2\|u_t\|_{\mathcal{H}}^2 + 2\left(1 - \frac{q-1}{M(p+1)(p-1)}\right)\int_\Omega uf(u)\mathrm{d}x \\ \geq & -2\|u\|_{H_0^1}^2 - \frac{p-q}{M(p-1)}\|u\|^2 + 2\|u_t\|_{\mathcal{H}}^2 + 2\left(p+1 - \frac{q-1}{M(p-1)}\right)\int_\Omega F(u)\mathrm{d}x \\ \geq & \left[(K(M)-2)\lambda_1 - \frac{p-q}{M(p-1)}\right]\|u\|_{\mathcal{H}}^2 + [K(M)+2]\|u_t\|_{\mathcal{H}}^2 - 2K(M)E(t), \end{aligned}$$

where $K(M) = p + 1 - \frac{q-1}{M(p-1)} > 0$ and

$$K_1(M) \triangleq (K(M)-2)\lambda_1 - \frac{p-q}{(p-1)M} > 0.$$

By a similar argument to that in Case1, we can obtain (5.14), and in order to avoid redundancy, we omit it here.

On the other hand, it follows from the Hölder inequality and the Poincaré inequality that

$$\begin{aligned} \|u\|_{\mathcal{H}} \leq & \|u_0\|_{\mathcal{H}} + \int_0^t \|u_\tau\|_{\mathcal{H}}\mathrm{d}\tau \leq \|u_0\|_{\mathcal{H}} + \left(1 + \frac{1}{\sqrt{\lambda_1}}\right)\int_0^t \|u_\tau\|\mathrm{d}\tau \\ \leq & \|u_0\|_{\mathcal{H}} + C\int_0^t \|u_\tau\|_{L^{q+1}}\mathrm{d}\tau \leq \|u_0\|_{\mathcal{H}} + Ct^{\frac{q}{q+1}}\left(\int_0^t \|u_\tau\|_{L^{q+1}}^{q+1}\mathrm{d}\tau\right)^{\frac{1}{q+1}} \\ \leq & \|u_0\|_{\mathcal{H}} + Ct^{\frac{q}{q+1}}E(0)^{\frac{1}{q+1}}, \end{aligned}$$

which is a contradiction with (5.14). The proof of Theorem 5.3 is completed.

Remark 5.1. *Asymptotic behavior of solutions for the problem* (1.1) *is also an interesting and important work, which is the further work to be considered.*

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Conflict of interest

The authors declare there is no conflicts of interest.

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