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Research article

On a class of double phase problem with nonlinear boundary conditions

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Abstract: The existence of nontrivial solutions of the double phase problem with nonlinear boundary value condition is an important quasilinear problem: we use variational techniques and sum decomposition of a space $W_0^{1,\xi}(\Omega)$ to prove the existence of infinitely many solutions of the problem considered. Moreover, our conditions are suitable and different from those considered previously.

Keywords: double phase problem; nonlinear boundary condition; variational method; infinitely many solutions; sign-changing potential

1. Introduction

The interest in variational problems with double phase operator is founded on their popular in various fields of mathematical physics, such as plasma physics, biophysics and chemical reactions, strongly anisotropic materials, Lavrentiev's phenomenon, etc.; we refer the reader to [1–6] and references therein.

In this paper, we study of the following double phase problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) + |u|^{p-2}u + \mu(x)|\nabla u|^{q-2}u = k(x,u), & \text{in } \Omega, \\ (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot \nu = h(x,u), & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain, ν is the outer normal unit vector on $\partial\Omega$, $-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)$ is a double phase operator,

$$\begin{aligned} &1 < p < N, p < q < p_*, \\ &\mu \in L^1(\Omega),\ \mu(x) \geq 0 \text{ for a.e. } x \in \Omega, \end{aligned} \tag{1.1}$$

k and h are Carathéodory functions on $\Omega\times\mathbb{R}$ and $\partial\Omega\times\mathbb{R}$, respectively. The double phase operator has been widely studied by many scholars; see [7–10] and references therein. Special cases of the double phase operator, studied extensively in the literature, occur when $\mu(x) \equiv 1$ (the (p,q)-Laplacian) or when $\mu(x) \equiv 0$ (the p-Laplace differential operator). For the existence results on quasilinear equations with the (p,q)-Laplace differential operator we refer to the papers of Baldelli et al. [11], Baldelli and Filippucci [12], and the references therein.

In the context of double phase problems with different boundary conditions we refer to the papers of [13–21] for the Dirichlet boundary condition, Papageorgiou et al. [22, 23] for the Robin boundary condition, Crespo-Blanco et al. [24] and Farkas et al. [25] for the Neumann type boundary condition.

Recently, problem (P) has begun to receive more and more attention, see, for example [26, 27]. In particular, Gasinski and Winkert [26] studied the existence of three nontrivial solutions for (P) by assuming the following conditions:

(H) $k : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are Carathéodory functions and satisfies the following conditions:

(i) for some $q < r_1 < p^* := \frac{Np}{N-p}$, $q < r_2 < p_* := \frac{(N-1)p}{N-p}$, there exist constants $c_1, c_2 > 0$ such that

$$|k(x,t)| \leq c_1(1 + |t|^{r_1-1}),\ \forall x \in \Omega, t \in \mathbb{R},$$

$$|h(x,t)| \leq c_2(1 + |t|^{r_2-1}),\ \forall x \in \partial\Omega, t \in \mathbb{R}.$$

(ii) $\frac{k(x,t)}{|t|^{q-2}t} \to +\infty$ as $|t| \to +\infty$ uniformly in $x \in \Omega$, $\frac{h(x,t)}{|t|^{q-2}t} \to +\infty$ as $|t| \to +\infty$ uniformly in $x \in \partial\Omega$.

(iii) $\frac{k(x,t)}{|t|^{p-2}t} \to 0$ as $t \to 0$ uniformly in $x \in \Omega$, $\frac{h(x,t)}{|t|^{p-2}t} \to 0$ as $t \to 0$ uniformly in $x \in \partial\Omega$.

(iv) the functions $t \mapsto k(x,t)t - qK(x,t)$ and $t \mapsto h(x,t)t - qH(x,t)$ are nonincreasing on $\mathbb{R}_-$ and nondecreasing on $\mathbb{R}_+$ for all $x \in \Omega$ and for all $x \in \partial\Omega$, respectively, where $K(x,t) = \int_0^t k(x,s)ds$ and $H(x,t) = \int_0^t h(x,s)ds$.

(v) the functions $\frac{k(x,t)}{|t|^{q-1}}$ and $\frac{h(x,t)}{|t|^{q-1}}$ are strictly increasing on $(-\infty, 0)$ and on $(0, +\infty)$ for all $x \in \Omega$ and for all $x \in \partial\Omega$, respectively.

Moreover, Cui and Sun [27] studied the existence of infinitely many solutions of problem (P) when (H)-(i), (H)-(ii), the following assumptions:

(vi) $\frac{K(x,t)}{|t|^q} \to +\infty$ as $|t| \to +\infty$ uniformly in $x \in \Omega$; $\frac{H(x,t)}{|t|^q} \to +\infty$ as $|t| \to +\infty$ uniformly in $x \in \partial\Omega$.

(vii) there exist constants $c_3, c_4 > 0$ such that

$$\mathbb{K}(x,t) \leq \mathbb{K}(x,s) + c_3,\ x \in \Omega,\quad s < t < 0 \text{ or } 0 < t < s,$$

and

$$\mathbb{H}(x,t) \leq \mathbb{H}(x,s) + c_4,\ x \in \partial\Omega,\ s < t < 0 \text{ or } s < t < 0,$$

where $\mathbb{K}(x,t) := k(x,t)t - qK(x,t)$ and $\mathbb{H}(x,t) := h(x,t)t - qH(x,t)$.

(viii) $k(x,-t) = -k(x,t)$ for $(x,t) \in \Omega \times \mathbb{R}$, $h(x,-t) = -h(x,t)$ for $(x,t) \in \partial\Omega \times \mathbb{R}$.

Assumption (vii) is originally due to Miyagaki and Souto [28] in the case $p = 2$ and $\mu \equiv 0$ and it is a weaker condition than (iv) or (v). Motivated by all results mentioned above, we will further study the existence of infinitely many nontrivial solutions of problem (P) under weaker conditions, which improves and develops the result of [26, 27]. The basic idea of proving our main results is motivated by the arguments used in [29]. To state our main result, we assume that k and h satisfy the following conditions:

(h_1) $\frac{K(x,t)}{|t|^q}, \frac{H(y,t)}{|t|^q} \to +\infty$ as $|t| \to \infty$ uniformly in $x \in \Omega$, $y \in \partial\Omega$, and there is a constant $r_0 > 0$ such that

$$K(x,t), H(y,t) \geq 0,\ \forall x \in \Omega, y \in \partial\Omega, t \in \mathbb{R},\ |t| \geq r_0;$$

(h_2) $\mathbb{K}(x,t) \geq 0$, $\mathbb{H}(x,t) \geq 0$ and for some $\sigma > \frac{N}{p}$ and $\hat{\sigma} > \frac{N-1}{p-1}$, there exist constants $c_3, c_4 > 0$ such that

$$|K(x,t)|^{\sigma} \leq c_3|t|^{p\sigma}\mathbb{K}(x,t),\ \forall x \in \Omega, t \in \mathbb{R},\ |t| \geq r_0,$$
$$|H(x,t)|^{\hat{\sigma}} \leq c_4|t|^{p\hat{\sigma}}\mathbb{H}(x,t),\ \forall x \in \partial\Omega, t \in \mathbb{R},\ |t| \geq r_0;$$

(h_3) for some $\vartheta > q$, there exist constants $\theta_1, \theta_2 > 0$ such that

$$\vartheta K(x,t) \leq tk(x,t) + \theta_1|t|^p,\ \forall x \in \Omega, t \in \mathbb{R},$$

$$\vartheta H(x,t) \leq th(x,t) + \theta_2|t|^p,\ \forall x \in \partial\Omega, t \in \mathbb{R}.$$

Now we state our main result.

Theorem 1.1. *If assumptions* (H)-(i), (H)-(viii)*, and* $(h_1)-(h_2)$ *are satisfied, then problem* (P) *possesses infinitely many nontrivial solutions.*

Theorem 1.2. *If the condition* (h_3) *is used in place of* (h_2)*, then the conclusion of Theorem 1.1 holds.*

Remark 1.1. *It is important to point out that* $K(x,t)$ *is allowed to be sign-changing under assumptions on* k *of Theorems 1.1 and 1.2, which generalizes Theorem 1.2 in [27]. We use* (h_1) *and* (h_2) *to replace (H)-(vi), or (H)-(ii) and (H)-(vii), which are essential in the study [27]. Another point was that the usual assumption (iii) is removed. The main difficulty in treating problem* (P) *is to verify the boundedness of the Cerami sequences. To overcome this difficulty, we use a method of decomposing regions and combining growth conditions.*

The remainder of this paper is organized as follows. In Section 2, we recall some basic facts regarding the Musielak-Orlicz Lebesgue space and Musielak-Orlicz Sobolev space which we will use later. In Section 3, we prove Theorems 1.1 and 1.2 by applying the fountain theorem.

2. Preliminaries

In this section, we will recall some necessary facts about the Musielak-Orlicz space. For the details we refer to [7, 29–32] and the references therein.

Let hypothesis (1.1) be satisfied and let us define the function by

$$\xi(x,t) = t^p + \mu(x)t^q,\ \forall (x,t) \in \Omega \times [0,+\infty).$$

Based on this we can define the modular function given by

$$\rho(u) = \int_{\Omega} \xi(x,|u|)dx.$$

Then we define the Musielak-Orlicz function space by

$$L^{\xi}(\Omega) = \Big\{u|u : \Omega \to \mathbb{R} \text{ is measurable and } \rho(u) < +\infty\Big\},$$

equipped with the Luxemburg norm $|u|_{\xi} = \inf\{\lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1\}$. Moreover, the corresponding Sobolev space $W^{1,\xi}(\Omega)$ is defined by

$$W^{1,\xi}(\Omega) = \{u \in L^{\xi}(\Omega) : |\nabla u| \in L^{\xi}(\Omega)\}$$

with the norm $\|u\| = |u|_\xi + |\nabla u|_\xi$. With these norm, the Sobolev space $W^{1,\xi}(\Omega)$ is separable reflexive Banach space; see Colasuonno and Squassina [7, Proposition 2.14] for the details.

We denote by $L^s(\Omega)$ and $L^s(\partial\Omega)$ the usual Lebesgue spaces endowed with the norm $|\cdot|_s$ and $|\cdot|_{s,\partial\Omega}$, respectively, for $1 \le s < +\infty$. Then from Proposition 2.15 of Colasuonno and Squassina [7], we known that the embedding

$$W^{1,\xi}(\Omega) \hookrightarrow L^\gamma(\Omega) \text{ and } W^{1,\xi}(\Omega) \hookrightarrow L^{\gamma'}(\partial\Omega) \tag{2.1}$$

are compact whenever $1 \le \gamma < p^*$ and $1 \le \gamma' < p_*$, where p_* and p^* are given in $(H) - (i)$. From Liu and Dai [10, Proposition 2.1], we directly obtain that

$$\min\{|u|_\xi^p, |u|_\xi^q\} \le \rho(u) \le \max\{|u|_\xi^p, |u|_\xi^q\}$$

for all $u \in L^\xi(\Omega)$. Analogously, if we define the modular function

$$\widehat{\rho}(u) = \rho(\nabla u) + \rho(u), \ \forall u \in W^{1,\xi}(\Omega),$$

then we have the following relations

$$\min\{\|u\|^p, \|u\|^q\} \le \widehat{\rho}(u) \le \max\{\|u\|^p, \|u\|^q\} \tag{2.2}$$

for all $u \in W^{1,\xi}(\Omega)$, see Gasinski and Winkert [26, Proposition 2.3].

Throughout this paper, we write $E := W^{1,\xi}(\Omega)$ and E^* is the dual space of E. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the dual pairing of E and its dual E^*.

Next, we consider the operator $L : E \to E^*$ defined by

$$\langle L(u), v \rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) \cdot \nabla v dx + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v dx.$$

From Manouni et al. [33, Proposition 2.6], we know that the operator L is bounded, continuous, monotone and of type (S_+), that is,

$$u_n \rightharpoonup u \text{ in } E \text{ and } \limsup_{n\to+\infty} \langle L(u_n), u_n - u \rangle \le 0,$$

imply $u_n \to u$ in E.

3. The proof of Theorems 1.1 and 1.2

In this section, we will discuss the existence of infinitely many solutions for (P) under suitable assumptions. For this purpose, we introduce the energy functional $I : E \to \mathbb{R}$ of problem (P) given by

$$\begin{aligned} I(u) = &\int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{\mu(x)}{q}|\nabla u|^q) dx + \int_\Omega (\frac{1}{p}|u|^p + \frac{\mu(x)}{q}|u|^q) dx \\ &- \int_\Omega K(x,u) dx - \int_{\partial\Omega} H(x,u) d\sigma. \end{aligned} \tag{3.1}$$

Firstly, we can introduce the following definition of weak solutions to problem (P).

Definition 3.1. A function $u \in E$ is called a weak solution of problem (P) if

$$\begin{aligned}\int_\Omega (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot \nabla v dx + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v dx \\ = \int_\Omega k(x,u)v dx + \int_{\partial\Omega} h(x,u)v d\sigma,\end{aligned} \tag{3.2}$$

is satisfied for all test functions $v \in E$.

The following assertion can be found in [7, Proposition 2.1].

Lemma 3.1. Under assumption $(H)-(i)$, the functional $I \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle I'(u), v\rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u)\cdot \nabla v dx + \int_\Omega (|u|^{p-2}u + \mu(x)|u|^{q-2}u)v dx \\ - \int_\Omega k(x,u)v dx - \int_{\partial\Omega} h(x,u)v d\sigma.\end{aligned} \tag{3.3}$$

for all $u, v \in E$. Moreover, $\psi' : E \to E^*$ is weakly-strongly continuous, namely, $u_n \rightharpoonup u$ implies $\psi'(u_n) \to \psi'(u_n)$, where $\psi(u) = \int_\Omega K(x,u)dx - \int_{\partial\Omega} H(x,u)d\sigma$.

Let us recall the definition of Cerami condition.

Definition 3.2. We say that $I \in C^1(E,\mathbb{R})$ satisfies the Cerami condition ($(C)_c$-condition for short) if every sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \text{ and } \|I'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0 \tag{3.4}$$

admits a strongly convergent subsequence. Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$-sequence for short.

To state the Fountain Theorem of Bartsch, we introduce some notations. Let X be a reflexive and separable Banach space, then there are $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that

$$X = \overline{\text{span}\{e_i : i \in N\}} \text{ and } X^* = \overline{\text{span}\{e_i^* : i \in N\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let us define $X_i = \mathbb{R}e_i$, $Y_n = \oplus_{i=1}^n X_i$ and $Z_n = \overline{\oplus_{i\geq n} X_i}$.

Now, let's remember the well known Fountain Theorem.

Lemma 3.2. ([35, Theorem 2.5]) Let $I \in C^1(X,\mathbb{R})$ be a functional satisfying the $(C)_c$-condition for all $c > 0$ and let $I(-u) = I(u)$. If for each $n \in N$, there exist $\tilde{\rho}_n > \tilde{\delta}_n > 0$ such that the following assumptions hold:

(A_1) $b_n := \inf\{I(u) : u \in Z_n, \|u\| = \tilde{\delta}_n\} \to +\infty$ as $n \to +\infty$;

(A_2) $a_n := \max\{I(u) : u \in Y_n, \|u\| = \tilde{\rho}_n\} \leq 0$,

then I has a sequence of critical values tending to $+\infty$.

Remark 3.1. In [34, 35], the Fountain Theorem is established under the Palais-Smale (PS) condition. It is known that the Cerami condition is weaker than the (PS) condition. However, the Deformation Theorem is still valid under the Cerami condition, see [36] for the details, as a consequence, we can also get the Fountain Theorem under the Cerami condition.

Lemma 3.3. Assume that $(H)-(i)$, (h_1) and (h_2) hold. If $\{u_n\} \subset E$ is a $(C)_c$ sequence of I, then $\{u_n\}$ is bounded in E.

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence of the functional I, that is, a sequence satisfying (3.4). We will argue by contradiction. If Thus, by (h_1), for n large we have

$$\begin{aligned} c+1 \geq & I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\ =& (\frac{1}{p} - \frac{1}{q}) \int_\Omega (|\nabla u_n|^p + |u_n|^p)dx + \frac{1}{q}\int_\Omega \mathbb{K}(x, u_n)dx + \frac{1}{q}\int_{\partial\Omega} \mathbb{H}(x, u_n)d\sigma \\ \geq & \frac{1}{q}\int_\Omega \mathbb{K}(x, u_n)dx + \frac{1}{q}\int_{\partial\Omega} \mathbb{H}(x, u_n)d\sigma. \end{aligned} \tag{3.5}$$

It follows from (2.2) and (3.4) that

$$\begin{aligned} \frac{c+o(1)}{\|u_n\|^p} = \frac{I(u_n)}{\|u_n\|^p} \geq & \frac{1}{q}\frac{\int_\Omega [|\nabla u_n|^p + |u_n|^p + \mu(x)(|\nabla u_n|^q + |u_n|^q)]dx}{\|u_n\|^p} \\ & - \int_\Omega \frac{K(x, u_n)}{\|u_n\|^p}dx - \int_{\partial\Omega} \frac{H(x, u_n)}{\|u_n\|^p}d\sigma \\ \geq & \frac{1}{q} - \int_\Omega \frac{K(x, u_n)}{\|u_n\|^p}dx - \int_{\partial\Omega} \frac{H(x, u_n)}{\|u_n\|^p}d\sigma \\ \geq & \frac{1}{q} - \int_\Omega \frac{|K(x, u_n)|}{\|u_n\|^p}dx - \int_{\partial\Omega} \frac{|H(x, u_n)|}{\|u_n\|^p}d\sigma \end{aligned}$$

and consequently,

$$\frac{1}{q} \leq \limsup_{n\to\infty}\Big(\int_\Omega \frac{|K(x, u_n)|}{\|u_n\|^p}dx + \int_{\partial\Omega} \frac{|H(x, u_n)|}{\|u_n\|^p}d\sigma\Big). \tag{3.6}$$

For $0 \leq \alpha < \beta$, let $\Lambda_n(\alpha,\beta) = \{x \in \Omega : \alpha \leq |u_n(x)| < \beta\}$ and $\partial\Lambda_n(\alpha,\beta) = \{x \in \partial\Omega : \alpha \leq |u_n(x)| < \beta\}$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Set $\sigma' = \frac{\sigma}{\sigma-1}$ and $\hat{\sigma}' = \frac{\hat{\sigma}}{\hat{\sigma}-1}$. Then it is clear that $p\sigma' \in (1, p^*)$ and $p\hat{\sigma}' \in (1, p_*)$, because $\sigma > \frac{N}{p}$ and $\hat{\sigma} > \frac{N-1}{p-1}$. Due to (2.1), we may ssume that, up to a subsequence, $v_n \rightharpoonup v$ in E and

$$\begin{aligned} & v_n \to v \text{ in } L^{p\sigma'}(\Omega),\ v_n \to v \text{ in } L^{p\hat{\sigma}'}(\partial\Omega), \\ & v_n(x) \to v(x) \text{ a.e. on } \Omega. \end{aligned} \tag{3.7}$$

We first consider the case $v = 0$. It follows from (3.7) that $v_n \to 0$ in $L^{p\sigma'}(\Omega)$ and $L^{p\hat{\sigma}'}(\partial\Omega)$ and $v_n(x) \to 0$ a.e. on Ω. On the one hand, it follows from $(H) - (i)$ that

$$\begin{aligned} \int_{\Lambda_n(0,r_0)} \frac{|K(x, u_n)|}{\|u_n\|^p}dx \leq \frac{c_1(r_0 + r_0^{r_1})\text{meas}(\Omega)}{\|u_n\|^p} \to 0 \text{ as } n \to +\infty, \\ \int_{\partial\Lambda_n(0,r_0)} \frac{|H(x, u_n)|}{\|u_n\|^p}d\sigma \leq \frac{c_2(r_0 + r_0^{r_2})\text{meas}(\partial\Omega)}{\|u_n\|^p} \to 0 \text{ as } n \to +\infty. \end{aligned} \tag{3.8}$$

On the other hand, from Hölder inequality, (h_2) and (3.5) we obtain that

$$\begin{aligned}
&\int_{\Lambda_n(r_0,+\infty)}\frac{|K(x,u_n)|}{|u_n|^p}|v_n|^p dx\\
\le&\Big(\int_{\Lambda_n(r_0,+\infty)}\frac{|K(x,u_n)|^\sigma}{|u_n|^{p\sigma}}dx\Big)^{\frac{1}{\sigma}}\Big(\int_{\Lambda_n(r_0,+\infty)}|v_n|^{p\sigma'}dx\Big)^{\frac{1}{\sigma'}}\\
\le&\Big(\int_{\Lambda_n(r_0,+\infty)}\frac{|K(x,u_n)|^\sigma}{|u_n|^{p\sigma}}dx\Big)^{\frac{1}{\sigma}}\Big(\int_{\Omega}|v_n|^{p\sigma'}dx\Big)^{\frac{1}{\sigma'}}\\
\le&c_3^{\frac{1}{\sigma}}\Big(\int_{\Lambda_n(r_0,+\infty)}\mathbb{K}(x,u_n)dx\Big)^{\frac{1}{\sigma}}\Big(\int_{\Omega}|v_n|^{p\sigma'}dx\Big)^{\frac{1}{\sigma'}}\\
\le&[qc_3(c+1)]^{\frac{1}{\sigma}}\Big(\int_{\Omega}|v_n|^{p\sigma'}dx\Big)^{\frac{1}{\sigma'}}\to 0,\ \text{as } n\to\infty
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\int_{\partial\Lambda_n(r_0,+\infty)}\frac{|H(x,u_n)|}{|u_n|^p}|v_n|^p d\sigma\\
\le&\Big(\int_{\partial\Lambda_n(r_0,+\infty)}\frac{|H(x,u_n)|^{\hat\sigma}}{|u_n|^{p\hat\sigma}}d\sigma\Big)^{\frac{1}{\hat\sigma}}\Big(\int_{\partial\Lambda_n(r_0,+\infty)}|v_n|^{p\hat\sigma'}dx\Big)^{\frac{1}{\hat\sigma'}}\\
\le&\Big(\int_{\partial\Lambda_n(r_0,+\infty)}\frac{|H(x,u_n)|^{\hat\sigma}}{|u_n|^{p\hat\sigma}}d\sigma\Big)^{\frac{1}{\hat\sigma}}\Big(\int_{\partial\Omega}|v_n|^{p\hat\sigma'}d\sigma\Big)^{\frac{1}{\hat\sigma'}}\\
\le&c_4^{\frac{1}{\hat\sigma}}\Big(\int_{\partial\Lambda_n(r_0,+\infty)}\mathbb{H}(x,u_n)dx\Big)^{\frac{1}{\hat\sigma}}\Big(\int_{\partial\Omega}|v_n|^{p\hat\sigma'}dx\Big)^{\frac{1}{\hat\sigma'}}\\
\le&[qc_4(c+1)]^{\frac{1}{\hat\sigma}}\Big(\int_{\partial\Omega}|v_n|^{p\hat\sigma'}dx\Big)^{\frac{1}{\hat\sigma'}}\to 0,\ \text{as } n\to\infty.
\end{aligned}\tag{3.10}$$

Therefore, using (3.8)–(3.10), one has

$$\begin{aligned}
&\int_{\Omega}\frac{|K(x,u_n)|}{\|u_n\|^p}dx+\int_{\partial\Omega}\frac{|H(x,u_n)|}{\|u_n\|^p}d\sigma\\
=&\int_{\Lambda_n(0,r_0)}\frac{|H(x,u_n)|}{\|u_n\|^p}dx+\int_{\Lambda_n(r_0,+\infty)}\frac{|K(x,u_n)|}{\|u_n\|^p}dx\\
&+\int_{\partial\Lambda_n(0,r_0)}\frac{|H(x,u_n)|}{\|u_n\|^p}d\sigma+\int_{\partial\Lambda_n(r_0,+\infty)}\frac{|H(x,u_n)|}{\|u_n\|^p}d\sigma\\
=&\int_{\Lambda_n(0,r_0)}\frac{|K(x,u_n)|}{\|u_n\|^p}dx+\int_{\Lambda_n(r_0,+\infty)}\frac{|K(x,u_n)|}{|u_n|^p}|v_n|^p dx\\
&+\int_{\partial\Lambda_n(0,r_0)}\frac{|H(x,u_n)|}{\|u_n\|^p}d\sigma+\int_{\partial\Lambda_n(r_0,+\infty)}\frac{|H(x,u_n)|}{|u_n|^p}|v_n|^p d\sigma\\
\to&0,\ \text{as } n\to\infty.
\end{aligned}\tag{3.11}$$

This contradicts (3.6).

For the second case $v\neq 0$, the set $\Lambda_{\neq} := \{x\in\Omega : v(x)\neq 0\}$ and $\partial\Lambda_{\neq} := \{x\in\partial\Omega : v(x)\neq 0\}$ have positive Lebesgue measure. It is obvious that $\lim_{n\to\infty}|u_n(x)| = +\infty$ for a.e. $x\in\Lambda_{\neq}\cup\partial\Lambda_{\neq}$. Therefore, we get

$$\Lambda_{\neq}\subset\Lambda_n(r_0,\infty) \text{ and } \partial\Lambda_{\neq}\subset\partial\Lambda_n(r_0,\infty)$$

for n large enough.

Similar to the proof of (3.8), we can show that

$$\begin{aligned} \int_{\Lambda_n(0,r_0)} \frac{|K(x,u_n)|}{\|u_n\|^q}dx &\leq \frac{c_1(r_0+r_0^{r_1})\text{meas}(\Omega)}{\|u_n\|^q} \to 0 \text{ as } n\to+\infty, \\ \int_{\partial\Lambda_n(0,r_0)} \frac{|H(x,u_n)|}{\|u_n\|^q}d\sigma &\leq \frac{c_2(r_0+r_0^{r_2})\text{meas}(\partial\Omega)}{\|u_n\|^q} \to 0 \text{ as } n\to+\infty. \end{aligned} \tag{3.12}$$

Hence, by $(H)-(i)$, (h_1), (3.12) and Fatou's Lemma, one has

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^q} = \lim_{n\to\infty}\frac{I(u_n)}{\|u_n\|^q} \\ &\leq \lim_{n\to\infty}\Big[\frac{1}{p}\frac{\int_\Omega[|\nabla u_n|^p+|u_n|^p+\mu(x)(|\nabla u_n|^q+\mu(x)|u_n|^q)]dx}{\|u_n\|^q} \\ &\quad -\int_\Omega \frac{K(x,u_n)}{\|u_n\|^q}dx - \int_{\partial\Omega}\frac{H(x,u_n)}{\|u_n\|^q}d\sigma\Big] \\ &= \lim_{n\to\infty}\Big[\frac{1}{p}\frac{\int_\Omega(|\nabla u_n|^p+\mu(x)|\nabla u_n|^q+|u_n|^p+\mu(x)|u_n|^q)dx}{\|u_n\|^q} \\ &\quad -\int_{\Lambda_n(0,r_0)}\frac{K(x,u_n)}{\|u_n\|^q}dx-\int_{\Lambda_n(r_0,+\infty)}\frac{K(x,u_n)}{\|u_n\|^q}dx\Big] \\ &\quad -\int_{\partial\Lambda_n(0,r_0)}\frac{H(x,u_n)}{\|u_n\|^q}d\sigma-\int_{\partial\Lambda_n(r_0,+\infty)}\frac{H(x,u_n)}{\|u_n\|^q}d\sigma\Big] \\ &= \lim_{n\to\infty}\Big[\frac{1}{p}\frac{\int_\Omega[|\nabla u_n|^p+|u_n|^p+\mu(x)(|\nabla u_n|^q+|u_n|^q)]dx}{\|u_n\|^q} \\ &\quad -\int_{\Lambda_n(r_0,+\infty)}\frac{K(x,u_n)}{\|u_n\|^q}dx-\int_{\partial\Lambda_n(r_0,+\infty)}\frac{H(x,u_n)}{\|u_n\|^q}d\sigma\Big] \\ &\leq \limsup_{n\to\infty}\Big[\frac{1}{p}-\int_{\Lambda_n(r_0,+\infty)}\frac{K(x,u_n)}{\|u_n\|^q}dx-\int_{\partial\Lambda_n(r_0,+\infty)}\frac{H(x,u_n)}{\|u_n\|^q}d\sigma\Big] \\ &=\frac{1}{p}-\liminf_{n\to\infty}\Big[\int_{\Lambda_n(r_0,+\infty)}\frac{K(x,u_n)}{|u_n|^q}|v_n|^q dx+\int_{\partial\Lambda_n(r_0,+\infty)}\frac{H(x,u_n)}{\|u_n\|^q}d\sigma\Big] \\ &\leq\frac{1}{p}-\liminf_{n\to\infty}\int_\Omega\frac{K(x,u_n)}{|u_n|^q}\chi_{\Lambda_n(r_0,+\infty)}(x)|v_n|^q dx \\ &\quad -\liminf_{n\to\infty}\int_{\partial\Omega}\frac{H(x,u_n)}{|u_n|^q}\chi_{\partial\Lambda_n(r_0,+\infty)}(x)|v_n|^q d\sigma \\ &\leq\frac{1}{p}-\int_\Omega\liminf_{n\to\infty}\frac{K(x,u_n)}{|u_n|^q}\chi_{\Lambda_n(r_0,+\infty)}(x)|v_n|^q dx \\ &\quad -\int_{\partial\Omega}\liminf_{n\to\infty}\frac{H(x,u_n)}{|u_n|^q}\chi_{\partial\Lambda_n(r_0,+\infty)}(x)|v_n|^q d\sigma \to -\infty. \end{aligned} \tag{3.13}$$

This is impossible. Therefore we have proved that $\{u_n\}\subset E$ is bounded, which concludes the proof of Lemma 3.3.

Lemma 3.4. Under assumption $(H)-(i)$, (h_1) and (h_2) hold. Then the functional I satisfies the $(C)_c$ condition.

Proof. Suppose that $\{u_n\} \subset E$ is a $(C)_c$ sequence of I. According to Lemma 3.3, we deduce that $\{u_n\}$ is bounded in E. Up to a subsequence we may assume that $u_n \rightharpoonup u$ in E. Consequently, we know from (2.1) that $u_n \to u$ in $L^s(\Omega)$ and $u_n \to u$ in $L^{\hat{s}}(\partial\Omega)$, where $1 \le s < p^*$ and $1 \le \hat{s} < p_*$. By a simple calculation, in view of (H)-(i), we have

$$\begin{aligned}
&\int_\Omega |k(x,u) - k(x,u_n)||u_n - u|dx \\
\le& \int_\Omega (|k(x,u)| + |k(x,u_n)|)|u_n - u|dx \\
\le& \int_\Omega [c_1(1 + |u|^{r_1-1}) + c_1(1 + |u_n|^{r_1-1})]|u_n - u|dx \\
\le& 2c_1 \int_\Omega |u_n - u|dx + c_1 \int_\Omega |u|^{r_1-1}|u_n - u|dx + c_1 \int_\Omega |u_n|^{r_1-1}|u_n - u|dx \\
\le& 2c_1 \int_\Omega |u_n - u|dx + c_1\Big(\int_\Omega |u|^{(r_1-1)r_1'}dx\Big)^{\frac{1}{r_1'}}\Big(\int_\Omega |u_n - u|^{r_1}dx\Big)^{\frac{1}{r_1}} \\
&+ c_1\Big(\int_\Omega |u_n|^{(r_1-1)r_1'}dx\Big)^{\frac{1}{r_1'}}\Big(\int_\Omega |u_n - u|^{r_1}dx\Big)^{\frac{1}{r_1}} \\
=& 2c_1 \int_\Omega |u_n - u|dx + c_1\Big(\int_\Omega |u_n|^{r_1}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_n - u|^{r_1}dx\Big)^{\frac{1}{r_1}} \\
&+ c_1\Big(\int_\Omega |u|^{r_1}dx\Big)^{\frac{r_1-1}{r_1}}\Big(\int_\Omega |u_n - u|^{r_1}dx\Big)^{\frac{1}{r_1}} \\
=& 2c_1|u_n - u|_1 + c_1|u_n|_{r_1}^{r_1-1}|u_n - u|_{r_1} + c_1|u|_{r_1}^{r_1-1}|u_n - u|_{r_1} \\
\to& 0, \text{ as } n \to \infty
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\int_{\partial\Omega} |h(x,u) - h(x,u_n)||u_n - u|d\sigma \\
\le& \int_{\partial\Omega} (|h(x,u)| + |h(x,u_n)|)|u_n - u|d\sigma \\
\le& \int_{\partial\Omega} [c_2(1 + |u|^{r_2-1}) + c_2(1 + |u_n|^{r_2-1})]|u_n - u|d\sigma \\
\le& 2c_2 \int_{\partial\Omega} |u_n - u|d\sigma + c_2 \int_{\partial\Omega} |u|^{r_2-1}|u_n - u|d\sigma + c_2 \int_{\partial\Omega} |u_n|^{r_2-1}|u_n - u|d\sigma \\
\le& 2c_2 \int_{\partial\Omega} |u_n - u|d\sigma + c_2\Big(\int_{\partial\Omega} |u|^{(r_2-1)r_2'}d\sigma\Big)^{\frac{1}{r_2'}}\Big(\int_{\partial\Omega} |u_n - u|^{r_2}d\sigma\Big)^{\frac{1}{r_2}} \\
&+ c_2\Big(\int_{\partial\Omega} |u_n|^{(r_2-1)r_2'}d\sigma\Big)^{\frac{1}{r_2'}}\Big(\int_{\partial\Omega} |u_n - u|^{r_2}d\sigma\Big)^{\frac{1}{r_2}} \\
=& 2c_2 \int_{\partial\Omega} |u_n - u|d\sigma + c_2\Big(\int_{\partial\Omega} |u|^{r_2}d\sigma\Big)^{\frac{r_2-1}{r_2}}\Big(\int_{\partial\Omega} |u_n - u|^{r_2}d\sigma\Big)^{\frac{1}{r_2}} \\
&+ c_2\Big(\int_{\partial\Omega} |u_n|^{r_2}d\sigma\Big)^{\frac{r_2-1}{r_2}}\Big(\int_{\partial\Omega} |u_n - u|^{r_2}d\sigma\Big)^{\frac{1}{r_2}} \\
=& 2c_2|u_n - u|_{1,\partial\Omega} + c_2|u_n|_{r_2,\partial\Omega}^{r_2-1}|u_n - u|_{r_2,\partial\Omega} + c_2|u|_{r_2,\partial\Omega}^{r_2-1}|u_n - u|_{r_2,\partial\Omega} \\
\to& 0, \text{ as } n \to \infty,
\end{aligned} \tag{3.15}$$

where $\frac{1}{r_i}+\frac{1}{r_i'}=1$, $i=1,2$. Observe that

$$\begin{aligned}\langle L(u_n)-L(u),u_n-u\rangle&=\langle I'(u_n)-I'(u),u_n-u\rangle\\&+\int_\Omega(k(x,u_n)-k(x,u))(u_n-u)dx+\int_{\partial\Omega}(h(x,u_n)-h(x,u))(u_n-u)d\sigma.\end{aligned}\tag{3.16}$$

First, it follows from (3.4) that

$$\lim_{n\to\infty}\langle I'(u_n)-I'(u),u_n-u\rangle=0.\tag{3.17}$$

In view of (3.14)–(3.17), we have

$$\lim_{n\to\infty}\langle L(u_n)-L(u),u_n-u\rangle=0.$$

Consequently, using the fact that L is of type $(S)_+$, we can conclude $u_n\to u$ in E. This completes the proof.

Lemma 3.5. Under assumption $(H)-(i)$, (h_1) and (h_3) hold. Then the functional I satisfies the $(C)_c$ condition.

Proof. Like in the proof of Lemma 3.4, it need only be proved that $\{u_n\}\subset E$ is bounded. To this end, arguing by contradiction, it is assumed that $\|u_n\|>1$ and $\|u_n\|\to\infty$ as $n\to\infty$. Denote $v_n=\frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$. Up to a subsequence, still denoted by $\{v_n\}$, we may assume that there is a $v\in E$ such that

$$v_n\to v \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega).\tag{3.18}$$

From (2.2) and (h_3), we deduce that

$$\begin{aligned}c+1\geq&I(u_n)-\frac{1}{\vartheta}\langle I'(u_n),u_n\rangle\\\geq&\frac{\vartheta-q}{q\vartheta}\int_\Omega[|\nabla u_n|^p+|u_n|^p+\mu(x)(|\nabla u_n|^q+|u_n|^q)]dx\\&-\frac{\theta_1}{\vartheta}\int_\Omega|u_n|^pdx-\frac{\theta_2}{\vartheta}\int_{\partial\Omega}|u_n|^pd\sigma\\\geq&\frac{\vartheta-q}{q\vartheta}\|u_n\|^p-\frac{\theta_1}{\vartheta}|u_n|_p^p-\frac{\theta_2}{\vartheta}|u_n|^p_{p,\partial\Omega}.\end{aligned}\tag{3.19}$$

It follows that

$$1\leq\frac{q}{\vartheta-q}\liminf_{n\to\infty}\Big(\theta_1|v_n|_p^p+\theta_2|v_n|^p_{p,\partial\Omega}\Big).\tag{3.20}$$

Hence, we obtain from (3.18) that $v\neq0$. Similar to the process of verifying the (3.13) in the proof of Lemma 3.3, we can yield a contradiction. Thus, $\{u_n\}$ is bounded in E. The rest of the proof is the same as that in Lemma 3.4.

Proof of Theorem 1.1. Let $X=E$. Then according to Lemma 3.4 and the oddness of k and h, we have that I satisfies the $(C)_c$ condition and $I(u)=I(-u)$. Next, we verify that $I(u)$ satisfies the other conditions of Lemma 3.2.

First, we verify that $I(u)$ satisfies (A_1). For each $s \in [1, p^*)$ and $\hat{s} \in [1, p_*)$, taking

$$\beta_{s,n} := \sup_{u \in Z_n,\ \|u\|=1} |u|_s \text{ and } \hat{\beta}_{\hat{s},n} := \sup_{u \in Z_n,\ \|u\|=1} |u|_{\hat{s},\partial\Omega},$$

one has $\beta_{s,n} \to 0$ and $\hat{\beta}_{\hat{s},n} \to 0$ as $n \to +\infty$ (see [27, Lemma 3.4]).

Taking $u \in Z_n$ with $\|u\| > 1$, recalling the definitions of $\beta_{s,n}$ and $\hat{\beta}_{\hat{s},n}$, we obtain

$$|u|_s \leq \beta_{s,n}\|u\|,\ |u|_{\hat{s},\partial\Omega} \leq \beta_{\hat{s},n}\|u\|. \tag{3.21}$$

Now, let us define functions $\pi, \hat{\pi} : [0, +\infty) \to \mathbb{R}$ by

$$\pi(t) = \frac{1}{4q}t^p - c_1\beta_{r_1,n}^{r_1}t^{r_1},\ \hat{\pi}(t) = \frac{1}{4q}t^p - c_2\hat{\beta}_{r_2,n}^{r_2}t^{r_2}.$$

A simple calculation shows that

$$\pi(\delta_n) = \hat{\pi}(\hat{\delta}_n) = 0,$$

where $\delta_n = (4qc_1\beta_{r_1,n}^{r_1})^{\frac{1}{p-r_1}}$ and $\hat{\delta}_n = (4qc_2r_2\hat{\beta}_{r_2,n}^{r_2})^{\frac{1}{p-r_2}}$. Since $1 < p < q < r_1, r_2$, together with $\lim_{n\to+\infty} \beta_{r_1,n} = \lim_{n\to+\infty} \hat{\beta}_{r_2,n} = 0$, we obtain that

$$\lim_{n\to+\infty} \delta_n = \lim_{n\to+\infty} \hat{\delta}_n = +\infty. \tag{3.22}$$

Choosing $\tilde{\delta}_n := \min\{\delta_n, \hat{\delta}_n\}$, we have

$$\pi(\tilde{\delta}_n) \geq 0, \hat{\pi}(\tilde{\delta}_n) \geq 0. \tag{3.23}$$

Thus, for each $u \in Z_n$ with $\|u\| = \tilde{\delta}_n := \min\{\delta_n, \hat{\delta}_n\}$, using (H)–(i), (3.22) and (3.23), we deduce

$$\begin{aligned}
I(u) = &\int_\Omega [\frac{1}{p}(|\nabla u|^p + |u|^p) + \frac{\mu(x)}{q}(|\nabla u|^q + |u|^q)]dx \\
&- \int_\Omega K(x,u)dx - \int_{\partial\Omega} H(x,u)d\sigma \\
\geq &\frac{1}{q}\|u\|^p - c_1(\int_\Omega |u|dx + \int_\Omega |u|^{r_1}dx) - c_2(\int_{\partial\Omega} |u|dx + \int_{\partial\Omega} |u|^{r_2}dx) \\
\geq &\frac{1}{q}\|u\|^p - c_1|u|_1 - c_1\beta_{r_1,n}^{r_1}\|u\|^{r_1} - c_2|u|_{1,\partial\Omega} - c_2\hat{\beta}_{r_2,n}^{r_2}\|u\|^{r_2} \\
\geq &\frac{1}{q}\|u\|^p - c_1\beta_{r_1,n}^{r_1}\|u\|^{r_1} - Cc_1\|u\| - c_2\hat{\beta}_{r_2,n}^{r_2}\|u\|^{r_2} - Cc_2\|u\| \\
= &\frac{1}{2q}\|u\|^p + (\frac{1}{4q}\|u\|^p - c_1\beta_{r_1,n}^{r_1}\|u\|^{r_1}) + (\frac{1}{4q}\|u\|^p - c_2\hat{\beta}_{r_2,n}^{r_2}\|u\|^{r_2}) - Cc_1\|u\| - Cc_2\|u\| \\
\geq &\frac{1}{2q}\tilde{\delta}_n^p - Cc_1\tilde{\delta}_n - Cc_2\tilde{\delta}_n \\
\to &+\infty,\ \text{as } n \to +\infty.
\end{aligned} \tag{3.24}$$

This shows relation (A_1).

Secondly, we prove that $I(u)$ satisfies (A_2). We will argue by contradiction. Assume that (A_2) is not satisfied for some $n_0 \in N$, then there exists a sequence $\{u_k\} \subset Y_{n_0}$ such that

$$\|u_k\| \to +\infty \text{ as } k \to +\infty \text{ and } I(u_k) \geq 0. \tag{3.25}$$

Set $w_k = \frac{u_k}{\|u_k\|}$. Then $\|w_k\| = 1$. Since $\dim Y_{n_0} < +\infty$, going if necessary to a subsequence, we can assume that there exists $w \in Y_{n_0} \setminus \{0\}$ such

$$\|w_k - w\| \to 0 \text{ and } w_k(x) \to w(x) \text{ a.e. } x \in \overline{\Omega} \text{ as } k \to +\infty,$$

and $\|w\| = 1$. Consequently, $\lim_{k\to\infty} |u_k(x)| = +\infty$ for a.e. $x \in \Theta_{\neq} \cup \partial\Theta_{\neq}$, where $\Theta_{\neq} := \{x \in \Omega : w(x) \neq 0\}$ and $\partial\Theta_{\neq} := \{x \in \partial\Omega : w(x) \neq 0\}$. Therefore, we have

$$\begin{aligned}
I(u_k) &= \int_\Omega (\frac{1}{p}|\nabla u_k|^p + \frac{\mu(x)}{q}|\nabla u_k|^q + \frac{1}{p}|u_k|^p + \frac{\mu(x)}{q}|u_k|^q)dx - \int_\Omega K(x,u_k)dx - \int_{\partial\Omega} H(x,u_k)dx \\
&= \int_\Omega (\frac{1}{p}|\nabla u_k|^p + \frac{\mu(x)}{q}|\nabla u_k|^q + \frac{1}{p}|u_k|^p + \frac{\mu(x)}{q}|u_k|^q)dx - \int_{\Lambda_k(0,r_0)} K(x,u_k)dx - \int_{\Lambda_k(r_0,+\infty)} K(x,u_k)dx \\
&\quad - \int_{\partial\Lambda_k(0,r_0)} H(x,u_k)dx - \int_{\partial\Lambda_k(r_0,+\infty)} H(x,u_k)dx \\
&\leq \frac{1}{p}\|u_k\|^q + C_1 \int_{\Lambda_k(0,r_0)} (r_0 + r_0^{r_1})dx - \int_{\Lambda_k(r_0,+\infty)} K(x,u_k)dx \\
&\quad + C_2 \int_{\partial\Lambda_k(0,r_0)} (r_0 + r_0^{r_2})dx - \int_{\partial\Lambda_k(r_0,+\infty)} H(x,u_k)dx \\
&\leq \frac{1}{p}\|u_k\|^q + C_1(r_0 + r_0^{r_1})\text{meas}(\Omega) - \int_{\Lambda_k(r_0,+\infty)\cap\Theta_{\neq}} K(x,u_k)dx \\
&\quad + C_2(r_0 + r_0^{r_2})\text{meas}(\partial\Omega) - \int_{\partial\Lambda_k(r_0,+\infty)\cap\partial\Theta_{\neq}} H(x,u_k)dx \\
&\leq \|u_k\|^q \Big(\frac{1}{p} + \frac{C_1(r_0 + r_0^{r_1})\text{meas}(\Omega) + C_2(r_0 + r_0^{r_2})\text{meas}(\partial\Omega)}{\|u_k\|^q} \\
&\quad - \int_{\Lambda_k(r_0,+\infty)\cap\Theta_{\neq}} \frac{K(x,u_k)}{\|u_k\|^q}dx - \int_{\Lambda_k(r_0,+\infty)\cap\partial\Theta_{\neq}} \frac{H(x,u_k)}{\|u_k\|^q}dx\Big) \\
&\to -\infty, \text{ as } k \to +\infty,
\end{aligned}$$

we can obtain a contradiction with (3.25). Hence, all conditions of Lemma 3.2 are satisfied. Conclusion of Theorem 1.2 is reached by Lemma 3.2.

Proof of Theorem 1.2. Obviously, $I(u) = I(-u)$. According to Lemma 3.5, we have that I satisfies the $(C)_c$ condition. The rest of the proof is the same as in the proof of Theorem 1.1.

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Conflict of interest

The authors declare there is no conflicts of interest.

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