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ERA, 31(1): 421–441.
DOI: 10.3934/era.2023021
Received: 28 August 2022
Revised: 12 October 2022
Accepted: 27 October 2022
Published: 08 November 2022

Research article

Spatial properties and the influence of the Soret coefficient on the solutions of time-dependent double-diffusive Darcy plane flow

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Abstract: This paper investigates time-dependent double-diffusive Darcy flow which is defined in a semi-infinite strip pipe, where the generatrix of the pipe is not parallel to the coordinate axis any more. By using several results which have been derived in the literature, the spatial properties and the influence of the Soret coefficient on the solutions are both obtained. We also give some concrete examples.

Keywords: convergence result; Darcy plane flow; Soret coefficient

1. Introduction

The porous media fluid defined in a two-dimensional semi-infinite pipe has received extensive attention. Liu et al. [1] defined a semi-infinite strip pipe whose generatrix is parallel to the coordinate axis (see Figure 1) and obtained the Phragmén-Lindelöf alternative result of shallow water equations.

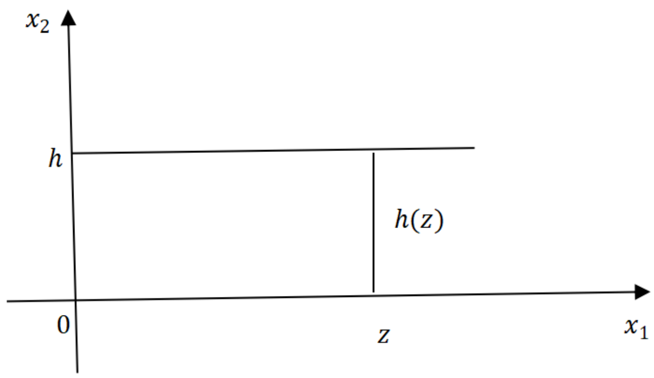

Figure 1. Cylindrical pipe 1.

Payne and Schaefer [2] considered the case that the generatrix is not parallel to the coordinate axis (see Figure 2) and obtained the Phragmén-Lindelöf alternative for the biharmonic equation.

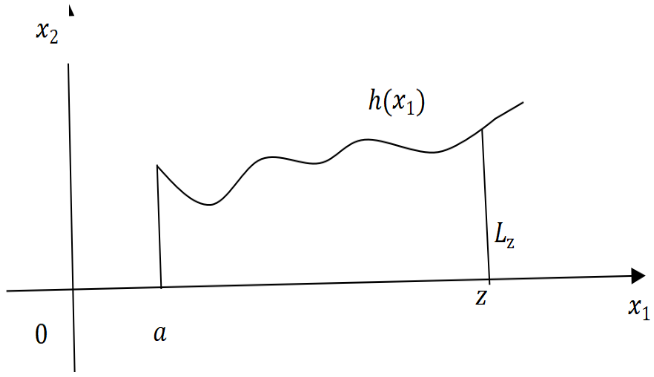

Figure 2. Cylindrical pipe 2.

Recently, Li and Chen [3] considered the Darcy equations on a semi-infinite channel which was defined as

$$R = \Big\{(x_1, x_2)|x_1 > a, 0 < x_2 < h(x_1)\Big\},$$

where $a > 0$ and $h(x_1)$ is a smooth curve in the plane. They proved that the solutions of Darcy equations grow polynomially or decay exponentially as a spatial variable $x_1 \to \infty$. For more on such studies, one can see [2, 4–6].

In this paper, we consider the convergence result on the Soret coefficient of the Darcy model in R. The importance of this type of convergence result was discussed by Hirsch and Smale [7] and there have been a lot of results. At first, people mainly focused on the structural stability of the solutions of various systems of partial differential equations in bounded domains (see [8–14]). Later, many scholars extended the study of structural stability to the case that there are two kinds of interface links in a bounded region (see [15–18]). Li et al. [19, 20] considered the structural stability of Brinkman-Forechheimer equations and the thermoelastic equations of type III on a three-dimensional semi-infinite cylinder, respectively. The generatrix of the cylinder was parallel to the coordinate axis. However, the stability of partial differential equations on a two-dimensional pipe has not received enough attention. It is especially emphasized that the generatrix of the pipe considered in this paper is not parallel to the coordinate axis.

We investigate the following double-diffusive Darcy flow of a fluid through a porous medium in R which can be written as (see [21])

$$u_\alpha = -p_{,\alpha} + g_\alpha T + h_\alpha C, \ in\ R \times (0, \tau), \tag{1.1}$$

$$u_{\alpha,\alpha} = 0, \ in\ R \times (0, \tau), \tag{1.2}$$

$$\partial_t T + u_\alpha T_{,\alpha} = \Delta T, \ in\ R \times (0, \tau), \tag{1.3}$$

$$\partial_t C + u_\alpha C_{,\alpha} = \Delta C + \sigma \Delta T, \ in\ R \times (0, \tau), \tag{1.4}$$

where $\alpha = 1, 2$. u_α, p, T and C represent the velocity, pressure, temperature, and concentration of the flow, respectively. g_α and h_α are bounded functions. $\sigma > 0$ is the Soret coefficient. For simplicity, we assume $\boldsymbol{g}$ and $\boldsymbol{h}$ satisfy $|\boldsymbol{g}|,\ |\boldsymbol{h}| \leq 1$. In this paper, we also use the summation convention summed from 1 to 2, and a comma is used to indicate differentiation. e.g., $u_{\alpha,\beta}u_{\alpha,\beta} = \sum_{\alpha,\beta=1}^{2}\left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2$.

The initial-boundary conditions can be written as

$$u_\alpha(x_1,0,t) = u_\alpha(x_1,h(x_1),t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{1.5}$$

$$T(x_1,0,t) = T(x_1,h(x_1),t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{1.6}$$

$$C(x_1,0,t) = C(x_1,h(x_1),t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{1.7}$$

$$u_\alpha(a,x_2,t) = F_\alpha(x_2,t),\ 0 < x_2 < h(a), 0 < t < \tau, \tag{1.8}$$

$$T(a,x_2,t) = H(x_2,t), C(a,x_2,t) = \widetilde{H}(x_2,t),\ 0 < x_2 < h(a), 0 < t < \tau, \tag{1.9}$$

$$T(x_1,x_2,0) = T_0(x_1,x_2),\ C(x_1,x_2,0) = C_0(x_1,x_2),\ (x_1,x_2) \in R, \tag{1.10}$$

where T_0 and C_0 are given functions. F_α, H and $\widetilde{H}$ are differentiable functions which are assumed to satisfy the appropriate compatibility conditions

$$F_\alpha(h(a),t) = H(h(a),t) = \widetilde{H}(h(a),t) = 0.$$

Now, we let $v(x_1,x_2,t)$ denote a stream function which satisfies

$$u_1 = v_{,2}, u_2 = -v_{,1}.$$

Equations (1.1)–(1.8) can be converted to

$$\Delta v = -\nabla^\perp \cdot \boldsymbol{g}T - \boldsymbol{g} \cdot \nabla^\perp T - \nabla^\perp \cdot \boldsymbol{h}C - \boldsymbol{h} \cdot \nabla^\perp C,\ in\ R \times (0,\tau), \tag{1.11}$$

$$\partial_t T + \nabla^\perp v \cdot \nabla T = \Delta T,\ in\ R \times (0,\tau), \tag{1.12}$$

$$\partial_t C + \nabla^\perp v \cdot \nabla C = \Delta C + \sigma\Delta T,\ in\ R \times (0,\tau), \tag{1.13}$$

$$v(x_1,0,t) = v(x_1,h(x_1),t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{1.14}$$

$$v_n(x_1,0,t) = v_n(x_1,h(x_1),t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{1.15}$$

$$T(x_1,0,t) = T(x_1,h(x_1),t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{1.16}$$

$$C(x_1,0,t) = C(x_1,h(x_1),t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{1.17}$$

$$v(a,x_2,t) = \widetilde{F}_1(x_2,t) = \int_0^{x_2} F_1(s,t)ds,\ 0 < x_2 < h(a), 0 < t < \tau, \tag{1.18}$$

$$v_{,1}(a,x_2,t) = \widetilde{F}_2(x_2,t) = -F_2(x_2,t),\ 0 < x_2 < h(a), 0 < t < \tau, \tag{1.19}$$

$$T(a,x_2,t) = H(x_2,t),\ C(a,x_2,t) = \widetilde{H}(x_2,t),\ 0 < x_2 < h(a), 0 < t < \tau, \tag{1.20}$$

$$T(x_1,x_2,0) = T_0(x_1,x_2),\ C(x_1,x_2,0) = C_0(x_1,x_2),\ (x_1,x_2) \in R, \tag{1.21}$$

where v_n is the outward normal derivative of v, $\boldsymbol{g} = (g_1,g_2)$, $\boldsymbol{h} = (h_1,h_2)$ and $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$.

In the next section, we give several lemmas that have been derived in the literature. In Section 3, we derive an important lemma that can be used to derive our main result. In Section 4, we obtain the convergence result on the Soret coefficient and give some concrete examples. Section 5 shows the summary and outlook of this paper.

2. Preliminary

We also introduce the notations

$$R_z = \Big\{(x_1, x_2)|x_1 \geq z > a, 0 < x_2 < h(x_1)\Big\},$$

$$L_z = \Big\{(x_1, x_2)|x_1 = z \geq a, 0 < x_2 < h(z)\Big\},$$

where z is a running variable along the x_1 axis.

Here are some lemmas that will be often used in this paper.

Lemma 2.1 (see [5,22]) If $w(x_1, 0) = w(x_1, h) = 0$ and $w_n(x_1, 0) = w_n(x_1, h) = 0$, then the following Wirtinger type inequality holds

$$\int_{L_z} w^2 dx_2 \leq \frac{h^2}{\pi^2}\int_{L_z}(w_{,2})^2 dx_2,$$

where $z > 0$ is a moving point on the x_1 axis.

Lemma 2.2 If $\varphi(x_1, 0) = \varphi(x_1, h) = 0$ and $\varphi \to 0$, as $x_1 \to \infty$, then

$$\int_{R_z}\varphi^4 dx_2 d\xi \leq \Big(\int_{R_z}\varphi^2 dx_2 d\xi\Big)\Big(\int_{R_z}\varphi_{,\alpha}\varphi_{,\alpha} dx_2 d\xi\Big).$$

Proof. Since $\varphi(x_1, 0) = \varphi(x_1, h) = 0$, we have

$$\varphi^2(x_1, x_2) = 2\int_0^{x_2}\varphi\frac{\partial}{\partial\zeta}\varphi(x_1,\zeta)d\zeta = -2\int_{x_2}^{h}\varphi\frac{\partial}{\partial\zeta}\varphi(x_1,\zeta)d\zeta \leq \int_0^h\Big|\varphi\frac{\partial}{\partial x_2}\varphi(x_1,x_2)\Big|dx_2. \tag{2.1}$$

Since $\varphi \to 0$, as $x_1 \to \infty$, we have

$$\varphi^2(x_1, x_2) = -2\int_{x_1}^{\infty}\varphi\frac{\partial}{\partial\xi}\varphi(\xi, x_2)d\xi \leq 2\int_{x_1}^{\infty}\Big|\varphi\frac{\partial}{\partial\xi}\varphi(\xi,x_2)\Big|d\xi. \tag{2.2}$$

Combining Eqs (2.1) and (2.2) and integrating over R_z, we obtain

$$\begin{aligned}\int_{R_z}\varphi^4(\xi, x_2)dx_2 d\xi &\leq 2\Big[\int_{R_z}\Big|\varphi\frac{\partial}{\partial x_2}\varphi(\xi,x_2)\Big|dx_2 d\xi\Big]\Big[\int_{R_z}\Big|\varphi\frac{\partial}{\partial\xi}\varphi(\xi,x_2)\Big|dx_2 d\xi\Big]\\ &\leq 2\int_{R_z}\varphi^2 dx_2 d\xi\Big(\int_{R_z}\Big(\frac{\partial\varphi}{\partial\xi}\Big)^2 dx_2 d\xi\Big)^{\frac{1}{2}}\Big(\int_{R_z}\Big(\frac{\partial\varphi}{\partial x_2}\Big)^2 dx_2 d\xi\Big)^{\frac{1}{2}}.\end{aligned}$$

Using Young's inequality, we can obtain Lemma 2.2.

Using a similar method of papers [9,23,24], we can have the following lemma.

Lemma 2.3 Assume that $T_0, H \in L^\infty$, then

$$\sup_{[0,\tau]}\|T\|_\infty \leq T_m,$$

where $T_m = \max\Big\{\|T_0\|_\infty,\ \sup_{[0,\tau]} H_\infty(\eta)\Big\}$.

If $C_0, \widetilde{H} \in L^\infty$ and $\sigma = 0$ in (1.4), we can also have

$$\sup_{[0,\tau]} \|C\|_\infty \le C_m, \tag{2.3}$$

where $C_m = \max\left\{\|C_0\|_\infty,\ \sup_{[0,\tau]} \widetilde{H}_\infty(\eta)\right\}$.

To obtain our main result, we shall use the following result which can be written as follows.

Lemma 2.4 (see [3]) Let (v, T, C) be a solution of the Eqs (1.1)–(1.10) in R and $\forall\ z \ge a$ such that $F(z, t) < 0$. Then for any fixed t

$$\begin{aligned}\int_0^t \int_{R_z} & e^{-\omega\eta}\Big[\frac{1}{4}\beta_2\omega T^2 + \frac{1}{4}\omega C^2 + \frac{1}{2}\beta_1 v_{,\alpha}v_{,\alpha} + \frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha} + \frac{1}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2 d\xi d\eta \\ &+ \frac{1}{2}e^{-\omega t}\int_z^\infty \int_{L_\xi}\Big[\beta_2 T^2 + C^2\Big]dx_2 d\xi \\ &\le m_1 e^{-2m_3\int_a^z \frac{1}{h(\zeta)}d\zeta} + m_2 e^{-m_3\int_a^z \frac{1}{h(\zeta)}d\zeta}\end{aligned}$$

holds, where $\beta_1, \beta_2, \omega, m_1, m_2$ and m_3 are positive constants which depends on the middle parameter σ and boundary conditions of the equation; also $F(z, t)$ has been defined as

$$\begin{aligned}F(z,t) = &\int_0^t \int_{L_z} e^{-\omega\eta}\Big[\beta_1 v v_{,1} + \beta_2 T T_{,1} + C C_{,1}\Big]dx_2 d\eta \\ &+ \beta_1 \int_0^t \int_{L_z} e^{-\omega\eta}\Big[g_2 T v + h_2 C v\Big]dx_2 d\eta \\ &+ \int_0^t \int_{L_z} e^{-\omega\eta}\Big[-\frac{1}{2}\beta_2 T^2 v_{,2} - \frac{1}{2}C^2 v_{,2} + \sigma C T_{,1}\Big]dx_2 d\eta.\end{aligned}$$

Remark 2.1 Lemma 2.4 shows that the solution of Eqs (1.11)–(1.21) decays exponentially with $z \to \infty$. Only in this case, the study of structural stability is meaningful. Lemma 2.4 will also provide a priori bounds for the estimate of nonlinear terms (see e.g., (3.55) and (3.56)).

Remark 2.2 Li and Chen [3] considered several special cases of $h(z)$, e.g., $h(z) = h$ a constant, $h(z) \le k_1 z^{\tau_1}$ and $h(z) \le k_2 z(lnz)^{\tau_2}$, where $k_1, k_2 > 0$ and $0 < \tau_1, \tau_2 \le 1$. In the fourth section, we will also consider several special cases with $h(z) = h$ as a constant, $\tau_1 = \frac{1}{2}, k_1 = 1$ and $\tau_1 = \frac{2}{3}, k_1 = 1$.

Lemma 2.5 (see [3]) Assume that (v, T, C) are solutions of Eqs (1.1)–(1.10). If $F(z, t) < 0$ for any $z \ge a$, then

$$\begin{aligned}\int_0^t \int_R & e^{-\omega\eta}\Big[\frac{1}{4}\beta_2\omega T^2 + \frac{1}{4}\omega C^2 + \frac{1}{2}\beta_1 v_{,\alpha}v_{,\alpha} + \frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha} + \frac{1}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2 d\xi d\eta \\ &+ \frac{1}{2}e^{-\omega t}\int_R \Big[\beta_2 T^2 + C^2\Big]dx_2 d\xi \le r(t),\end{aligned}$$

where $r(t)$ is a positive known function which depends only on t.

Now, we derive a bound of $\int_R v_{,\alpha}v_{,\alpha}dx_2 d\xi$.

Lemma 2.6 Assume that $\widetilde{F}_1, \widetilde{F}_2, H, \widetilde{H} \in L^\infty(R)$, then

$$\int_R v_{,\alpha}v_{,\alpha}dx_2 d\xi \le \epsilon_1, \tag{2.4}$$

where

$$\epsilon_1 = 2\sup_{[0,\tau]}\Big\{\int_{L_a}|\widetilde{F}_1\widetilde{F}_2|dx_2d\xi + \int_{L_a}|g_2H\widetilde{F}_1|dx_2d\xi + \int_{L_a}|h_2\widetilde{H}\widetilde{F}_1|dx_2d\xi\Big\} + 4\max\{\frac{1}{\beta_2},1\}r^*e^{\omega\tau}.$$

Additionally, r^* is the maximum value of $r(t)$ in $[0,\tau]$.

Proof. Using Eq (1.11), we have

$$\int_R\Big[\Delta v + \nabla^\perp\cdot\boldsymbol{g}T + \boldsymbol{g}\cdot\nabla^\perp T + \nabla^\perp\cdot\boldsymbol{h}C + \boldsymbol{h}\cdot\nabla^\perp C\Big]vdx_2d\xi = 0.$$

Using Eqs (1.18)–(1.20) it follows that

$$\begin{aligned}\int_R v_{,\alpha}v_{,\alpha}dx_2d\xi = &-\int_{L_a}\widetilde{F}_1\widetilde{F}_2dx_2d\xi - \int_{L_a}g_2H\widetilde{F}_1dx_2d\xi - \int_{L_a}h_2\widetilde{H}\widetilde{F}_1dx_2d\xi\\ &-\int_R\nabla^\perp v\cdot\boldsymbol{g}Tdx_2d\xi - \int_R\nabla^\perp v\cdot\boldsymbol{h}Cdx_2d\xi\\ &\leq\int_{L_a}|\widetilde{F}_1\widetilde{F}_2|dx_2d\xi + \int_{L_a}|g_2H\widetilde{F}_1|dx_2d\xi + \int_{L_a}|h_2\widetilde{H}\widetilde{F}_1|dx_2d\xi\\ &+\frac{1}{2}\int_R v_{,\alpha}v_{,\alpha}dx_2d\xi + \int_R T^2dx_2d\xi + \int_R C^2dx_2d\xi.\end{aligned} \tag{2.5}$$

Using Lemma 2.5 in Eq (2.5), we can get Lemma 2.6.

3. Important lemma

In this section, we derive the convergence result when the Soret coefficient $\sigma \to 0$. To do this, we let (v, T, C) be the solution of Eqs (1.11)–(1.21). Furthermore, let (v^*, T^*, C^*) be the solution to the following equations

$$\Delta v^* = -\nabla^\perp\cdot\boldsymbol{g}T^* - \boldsymbol{g}\cdot\nabla^\perp T^* - \nabla^\perp\cdot\boldsymbol{h}C^* - \boldsymbol{h}\cdot\nabla^\perp C^*,\ in\ R\times(0,\tau), \tag{3.1}$$

$$\partial_t T^* + \nabla^\perp v^*\cdot\nabla T^* = \Delta T^*,\ in\ R\times(0,\tau), \tag{3.2}$$

$$\partial_t C^* + \nabla^\perp v^*\cdot\nabla C^* = \Delta C^*,\ in\ R\times(0,\tau), \tag{3.3}$$

$$v^*(x_1,0,t) = v^*(x_1,h(x_1),t) = 0,\ x_1\geq a, 0<t<\tau, \tag{3.4}$$

$$v^*_n(x_1,0,t) = v^*_n(x_1,h(x_1),t) = 0,\ x_1\geq a, 0<t<\tau, \tag{3.5}$$

$$T^*(x_1,0,t) = T^*(x_1,h(x_1),t) = 0,\ x_1\geq a, 0<t<\tau, \tag{3.6}$$

$$C^*(x_1,0,t) = C^*(x_1,h(x_1),t) = 0,\ x_1\geq a, 0<t<\tau, \tag{3.7}$$

$$v^*(a,x_2,t) = \widetilde{F}_1(x_2,t) = \int_0^{x_2}F_1(s,t)ds,\ 0<x_2<h(a), 0<t<\tau, \tag{3.8}$$

$$v^*_{,1}(a,x_2,t) = \widetilde{F}_2(x_2,t) = -F_2(x_2,t),\ 0<x_2<h(a), 0<t<\tau, \tag{3.9}$$

$$T^*(a,x_2,t) = H(x_2,t),\ C^*(a,x_2,t) = \widetilde{H}(x_2,t),\ 0<x_2<h(a), 0<t<\tau, \tag{3.10}$$

$$T^*(x_1,x_2,0) = T_0(x_1,x_2),\ C^*(x_1,x_2,0) = C_0(x_1,x_2),\ (x_1,x_2)\in R. \tag{3.11}$$

Remark 3.1 We note that Lemmas 2.4 and 2.5 also hold for (v^*, T^*, C^*).

Now, we let

$$w = v - v^*,\ \theta = T - T^*,\ \Sigma = C - C^*.$$

Then (w, θ, Σ) satisfies

$$\Delta w = -\nabla^{\perp} \cdot \boldsymbol{g}\theta - \boldsymbol{g} \cdot \nabla^{\perp}\theta - \nabla^{\perp} \cdot \boldsymbol{h}\Sigma - \boldsymbol{h} \cdot \nabla^{\perp}\Sigma,\ in\ R \times (0, \tau), \tag{3.12}$$

$$\partial_t\theta + \nabla^{\perp}w \cdot \nabla T + \nabla^{\perp}v^* \cdot \nabla\theta = \Delta\theta,\ in\ R \times (0, \tau), \tag{3.13}$$

$$\partial_t\Sigma + \nabla^{\perp}v \cdot \nabla\Sigma + \nabla^{\perp}w \cdot \nabla C^* = \Delta\Sigma + \sigma\Delta T,\ in\ R \times (0, \tau), \tag{3.14}$$

$$w(x_1, 0, t) = w(x_1, h(x_1), t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{3.15}$$

$$w_n(x_1, 0, t) = w_n(x_1, h(x_1), t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{3.16}$$

$$\theta(x_1, 0, t) = \theta(x_1, h(x_1), t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{3.17}$$

$$\Sigma(x_1, 0, t) = \Sigma(x_1, h(x_1), t) = 0,\ x_1 \geq a, 0 < t < \tau, \tag{3.18}$$

$$w(a, x_2, t) = w_{,1}(a, x_2, t) = 0,\ 0 < x_2 < h(a), 0 < t < \tau, \tag{3.19}$$

$$\theta(a, x_2, t) = \Sigma(a, x_2, t) = 0,\ 0 < x_2 < h(a), 0 < t < \tau, \tag{3.20}$$

$$\theta(x_1, x_2, 0) = \Sigma(x_1, x_2, 0) = 0,\ (x_1, x_2) \in R. \tag{3.21}$$

We establish the following auxiliary functions

$$\begin{aligned}
E_1(z, t) = &- \int_0^t \int_{R_z} e^{-\omega\eta} w w_{,1} dx_2 d\xi d\eta + \int_0^t \int_{R_z} e^{-\omega\eta} g_2 \theta w dx_2 d\xi d\eta \\
&+ \int_0^t \int_{R_z} e^{-\omega\eta} h_2 \Sigma w dx_2 d\xi d\eta \\
&\doteq A_1(z, t) + A_2(z, t) + A_3(z, t),
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
E_2(z, t) = &\int_0^t \int_{R_z} e^{-\omega\eta} \theta\theta_{,1} dx_2 d\xi d\eta - \frac{1}{2} \int_0^t \int_{R_z} e^{-\omega\eta} \theta^2 v^*_{,2} dx_2 d\xi d\eta \\
&+ \int_0^t \int_{R_z} e^{-\omega\eta} \theta T w_{,2} dx_2 d\xi d\eta + \int_0^t \int_{R_z} e^{-\omega\eta} (\xi - z) \nabla^{\perp} w \cdot \nabla\theta T dx_2 d\xi d\eta \\
&\doteq B_1(z, t) + B_2(z, t) + B_3(z, t) + B_4(z, t),
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
E_3(z, t) = &\int_0^t \int_{R_z} e^{-\omega\eta} \Sigma\Sigma_{,1} dx_2 d\xi d\eta - \frac{1}{2} \int_0^t \int_{R_z} e^{-\omega\eta} \Sigma^2 v_{,2} dx_2 d\xi d\eta \\
&- \int_0^t \int_{R_z} e^{-\omega\eta} w_{,2} C^* \Sigma dx_2 d\xi d\eta + \sigma \int_0^t \int_{R_z} e^{-\omega\eta} \Sigma T_{,1} dx_2 d\xi d\eta \\
&+ \int_0^t \int_{R_z} e^{-\omega\eta} (\xi - z) \nabla^{\perp} w \cdot \nabla\Sigma C^* dx_2 d\xi d\eta + \sigma \int_0^t \int_{R_z} e^{-\omega\eta} (\xi - z) T_{,\alpha} \Sigma_{,\alpha} dx_2 d\xi d\eta \\
&\doteq C_1(z, t) + C_2(z, t) + C_3(z, t) + C_4(z, t) + C_5(z, t) + C_6(z, t),
\end{aligned} \tag{3.24}$$

where ω is an arbitrary positive constant.

Using the divergence theorem and Eqs (3.12)-(3.21), we have

$$E_1(z,t) = -\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)_{,\alpha} w w_{,\alpha} dx_2 d\xi d\eta + \int_0^t \int_{R_z} e^{-\omega\eta} g_2 \theta w dx_2 d\xi d\eta$$
$$+ \int_0^t \int_{R_z} e^{-\omega\eta} h_2 \Sigma w dx_2 d\xi d\eta$$
$$= \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z) w_{,\alpha} w_{,\alpha} dx_2 d\xi d\eta + \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\boldsymbol{g} \cdot \nabla^{\perp} w \theta dx_2 d\xi d\eta$$
$$+ \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\boldsymbol{h} \cdot \nabla^{\perp} w \Sigma dx_2 d\xi d\eta. \tag{3.25}$$

Similarly, we have

$$E_2(z,t) = \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}\omega\theta^2 + \theta_{,\alpha}\theta_{,\alpha}\Big] dx_2 d\xi d\eta + \frac{1}{2} e^{-\omega t} \int_{R_z} (\xi - z)\theta^2 dx_2 d\xi, \tag{3.26}$$

and

$$E_3(z,t) = \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}\omega\Sigma^2 + \Sigma_{,\alpha}\Sigma_{,\alpha}\Big] dx_2 d\xi d\eta + \frac{1}{2} e^{-\omega t} \int_{R_z} e^{-\omega\eta}(\xi - z)\Sigma^2 dx_2 d\xi. \tag{3.27}$$

We also define

$$E(z,t) = E_1(z,t) + \delta_2 E_2(z,t) + \delta_3 E_3(z,t)$$
$$= \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{2}\delta_2\omega\theta^2 + \frac{1}{2}\delta_3\omega\Sigma^2 + w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big] dx_2 d\xi d\eta$$
$$+ \frac{1}{2} e^{-\omega t} \int_{R_z} (\xi - z)\Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big] dx_2$$
$$+ \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\boldsymbol{g} \cdot \nabla^{\perp} w \theta dx_2 d\xi d\eta + \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\boldsymbol{h} \cdot \nabla^{\perp} w \Sigma dx_2 d\xi d\eta, \tag{3.28}$$

where δ_2 and δ_3 are positive constants.

Using the Cauchy-Schwarz inequality, we have

$$\Big|\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\boldsymbol{g} \cdot \nabla^{\perp} w \theta dx_2 d\xi d\eta\Big| \le \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\theta^2 dx_2 d\xi d\eta$$
$$+ \frac{1}{4}\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z) w_{,\alpha} w_{,\alpha} dx_2 d\xi d\eta, \tag{3.29}$$
$$\Big|\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\boldsymbol{h} \cdot \nabla^{\perp} w \Sigma dx_2 d\xi d\eta\Big| \le \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Sigma^2 dx_2 d\xi d\eta$$
$$+ \frac{1}{4}\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z) w_{,\alpha} w_{,\alpha} dx_2 d\xi d\eta. \tag{3.30}$$

Inserting Eqs (3.29) and (3.30) into Eq (3.28) and choosing $\omega = \max\{\frac{4}{\delta_3}, \frac{4}{\delta_2}\}$, we have

$$E(z,t) \geq \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{4}\delta_2\omega\theta^2 + \frac{1}{4}\omega\delta_3\Sigma^2 + \frac{1}{2}w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+ \frac{1}{2}e^{-\omega t}\int_{R_z}(\xi - z)\Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big]dx_2d\xi. \tag{3.31}$$

From Eq (3.28), we have

$$-\frac{\partial}{\partial z}E(z,t) = \int_0^t \int_{R_z} e^{-\omega\eta}\Big[\frac{1}{2}\delta_2\omega\theta^2 + \frac{1}{2}\delta_3\omega\Sigma^2 + w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+ \frac{1}{2}e^{-\omega t}\int_{R_z}\Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big]dx_2d\xi$$
$$+ \int_0^t\int_{R_z} e^{-\omega\eta}\boldsymbol{g}\cdot\nabla^{\perp}w\theta dx_2d\xi d\eta + \int_0^t\int_{R_z} e^{-\omega\eta}\boldsymbol{h}\cdot\nabla^{\perp}w\Sigma dx_2d\xi d\eta. \tag{3.32}$$

Using the Cauchy-Schwarz inequality, we have

$$\Big|\int_0^t\int_{R_z} e^{-\omega\eta}\boldsymbol{g}\cdot\nabla^{\perp}w\theta dx_2d\xi d\eta\Big| \leq \int_0^t\int_{R_z} e^{-\omega\eta}\theta^2 dx_2d\xi d\eta + \frac{1}{4}\int_0^t\int_{R_z} e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta, \tag{3.33}$$
$$\Big|\int_0^t\int_{R_z} e^{-\omega\eta}\boldsymbol{h}\cdot\nabla^{\perp}w\Sigma dx_2d\xi d\eta\Big| \leq \int_0^t\int_{R_z} e^{-\omega\eta}\Sigma^2 dx_2d\xi d\eta + \frac{1}{4}\int_0^t\int_{R_z} e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta. \tag{3.34}$$

Inserting Eqs (3.33) and (3.34) into Eq (3.32), we have

$$-\frac{\partial}{\partial z}E(z,t) \leq \int_0^t \int_{R_z} e^{-\omega\eta}\Big[\frac{3}{4}\delta_2\omega\theta^2 + \frac{3}{4}\omega\delta_3\Sigma^2 + \frac{3}{2}w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+ \frac{1}{2}e^{-\omega t}\int_{R_z}\Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big]dx_2d\xi, \tag{3.35}$$

and

$$-\frac{\partial}{\partial z}E(z,t) \geq \int_0^t \int_{R_z} e^{-\omega\eta}\Big[\frac{1}{4}\delta_2\omega\theta^2 + \frac{1}{4}\omega\delta_3\Sigma^2 + \frac{1}{2}w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+ \frac{1}{2}e^{-\omega t}\int_{R_z}\Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big]dx_2d\xi. \tag{3.36}$$

Based on Eqs (3.22)–(3.24) and using Eq (3.36), we have the following lemma.

Lemma 3.1. Assume that $T_0, C_0, H, \widetilde{H} \in C^\infty$, $\frac{1}{h(z)} \in C(R)$ and the function $E(z,t)$ is defined in Eq (3.28). Then $E(z,t)$ satisfies

$$E(z,t) \leq n_1 h(z)\Big[-\frac{\partial}{\partial z}E(z,t)\Big]$$
$$+ \frac{4\sigma^2\delta_3}{\beta_3\beta_2\omega}m_1 e^{-2m_3\int_a^z \frac{1}{h(\zeta)}d\zeta} + \frac{4\sigma^2\delta_3}{\beta_3\beta_2\omega}m_2 e^{-m_3\int_a^z \frac{1}{h(\zeta)}d\zeta}$$
$$+ \frac{4}{\beta_2}\sigma^2 m_1\int_z^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi + \frac{4}{\beta_2}\sigma^2 m_2\int_z^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi, \tag{3.37}$$

where n_1 is a positive constant.

Proof. Using the Hölder inequality, Young's inequality and Lemma 2.1, we have

$$
\begin{aligned}
A_1(z,t) &\le \Big[\int_0^t\int_{R_z} e^{-\omega\eta} w^2 dx_2 d\xi d\eta \int_0^t\int_{R_z} e^{-\omega\eta}(w_{,1})^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}} \\
&\le \frac{h}{\pi}\Big[\int_0^t\int_{R_z} e^{-\omega\eta}(w_{,2})^2 dx_2 d\xi d\eta \int_0^t\int_{R_z} e^{-\omega\eta}(w_{,1})^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}} \\
&\le \frac{h}{2\pi}\int_0^t\int_{R_z} e^{-\omega\eta} w_{,\alpha}w_{,\alpha} dx_2 d\xi d\eta, \qquad (3.38)\\
A_2(z,t) &\le \Big[\int_0^t\int_{R_z} e^{-\omega\eta} w^2 dx_2 d\xi d\eta \int_0^t\int_{R_z} e^{-\omega\eta}\theta^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}} \\
&\le \frac{2\sqrt{2}h}{\pi\sqrt{\delta_2\omega}}\Big[\frac{1}{2}\int_0^t\int_{R_z} e^{-\omega\eta}(w_{,2})^2 dx_2 d\xi d\eta \cdot \frac{1}{4}\delta_2\omega\int_0^t\int_{R_z} e^{-\omega\eta}\theta^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}} \\
&\le \frac{\sqrt{2}h}{\sqrt{\delta_2\omega}\pi}\Big[\frac{1}{2}\int_0^t\int_{R_z} e^{-\omega\eta} w_{,\alpha}w_{,\alpha} dx_2 d\xi d\eta + \frac{1}{4}\delta_2\omega\int_0^t\int_{R_z} e^{-\omega\eta}\theta^2 dx_2 d\xi d\eta\Big], \qquad (3.39)\\
A_3(z,t) &\le \frac{\sqrt{2}h}{\sqrt{\delta_3\omega}\pi}\Big[\frac{1}{2}\int_0^t\int_{R_z} e^{-\omega\eta} w_{,\alpha}w_{,\alpha} dx_2 d\xi d\eta + \frac{1}{4}\omega\delta_3\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma^2 dx_2 d\xi d\eta\Big]. \qquad (3.40)
\end{aligned}
$$

Combining Eqs (3.22) and (3.38)–(3.40), we obtain

$$
\begin{aligned}
E_1(z,t) \le{}& \frac{h}{\pi}\Big[1+\frac{\sqrt{2}}{\sqrt{\delta_2\omega}}+\frac{\sqrt{2}}{\sqrt{\delta_3\omega}}\Big]\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{2}w_{,\alpha}w_{,\alpha} dx_2 d\xi d\eta \\
&+\frac{\sqrt{2}h}{\sqrt{\delta_2\omega}\pi}\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{4}\delta_2\omega\theta^2 dx_2 d\xi d\eta + \frac{\sqrt{2}h}{\sqrt{\delta_3\omega}\pi}\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{4}\omega\delta_3\Sigma^2 dx_2 d\xi d\eta. \qquad (3.41)
\end{aligned}
$$

Using the Hölder inequality, Young's inequality, Lemmas 2.1, 2.3, 2.5 and Eq (2.5), we have

$$
\begin{aligned}
B_1(z,t) &\le \Big[\int_0^t\int_{R_z} e^{-\omega\eta}\theta^2 dx_2 d\xi d\eta \int_0^t\int_{R_z} e^{-\omega\eta}(\theta_{,1})^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}} \\
&\le \frac{h}{\pi}\Big[\int_0^t\int_{R_z} e^{-\omega\eta}(\theta_{,2})^2 dx_2 d\xi d\eta \int_0^t\int_{R_z} e^{-\omega\eta}(\theta_{,1})^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}} \\
&\le \frac{h}{\pi\delta_2}\cdot\frac{1}{2}\delta_2\int_0^t\int_{R_z} e^{-\omega\eta}\theta_{,\alpha}\theta_{,\alpha} dx_2 d\xi d\eta, \qquad (3.42)\\
B_2(z,t) &\le \frac{1}{2}\int_0^t\Big[\int_{R_z} e^{-\omega\eta}(v^*_{,2})^2 dx_2 d\xi \int_{R_z} e^{-\omega\eta}\theta^4 dx_2 d\xi\Big]^{\frac{1}{2}} d\eta \\
&\le \frac{1}{2}\int_0^t\Big[\int_{R} e^{-\omega\eta}(v^*_{,2})^2 dx_2 d\xi \int_{R_z} e^{-\omega\eta}\theta^2 dx_2 d\xi\Big]^{\frac{1}{2}} d\eta \\
&\le \frac{\sqrt{\epsilon_1}}{2\delta_2\pi}\int_0^t\Big[\int_{R_z} e^{-\omega\eta}\theta^2 dx_2 d\xi \int_{R_z} e^{-\omega\eta}\theta_{,\alpha}\theta_{,\alpha} dx_2 d\xi\Big]^{\frac{1}{2}} d\eta \\
&\le \frac{h\sqrt{\epsilon_1}}{2\delta_2\pi}\int_0^t\int_{R_z} e^{-\omega\eta}\delta_2\theta_{,\alpha}\theta_{,\alpha} dx_2 d\xi d\eta, \qquad (3.43)\\
B_3(z,t) &\le T_m\Big[\int_0^t\int_{R_z} e^{-\omega\eta}(w_{,2})^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}}\Big[\int_0^t\int_{R_z} e^{-\omega\eta}\theta^2 dx_2 d\xi d\eta\Big]^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \frac{h}{\pi\sqrt{\delta_2}}T_m\Big[\frac{1}{2}\int_0^t\int_{R_z}e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta\Big]^{\frac{1}{2}}\Big[\delta_2\int_0^t\int_{R_z}e^{-\omega\eta}\theta_{,\alpha}\theta_{,\alpha}dx_2d\xi d\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{h}{\pi\sqrt{2\delta_2}}T_m\Big[\frac{1}{2}\int_0^t\int_{R_z}e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta+\delta_2\int_0^t\int_{R_z}e^{-\omega\eta}\theta_{,\alpha}\theta_{,\alpha}dx_2d\xi d\eta\Big]. \tag{3.44}$$

Using Lemma 2.3, we have

$$B_4(z,t)\leq \frac{1}{2}T_m^2\int_0^t\int_{R_z}e^{-\omega\eta}(\xi-z)w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta+\frac{1}{2}\int_0^t\int_{R_z}e^{-\omega\eta}(\xi-z)\theta_{,\alpha}\theta_{,\alpha}dx_2d\xi d\eta. \tag{3.45}$$

Inserting Eqs (3.42) and (3.45) into Eq (3.23), we obtain

$$\begin{aligned}E_2(z,t)\leq&\Big[\frac{h}{\pi\delta_2}+\frac{h}{\pi\sqrt{2\delta_2}}T_m+\frac{h\sqrt{\epsilon_1}}{2\delta_2\pi}\Big]\int_0^t\int_{R_z}e^{-\omega\eta}\delta_2\theta_{,\alpha}\theta_{,\alpha}dx_2d\xi d\eta\\&+\frac{h}{\pi\sqrt{2\delta_2}}T_m\Big[\frac{1}{2}\int_0^t\int_{R_z}e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta\Big]\\&+\frac{1}{2}T_m^2\int_0^t\int_{R_z}e^{-\omega\eta}(\xi-z)w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta+\frac{1}{2}\int_0^t\int_{R_z}e^{-\omega\eta}(\xi-z)\theta_{,\alpha}\theta_{,\alpha}dx_2d\xi d\eta.\end{aligned} \tag{3.46}$$

Similarly, we have

$$C_1(z,t)\leq \frac{h}{\pi}\Big[\int_0^t\int_{R_z}e^{-\omega\eta}(\Sigma_{,2})^2dx_2d\xi d\eta\int_0^t\int_{R_z}e^{-\omega\eta}(\Sigma_{,1})^2dx_2d\xi d\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{h}{2\delta_3\pi}\cdot\delta_3\int_0^t\int_{R_z}e^{-\omega\eta}\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2d\xi d\eta, \tag{3.47}$$
$$C_2(z,t)\leq \frac{1}{2}\int_0^t\Big[\int_{R_z}e^{-\omega\eta}(v^*_{,2})^2dx_2d\xi\int_{R_z}e^{-\omega\eta}\Sigma^4dx_2d\xi\Big]^{\frac{1}{2}}d\eta$$
$$\leq \frac{h\sqrt{\epsilon_1}}{2\delta_3\pi}\cdot\delta_3\int_0^t\int_{R_z}e^{-\omega\eta}\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2d\xi d\eta, \tag{3.48}$$
$$C_3(z,t)\leq C_m\Big[\int_0^t\int_{R_z}e^{-\omega\eta}(w_{,2})^2dx_2d\xi d\eta\Big]^{\frac{1}{2}}\Big[\int_0^t\int_{R_z}e^{-\omega\eta}\Sigma^2dx_2d\xi d\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{h}{\pi\sqrt{2\delta_3}}C_m\Big[\frac{1}{2}\int_0^t\int_{R_z}e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta+\delta_3\int_0^t\int_{R_z}e^{-\omega\eta}\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2d\xi d\eta\Big], \tag{3.49}$$
$$C_4(z,t)\leq \frac{1}{4}\omega\delta_3\int_0^t\int_{R_z}e^{-\omega\eta}\Sigma^2dx_2d\xi d\eta+\frac{\sigma^2}{\beta_3\omega}\int_0^t\int_{R_z}e^{-\omega\eta}(T_{,1})^2dx_2d\xi d\eta, \tag{3.50}$$
$$C_5(z,t)\leq \frac{1}{4\delta_3}\int_0^t\int_{R_z}e^{-\omega\eta}(\xi-z)w_{,\alpha}w_{,\alpha}dx_2d\xi d\eta+C_m^2\delta_3\int_0^t\int_{R_z}e^{-\omega\eta}(\xi-z)\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2d\xi d\eta \tag{3.51}$$

and

$$C_6(z,t) \le \frac{1}{\delta_3}\sigma^2 \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)T_{,\alpha}T_{,\alpha}dx_2 d\xi d\eta + \frac{1}{4}\delta_3 \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2 d\xi d\eta. \tag{3.52}$$

Inserting Eqs (3.47)–(3.52) into Eq (3.24), we obtain

$$\begin{aligned} E_3(z,t) \le & \Big[\frac{h}{2\delta_3\pi} + \frac{h}{\pi\sqrt{2\delta_3}}C_m + \frac{h\sqrt{\epsilon_1}}{2\delta_3\pi}\Big]\cdot\delta_3 \int_0^t \int_{R_z} e^{-\omega\eta}\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2 d\xi d\eta \\ & + \frac{h}{\pi\sqrt{2\delta_3}}C_m\Big[\frac{1}{2}\int_0^t \int_{R_z} e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2 d\xi d\eta\Big] + \frac{1}{4}\omega\delta_3 \int_0^t \int_{R_z} e^{-\omega\eta}\Sigma^2 dx_2 d\xi d\eta \\ & + \Big[\frac{1}{4}\delta_3 + C_m^2\delta_3\Big]\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2 d\xi d\eta + \frac{1}{4\delta_3}\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)w_{,\alpha}w_{,\alpha}dx_2 d\xi d\eta \\ & + \frac{\sigma^2}{\beta_3\omega}\int_0^t \int_{R_z} e^{-\omega\eta}(T_{,1})^2 dx_2 d\xi d\eta + \frac{1}{\delta_3}\sigma^2 \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)T_{,\alpha}T_{,\alpha}dx_2 d\xi d\eta. \end{aligned} \tag{3.53}$$

Now, inserting Eqs (3.41), (3.46) and (3.53) into Eq (3.28), choosing $\delta_2 < \frac{1}{2T_m^2}, \delta_3 < \frac{1}{4C_m^2}$ and noting Eqs (3.36) and (3.31), we obtain

$$\begin{aligned} E(z,t) \le & \frac{1}{2}E(z,t) + \frac{1}{2}n_1 h(z)\Big[-\frac{\partial}{\partial z}E(z,t)\Big] \\ & + \frac{\sigma^2\delta_3}{\beta_3\omega}\int_0^t \int_{R_z} e^{-\omega\eta}(T_{,1})^2 dx_2 d\xi d\eta + \sigma^2 \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)T_{,\alpha}T_{,\alpha}dx_2 d\xi d\eta, \end{aligned} \tag{3.54}$$

where

$$\begin{aligned} \frac{1}{2}n_1 \ge \max\Big\{ & \frac{1}{\pi}\Big[1 + \frac{\sqrt{2}}{\sqrt{\delta_2\omega}} + \frac{\sqrt{2}}{\sqrt{\delta_3\omega}}\Big] + \frac{T_m}{\pi\sqrt{2\delta_2}} + \frac{C_m}{\pi\sqrt{2\delta_3}}, 1 + \frac{\sqrt{2}}{\sqrt{\delta_2\omega\pi}}, \\ & \frac{1}{2\delta_3\pi} + \frac{1}{\pi\sqrt{2\delta_3}}C_m + \frac{\sqrt{\epsilon_1}}{2\delta_3\pi}, \frac{h}{\pi\delta_2} + \frac{1}{\pi\sqrt{2\delta_2}}T_m + \frac{\sqrt{\epsilon_1}}{2\delta_2\pi}\Big\}. \end{aligned}$$

From Lemma 2.4, we have

$$\int_0^t \int_{R_z} e^{-\omega\eta}T_{,\alpha}T_{,\alpha}dx_2 d\xi d\eta \le \frac{2}{\beta_2}m_1 e^{-2m_3\int_a^z \frac{1}{h(\zeta)}d\zeta} + \frac{2}{\beta_2}m_2 e^{-m_3\int_a^z \frac{1}{h(\zeta)}d\zeta}. \tag{3.55}$$

Integrating Eq (3.55) from z to ∞, we get

$$\begin{aligned} \int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)T_{,\alpha}T_{,\alpha}dx_2 d\xi d\eta \le & \frac{2}{\beta_2}m_1 \int_z^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi \\ & + \frac{2}{\beta_2}m_2 \int_z^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi. \end{aligned} \tag{3.56}$$

Combining Eqs (3.54)–(3.56), we can obtain Lemma 3.1.

4. Convergence results on the Soret coefficient

Based on Lemma 3.1, we derive our main results in this section. To do this, integrating Eq (3.37) from a to z, we obtain

$$
\begin{aligned}
E(z,t) \leq\ & E(a,t)e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta} \\
&+ \frac{4m_1\delta_3\sigma^2}{n_1\beta_3\beta_2\omega}e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta}\int_a^z \frac{1}{h(\xi)}e^{\int_a^\xi\{\frac{1}{n_1 h(\zeta)}-\frac{2m_3}{h(\zeta)}\}d\zeta}d\xi \\
&+ \frac{4m_2\sigma^2\delta_3}{n_1\beta_3\beta_2\omega}e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta}\int_a^z \frac{1}{h(\xi)}e^{\int_a^\xi\{\frac{1}{n_1 h(\zeta)}-\frac{m_3}{h(\zeta)}\}d\zeta}d\xi \\
&+ \frac{4m_1\sigma^2}{n_1\beta_2}e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta}\int_a^z \frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau \frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi d\tau \\
&+ \frac{4m_2\sigma^2}{n_1\beta_2}e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta}\int_a^z \frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau \frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi d\tau. \qquad (4.1)
\end{aligned}
$$

To get the convergence results on the Soret coefficient, we have to derive the upper bound for $E(a,t)$. To do this, we choose $z = a$ in Eq (3.37) to get

$$
\begin{aligned}
E(a,t) \leq\ & n_1 h(a)\Big[-\frac{\partial}{\partial z}E(a,t)\Big] + \frac{4\delta_3\sigma^2}{\beta_3\beta_2\omega}m_1 + \frac{4\delta_3\sigma^2}{\beta_3\beta_2\omega}m_2 \\
&+ \frac{4}{\beta_2}\sigma^2 m_1\int_a^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi + \frac{4}{\beta_2}\sigma^2 m_2\int_a^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi. \qquad (4.2)
\end{aligned}
$$

By differentiating Eqs (3.22)–(3.24), then choosing $z = a$ and using Eqs (3.19) and (3.20), it can be obtained that

$$
\begin{aligned}
-\frac{\partial}{\partial z}E(a,t) &= -\frac{\partial}{\partial z}E_1(a,t) - \delta_2\frac{\partial}{\partial z}E_2(a,t) - \delta_3\frac{\partial}{\partial z}E_3(a,t) \\
&= \delta_2\int_0^t\int_R e^{-\omega\eta}\nabla^\perp w\cdot\nabla\theta T\,dx_2 d\xi d\eta \\
&\quad + \delta_3\int_0^t\int_R e^{-\omega\eta}\nabla^\perp w\cdot\nabla\Sigma C^* dx_2 d\xi d\eta + \delta_3\sigma\int_0^t\int_R e^{-\omega\eta}T_{,\alpha}\Sigma_{,\alpha}dx_2 d\xi d\eta. \qquad (4.3)
\end{aligned}
$$

Using the Hölder inequality, Young's inequality and Lemmas 2.3 and 2.5, we have

$$
\begin{aligned}
&\delta_2\int_0^t\int_R e^{-\omega\eta}\nabla^\perp w\cdot\nabla\theta T\,dx_2 d\xi d\eta \\
&\qquad \leq \frac{1}{2}\delta_2 T_m^2\int_0^t\int_R e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2 d\xi d\eta + \frac{1}{2}\delta_2\int_0^t\int_R e^{-\omega\eta}\theta_{,\alpha}\theta_{,\alpha}dx_2 d\xi d\eta, \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
&\delta_3\int_0^t\int_R e^{-\omega\eta}\nabla^\perp w\cdot\nabla\Sigma C^* dx_2 d\xi d\eta \\
&\qquad \leq \delta_3 C_m^2\int_0^t\int_R e^{-\omega\eta}w_{,\alpha}w_{,\alpha}dx_2 d\xi d\eta + \frac{1}{4}\delta_3\int_0^t\int_R e^{-\omega\eta}\Sigma_{,\alpha}\Sigma_{,\alpha}dx_2 d\xi d\eta, \qquad (4.5)
\end{aligned}
$$

$$
\delta_3\sigma\int_0^t\int_R e^{-\omega\eta}T_{,\alpha}\Sigma_{,\alpha}dx_2 d\xi d\eta
$$

$$\leq \delta_3\sigma^2 \int_0^t \int_R e^{-\omega\eta} T_{,\alpha}T_{,\alpha} dx_2 d\xi d\eta + \frac{1}{4}\delta_3 \int_0^t \int_R e^{-\omega\eta}\Sigma_{,\alpha}\Sigma_{,\alpha} dx_2 d\xi d\eta$$
$$\leq \frac{2}{\beta_2}\delta_3\sigma^2 r(t) + \frac{1}{4}\delta_3 \int_0^t \int_R e^{-\omega\eta}\Sigma_{,\alpha}\Sigma_{,\alpha} dx_2 d\xi d\eta. \quad (4.6)$$

On the other hand, we choose $z = a$ in Eq (3.36) to get

$$-\frac{\partial}{\partial z}E(a,t) \geq \int_0^t \int_R e^{-\omega\eta}\Big[\frac{1}{4}\delta_2\omega\theta^2 + \frac{1}{4}\omega\delta_3\Sigma^2 + \frac{1}{2}w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big] dx_2 d\xi d\eta$$
$$+ \frac{1}{2}e^{-\omega t}\int_R \Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big] dx_2 d\xi. \quad (4.7)$$

Inserting Eqs (4.4)–(4.6) into Eq (4.3) and noting Eq (4.7), we get

$$-\frac{\partial}{\partial z}E(a,t) \leq \frac{1}{2}\Big[-\frac{\partial}{\partial z}E(a,t)\Big] + \frac{2}{\beta_2}\delta_3\sigma^2 r(t),$$

or

$$-\frac{\partial}{\partial z}E(a,t) \leq \frac{4}{\beta_2}\delta_3\sigma^2 r(t). \quad (4.8)$$

Inserting Eq (4.8) into Eq (4.2), we have

$$E(a,t) \leq n_2\sigma^2 + \frac{4}{\beta_2}\sigma^2 m_1 \int_a^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta} d\xi + \frac{4}{\beta_2}\sigma^2 m_2 \int_a^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta} d\xi, \quad (4.9)$$

where $n_2 = n_1(h(a) + h^{\frac{3}{2}}(a))\frac{4}{\beta_2}\delta_3 r(t) + \frac{4\delta_3}{\beta_3\beta_2\omega}m_1 + \frac{4\delta_3}{\beta_3\beta_2\omega}m_2$.

Now, inserting Eq (4.9) into Eq (4.1) and in light of Eq (3.31), we can obtain the following theorem.

Theorem 4.1 Letting (w,θ,Σ) is solution of Eqs (3.12)–(3.21) with $T_0, C_0, H, \widetilde{H} \in C^\infty$, then

$$(w,\theta,\Sigma) \to \mathbf{0},\ as\ \sigma \to 0.$$

Specifically

$$\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{4}\delta_2\omega\theta^2 + \frac{1}{4}\omega\delta_3\Sigma^2 + \frac{1}{2}w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big] dx_2 d\xi d\eta$$
$$+ \frac{1}{2}e^{-\omega t}\int_{R_z}(\xi - z)\Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big] dx_2 d\xi$$
$$\leq n_2\sigma^2 e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta}$$
$$+ \frac{4}{\beta_2}\sigma^2 m_1 \int_a^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta} d\xi e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta}$$
$$+ \frac{4}{\beta_2}\sigma^2 m_2 \int_a^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta} d\xi e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta}$$
$$+ \frac{4m_1\delta_3\sigma^2}{n_1\beta_3\beta_2\omega} e^{-\frac{1}{n_1}\int_a^z \frac{1}{h(\zeta)}d\zeta} \int_a^z \frac{1}{h(\xi)} e^{\int_a^\xi \{\frac{1}{n_1 h(\zeta)} - \frac{2m_3}{h(\zeta)}\}d\zeta} d\xi$$

$$+\frac{4m_2\sigma^2\delta_3}{n_1\beta_3\beta_2\omega}e^{-\frac{1}{n_1}\int_a^z\frac{1}{h(\zeta)}d\zeta}\int_a^z\frac{1}{h(\xi)}e^{\int_a^\xi\{\frac{1}{n_1h(\zeta)}-\frac{m_3}{h(\zeta)}\}d\zeta}d\xi$$
$$+\frac{4m_1\sigma^2}{n_1\beta_2}e^{-\frac{1}{n_1}\int_a^z\frac{1}{h(\zeta)}d\zeta}\int_a^z\frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau\frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-2m_3\int_a^\xi\frac{1}{h(\zeta)}d\zeta}d\xi d\tau$$
$$+\frac{4m_2\sigma^2}{n_1\beta_2}e^{-\frac{1}{n_1}\int_a^z\frac{1}{h(\zeta)}d\zeta}\int_a^z\frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau\frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-m_3\int_a^\xi\frac{1}{h(\zeta)}d\zeta}d\xi d\tau. \quad (4.10)$$

Next, we give some examples.
Remark 4.1 If $h(z)=h$ is a positive constant, then

$$\int_a^z\frac{1}{h(\zeta)}d\zeta=\frac{1}{h}(z-a).$$

Therefore, we have

$$\int_\tau^\infty e^{-m_3\int_a^\xi\frac{1}{h(\zeta)}d\zeta}d\xi=\frac{h}{m_3}e^{-\frac{m_3}{h}(\tau-a)},\ \int_\tau^\infty e^{-2m_3\int_a^\xi\frac{1}{h(\zeta)}d\zeta}d\xi=\frac{h}{2m_3}e^{-\frac{2m_3}{h}(\tau-a)}. \quad (4.11)$$

Choosing n_1 such that $\frac{1}{n_1}\neq 2m_3$ and $\frac{1}{n_1}\neq m_3$, then

$$\int_a^z\frac{1}{h(\xi)}e^{\int_a^\xi\{\frac{1}{n_1h(\zeta)}-\frac{m_3}{h(\zeta)}\}d\zeta}d\xi=\frac{1}{h}\int_a^z e^{[\frac{1}{n_1h}-\frac{m_3}{h}](\xi-a)}d\xi$$
$$=\frac{1}{[\frac{1}{n_1}-m_3]}\Big[e^{[\frac{1}{n_1h}-\frac{m_3}{h}](z-a)}-1\Big], \quad (4.12)$$
$$\int_a^z\frac{1}{h(\xi)}e^{\int_a^\xi\{\frac{1}{n_1h(\zeta)}-\frac{2m_3}{h(\zeta)}\}d\zeta}d\xi=\frac{1}{[\frac{1}{n_1}-2m_3]}\Big[e^{[\frac{1}{n_1h}-\frac{2m_3}{h}](z-a)}-1\Big], \quad (4.13)$$

and

$$\int_a^z\frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau\frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-2m_3\int_a^\xi\frac{1}{h(\zeta)}d\zeta}d\xi d\tau=\frac{h}{2m_3[\frac{1}{n_1}-2m_3]}\Big[e^{[\frac{1}{n_1h}-\frac{2m_3}{h}](z-a)}-1\Big], \quad (4.14)$$
$$\int_a^z\frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau\frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-m_3\int_a^\xi\frac{1}{h(\zeta)}d\zeta}d\xi d\tau=\frac{h}{m_3[\frac{1}{n_1}-m_3]}\Big[e^{[\frac{1}{n_1h}-\frac{m_3}{h}](z-a)}-1\Big]. \quad (4.15)$$

Inserting Eqs (4.11)–(4.15) into Eq (4.10), we get

$$\int_0^t\int_{R_z}e^{-\omega\eta}(\xi-z)\Big[\frac{1}{4}\delta_2\omega\theta^2+\frac{1}{4}\omega\delta_3\Sigma^2+\frac{1}{2}w_{,\alpha}w_{,\alpha}+\delta_2\theta_{,\alpha}\theta_{,\alpha}+\delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+\frac{1}{2}e^{-\omega t}\int_{R_z}(\xi-z)\Big[\delta_2\theta^2+\delta_3\Sigma^2\Big]dx_2d\xi$$
$$\leq n_4\sigma^2e^{-n_3(z-a)}+n_5\sigma^2\Big[e^{-\frac{2m_3}{h}(z-a)}-e^{-n_3(z-a)}\Big]$$
$$+n_6\sigma^2\Big[e^{-\frac{m_3}{h}(z-a)}-e^{-n_3(z-a)}\Big], \quad (4.16)$$

where

$$n_3=\frac{1}{n_1h},n_4=n_2+\frac{2m_1h}{m_3\beta_2}+\frac{4m_2h}{m_3\beta_2},$$

$$n_5 = \frac{2m_1\delta_3}{n_1\beta_3\beta_2\omega}\frac{1}{[\frac{1}{n_1} - m_3]} + \frac{2m_1}{n_1\beta_2}\frac{h}{2m_3[\frac{1}{n_1} - 2m_3]}$$
$$n_6 = \frac{4m_2\delta_3}{n_1\beta_3\beta_2\omega}\frac{1}{[\frac{1}{n_1} - m_3]} + \frac{4m_2}{n_1\beta_2}\frac{h}{m_3[\frac{1}{n_1} - m_3]}.$$

From Eq (4.16), we can conclude that Theorem 4.1 not only shows the convergence of the solutions of Eqs (3.12)–(3.21) on the coefficient σ, but it also shows a exponentially decay result as $z \to \infty$.

Remark 4.2. If $h(z) = \sqrt{z}$, then

$$\int_a^z \frac{1}{h(\zeta)}d\zeta = \int_a^z \frac{1}{\sqrt{\zeta}}d\zeta = 2(\sqrt{z} - \sqrt{a}).$$

Therefore

$$\begin{aligned}\int_z^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi &= \int_z^\infty e^{-2m_3[\sqrt{\xi}-\sqrt{a}]}d\xi \\ &= -\int_z^\infty \frac{1}{m_3}\sqrt{\xi}d\Big\{e^{-2m_3[\sqrt{\xi}-\sqrt{a}]}\Big\} \\ &\le \frac{1}{m_3}\sqrt{z}e^{-2m_3[\sqrt{z}-\sqrt{a}]}, \end{aligned} \tag{4.17}$$

$$\int_z^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi \le \frac{1}{2m_3}\sqrt{z}e^{-4m_3[\sqrt{z}-\sqrt{a}]}. \tag{4.18}$$

Moreover, we also have

$$\begin{aligned}\int_a^z \frac{1}{h(\xi)}e^{\int_a^\xi\{\frac{1}{n_1 h(\zeta)}-\frac{2m_3}{h(\zeta)}\}d\zeta}d\xi &= \int_a^z \frac{1}{\sqrt{\xi}}e^{2[\frac{1}{n_1}-2m_3][\sqrt{\xi}-\sqrt{a}]}d\xi \\ &= \frac{1}{\frac{1}{n_1}-2m_3}\Big[e^{2[\frac{1}{n_1}-2m_3][\sqrt{z}-\sqrt{a}]} - 1\Big], \end{aligned} \tag{4.19}$$

$$\int_a^z \frac{1}{h(\xi)}e^{\int_a^\xi\{\frac{1}{n_1 h(\zeta)}-\frac{m_3}{h(\zeta)}\}d\zeta}d\xi = \frac{1}{\frac{1}{n_1}-m_3}\Big[e^{2[\frac{1}{n_1}-m_3][\sqrt{z}-\sqrt{a}]} - 1\Big], \tag{4.20}$$

$$\begin{aligned}\int_a^z \frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau \frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi d\tau &= \frac{1}{2m_3}\int_a^z e^{2[\frac{1}{n_1}-2m_3][\sqrt{\tau}-\sqrt{a}]}d\tau \\ &\le \frac{1}{2m_3[\frac{1}{n_1}-2m_3]}\sqrt{z}\Big[e^{2[\frac{1}{n_1}-2m_3][\sqrt{z}-\sqrt{a}]} - 1\Big], \end{aligned} \tag{4.21}$$

and

$$\int_a^z \frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^\tau \frac{1}{h(\zeta)}d\zeta}\int_\tau^\infty e^{-m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi d\tau \le \frac{1}{m_3[\frac{1}{n_1}-m_3]}\sqrt{z}\Big[e^{2[\frac{1}{n_1}-m_3][\sqrt{z}-\sqrt{a}]} - 1\Big]. \tag{4.22}$$

Inserting Eqs (4.19)–(4.22) into Eq (4.10), we obtain

$$\int_0^t \int_{R_z} e^{-\omega\eta}(\xi - z)\Big[\frac{1}{4}\delta_2\omega\theta^2 + \frac{1}{4}\omega\delta_3\Sigma^2 + \frac{1}{2}w_{,\alpha}w_{,\alpha} + \delta_2\theta_{,\alpha}\theta_{,\alpha} + \delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+ \frac{1}{2}e^{-\omega t}\int_{R_z}(\xi - z)\Big[\delta_2\theta^2 + \delta_3\Sigma^2\Big]dx_2d\xi$$
$$\leq n_7\sigma^2 e^{-\frac{2}{n_1}(\sqrt{z}-\sqrt{a})}$$
$$+ \frac{4m_1\delta_3\sigma^2}{n_1\beta_3\beta_2\omega}\frac{1}{\frac{1}{n_1} - 2m_3}\Big[e^{-4m_3[\sqrt{z}-\sqrt{a}]} - e^{-\frac{2}{n_1}(\sqrt{z}-\sqrt{a})}\Big]$$
$$+ \frac{4m_2\sigma^2\delta_3}{n_1\beta_3\beta_2\omega}\frac{1}{\frac{1}{n_1} - m_3}\Big[e^{-2m_3[\sqrt{z}-\sqrt{a}]} - e^{-\frac{2}{n_1}(\sqrt{z}-\sqrt{a})}\Big]$$
$$+ \frac{4m_1\sigma^2}{n_1\beta_2}\frac{1}{2m_3[\frac{1}{n_1} - 2m_3]}\sqrt{z}\Big[e^{-4m_3[\sqrt{z}-\sqrt{a}]} - e^{-\frac{2}{n_1}(\sqrt{z}-\sqrt{a})}\Big]$$
$$+ \frac{4m_2\sigma^2}{n_1\beta_2}\frac{1}{m_3[\frac{1}{n_1} - m_3]}\sqrt{z}\Big[e^{-2m_3[\sqrt{z}-\sqrt{a}]} - e^{-\frac{2}{n_1}(\sqrt{z}-\sqrt{a})}\Big], \tag{4.23}$$

where

$$n_7 = n_2 + \frac{2m_1\sqrt{a}}{m_3\beta_2} + \frac{4m_2\sqrt{a}}{m_3\beta_2}.$$

From Eq (4.23), we can conclude that Theorem 4.1 not only shows the convergence of the solutions of Eqs (3.12)–(3.21) on the coefficient σ, but it also shows a exponentially decay result as $z \to \infty$. Obviously, the decay rate is slightly slower than that in Remark 3.1.

Remark 4.3. If $h(z)$ satisfies

$$h(z) = z^{\frac{2}{3}}, \tag{4.24}$$

then

$$\int_a^z \frac{1}{h}d\zeta = \int_a^z \frac{1}{\zeta^{\frac{2}{3}}}d\zeta = 3\Big[\sqrt[3]{z} - \sqrt[3]{a}\Big].$$

Therefore, we have

$$\int_\tau^\infty e^{-2m_3\int_a^\xi \frac{1}{h(\zeta)}d\zeta}d\xi = \int_\tau^\infty e^{-6m_3\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}d\xi$$
$$= -\frac{1}{2m_3}\int_\tau^\infty \xi^{\frac{2}{3}}d\Big[e^{-6m_3\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}\Big]$$
$$\leq -\frac{1}{2m_3}\tau^{\frac{2}{3}}\int_\tau^\infty d\Big[e^{-6m_3\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}\Big]$$
$$= \frac{1}{2m_3}\tau^{\frac{2}{3}}e^{-6m_3\left(\sqrt[3]{\tau}-\sqrt[3]{a}\right)}, \tag{4.25}$$

$$\int_{\tau}^{\infty} e^{-m_3\int_a^{\xi}\frac{1}{h(\zeta)}d\zeta}d\xi \leq \frac{1}{m_3}\tau^{\frac{2}{3}}e^{-3m_3\left(\sqrt[3]{\tau}-\sqrt[3]{a}\right)}. \tag{4.26}$$

Choosing $n_1 > 3$, we get

$$\int_a^z \frac{1}{h(\xi)}e^{\int_a^{\xi}\{\frac{1}{n_1 h(\zeta)}-\frac{2m_3}{h(\zeta)}\}d\zeta}d\xi = \int_a^z \frac{1}{\xi^{\frac{2}{3}}}e^{[\frac{3}{n_1}-6m_3]\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}d\xi$$

$$= \frac{3}{\frac{3}{n_1}-6m_3}\Big[e^{[\frac{3}{n_1}-6m_3]\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}-1\Big], \tag{4.27}$$

$$\int_a^z \frac{1}{h(\xi)}e^{\int_a^{\xi}\{\frac{1}{n_1 h(\zeta)}-\frac{m_3}{h(\zeta)}\}d\zeta}d\xi \leq \frac{3}{\frac{3}{n_1}-3m_3}\Big[e^{[\frac{3}{n_1}-3m_3]\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}-1\Big], \tag{4.28}$$

and

$$\int_a^z \frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^{\tau}\frac{1}{h(\zeta)}d\zeta}\int_{\tau}^{\infty} e^{-2m_3\int_a^{\xi}\frac{1}{h(\zeta)}d\zeta}d\xi d\tau$$

$$= \frac{1}{2m_3}\int_a^z e^{[\frac{3}{n_1}-6m_3]\left(\sqrt[3]{\tau}-\sqrt[3]{a}\right)}d\tau$$

$$\leq \frac{1}{2m_3[\frac{3}{n_1}-6m_3]}z^{\frac{2}{3}}\Big[e^{[\frac{3}{n_1}-6m_3]\left(\sqrt[3]{\tau}-\sqrt[3]{a}\right)}-1\Big], \tag{4.29}$$

$$\int_a^z \frac{1}{h(\tau)}e^{\frac{1}{n_1}\int_a^{\tau}\frac{1}{h(\zeta)}d\zeta}\int_{\tau}^{\infty} e^{-m_3\int_a^{\xi}\frac{1}{h(\zeta)}d\zeta}d\xi d\tau$$

$$\leq \frac{1}{2m_3[\frac{3}{n_1}-3m_3]}z^{\frac{2}{3}}\Big[e^{[\frac{3}{n_1}-3m_3]\left(\sqrt[3]{\tau}-\sqrt[3]{a}\right)}-1\Big]. \tag{4.30}$$

Inserting Eqs (4.27)–(4.30) into Eq (4.10), we obtain

$$\int_0^t\int_{R_z} e^{-\omega\eta}(\xi-z)\Big[\frac{1}{4}\delta_2\omega\theta^2+\frac{1}{4}\omega\delta_3\Sigma^2+\frac{1}{2}w_{,\alpha}w_{,\alpha}+\delta_2\theta_{,\alpha}\theta_{,\alpha}+\delta_3\Sigma_{,\alpha}\Sigma_{,\alpha}\Big]dx_2d\xi d\eta$$

$$+\frac{1}{2}e^{-\omega t}\int_{R_z}(\xi-z)\Big[\delta_2\theta^2+\delta_3\Sigma^2\Big]dx_2d\xi$$

$$\leq n_8\sigma^2 e^{-\frac{3}{n_1}\left[\sqrt[3]{z}-\sqrt[3]{a}\right]}$$

$$+\frac{4m_1\delta_3\sigma^2}{n_1\beta_3\beta_2\omega}\frac{3}{\frac{3}{n_1}-6m_3}\Big[e^{-6m_3\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}-e^{-\frac{3}{n_1}\left[\sqrt[3]{z}-\sqrt[3]{a}\right]}\Big]$$

$$+\frac{4m_2\sigma^2\delta_3}{n_1\beta_3\beta_2\omega}\frac{3}{\frac{3}{n_1}-3m_3}\Big[e^{-3m_3\left(\sqrt[3]{\xi}-\sqrt[3]{a}\right)}-e^{-\frac{3}{n_1}\left[\sqrt[3]{z}-\sqrt[3]{a}\right]}\Big]$$

$$+\frac{4m_1\sigma^2}{n_1\beta_2}\frac{1}{2m_3[\frac{3}{n_1}-6m_3]}z^{\frac{2}{3}}\Big[e^{-6m_3\left(\sqrt[3]{z}-\sqrt[3]{a}\right)}-e^{-\frac{3}{n_1}\left[\sqrt[3]{z}-\sqrt[3]{a}\right]}\Big]$$

$$+\frac{4m_2\sigma^2}{n_1\beta_2}\frac{1}{m_3[\frac{3}{n_1}-3m_3]}z^{\frac{2}{3}}\Big[e^{-3m_3\left(\sqrt[3]{z}-\sqrt[3]{a}\right)}-e^{-\frac{3}{n_1}\left[\sqrt[3]{z}-\sqrt[3]{a}\right]}\Big], \tag{4.31}$$

where

$$n_8 = n_2 + \frac{2m_1 a^{\frac{2}{3}}}{m_3\beta_2} + \frac{4m_2 a^{\frac{2}{3}}}{\beta_2 m_3}.$$

In this case, the inequality (Eq 4.31) not only shows the convergence of the solutions of Eqs (3.12)–(3.21) on the coefficient σ, but it also shows a exponentially decay result as $z \to \infty$.

5. Conclusions

In this paper, the convergence of the solutions of Eqs (3.12)–(3.21) on the coefficient σ has been obtained and three examples have been given. Obviously, the convergence of various systems of partial differential equations defined on R is rare. But Eq (1.1) is linear. It will be meaningful to study nonlinear equations (e.g., Brinkman equations, Forchheimer equations) by using the method in this paper.

Acknowledgments

The authors express their heartfelt thanks to the editors and referees who have provided some important suggestions. This work is supported by the Tutor System Rroject of Guangzhou Huashang College (2021HSDS16).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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