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Research article

Uniqueness results for a mixed p-Laplacian boundary value problem involving fractional derivatives and integrals with respect to a power function

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Abstract: This paper is concerned with a mixed p-Laplacian boundary value problem involving right-sided and left-sided fractional derivatives and left-sided integral operators with respect to a power function. We prove the uniqueness of positive solutions for the given problem for the cases $1 < p \leq 2$ and $p > 2$ by applying an efficient novel approach together with the Banach contraction mapping principle. Estimates for Green's functions appearing in the solution of the problem at hand are also presented. Examples are given to illustrate the obtained results.

Keywords: derivatives and integrals with respect to a power function; p-Laplace operator; Green's function; existence; fixed point

1. Introduction

Fractional differential equations have received overwhelming interest in the recent years as such equations describe the natural phenomena in a more realistic manner. The p-Laplacian operator is found to be of great help in describing certain problems occurring in mechanics, nonlinear dynamics and many other fields. In consequence, the study of fractional differential equations together with the p-Laplace operator attracted the attention of many researchers. Let us now dwell on some recent works on p-Laplacian fractional boundary value problems.

Liu et al. [1] applied the method of lower and upper solutions to study the existence of solutions for the following problem:

$$\begin{cases} D_{0^+}^{\alpha}(\phi_p({}^cD_{0^+}^{\beta}y(x))) = f(x, y(x), {}^cD_{0^+}^{\beta}y(x)), \\ {}^cD_{0^+}^{\beta}y(0) = y'(0) = 0, \\ y(1) = r_1 y(\eta), \quad {}^cD_{0^+}^{\beta}y(1) = r_2 {}^cD_{0^+}^{\beta}y(\xi), \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $r_1, r_2 \geq 0$, ϕ_p is the p-Laplacian operator, $p > 1$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, and ${}^cD_{0^+}^{\beta}$ is the Caputo fractional derivative, $f \in C([0,1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$.

In [2], Bai investigated the existence of positive solutions with the aid of the properties of Green's functions for the following p-Laplacian problem:

$$\begin{cases} (\phi_p(D_{0^+}^{\alpha}y(x)))' + f(x, y(x)) = 0, \\ y(0) = D_{0^+}^{\beta}y(0) = 0, \ {}^cD_{0^+}^{\beta}y(0) = {}^cD_{0^+}^{\beta}y(1) = 0, \end{cases}$$

where $0 < \beta < 1$, $2 < \alpha < \beta + 2$, D_{0+}^{α} and ${}^cD_{0+}^{\beta}$ are the Riemann-Liouville fractional derivative and the Caputo fractional derivative of order α and β respectively, ϕ_p is the p-Laplacian operator, $p > 1$, and $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$.

Recently, Wang and Bai [3] discussed the existence and uniqueness of positive solutions to a mixed p-Laplacian fractional boundary value problem given by

$$\begin{cases} {}^cD_{1^-}^{\gamma}(\phi_p(D_{0^+}^{\delta}y(t))) = g(t, y(t), {}^cD_{0^+}^{\delta}y(t)), \\ y(0) = 0, \quad y(1) = r_1 y(\mu), \\ D_{0^+}^{\delta}y(1) = 0, \qquad \phi_p(D_{0^+}^{\delta}y(0)) = r_2 \phi_p(D_{0^+}^{\delta}y(\eta)), \end{cases}$$

where $\phi_p(t) = |t|^{p-2} \cdot t$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $0 < t < 1$, $1 < \gamma, \delta \leq 2$, $0 < \mu, \eta < 1, 0 \leq r_1 < \frac{1}{\mu^{\beta-1}}$, $0 \leq r_2 < \frac{1}{(1-\eta)}$, ${}^cD_{1^-}^{\gamma}$ is the right Caputo fractional derivative and $D_{0^+}^{\delta}$ is the left Riemann-Liouville fractional derivative and $g \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$. The authors in [3] proved the existence and uniqueness of the solutions to the above problem for $1 < p \leq 2$. However, they did not consider the case when $p > 2$.

For some recent results on p-Laplacian boundary value problems, for instance, see the articles [4–11]. The construction of the Green's function together with its properties is a useful tool to investigate the existence of positive solutions to the boundary value problems; for instance, see the text [12].

Let us now review some recent works dealing with a modified form of Caputo and Riemann-Liouville fractional derivatives. In [13], the authors studied the asymptotic stability of solutions of generalized Caputo fractional differential equations. Caputo modification of the generalized fractional derivatives was discussed in [14]. Some existence results for a nonlocal boundary value problem involving generalized Liouville-Caputo derivatives and generalized fractional integral were presented in [15]. The authors in [16] discussed the existence of solutions for generalized fractional differential equations and inclusions equipped with nonlocal generalized fractional integral boundary conditions. In [17], extremal solutions for an integro-initial value problem for generalized Caputo fractional differential equations were obtained. In [18], the authors introduced and studied a new class of coupled systems containing both Caputo and Riemann-Liouville generalized fractional derivatives. For some recent works on the problems involving generalized fractional derivatives, for example, see [19–24].

Motivated by the aforementioned studies on boundary value problems involving a p-Laplacian operator and modified versions of fractional derivatives, in this paper, we introduce a new class of mixed p-Laplacian fractional boundary value problems involving right-sided and left-sided fractional derivatives and left-sided integral operators with respect to a power function. In precise terms, we investigate the following problem:

$$\begin{cases} {}^\rho D^\alpha_{1^-}(\phi_p({}^\rho D^\beta_{0^+}y(t))) = \nu_1 f(t, y(t), {}^\rho D^\beta_{0^+}y(t)) + \nu_2 {}^\rho I^\zeta_{0+}g(t, y(t), {}^\rho D^\beta_{0^+}y(t)), \\ y(0) = 0, \quad y(1) = \lambda_1 y(\mu), \\ {}^\rho D^\beta_{0^+}y(1) = 0, \qquad \phi_p({}^\rho D^\beta_{0^+}y(0)) = \lambda_2\phi_p({}^\rho D^\beta_{0^+}y(\eta)), \end{cases} \tag{1.1}$$

where $\phi_p(t) = |t|^{p-2} \cdot t$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $0 < t < 1$, $1 < \alpha, \beta \leq 2$, $\rho > 0$, $\zeta > 0$, $0 < \mu, \eta < 1$, $0 \leq \lambda_1 < \frac{1}{\mu^{\rho(\beta-1)}}$, $0 \leq \lambda_2 < \frac{1}{(1-\eta^\rho)^{\alpha-1}}$, ${}^\rho D^\alpha_{1^-}$ and ${}^\rho D^\beta_{0^+}$ respectively denote the right and left fractional derivatives of orders α and β with respect to a power function (see Definitions 2.2), ${}^\rho I^\zeta_{0+}$ is the fractional integral operator of order ζ with respect to a power function (see Definitions 2.1), $\nu_1, \nu_2 \in \mathbb{R}$ and $f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

The remainder of the paper is arranged as follows. In Section 2, we present the background material related to our problem and prove some important lemmas that play a key role in the forthcoming analysis. Section 3 contains the main results for the given problem. In Section 4, we illustrate our results with the aid of examples. The paper concludes with certain interesting observations.

2. Preliminaries

Let us first recall that the concept of fractional calculus of a function with respect to another function can be found in the books by Samko et al. ([25]; Section 18.2) and Kilbas et al. ([26]; Section 2.5), while an article by Erdelyi [27] contains the first study of fractional integrals with respect to a power function (now mistakenly named after Katugampola).

Definition 2.1. *The fractional integral with respect to a power function (t^ρ/ρ) of order $\alpha > 0$ for a function $h \in X^p_c(a, b)$ for $-\infty < a < t < b < \infty$, is defined by*

$$({}^\rho I^\alpha_{a^+}h)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} h(s)ds, \; \rho > 0, \tag{2.1}$$

where $X^p_c(a, b)$ denotes the space of all complex-valued Lebesgue measurable functions ϕ on (a, b) equipped with the norm:

$$\|\phi\|_{X^p_c} = \Big(\int_a^b |x^c\phi(x)|^p \frac{dx}{x}\Big)^{1/p} < \infty, \;\; c \in \mathbb{R}, 1 \leq p \leq \infty.$$

Note that the integral in (2.1) is called the left-sided fractional integral. Similarly, we can define right-sided fractional integral ${}^\rho I^\alpha_{b^-}f$ as

$$({}^\rho I^\alpha_{b^-}h)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} h(s)ds. \tag{2.2}$$

Here we mention that the above definitions of fractional integrals follow from the integrals (2.5.1) and (2.5.2) on pages 99–100 in the text [26] by taking the power function $g(x) = x^\rho/\rho$.

Definition 2.2. *For $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$, the fractional derivatives with respect to a power function (t^ρ/ρ), associated with the fractional integrals (2.1) and (2.2) are defined, for $0 \le a < x < b < \infty$, by*

$$\begin{aligned}({}^\rho D^\alpha_{a+}g)(t) &= \Big(t^{1-\rho}\frac{d}{dt}\Big)^n ({}^\rho I^{n-\alpha}_{a+}g)(t)\\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\Big(t^{1-\rho}\frac{d}{dt}\Big)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}} g(s)ds, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}({}^\rho D^\alpha_{b-}g)(t) &= \Big(-t^{1-\rho}\frac{d}{dt}\Big)^n ({}^\rho I^{n-\alpha}_{b-}g)(t)\\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\Big(-t^{1-\rho}\frac{d}{dt}\Big)^n \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{\alpha-n+1}} g(s)ds. \end{aligned} \tag{2.4}$$

Note that the above definitions of fractional derivatives follow from the integrals (2.5.17) and (2.5.18) on page 101 in the text [26] by taking the power function $g(x) = x^\rho/\rho$.

Lemma 2.1. *[28] Let $1 < \alpha \le 2, \rho > 0$, $u \in X^p_c(0,T)$ and ${}^\rho I^{2-\alpha} u \in AC^2_\rho$, where $AC^2_\rho([a,b])$ denotes the space of absolutely continuous functions possessing the $t^{1-\rho}\frac{d}{dt}$-derivative defined by*

$$AC^2_\rho[a,b] = \Big\{f : [a,b] \to \mathbb{R} : (t^{1-\rho}\frac{d}{dt}f) \in AC[a,b]\Big\}.$$

Then the general solution of the fractional differential equation ${}^\rho D^\alpha_{0+}u(t) = 0$ is

$$u(t) = c_1 t^{\rho(\alpha-1)} + c_2 t^{\rho(\alpha-2)},$$

where $c_i \in \mathbb{R}, i = 1, 2$. Moreover,

$$({}^\rho I^\alpha_{0+}\, {}^\rho D^\alpha_{0+}u)(t) = u(t) + c_1 t^{\rho(\alpha-1)} + c_2 t^{\rho(\alpha-2)}.$$

Lemma 2.2. *For any $\psi \in C([0,1],\mathbb{R})$, the integral representation of the solution for the following non-local p-Laplacian boundary value problem involving right-sided and left-sided fractional derivatives with respect to a power function:*

$$\begin{cases} {}^\rho D^\alpha_{1^-}(\phi_p({}^\rho D^\beta_{0^+}y(t))) = \psi(t),\\ y(0) = 0, \quad y(1) = \lambda_1 y(\mu),\\ {}^\rho D^\beta_{0^+}y(1) = 0, \qquad \phi_p({}^\rho D^\beta_{0^+}y(0)) = \lambda_2\phi_p({}^\rho D^\beta_{0^+}y(\eta)),\end{cases} \tag{2.5}$$

is given by

$$y(t) = \int_0^1 \tau^{\rho-1} G_1(t,\tau)\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big)d\tau, \tag{2.6}$$

where

$$G_1(t,\tau) = \frac{\rho^{1-\beta}}{\Gamma(\beta)}\begin{cases} \Lambda_1[(1-\tau^\rho)^{\beta-1} - \lambda_1(\mu^\rho - \tau^\rho)^{\beta-1}] - (t^\rho - \tau^\rho)^{\beta-1}, & 0 \le \tau \le \min\{t,\mu\},\\ \Lambda_1(1-\tau^\rho)^{\beta-1} - (t^\rho - \tau^\rho)^{\beta-1}, & \mu \le \tau \le t,\\ \Lambda_1[(1-\tau^\rho)^{\beta-1} - \lambda_1(\mu^\rho - \tau^\rho)^{\beta-1}], & t \le \tau \le \mu,\\ \Lambda_1(1-\tau^\rho)^{\beta-1}, & \max\{t,\mu\} \le \tau \le 1,\end{cases} \tag{2.7}$$

and

$$G_2(\tau,s)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\begin{cases}\Lambda_2 s^{\rho(\alpha-1)}, & 0\le s\le \min\{\tau,\eta\},\\ \Lambda_2[s^{\rho(\alpha-1)}-\lambda_2(s^\rho-\eta^\rho)^{\alpha-1}], & \eta\le s\le\tau,\\ \Lambda_2 s^{\rho(\alpha-1)}-(s^\rho-\tau^\rho)^{\alpha-1}, & \tau\le s\le\eta,\\ \Lambda_2[s^{\rho(\alpha-1)}-\lambda_2(s^\rho-\eta^\rho)^{\alpha-1}]-(s^\rho-t^\rho)^{\alpha-1}, & \max\{\tau,\eta\}\le s\le 1,\end{cases} \tag{2.8}$$

with

$$\Lambda_1=\frac{t^{\rho(\beta-1)}}{1-\lambda_1\mu^{\rho(\beta-1)}},\ \Lambda_2=\frac{(1-\tau^\rho)^{\alpha-1}}{1-\lambda_2(1-\eta^\rho)^{\alpha-1}}.$$

Proof. Letting $-\phi_p({}^\rho D^\beta_{0^+}y(t))=\mathcal{H}(t)$, we decompose the mixed boundary value problem (2.5) as

$$\begin{cases}{}^\rho D^\alpha_{1^-}\mathcal{H}(t)=-\psi(t),\\ \mathcal{H}(1)=0, \qquad \mathcal{H}(0)=\lambda_2\,\mathcal{H}(\eta),\end{cases} \tag{2.9}$$

and

$$\begin{cases}{}^\rho D^\beta_{0^+}y(t)=-\phi_q(\mathcal{H}(t)),\\ y(0)=0, \quad y(1)=\lambda_1 y(\mu).\end{cases} \tag{2.10}$$

Solving the equation ${}^\rho D^\alpha_{1^-}\mathcal{H}(t)=-\psi(t)$, we get

$$\mathcal{H}(t) = -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^1 s^{\rho-1}(s^\rho-t^\rho)^{\alpha-1}\psi(s)ds+c_0(1-t^\rho)^{\alpha-2}+c_1(1-t^\rho)^{\alpha-1}, \tag{2.11}$$

where c_0 and c_1 are arbitrary constants. Using the condition $\mathcal{H}(1)=0$ in (2.11) yields $c_0=0$. Then, inserting (2.11) with $c_0=0$ in the condition: $\mathcal{H}(0)=\lambda_2\mathcal{H}(\eta)$, the value of c_1 is found to be

$$c_1=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})}\Big(\int_0^1 s^{\rho\alpha-1}\psi(s)ds-\lambda_2\int_\eta^1 s^{\rho-1}(s^\rho-t^\rho)^{\alpha-1}\psi(s)ds\Big). \tag{2.12}$$

So (2.11) becomes

$$\begin{aligned}\mathcal{H}(t) &= -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^1 s^{\rho-1}(s^\rho-t^\rho)^{\alpha-1}\psi(s)ds\\ &\quad+\frac{\rho^{1-\alpha}(1-t^\rho)^{\alpha-1}}{\Gamma(\alpha)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})}\Big(\int_0^1 s^{\rho\alpha-1}\psi(s)ds-\lambda_2\int_\eta^1 s^{\rho-1}(s^\rho-\eta^\rho)^{\alpha-1}\psi(s)ds\Big)\\ &= \int_0^1 s^{\rho-1}G_2(t,s)\psi(s)ds,\end{aligned} \tag{2.13}$$

where $G_2(t,s)$ is given in (2.8). Applying the integral operator ${}^\rho I^\beta_{0+}$ on both sides of the differential equation in (2.10), we have

$$y(t) = -\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\beta-1}\phi_q(\mathcal{H}(s))ds+d_0t^{\rho(\beta-2)}+d_1t^{\rho(\beta-1)}, \tag{2.14}$$

where d_0 and d_1 are arbitrary constants.
Using (2.14) in the boundary conditions of (2.10), we obtain $d_0 = 0$ and

$$\begin{aligned} d_1 &= \frac{\rho^{1-\beta}}{\Gamma(\beta)(1-\lambda_1\mu^{\rho(\beta-1)})}\Big(\int_0^1 s^{\rho-1}(1-t^\rho)^{\alpha-1}\phi_q(\mathcal{H}(s))ds \\ &\quad -\lambda_1\int_0^\mu s^{\rho-1}(\mu^\rho - s^\rho)^{\alpha-1}\phi_q(\mathcal{H}(s))ds\Big). \end{aligned} \tag{2.15}$$

Thus, (2.14) takes the form:

$$y(t) = \int_0^1 s^{\rho-1}G_1(t,s)\phi_q(\mathcal{H}(s))ds,$$

where $G_1(t,s)$ and $\mathcal{H}(.)$ are respectively given in (2.7) and (2.13). □

Lemma 2.3. *The functions $G_1(t,s)$ and $G_2(t,s)$ given in (2.7) and (2.8) respectively, are continuous and possess the following properties:*

(*i*) $G_1(t,s) > 0$, $G_2(t,s) > 0$, $\forall\, t,s \in (0,1)$;

(*ii*) $\dfrac{Q_1\,\rho^{1-\beta}\,t^{\rho(\beta-1)}\,s^\rho\,(1-s^\rho)^{\beta-1}}{\Gamma(\beta)(1-\lambda_1\mu^{\rho(\beta-1)})} \le G_1(t,s) \le \dfrac{\rho^{1-\beta}\,t^{\rho(\beta-1)}\,(1-s^\rho)^{\beta-1}}{\Gamma(\beta)(1-\lambda_1\mu^{\rho(\beta-1)})}$, $\forall\, t,s \in (0,1)$;

(*iii*) $\dfrac{Q_2\,\rho^{1-\alpha}\,s^{\rho(\alpha-1)}\,(1-t^\rho)^{\alpha-1}\,(1-s^\rho)}{\Gamma(\alpha)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})} \le G_2(t,s) \le \dfrac{\rho^{1-\alpha}\,s^{\rho(\alpha-1)}\,(1-t^\rho)^{\alpha-1}}{\Gamma(\alpha)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})}$, $\forall\, t,s \in (0,1)$,

where

$$0 < Q_1 := \min\{1-\lambda_1\mu^{\rho(\beta-1)}, \lambda_1\mu^{\rho(\beta-2)}(1-\mu^\rho), \lambda_1\mu^{\rho(\beta-1)}\} < 1,$$

and

$$0 < Q_2 := \min\{1-\lambda_2(1-\eta^\rho)^{\alpha-1}, \lambda_2\eta^\rho(1-\eta^\rho)^{\alpha-2}, \lambda_2(1-\eta^\rho)^{\alpha-1}\} < 1.$$

Proof. Let us first prove part (*i*) with different cases.

Case 1. If $0 \le \tau \le \min\{t,\mu\} < 1$, then we have $\frac{\tau^\rho}{\mu^\rho} \ge \tau^\rho$ since $\mu^\rho < 1$, which implies $1 - \frac{\tau^\rho}{\mu^\rho} \le 1 - \tau^\rho$. Hence, we find that

$$\begin{aligned} &\frac{t^{\rho(\beta-1)}}{1-\lambda_1\mu^{\rho(\beta-1)}}\Big[(1-\tau^\rho)^{\beta-1} - \lambda_1(\mu^\rho-\tau^\rho)^{\beta-1}\Big] \\ =\ & \frac{t^{\rho(\beta-1)}}{1-\lambda_1\mu^{\rho(\beta-1)}}\Big[(1-\tau^\rho)^{\beta-1} - \lambda_1\mu^{\rho(\beta-1)}(1-\frac{\tau^\rho}{\mu^\rho})^{\beta-1}\Big] \\ \ge\ & \frac{t^{\rho(\beta-1)}}{1-\lambda_1\mu^{\rho(\beta-1)}}\Big[(1-\tau^\rho)^{\beta-1} - \lambda_1\mu^{\rho(\beta-1)}(1-\tau^\rho)^{\beta-1}\Big] \\ =\ & t^{\rho(\beta-1)}(1-\tau^\rho)^{\beta-1} \\ \ge\ & t^{\rho(\beta-1)}(1-\frac{\tau^\rho}{t^\rho})^{\beta-1} = (t^\rho-\tau^\rho)^{\beta-1}, \end{aligned}$$

which means that

$$\Lambda_1\big[(1-\tau^\rho)^{\beta-1} - \lambda_1(\mu^\rho-\tau^\rho)^{\beta-1}\big] - (t^\rho-\tau^\rho)^{\beta-1} \ge 0.$$

Case 2. For $\mu \le \tau \le t$, we have

$$\frac{t^{\rho(\beta-1)}}{1-\lambda_1\mu^{\rho(\beta-1)}}(1-\tau^\rho)^{\beta-1} \ \ge\ t^{\rho(\beta-1)}(1-\tau^\rho)^{\beta-1} \ge t^{\rho(\beta-1)}(1-\frac{\tau^\rho}{t^\rho})^{\beta-1} = (t^\rho-\tau^\rho)^{\beta-1}.$$

Thus, $\Lambda_1(1-\tau^\rho)^{\beta-1}-(t^\rho-\tau^\rho)^{\beta-1}\ge 0$.

Case 3. For $t\le\tau\le\mu$, we have

$$(1-\tau^\rho)^{\beta-1} \ \ge\ \lambda_1\mu^{\rho(\beta-1)}(1-\tau^\rho)^{\beta-1}\ge\lambda_1\mu^{\rho(\beta-1)}(1-\frac{\tau^\rho}{\mu^\rho})^{\beta-1}=\lambda_1(\mu^\rho-\tau^\rho)^{\beta-1}.$$

So, $\Lambda_1[(1-\tau^\rho)^{\beta-1}-\lambda_1(\mu^\rho-\tau^\rho)^{\beta-1}]\ge 0$.

Case 4. For $\max\{t,\mu\}\le\tau\le 1$, it is obvious that $\Lambda_1(1-\tau^\rho)^{\beta-1}\ge 0$.

Consequently, we get $G_1(t,\tau)\ge 0$. By a similar argument, one can show that $G_2(t,s)\ge 0$.

In order to establish (ii), let $\Gamma(\beta)\rho^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})G_1(t,s)=g_1(t,s)$. Then, for $0\le s\le\min\{t,\mu\}$, we have

$$\begin{aligned}
g_1(t,s) &= t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-\lambda_1 t^{\rho(\beta-1)}(\mu^\rho-s^\rho)^{\beta-1}-(t^\rho-s^\rho)^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})\\
&= t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-\lambda_1 t^{\rho(\beta-1)}\mu^{\rho(\beta-1)}(1-\frac{s^\rho}{\mu^\rho})^{\beta-1}-t^{\rho(\beta-1)}(1-\frac{s^\rho}{t^\rho})^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})\\
&\ge t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-\lambda_1 t^{\rho(\beta-1)}\mu^{\rho(\beta-1)}(1-\frac{s^\rho}{\mu^\rho})^{\beta-1}-t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})\\
&= \lambda_1\mu^{\rho(\beta-1)}t^{\rho(\beta-1)}\Big(-(1-\frac{s^\rho}{\mu^\rho})^{\beta-1}+(1-s^\rho)^{\beta-1}\Big)\\
&\ge \lambda_1\mu^{\rho(\beta-1)}t^{\rho(\beta-1)}(1-s^\rho)^{\beta-2}\Big(-(1-\frac{s^\rho}{\mu^\rho})+(1-s^\rho)\Big)\\
&\ge \lambda_1\mu^{\rho(\beta-1)}t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}s^\rho\Big(\frac{1-\mu^\rho}{\mu^\rho}\Big)\\
&\ge Q_1 t^{\rho(\beta-1)}s^\rho(1-s^\rho)^{\beta-1}.
\end{aligned}$$

For $\mu\le s\le t$, let $\Gamma(\beta)\rho^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})G_1(t,s)=g_2(t,s)$. Then, we get

$$\begin{aligned}
g_2(t,s) &= t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-(t^\rho-s^\rho)^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})\\
&= t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-t^{\rho(\beta-1)}(1-\frac{s^\rho}{t^\rho})^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})\\
&\ge t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})\\
&= \lambda_1\mu^{\rho(\beta-1)}t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}\\
&\ge Q_1 t^{\rho(\beta-1)}s^\rho(1-s^\rho)^{\beta-1}.
\end{aligned}$$

For $t\le s\le\mu$, let $\Gamma(\beta)\rho^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})G_1(t,s)=g_3(t,s)$. Then, we obtain

$$\begin{aligned}
g_3(t,s) &= t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-\lambda_1 t^{\rho(\beta-1)}(\mu^\rho-s^\rho)^{\beta-1}\\
&= t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-\lambda_1 t^{\rho(\beta-1)}\mu^{\rho(\beta-1)}(1-\frac{s^\rho}{\mu^\rho})^{\beta-1}\\
&\ge t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}-\lambda_1 t^{\rho(\beta-1)}\mu^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}\\
&= t^{\rho(\beta-1)}(1-\lambda_1\mu^{\rho(\beta-1)})(1-s^\rho)^{\beta-1}\\
&\ge Q_1 t^{\rho(\beta-1)}s^\rho(1-s^\rho)^{\beta-1}.
\end{aligned}$$

Lastly, when $\max\{t,\mu\}\le s\le 1$, it is clear that

$$\Gamma(\beta)\rho^{\beta-1}(1-\lambda_1\mu^{\rho(\beta-1)})G_1(t,s)=t^{\rho(\beta-1)}(1-s^\rho)^{\beta-1}\ge Q_1 t^{\rho(\beta-1)}s^\rho(1-s^\rho)^{\beta-1}.$$

Thus,

$$G_1(t,s) \geq \frac{Q_1\rho^{1-\beta}t^{\rho(\beta-1)}s^{\rho}(1-s^{\rho})^{\beta-1}}{\Gamma(\beta)(1-\lambda_1\mu^{\rho(\beta-1)})}.$$

On the other hand, it easy to show that

$$G_1(t,s) \leq \frac{\rho^{1-\beta}t^{\rho(\beta-1)}(1-s^{\rho})^{\beta-1}}{\Gamma(\beta)(1-\lambda_1\mu^{\beta-1})}, \ \ \forall(t,s)\in(0,1)\times(0,1).$$

Now, for (iii), consider $\Gamma(\alpha)\rho^{\alpha-1}(1-\lambda_2(1-\eta^{\rho})^{\alpha-1})G_2(t,s)$. If $0\leq s\leq \min\{t,\eta\}$, then we find that

$$(1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)} \geq Q_2(1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}(1-s^{\rho}).$$

When $t\leq s\leq\eta$, let $\Gamma(\alpha)\rho^{\alpha-1}(1-\lambda_2(1-\eta^{\rho}))G_2(t,s)=h_1(s,t)$; then we have

$$\begin{aligned}
h_1(t,s) &= (1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}-(s^{\rho}-t^{\rho})^{\alpha-1}(1-\lambda_2(1-\eta^{\rho})^{\alpha-1})\\
&= (1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}-s^{\rho(\alpha-1)}(1-\frac{t^{\rho}}{s^{\rho}})^{\alpha-1}(1-\lambda_2(1-\eta^{\rho})^{\alpha-1})\\
&\geq s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-1+\lambda_2(1-\eta^{\rho})^{\alpha-1})\\
&\geq s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}\lambda_2(1-\eta^{\rho})^{\alpha-1}\\
&\geq Q_2s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-s^{\rho}).
\end{aligned}$$

When $\eta\leq s\leq t$, let $\Gamma(\alpha)\rho^{\alpha-1}(1-\lambda_2(1-\eta^{\rho}))G_2(t,s)=h_2(t,s)$; then we have

$$\begin{aligned}
h_2(t,s) &= (1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}-\lambda_2(1-t^{\rho})^{\alpha-1}(s^{\rho}-\eta^{\rho})^{\alpha-1}\\
&= (1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}-\lambda_2(1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}(1-\frac{\eta^{\rho}}{s^{\rho}})^{\alpha-1}\\
&\geq s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-\lambda_2(1-\eta^{\rho})^{\alpha-1})\\
&\geq Q_2s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-s^{\rho}).
\end{aligned}$$

When $\max\{\eta,t\}\leq s\leq 1$, let $\Gamma(\alpha)\rho^{\alpha-1}(1-\lambda_2(1-\eta^{\rho}))G_2(t,s)=h_3(t,s)$. Then, we obtain

$$\begin{aligned}
h_3(t,s) &= (1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}-\lambda_2(1-t^{\rho})^{\alpha-1}(s^{\rho}-\eta^{\rho})^{\alpha-1}-(s^{\rho}-t^{\rho})^{\alpha-1}(1-\lambda_2(1-\eta^{\rho})^{\alpha-1})\\
&= (1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}-\lambda_2(1-t^{\rho})^{\alpha-1}s^{\rho(\alpha-1)}(1-\frac{\eta^{\rho}}{s^{\rho}})^{\alpha-1}\\
&\quad -s^{\rho(\alpha-1)}(1-\frac{t^{\rho}}{s^{\rho}})^{\alpha-1}(1-\lambda_2(1-\eta^{\rho})^{\alpha-1})\\
&\geq s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}\Big(1-\lambda_2(1-\frac{\eta^{\rho}}{s^{\rho}})^{\alpha-1}-1+\lambda_2(1-\eta^{\rho})^{\alpha-1}\Big)\\
&\geq \lambda_2s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-\eta^{\rho})^{\alpha-2}\Big((1-\eta^{\rho})-(1-\frac{\eta^{\rho}}{s^{\rho}})\Big)\\
&\geq \lambda_2s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-\eta^{\rho})^{\alpha-2}\eta^{\rho}s^{-\rho}(1-s^{\rho}))\\
&= \lambda_2(1-\eta^{\rho})^{\alpha-2}\eta^{\rho}s^{\rho(\alpha-2)}(1-t^{\rho})^{\alpha-1}(1-s^{\rho})\\
&\geq Q_2s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-s^{\rho}).
\end{aligned}$$

Hence,

$$G_2(t,s) \geq \frac{Q_2\rho^{1-\alpha}s^{\rho(\alpha-1)}(1-t^{\rho})^{\alpha-1}(1-s^{\rho})}{\Gamma(\alpha)(1-\lambda_2(1-\eta^{\rho})^{\alpha-1})}.$$

On the other hand, it is easy to show that

$$G_2(t,s) \le \frac{\rho^{1-\alpha} s^{\rho(\alpha-1)}(1-t^\rho)^{\alpha-1}}{\Gamma(\alpha)(1-\lambda_2(1-\eta^\rho)^{\beta-1})},\ \ \forall (t,s)\in(0,1)\times(0,1).$$

Thus, the proof is completed. □

Remark 2.1. *When $\rho \to 1$, (ii) is similar to the result presented in Theorem 1 in [29].*

Lemma 2.4. *Let $\psi \in C([0,1],\mathbb{R})$ and*

$$\omega(\tau) = -\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big),\ \ y(t) = -\int_0^1 \tau^{\rho-1}G_1(t,\tau)\omega(\tau)d\tau.$$

Then, for $t,\tau,s\in[0,1]$ and $\frac{1}{p}+\frac{1}{q}=1,\ p,q\ge 1$, the following results hold:

(*i*) $\|\omega\| \le \Omega_1^{q-1}\|\psi\|^{q-1},\ \ \|y\| \le \Omega_2\Omega_1^{q-1}\|\psi\|^{q-1}$, *where*

$$\Omega_1 := \frac{1}{\rho^\alpha\Gamma(\alpha+1)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})},\ \ \Omega_2 := \frac{1}{\rho^\beta\Gamma(\beta+1)(1-\lambda_1\mu^{\rho(\beta-1)})};$$

(*ii*) $\omega(\tau) \le -m^{q-1}\Omega_3^{q-1}(1-\tau^\rho)^{(\alpha-1)(q-1)},\ \ y(t) \ge m^{q-1}\Omega_3^{q-1}\Omega_4 t^{\rho(\beta-1)}$, *for $\psi(t)\ge m>0$ and $\forall t\in[0,1]$, where*

$$\Omega_3 := \frac{Q_2}{\rho^\alpha\Gamma(\alpha+2)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})},$$
$$\Omega_4 := \frac{Q_1}{\rho^\beta\Gamma(\beta)(\beta+(\alpha-1)(q-1))(\beta+(\alpha-1)(q-1)+1)(1-\lambda_1\mu^{\rho(\beta-1)})}.$$

Proof. By Lemma 2.3, we have

$$\frac{Q_1\ t^{\rho(\beta-1)}}{\rho^\beta\Gamma(\beta+2)(1-\lambda_1\mu^{\rho(\beta-1)})} \le \int_0^1 \tau^{\rho-1}G_1(t,\tau)d\tau \le \frac{t^{\rho(\beta-1)}}{\rho^\beta\Gamma(\beta+1)(1-\lambda_1\mu^{\rho(\beta-1)})},$$

and

$$\frac{Q_2(1-\tau^\rho)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha+2)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})} \le \int_0^1 s^{\rho-1}G_2(\tau,s)ds \le \frac{(1-\tau^\rho)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha+1)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})}.$$

Consequently,

$$\begin{aligned}
\Big|\omega(\tau)\Big| &= \Big|-\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big)\Big| \\
&= \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)|\psi(s)|ds\Big) \\
&\le \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)ds\|\psi\|\Big) \\
&\le \Big(\frac{1}{\Gamma(\alpha+1)\rho^\alpha(1-\lambda_2(1-\eta^\rho)^{\alpha-1}))}\Big)^{q-1}\|\psi\|^{q-1} \\
&= \Omega_1^{q-1}\|\psi\|^{q-1}.
\end{aligned}$$

Thus, $\|\omega\| \leq \Omega_1^{q-1}\|\psi\|^{q-1}$. Similarly, we have that $\|y\| \leq \Omega_2\Omega_1^{q-1}\|\psi\|^{q-1}$. This establishes (i).
For (ii), we have

$$\begin{aligned}\omega(\tau) &= -\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big)\\ &\leq -\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)m\,ds\Big)\\ &\leq -\Big(\frac{Q_2(1-\tau^\rho)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha+2)(1-\lambda_2(1-\eta^\rho)^{\alpha-1})}\Big)^{q-1}m^{q-1}\\ &= -\Omega_3^{q-1}m^{q-1}(1-\tau^\rho)^{(\alpha-1)(q-1)}.\end{aligned}$$

Likewise, we have that $y(t) \geq m^{q-1}\Omega_3^{q-1}\Omega_4 t^{\rho(\beta-1)}$. This completes the proof. □

The following lemma describes some properties of the p-Laplace operator which can easily be proved by using the mean value theorem when the function $\phi_p(k) = |k|^{p-2}k$ is differentiable at all k except $k = 0$ and $|\frac{\partial\phi(k)}{\partial k}|$ is bounded by $(p-1)\max|k|^{p-2}$ for $p > 2$ and bounded by $(p-1)\min|k|^{p-2}$ for $1 < p < 2$.

Lemma 2.5. *(see (2.1) and (2.2) on page 3268 in [30]) The following relations hold for the p-Laplace operator:*

(i) *For* $1 < p \leq 2$, $|k_1|, |k_2| \geq \Delta_1 > 0$ *and* $k_1k_2 > 0$, $|\phi_p(k_2) - \phi_p(k_1)| \leq (p-1)\Delta_1^{p-2}|k_2 - k_1|$;
(ii) *For* $p > 2$, $|k_1|, |k_2| \leq \Delta_2$ *and* $k_1k_2 > 0$, $|\phi_p(k_2) - \phi_p(k_1)| \leq (p-1)\Delta_2^{p-2}|k_2 - k_1|$.

3. Existence and uniqueness results

In this section, we discuss the existence and uniqueness of the solutions to the problem (1.1). For a given number $\mathcal{M} > 0$, let us consider the following set

$$\Upsilon_{\mathcal{M}} = \{(t,y,\omega) : 0 \leq t \leq 1, \|y\| \leq \Omega_2\Omega_1^{q-1}\mathcal{M}^{q-1}, \|\omega\| \leq \Omega_1^{q-1}\mathcal{M}^{q-1}\};$$

and denote by $\Gamma[O,\mathcal{M}]$ a closed ball in the space of the continuous function $C[0,1]$.

Theorem 3.1. *Assume that* $1 < p \leq 2$ *and there exist positive constants* M_1, M_2, C_1, C_2, κ_1*and*κ_2 *such that*

(A_1) $|f(t,y,\omega)| \leq M_1$, $|g(t,y,\omega)| \leq M_2$ *for* $(t,y,\omega) \in \Upsilon_{\mathcal{M}}$;
(A_2) $|f(t,y_1,\omega_1) - f(t,y_2,\omega_2)| \leq C_1|y_2 - y_1| + C_2|\omega_2 - \omega_1|$, *for* $(t,y_i,\omega_i) \in \Upsilon_{\mathcal{M}}$, $i = 1,2$;
(A_3) $|g(t,y_1,\omega_1) - g(t,y_2,\omega_2)| \leq \kappa_1|y_2 - y_1| + \kappa_2|\omega_2 - \omega_1|$, *for* $(t,y_i,\omega_i) \in \Upsilon_{\mathcal{M}}$, $i = 1,2$;
(A_4) $L_1 := (q-1)\mathcal{M}^{q-2}\Omega_1^{q-1}\Big\{|\nu_1|\Big(C_1\Omega_2 + C_2\Big) + \frac{|\nu_2|}{\rho^\zeta\Gamma(\zeta+1)}\Big(\kappa_1\Omega_2 + \kappa_2\Big)\Big\} < 1$.

Then the mixed boundary value problem (1.1) has a unique solution satisfying the following inequalities:

$$|y(t)| \leq \Omega_2\Omega_1^{q-1}\mathcal{M}^{q-1},\quad |{}^\rho D_{0+}^\beta y(t)| \leq \Omega_1^{q-1}\mathcal{M}^{q-1}, \textit{for all } t \in [0,1], \tag{3.1}$$

where $\mathcal{M} \geq |\nu_1|M_1 + |\nu_2|\frac{M_2}{\rho^\zeta\Gamma(\zeta+1)}$.

Proof. Define an operator $\mathcal{G}: C[0,1] \to C[0,1]$ by

$$\begin{aligned}(\mathcal{G}\psi)(t) &= \nu_1 f\Big(t, \int_0^1 \tau^{\rho-1}G_1(t,\tau)\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big)d\tau, -\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big)\Big)\\ &+ \nu_2\int_0^t \frac{r^{\rho-1}(t^\rho - r^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)} g\Big(r, \int_0^1 \tau^{\rho-1}G_1(r,\tau)\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big)d\tau,\\ &-\phi_q\Big(\int_0^1 s^{\rho-1}G_2(r,s)\psi(s)ds\Big)\Big)dr. \end{aligned} \tag{3.2}$$

Observe that the continuity of $G_1(t,\tau)$, $G_2(\tau,s)$, $f(t,y,\omega)$ and $g(t,y,\omega)$ leads to that of the operator $\mathcal{G}$. Moreover, if $y(t)$ is a solution to the problem (1.1), then $\psi(t) = {}^\rho D_{1^-}^\alpha(\phi_p({}^\rho D_{0^+}^\beta y(x)))$ is the fixed point of the operator $\mathcal{G}$. Conversely, if $\psi(t)$ is a fixed point of the operator $\mathcal{G}$, then

$$y(t) = \int_0^1 \tau^{\rho-1}G_1(t,\tau)\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds\Big)d\tau,$$

is a solution to the problem (1.1).
Next, we need to show that the operator $\mathcal{G}$ maps $\Gamma[O,\mathcal{M}]$ into itself. Let $\psi \in \Gamma[O,\mathcal{M}]$; then, by Lemma 2.3, we have

$$|\omega(t)| \le \Omega_1^{q-1}\mathcal{M}^{q-1}, \quad |y(t)| \le \Omega_2\Omega_1^{q-1}\mathcal{M}^{q-1}.$$

Consequently, for any $t \in [0,1]$, there is $(t, y(t), \omega(t)) \in \Upsilon_\mathcal{M}$. So, from (A_1), we have

$$|(\mathcal{G}\psi)(t)| = |\nu_1 f(t,y(t),\omega(t)) + \nu_2 {}^\rho I_{0+}^\zeta g(t,y(t),\omega(t))| \le |\nu_1|M_1 + |\nu_2|\frac{M_2}{\Gamma(\zeta+1)} \le \mathcal{M}.$$

Therefore, $(\mathcal{G}\psi)(t) \in \Gamma[O,\mathcal{M}]$. Thus, the operator $\mathcal{G}$ maps $\Gamma[O,\mathcal{M}]$ into itself. Now, we show that the operator $\mathcal{G}: \Gamma[O,\mathcal{M}] \to \Gamma[O,\mathcal{M}]$ is a contraction. From (A_2), (A_3), Lemma 2.3, Lemma 2.4 and (ii) of Lemma 2.5, there is $\Delta_2 := \Omega_1\mathcal{M} \ge |\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds|$ for each $\psi_1(t), \psi_2(t) \in \Gamma[O,\mathcal{M}]$, and $1 < p \le 2$ (that is, $q \ge 2$). Thus, we obtain

$$\begin{aligned}&\Big|(\mathcal{G}\psi_2)(t) - (\mathcal{G}\psi_1)(t)\Big|\\ \le\ & |\nu_1|\Big|f(t,y_2(t),\omega_2(t)) - f(t,y_1(t),\omega_1(t))\Big| + |\nu_2|\Big|{}^\rho I_{0+}^\zeta g(t,y_2(t),\omega_2(t)) - {}^\rho I_{0+}^\zeta g(t,y_1(t),\omega_1(t))\Big|\\ \le\ & |\nu_1|\Big\{C_1\Big|y_2(t)-y_1(t)\Big| + C_2\Big|\omega_2(t)-\omega_1(t)\Big|\Big\} + |\nu_2|\Big\{\kappa_1 {}^\rho I_{0+}^\zeta|y_2(t)-y_1(t)| + \kappa_1 {}^\rho I_{0+}^\zeta|\omega_2(t)-\omega_1(t)|\Big\}\\ =\ & |\nu_1|\Big\{C_1\Big|\int_0^1 \tau^{\rho-1}G_1(t,\tau)\Big[\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big) - \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big]d\tau\Big|\\ &+C_2\Big|\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big) - \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big|\Big\}\\ &+|\nu_2|\Big\{\kappa_1\int_0^t \frac{r^{\rho-1}(t^\rho-r^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\Big|\int_0^1 \tau^{\rho-1}G_1(r,\tau)\Big[\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big)\\ &-\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big]d\tau\Big|dr\\ &+\kappa_2\int_0^t \frac{\tau^{\rho-1}(t^\rho-\tau^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\Big|\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big) - \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big|d\tau\Big\}\\ \le\ & |\nu_1|\Big\{C_1(q-1)(\mathcal{M}\Omega_1)^{q-2}\Big[\int_0^1 \tau^{\rho-1}G_1(t,\tau)\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|d\tau\Big]\end{aligned}$$

$$
\begin{aligned}
&+C_2(q-1)(\mathcal{M}\Omega_1)^{q-2}\Big|\int_0^1 G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|\Big\}\\
&+|\nu_2|\Big\{\kappa_1(q-1)(\mathcal{M}\Omega_1)^{q-2}\Big[\int_0^t \frac{r^{\rho-1}(t^\rho-r^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_0^1 \tau^{\rho-1}G_1(r,\tau)\times\\
&\times\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|d\tau dr\Big]\\
&+\kappa_2(q-1)(\mathcal{M}\Omega_1)^{q-2}\int_0^t \frac{\tau^{\rho-1}(t^\rho-\tau^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\Big|\int_0^1 G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|d\tau\Big\}\\
\leq\ & |\nu_1|\Big\{C_1(q-1)(\mathcal{M}\Omega_1)^{q-2}\Big[\int_0^1 \tau^{\rho-1}G_1(t,\tau)\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)ds\Big|d\tau\|\psi_2-\psi_1\|\Big]\\
&+C_2(q-1)(\mathcal{M}\Omega_1)^{q-2}\Big[\int_0^1 s^{\rho-1}G_2(\tau,s)ds\|\psi_2-\psi_1\|\Big]\Big\}\\
&+|\nu_2|\Big\{\kappa_1(q-1)(\mathcal{M}\Omega_1)^{q-2}\Big[\int_0^t \frac{r^{\rho-1}(t^\rho-r^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_0^1 \tau^{\rho-1}G_1(r,\tau)\times\\
&\times\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)ds\Big|d\tau dr\|\psi_2-\psi_1\|\Big]\\
&+\kappa_2(\mathcal{M}\Omega_1)^{q-2}\Big[\int_0^t \frac{\tau^{\rho-1}(t^\rho-\tau^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_0^1 s^{\rho-1}G_2(\tau,s)ds\|\psi_2-\psi_1\|d\tau\Big]\Big\}\\
\leq\ & |\nu_1|(q-1)\mathcal{M}^{q-2}\Omega_1^{q-1}\big(C_1\Omega_2+C_2\big)\|\psi_2-\psi_1\|\\
&+|\nu_2|(q-1)\mathcal{M}^{q-2}\frac{\Omega_1^{q-1}}{\Gamma(\zeta+1)}\big(\kappa_1\Omega_2+\kappa_2\big)\|\psi_2-\psi_1\|\\
=\ & (q-1)\mathcal{M}^{q-2}\Omega_1^{q-1}\Big\{|\nu_1|\big(C_1\Omega_2+C_2\big)+\frac{|\nu_2|}{\Gamma(\zeta+1)}\big(\kappa_1\Omega_2+\kappa_2\big)\Big\}\|\psi_2-\psi_1\|\\
=\ & L_1\|\psi_2-\psi_1\|,
\end{aligned}
$$

which, on taking the norm for $[0,1]$, yields

$$\|(\mathcal{G}\psi_2)-(\mathcal{G}\psi_1)\|\leq L_1\|\psi_2-\psi_1\|.$$

Since $L_1<1$ by (A_4), the operator $\mathcal{G}$ is a contraction. So, we deduce by Banach's contraction mapping principle that $\psi(t)$ is the unique fixed point of the operator $\mathcal{G}$. Hence, there exists a unique solution to the mixed boundary value problem (1.1) satisfying (3.1). The proof is completed. □

In the following result, we consider a special case of $\Upsilon_{\mathcal{M}}$. With the aid of Lemma 2.4, for $m>0$ with $\mathcal{M}>m$, we define the sets

$$\Upsilon^+_{\mathcal{M}}=\left\{(t,y,\omega):\ \begin{array}{c}0\leq t\leq 1,\ \Omega_4\Omega_3^{q-1}m^{q-1}t^{\rho(\beta-1)}\leq y(t)\leq\Omega_2\Omega_1^{q-1}\mathcal{M}^{q-1},\\ -\Omega_1^{q-1}\mathcal{M}^{q-1}\leq\omega(t)\leq-\Omega_3^{q-1}m^{q-1}(1-t^\rho)^{(\alpha-1)(q-1)},\end{array}\right\}$$

and

$$\Gamma_m=\{\psi(t)\in C[0,1]: m<\psi(t)<\mathcal{M}\}.$$

The following theorem is concerned with the existence of a unique solution to the problem (1.1) when $p>2$.

Theorem 3.2. *Assume that $p>2$ and there exist positive numbers m_1, m_2, M_1, M_2, C_1 and C_2 such that*

(B_1) $m_1 \leq f(t,y,\omega) \leq M_1 and\, m_2 \leq g(t,y,\omega) \leq M_2,$ *for* $(t,y,\omega) \in \Upsilon^+_{\mathcal{M}};$
(B_2) $|f(t,y_1,\omega_1) - f(t,y_2,\omega_2)| \leq C_1|y_2 - y_1| + C_2|\omega_2 - \omega_2|,$*for* $(t,y_i,\omega_i) \in \Upsilon^+_{\mathcal{M}},\ i = 1,2;$
(B_3) $|g(t,y_1,\omega_1) - g(t,y_2,\omega_2)| \leq \kappa_1|y_2 - y_1| + \kappa_2|\omega_2 - \omega_1|,$ *for* $(t,y_i,\omega_i) \in \Upsilon^+_{\mathcal{M}},\ i = 1,2;$
(B_4) $L_2 := (q-1)m^{q-2}\Omega_3^{q-2}\Omega_1\Big\{|\nu_1|\Big(C_1\Omega_2 + C_2\Big) + \frac{|\nu_2|}{\rho^{\zeta}\Gamma(\zeta+1)}\Big(\kappa_1\Omega_2 + \kappa_2\Big)\Big\} < 1.$

Then the mixed boundary value problem (1.1) has a unique solution satisfying the following inequalities

$$\Omega_4\Omega_3^{q-1}m^{q-1}t^{\rho(\beta-1)} \leq y(t) \leq \Omega_2\Omega_1^{q-1}\mathcal{M}^{q-1}, \textit{for all } t \in [0,1],$$
$$-\Omega_1^{q-1}\mathcal{M}^{q-1} \leq {}^{\rho}D^{\beta}_{0+}y(t) \leq -\Omega_3^{q-1}m^{q-1}(1-t^{\rho})^{(\alpha-1)(q-1)}, \textit{for all } t \in [0,1], \tag{3.3}$$

where $\mathcal{M} \geq |\nu_1|M_1 + |\nu_2|\frac{M_2}{\rho^{\zeta}\Gamma(\zeta+1)}$ *and* $0 < m \leq \nu_1 m_1 + \nu_2\frac{m_2}{\rho^{\zeta}\Gamma(\zeta+1)}$.

Proof. As argued in the proof of the last theorem, the operator $\mathcal{G}$ defined by (3.2) is continuous and $\mathcal{G}$ maps any $\psi \in \Gamma_m$ into itself.
Now, from (B_2), (B_3), Lemma 2.3, Lemma 2.4, and (i) of Lemma 2.5, there exists $\Delta_1 := m\,\Omega_3(1-\tau^{\rho})^{\alpha-1} \leq |\int_0^1 s^{\rho-1}G_2(\tau,s)\psi(s)ds|$. Then, for all $\psi_1,\psi_2 \in \Gamma_m$, and $p > 2$ (that is, $1 < q \leq 2$), we find that

$$\begin{aligned}
&\Big|(\mathcal{G}\psi_2)(t) - (\mathcal{G}\psi_1)(t)\Big| \\
\leq\ & |\nu_1|\Big|f(t,y_2(t),\omega_2(t)) - f(t,y_1(t),\omega_1(t))\Big| + |\nu_2|\Big|{}^{\rho}I^{\zeta}_{0+}g(t,y_2(t),\omega_2(t)) - {}^{\rho}I^{\zeta}_{0+}g(t,y_1(t),\omega_1(t))\Big| \\
\leq\ & |\nu_1|\Big\{C_1\Big|y_2(t) - y_1(t)\Big| + C_2\Big|\omega_2(t) - \omega_1(t)\Big|\Big\} + |\nu_2|\Big\{\kappa_1{}^{\rho}I^{\zeta}_{0+}|y_2(t) - y_1(t)| + \kappa_1{}^{\rho}I^{\zeta}_{0+}|\omega_2(t) - \omega_1(t)\Big|\Big\} \\
=\ & |\nu_1|\Big\{C_1\Big|\int_0^1 \tau^{\rho-1}G_1(t,\tau)\Big[\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big) - \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big]d\tau\Big| \\
& +C_2\Big|\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big) - \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big|\Big\} \\
& +|\nu_2|\Big\{\kappa_1\int_0^t \frac{r^{\rho-1}(t^{\rho}-r^{\rho})^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\Big|\int_0^1 \tau^{\rho-1}G_1(r,\tau)\Big[\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big) \\
& -\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big]d\tau\Big|dr \\
& +\kappa_2\int_0^t \frac{\tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\Big|\phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_2(s)ds\Big) - \phi_q\Big(\int_0^1 s^{\rho-1}G_2(\tau,s)\psi_1(s)ds\Big)\Big|d\tau\Big\} \\
\leq\ & |\nu_1|\Big\{C_1(q-1)\int_0^1 \tau^{\rho-1}G_1(t,\tau)m^{q-2}\Omega_3^{q-2}(1-\tau^{\rho})^{(\alpha-1)(q-2)}\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|d\tau \\
& +C_2(q-1)m^{q-2}\Omega_3^{q-2}(1-\tau^{\rho})^{(\alpha-1)(q-2)}\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|\Big\} \\
& +|\nu_2|\Big\{\kappa_1(q-1)\int_0^t \frac{r^{\rho-1}(t^{\rho}-r^{\rho})^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_0^1 \tau^{\rho-1}G_1(r,\tau)m^{q-2}\Omega_3^{q-2}(1-\tau^{\rho})^{(\alpha-1)(q-2)} \times \\
& \times\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|d\tau dr \\
& +\kappa_2(q-1)m^{q-2}\Omega_3^{q-2\rho}\int_0^t \frac{\tau^{\rho-1}(t^{\rho}-\tau^{\rho})^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}(1-\tau^{\rho})^{(\alpha-1)(q-2)} \times \\
& \times\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)(\psi_2(s)-\psi_1(s))ds\Big|d\tau\Big\}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & |\nu_1|\Big\{C_1(q-1)m^{q-2}\Omega_3^{q-2}\Big[\int_0^1 \tau^{\rho-1}G_1(t,\tau)(1-\tau^\rho)^{(\alpha-1)(q-2)}\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)ds\Big|d\tau\|\psi_2-\psi_1\|\Big] \\
& +C_2(q-1)m^{q-2}\Omega_3^{q-2}(1-\tau^\rho)^{(\alpha-1)(q-2)}\Big[\int_0^1 s^{\rho-1}G_2(\tau,s)ds\|\psi_2-\psi_1\|\Big]\Big\} \\
& +|\nu_2|\Big\{\kappa_1(q-1)m^{q-2}\Omega_3^{q-2}\Big[\int_0^t \frac{r^{\rho-1}(t^\rho-r^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_0^1 \tau^{\rho-1}G_1(r,\tau)(1-\tau^\rho)^{(\alpha-1)(q-2)}\times \\
& \times\Big|\int_0^1 s^{\rho-1}G_2(\tau,s)ds\Big|d\tau dr\|\psi_2-\psi_1\|\Big] \\
& +\kappa_2(q-1)m^{q-2}\Omega_3^{q-2}\Big[\int_0^t \frac{\tau^{\rho-1}(t^\rho-\tau^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}(1-\tau^\rho)^{(\alpha-1)(q-2)}\int_0^1 s^{\rho-1}G_2(\tau,s)dsd\tau\|\psi_2-\psi_1\|\Big]\Big\} \\
\leq\ & |\nu_1|\Big\{C_1(q-1)m^{q-2}\Omega_3^{q-2}\Big[\int_0^1 \tau^{\rho-1}G_1(t,\tau)(1-\tau^\rho)^{(\alpha-1)(q-2)}\Omega_1(1-\tau^\rho)^{\alpha-1}d\tau\|\psi_2-\psi_1\|\Big] \\
& +C_2(q-1)m^{q-2}\Omega_3^{q-2}(1-\tau^\rho)^{(\alpha-1)(q-2)}\Big[\Omega_1(1-\tau^\rho)^{\alpha-1}\|\psi_2-\psi_1\|\Big]\Big\} \\
& +|\nu_2|\Big\{\kappa_1(q-1)m^{q-2}\Omega_3^{q-2}\Big[\int_0^t \frac{r^{\rho-1}(t^\rho-r^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}\int_0^1 \tau^{\rho-1}G_1(r,\tau)(1-\tau^\rho)^{(\alpha-1)(q-2)}\times \\
& \times\Omega_1(1-\tau^\rho)^{\alpha-1}d\tau dr\|\psi_2-\psi_1\|\Big] \\
& +\kappa_2(q-1)m^{q-2}\Omega_3^{q-2}\Big[\int_0^t \frac{\tau^{\rho-1}(t^\rho-\tau^\rho)^{\zeta-1}}{\rho^{\zeta-1}\Gamma(\zeta)}(1-\tau^\rho)^{(\alpha-1)(q-2)}\Omega_1(1-\tau^\rho)^{\alpha-1}d\tau\|\psi_2-\psi_1\|\Big]\Big\} \\
\leq\ & (q-1)m^{q-2}\Omega_3^{q-2}\Omega_1\Big\{|\nu_1|\Big(C_1\Omega_2+C_2\Big)+\frac{|\nu_2|}{\rho^\zeta\Gamma(\zeta+1)}\Big(\kappa_1\Omega_2+\kappa_2\Big)\Big\}\|\psi_2-\psi_1\| \\
=\ & L_2\|\psi_2-\psi_1\|,
\end{aligned}
$$

which, after taking the norm for $t\in[0,1]$, takes the form:

$$\|(\mathcal{G}\psi_2)-(\mathcal{G}\psi_1)\|\leq L_2\|\psi_2-\psi_1\|,$$

with $L_2<1$ by the condition (B_4). Consequently, the operator $\mathcal{G}$ is a contraction. Hence, by Banach's contraction mapping principle, $\psi(t)$ is the unique fixed point of the operator $\mathcal{G}$. Therefore, there exists a unique solution to the problem (1.1) satisfying (3.3). This finishes the proof.

4. Examples

Consider the following problem

$$
\begin{cases}
{}^{1/2}D_{1^-}^{5/3}(\phi_p({}^{1/2}D_{0^+}^{3/2}y(x)))=\frac{3}{4}f(t,y(t),{}^{1/2}D_{0^+}^{3/2}y(t))-\frac{2}{3}\,{}^{1/2}I_{0+}^{1/4}g(t,y(t),{}^{1/2}D_{0^+}^{3/2}y(t)),\\
y(0)=0,\quad y(1)=\frac{1}{2}y(3/4),\ {}^{1/2}D_{0^+}^{3/2}y(1)=0,\ \phi_p({}^{1/2}D_{0^+}^{3/2}y(0))=\phi_p({}^{1/2}D_{0^+}^{3/2}y(1/2)),
\end{cases}
\tag{4.1}
$$

where $\alpha=5/3$, $\beta=3/2$, $\rho=1/2$, $\lambda_1=1/2$, $\lambda_2=1$, $\mu=3/4$, $\eta=1/2$, $\nu_1=3/4$ *and* $\nu_2=-2/3$, and p, $f(t,y,\omega)$ and $g(t,y,\omega)$ will be fixed later.

From the given data, we have

$$G_1(t,s) \approx \frac{\sqrt{2}}{\Gamma(3/2)} \begin{cases} (1.87022)t^{1/4}\Big[(1-s^{1/2})^{1/2} - 1/2((3/4)^{1/2} - s^{1/2})^{1/2}\Big] - (t^{1/2} - s^{1/2})^{1/2}, \\ \qquad 0 \le s \le \min\{t, 3/4\}; \\ (1.87022)t^{1/4}(1-s^{1/2})^{1/2} - (t^{1/2} - s^{1/2})^{1/2}, \qquad 3/4 \le s \le t; \\ (1.87022)t^{1/4}\Big[(1-s^{1/2})^{1/2} - 1/2((3/4)^{1/2} - s^{1/2})^{1/2}\Big], \qquad t \le s \le 3/4; \\ (1.87022)t^{1/4}(1-s^{1/2})^{1/2}, \qquad \max\{t, 3/4\} \le s \le 1, \end{cases}$$

and

$$G_2(t,s) \approx \frac{2^{2/3}}{\Gamma(5/3)} \begin{cases} 1.78902(1-t^{1/2})^{2/3} s^{1/3}, \qquad 0 \le s \le \min\{t, 1/2\}; \\ 1.78902(1-t^{1/2})^{2/3}[s^{1/3} - (s^{1/2} - (1/2)^{1/2})^{2/3}], \qquad 1/2 \le s \le t; \\ 1.78902(1-t^{1/2})^{2/3} s^{1/3} - (s^{1/2} - t^{1/2})^{2/3}, \qquad t \le s \le 1/2; \\ 1.78902(1-t^{1/2})^{2/3}[s^{1/3} - (s^{1/2} - (1/2)^{1/2})^{2/3}] - (s^{1/2} - t^{1/2})^{2/3}, \\ \qquad \max\{t, 1/2\} \le s \le 1, \end{cases}$$

which satisfy the properties expressed in Lemma 2.3. Moreover, for $t, s \in (0,1)$, we have $G_1(t,s) > 0, G_2(t,s) > 0$ and

$$\frac{\sqrt{2}}{\Gamma(3/2)}(0.8702161)\, t^{1/4}\, s^{1/2}\, (1-s^{1/2})^{1/2} \le G_1(t,s) \le \frac{\sqrt{2}}{\Gamma(3/2)}(1.87022)\, t^{1/4}\, (1-s^{1/2})^{1/2},$$

$$\frac{2^{2/3}}{\Gamma(5/3)}(0.7890202)\, s^{1/3}\, (1-t^{1/2})^{2/3}\, (1-s^{1/2}) \le G_2(t,s) \le \frac{2^{2/3}}{\Gamma(5/3)}(1.78902)\, s^{1/3}\, (1-t^{1/2})^{2/3},$$

with $Q_1 \approx 0.465302$ and $Q_2 \approx 0.4410348$.
For illustrating Theorem 3.1, let us take

$$f(t,u,\omega) = \frac{e^{-t}}{36\sqrt{900+t}}\Big(y^2 + \frac{|\omega|}{2} + \cos t\Big), \tag{4.2}$$

$$g(t,u,\omega) = \frac{\tan^{-1} y + \omega^2}{(t+25)^2}, \tag{4.3}$$

and $p = 3/2$ (that is, $q = 3$). Using the given values, it is found that $\Omega_1 = 3.7750084$, $\Omega_2 = 3.9792441$, $\Omega_3 = 0.6243413 and\, \Omega_4 = 0.2557131$. Also, $\mathcal{M}$ satisfies the following relations:

$$2.977481684\, \mathcal{M}^4 + 0.1319508185\, \mathcal{M}^2 + 0.9259259259 \le M_1,$$
$$0.2513274123 + 0.3249313918\, \mathcal{M}^4 \le M_2,\ 3/4M_1 + 2/3\frac{M_2}{1/2^{1/4}\Gamma(1/4+1)} \le \mathcal{M},$$

with $\|y\| \le 56.706968\, \mathcal{M}^2$ and $\|\omega\| \le 14.250688\, \mathcal{M}^2$. Choosing $M_1 = 0.3$, $M_2 = 0.08$ and $\mathcal{M} = 0.4$, it can easily be verified that the functions $f(t,y,\omega)$ and $g(t,y,\omega)$ given by (4.2) and (4.3) respectively, satisfy the condition (A_1). Furthermore, on the domain:

$$\Upsilon_{0.4} := \{(t,y,\omega), 0 \le t \le 1, |y| \le 9.073114878, |\omega| \le 2.280110144\},$$

we find that

$$|f_y| = \left|\frac{2e^{-t}y}{36\sqrt{900+t}}\right| \le 0.01680206459,\ |f_\omega| = \left|\frac{e^{-t}}{72\sqrt{900+t}}\right| \le 0.001055606548,$$
$$|g_y| = \left|\frac{1}{(1+y^2)(t+25)^2}\right| \le 0.0016,\ |g_\omega| = \left|\frac{2\omega}{(t+25)^2}\right| \le 0.00729635.$$

Obviously the conditions (A_2) and (A_3) are satisfied with $C_1 = 0.01680206459, C_2 = 0.001055606548$, $\kappa_1 = 0.0016 and \kappa_2 = 0.00729635246$. Also, $L_1 = 0.7169476783 < 1$. Thus all of the conditions of Theorem 3.1 are satisfied and hence the problem (4.1) has a unique solution on $\Upsilon_{0.4}$.

We illustrate Theorem 3.2 by choosing

$$f(t,y,\omega) = \frac{1}{120}\Big(y + \omega^2 + 5(t+3)\Big), \tag{4.4}$$
$$g(t,y,\omega) = \frac{e^{-t}}{3\sqrt{t+900}}\Big(y + 3\omega^2/4\Big) \tag{4.5}$$

and $p = 4$ (that is, $q = 4/3$). Here the values of Ω_1, $\Omega_2 and \Omega_3$ are the same as those found in the first example and $\Omega_4 = 0.592399$. Letting $m = 0.05 and \mathcal{M} = 0.4$, as argued in the first example, we find that the functions $f(t,y,\omega)$ and $g(t,y,\omega)$ given by (4.4) and (4.5) respectively, satisfy the condition (B_1) in the following domain: $\Upsilon^+_{0.4} := \{(t,y,\omega), 0 \le t \le 1, 0.1865284535\ t^{1/4} \le y(t) \le 4.565200828, -1.147253270 \le \omega \le -0.3148695697\ (1-\sqrt{t})^{2/9}\}$.

Moreover, the conditions (B_2) and (B_3) hold true with $C_1 = 1/120$, $C_2 = 0.01912088784$, $\kappa_1 = 1/90$, $\kappa_2 = 0.005247826157$ and $L_2 = 0.3153939012 < 1$. Thus all of the conditions of Theorem 3.2 are satisfied and hence the problem (4.1) has a unique solution on $\Upsilon^+_{0.4}$.

5. Conclusions

In this paper, we have investigated the criteria for ensuring the uniqueness of positive solutions for a class of fractional integro-differential equations with a p-Laplacian operator, complemented with nonlocal boundary conditions involving fractional derivatives and the p-Laplacian operator. Using a method employed in [31] together with the properties of the associated Green's functions established for the given problem, we proved two uniqueness results for the cases $1 < p \le 2$ and $p > 2$, respectively. Illustrative examples demonstrating application of the obtained results are presented. It is worthwhile to note that our results are new in the given configuration and enrich the literature on p-Laplacian fractional boundary value problems involving right-sided and left-sided fractional derivative operators, as well as left-sided fractional integral operators with respect to the power function.

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. X. Liu, M. Jia, W. Ge, The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator, *Appl. Math. Lett.*, **65** (2017), 56–62. https://doi.org/10.1016/j.aml.2016.10.001
2. C. Bai, Existence and uniqueness of solutions for fractional boundary value problems with p-Laplacian operator, *Adv. Differ. Equations*, **4** (2018), 12. https://doi.org/10.1186/s13662-017-1460-3
3. S. Wang, Z. Bai, Existence and uniqueness of solutions for a mixed p-Laplace boundary value problem involving fractional derivatives, *Adv. Differ. Equations*, **694** (2020), 9. https://doi.org/10.1186/s13662-020-03154-2
4. J. Tan, M. Li, Solutions of fractional differential equations with p-Laplacian operator in Banach spaces, *Bound. Value Probl.*, **15** (2018), 13. https://doi.org/10.1186/s13661-018-0930-1
5. M. M. Matar, A. A. Lubbad, J. Alzabut, On p-Laplacian boundary value problems involving Caputo–Katugampula fractional derivatives, *Math. Methods Appl. Sci.*, (2020). https://doi.org/10.1002/mma.6534
6. M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, S. Rezapour, Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives, *Adv. Differ. Equations*, **68** (2021), 18. https://doi.org/10.1186/s13662-021-03228-9
7. W. Dai, Z. Liu, P. Wang, Monotonicity and symmetry of positive solutions to fractional p-Laplacian equation, *Commun. Contemp. Math.*, **24** (2022), 17. https://doi.org/10.1142/S021919972150005X
8. R. Luca, On a system of fractional differential equations with p-Laplacian operators and integral boundary conditions, *Rev. Roumaine Math. Pures Appl.*, **66** (2021), 749–766.
9. Q. Lou, Y. Qin, F. Liu, The existence of constrained minimizers related to fractional p-Laplacian equations, *Topol. Methods Nonlinear Anal.*, **58** (2021), 657–676. https://doi.org/10.12775/TMNA.2020.079
10. J. R. Graef, S. Heidarkhani, L. Kong, S. Moradi, Three solutions for impulsive fractional boundary value problems with p-Laplacian, *Bull. Iran. Math. Soc.*, **48** (2022), 1413–1433. https://doi.org/10.1007/s41980-021-00589-5
11. B. Sun, W. Jiang, S. Zhang, Solvability of fractional differential equations with p-Laplacian and functional boundary value conditions at resonance, *Mediterr. J. Math.*, **19** (2022), 18. https://doi.org/10.1007/s00009-021-01753-1
12. B. Ahmad, J. Henderson, R. Luca, *Boundary Value Problems for Fractional Differential Equations and Systems, Trends in Abstract and Applied Analysis*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2021.

13. D. Baleanu, G. C. Wu, S.D Zeng, Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, *Chaos Solitons Fractals*, **102** (2017), 99–105. https://doi.org/10.1016/j.chaos.2017.02.007

14. F. Jarad, T. Abdeljawad, D. Baleanu, On the generalized fractional derivatives and their Caputo modification, *J. Nonlinear Sci. Appl.*, **10** (2017), 2607–2619. https://doi.org/10.22436/jnsa.010.05.27

15. B. Ahmad, M. Alghanmi, A. Alsaedi, H. M. Srivastava, S. K. Ntouyas, The Langevin equation in terms of generalized Liouville-Caputo derivatives with nonlocal boundary conditions involving a generalized fractional integral, *Mathematics*, **7** (2019), 533. https://doi.org/10.3390/math7060533

16. S. K. Ntouyas, B. Ahmad, M. Alghanmi, A. Alsaedi, Generalized fractional differential equations and inclusions equipped with nonlocal generalized fractional integral boundary conditions, *Topol. Method. Nonlinear Anal.*, **54** (2019), 1051–1073. https://doi.org/10.12775/TMNA.2019.035

17. A. Alsaedi, B. Ahmad, M. Alghanmi, Extremal solutions for generalized Caputo fractional differential equations with Steiltjes-type fractional integro-initial conditions, *Appl. Math. Lett.*, **91** (2019), 113–120. https://doi.org/10.1016/j.aml.2018.12.006

18. B. Ahmad, M. Alghanmi, A. Alsaedi, Existence results for a nonlinear coupled system involving both Caputo and Riemann-Liouville generalized fractional derivatives and coupled integral boundary conditions, *Rocky Mountain J. Math.*, **50** (2020), 1901–1922. https://doi.org/10.1216/rmj.2020.50.1901

19. Y. Li, Y. Liu, Multiple solutions for a class of boundary value problems of fractional differential equations with generalized Caputo derivatives, *AIMS Math.*, **6** (2021), 13119–13142. https://doi.org/10.3934/math.2021758

20. T. V. An, H. Vu, N. V. Hoa, Finite-time stability of fractional delay differential equations involving the generalized Caputo fractional derivative with non-instantaneous impulses, *Math. Methods Appl. Sci.*, **45** (2022), 4938–4955. https://doi.org/10.1002/mma.8084

21. R. Singh, A. Wazwaz, An efficient method for solving the generalized Thomas-Fermi and Lane-Emden-Fowler type equations with nonlocal integral type boundary conditions, *Int. J. Appl. Comput. Math.*, **8** (2022), 22. https://doi.org/10.1007/s40819-022-01280-x

22. N. M. Dien, J. J. Nieto, Lyapunov-type inequalities for a nonlinear sequential fractional BVP in the frame of generalized Hilfer derivatives, *Math. Inequal. Appl.*, **25** (2022), 851–867. https://doi.org/10.7153/mia-2022-25-54

23. J. Li, B. Li, Y. Meng, Solving generalized fractional problem on a funnel-shaped domain depicting viscoelastic fluid in porous medium, *Appl. Math. Lett.*, **134** (2022), 108335. https://doi.org/10.1016/j.aml.2022.108335

24. T. V. An, N. D. Phu, N. V. Hoa, A survey on non-instantaneous impulsive fuzzy differential equations involving the generalized Caputo fractional derivative in the short memory case, *Fuzzy Set. Syst.*, **443** (2022), 160–197. https://doi.org/10.1016/j.fss.2021.10.008

25. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science, Yverdon, 1993.

26. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of the Fractional Differential Equations*, Elsevier: Amsterdam, The Netherlands, 2006.
27. A. Erdelyi, An integral equation involving Legendre functions, *J. Soc. Indust. Appl. Math.*, **12** (1964), 15–30. https://doi.org/10.1137/0112002
28. B. Lupinska, T. Odzijewicz, A Lyapunov-type inequality with the Katugampola fractional derivative, *Math. Methods Appl. Sci.*, **41** (2018), 8985–8996. https://doi.org/10.1002/mma.4782
29. F. Jiang, X. Xu, Z. Cao, The positive properties of Green's function for fractional differential equations and its applications, *Abstr. Appl. Anal.*, **2013** (2013), 12. https://doi.org/10.1155/2013/531038
30. X. Liu, M. Jia, X. Xiang, On the solvability of a fractional differential equation model involving the *p*-Laplacian operator, *Comput. Math. Appl.*, **64** (2012), 3267–3275. https://doi.org/10.1016/j.camwa.2012.03.001
31. Q. A. Dang, D. Q. Long, T. K. Q. Ngo, A novel efficient method for nonlinear boundary value problems, *Numer. Algorithms*, **76** (2017), 427–439. https://doi.org/10.1007/s11075-017-0264-6