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Research article

Boundedness criteria for the quasilinear attraction-repulsion chemotaxis system with nonlinear signal production and logistic source

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Abstract: This paper deals with the following quasilinear attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \nabla\cdot((u+1)^m\nabla u - \chi u(u+1)^{\theta-1}\nabla v + \xi u(u+1)^{l-1}\nabla w) + au - bu^{\kappa}, & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u^{\gamma_1} - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u^{\gamma_2} - \delta w, & x\in\Omega,\ t>0, \end{cases}$$

with homogeneous Neumann boundary conditions in a bounded, smooth domain $\Omega\subset\mathbb{R}^n (n\geq 1)$, where $m,\theta,l\in\mathbb{R}$ and $\chi,\xi,a,b,\alpha,\beta,\gamma,\delta,\gamma_1,\gamma_2>0,\kappa>1$. It is proved that if the nonlinear exponents of the system satisfy $\theta+\gamma_1<\max\left\{l+\gamma_2,\kappa,m+\frac{2}{n}+1\right\}$, then the system has globally bounded classical solutions. Furthermore, assume that $\theta+\gamma_1=\max\{l+\gamma_2,\kappa\}\geq m+\frac{2}{n}+1$, if one of the following conditions holds:

(a) when $\theta+\gamma_1=l+\gamma_2=\kappa$, if $\theta\geq l\geq 1$ and $\dfrac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n}<b$ or if $l\geq\theta\geq 1$ and $\dfrac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}<b$;

(b) when $\theta+\gamma_1=l+\gamma_2>\kappa$, if $\theta\geq l\geq 1$ and $2\alpha\chi\leq\gamma\xi$;

(c) when $\theta+\gamma_1=\kappa>l+\gamma_2$, if $\theta\geq 1$ and $\dfrac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}<b$,

then the classical solutions of the system would be globally bounded. The global boundedness criteria generalize the results established by previous researchers.

Keywords: boundedness criteria; attraction-repulsion system; nonlinear signal production; logistic source

1. Introduction

Chemotaxis is one of the basic physiological reactions of organisms, which refers to the directional movement of biological cells or organisms along the concentration gradient of stimulants under the stimulation of some chemicals in the environment. The chemotaxis phenomenon has been described in the pioneering work proposed by Keller and Segel [1], which is given by

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, $\chi > 0$, $\tau \in \{0, 1\}$, $u(x,t)$ and $v(x,t)$ denote the density of cells and the concentration of the chemical signal, respectively. It is well known that chemotaxis research has many important applications in biology and medicine, such as in bacterial colonies [2], tumor invasion processes [3,4] and embryonic development [5], so that it has been one of the hottest research focuses in applied mathematics nowadays. In the past few decades, a large number of valuable theoretical results have been obtained by scholars [6–8]. Among them, one of the main issues related to (1.1) is to study whether there is a globally in-time bounded solution or when blow-up occurs. For $\tau = 1$, it has been shown that the system (1.1) has globally bounded classical solution when $n = 1$ [9] or $n = 2$ and $\int_\Omega u_0 dx < \frac{4\pi}{\chi}$ [10, 11], whereas the system (1.1) has finite time blow-up solution in the case of $n = 2$ and $\int_\Omega u_0 dx > \frac{4\pi}{\chi}$ [12, 13] or in the case of $n \geq 3$ [14, 15]. When the chemical substance diffuses much faster than the diffusion of cells [16], model (1.1) can be reduced to the simplified parabolic-elliptic model, namely, the second equation in system (1.1) is replaced by $0 = \Delta v + \mu - v$ with $\mu = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$. Compared with the fully parabolic version of system (1.1), the similar results on global boundedness and blow-up of solutions can be found in [17–20], which still depend on the dimensions of space.

As described in system (1.1), the term of signal production is a linear function of the cell density in the classical Keller-Segel model. Nevertheless, the mechanism of signal production might be very complex, particularly, it could be in a nonlinear form. When the second equation in system (1.1) is replaced by $v_t = \Delta v - v + g(u)$ with $g(u) \in C^1([0, +\infty))$ and $0 \leq g(u) \leq Ku^\alpha$ for some constants $K, \alpha > 0$, Liu and Tao [21] proved that if $0 < \alpha < \frac{2}{n}$, the system (1.1) possesses a globally bounded classical solution. When the second equation in system (1.1) degenerates into an elliptic equation, u is replaced by $g(u)$ and v is replaced by $\mu(t) := \frac{1}{|\Omega|}\int_\Omega g(u(\cdot,t))$, $g(u) = u^\kappa$ with $\kappa > 0$, Winkler [22] derived a blow-up critical exponent $\kappa = \frac{2}{n}$, which asserted that the radially symmetric solution blows up in finite time if the parameter κ satisfies $\kappa > \frac{2}{n}$. Conversely, when $\kappa < \frac{2}{n}$ they proved that there existed suitable initial data u_0 such that the system has globally bounded classical solutions. In many biological processes, the proliferation and death of cells should be considered, from which one can derived the related chemotaxis-growth model

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + g(u), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & x \in \Omega. \end{cases} \tag{1.2}$$

Here it is worth mentioning that logistic source term $f(u)$ should somewhat decrease the possibility of blow-up. When $\tau = 0$, Tello and Winkler [23] considered the system (1.2) with $f(u) \leq u(a - bu)$ and $g(u) = u$ for $a, b > 0$ and they proved that the system has globally bounded classical solution whenever $\frac{n-2}{n}\chi < b$. For the more general case $f(u) \leq u(a - bu^s)$ and $g(u) = u^k$ with $k, s > 0$, Wang and Xiang [24] showed that the system (1.2) has globally bounded classical solutions if either $s > k$ or $s = k$ with $\frac{kn-2}{kn}\chi < b$. For $f(u) = au - bu^s$ and $g(u) = u$ with $s > 1, a \geq 0, b > 0$, Winkler [25] proved global existence of very weak solutions of (1.2) under the assumption that $s > 2 - \frac{1}{n}$, moreover, boundedness properties of the constructed solutions were studied. When $\tau = 1$, $g(u) = u$ and $f(u)$ is controlled by $-c_0(u + u^s)$ and $a - bu^s$, respectively, for all $u \geq 0$ with some $s > 1, b, c_0 > 0$ and $a \geq 0$, by an appropriate definition of very weak solution, Viglialoro [26] constructed the such global solutions under the assumptions that $n \geq 2$ and $s > 1 - \frac{2}{n}$, and in [27], a relaxation of these hypotheses could be achieved so as to ensure solvability even for any $s > \frac{2n+4}{n+4}$ when $n \geq 2$. Furthermore, when $f(u) = au - bu^s$ and $g(u) = u$, Winkler [28] proved that if $s \geq 2 - \frac{2}{n}$, under an appropriate smallness assumption on χ any such solution at least asymptotically exhibits relaxation by approaching the nontrivial spatially homogeneous steady state $\left((\frac{a}{b})^{\frac{1}{s-1}}, (\frac{a}{b})^{\frac{1}{s-1}}\right)$ in the large time limit. Continuing the research initiated in [26], Viglialoro and Woolley [29] studied the boundedness and regularity of these solutions after some time.

However, not all logistic source terms can guarantee the global existence of solutions. When $g(u) = u$ and v is replaced by $m(t) := \frac{1}{|\Omega|}\int_\Omega u(\cdot, t)$ in the second equation of system (1.2), f satisfies the set of hypotheses: (i) $f \in C^0([0, \infty)) \cap C^1((1, \infty))$; (ii) $f(u) \geq -\mu u^\kappa$ for all $u \geq 0$ and some $\mu \geq 0$ and $\kappa > 1$; (iii) $f(u) \leq A(u + 1)$ for all $u \geq 0$ with some $A \geq 0$. Winkler [30] proved that if $n \geq 5$ and $\kappa < \frac{3}{2} + \frac{1}{2n-2}$, then there exist initial data such that the smooth local-in-time solution of (1.2) blows up in finite time. When $f(u) = \lambda u - \mu u^2$ with $\lambda \geq 0$, $\mu > 0$ and Δu in the first equation of system (1.2) is replaced by $\varepsilon\Delta u$ with $\varepsilon \to 0$, Winkler [31] proved that if $\mu < 1$ then some solutions blow up in finite time. Later, Lankiet [32] extended this result to higher-dimensional cases. Kang [33] improved the results in [31, 32] to $\mathbb{R}^n (n > 1)$ and $f(u) = \lambda u - \mu u^\beta$ with $\beta > 0$. When $f(u) = \lambda u - \mu u^\kappa$ with $\lambda \in \mathbb{R}, \mu > 0$ and $\kappa > 1$, Winkler [34] obtained a condition on initial data to ensure the occurrence of finite-time blow-up to system (1.2) for

$$\kappa < \begin{cases} \frac{7}{6}, & \text{if } n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)}, & \text{if } n \geq 5. \end{cases}$$

When $f(u) = \lambda u - \mu u^\alpha$ for $\alpha > 1$ and the second equation in system (1.2) is replaced by $0 = \Delta v - m(t) + g(u)$ with $g(u) \geq ku^\kappa$ for $k, \kappa > 0$ and $m(t) = \frac{1}{|\Omega|}\int_\Omega g(u(\cdot, t))$, Yi et al. [35] proved that if κ and α satisfy $\kappa + 1 > \alpha(1 + \frac{2}{n})$, then the corresponding solutions to system blow up in finite time. Besides, for the fully parabolic case, Winkler [36] revealed an unboundeness phenomenon, possibly transient in time, despite logistic growth restrictions.

From a physical point of view, the equation modeling the migration of cells should rather be regarded as nonlinear diffusion [37], especially the slow diffusion with finite propagation property, which reads

$$\begin{cases} u_t = \nabla \cdot (D(u)u) - \nabla \cdot (uS(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where the positive functions $D(u)$ and $S(u)$ are used to describe the strength of diffusion and chemoattractant, respectively. When $\tau = 1, n \geq 2$, $\Omega \subset \mathbb{R}^n$ is a ball and $f(u) = 0$, Winkler [38] proved that if

$\frac{D(u)}{S(u)}$ grows faster than $u^{\frac{2}{n}}$ as $u \to \infty$ and some further technical conditions are fulfilled, then there exist solutions that blow up in either finite or infinite time, which implies that the result is optimal. Inter alia, there still exist many results on global boundedness and blow-up in (1.3), please see [39–42]. For $\tau = 0$, when $f(u) = au - bu^{\kappa}$ for all $u \geq 0$ with $a \geq 0, b > 0$ and $\kappa > 1$, and the second equation in (1.3) is replaced by $0 = \Delta v - \mu(t) + u$ with $\mu(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$. In [43], for $D(u) \geq D_0 u^{-m}$ and $S(u) = \chi$ for all $u > 0$ with some $m \in \mathbb{R}, \chi, D_0 > 0$, it was shown that the system (1.3) possesses a unique globally bounded classical solution for any initial data $u_0 \in C^0(\overline{\Omega})$ and $n \geq 2$ if $\kappa > \max\{m + 3 - \frac{4}{n+2}, 2\}$; and for $D(u) = D_0 u^{-m}$ and $S(u) = \chi$ with $\frac{4}{n} - 1 < m \leq 0$, the system (1.3) blows up in finite time in a ball if $\kappa \in \left(1, \frac{(3-m)n-2}{2n-2}\right)$ and $n \geq 5$. For more boundedness results and blow-up of solutions to system (1.3) with or without logistic source, we refer the interested readers to [44–48].

As stated in [49], studies have shown that the reaction of one species to multiple stimuli is given by the motion of microglia in Alzheimer disease tethered to a glass side in a conflict situation involving β-amyloid (an attractant) and tumor necrosis factor α (a repellent). To model such biological processes, the following attraction-repulsion chemotaxis system was proposed in [50]

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \chi \nabla v + \xi \nabla w) + f(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \beta v + g_1(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta w - \delta w + g_2(u), & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

where u, v and w represent the density of cell, chemical concentration of attractant and chemical concentration of repellent, respectively, the parameters $\chi, \xi > 0$. To better understand system (1.4), let us mention some previous contributions in this direction. We first introduce some global boundedness of classical solutions related to system (1.4). For the case of $f(u) \leq u(a - bu)$, when $g_1(u) = \alpha u$ and $g_2(u) = \gamma u$ with $a, b, \alpha, \gamma > 0$, Zhang et al. [51] obtained that for any nonnegative $u_0(x) \in C^0(\overline{\Omega})$, if one of the following conditions holds

$$\text{(a) } \alpha\chi - \gamma\xi \leq b; \text{ (b) } n \leq 2; \text{ (c) } \frac{n-2}{n}(\alpha\chi - \gamma\xi) < b \text{ with } n \geq 3, \tag{1.5}$$

then the system (1.4) has a globally bounded classical solution. For the case of $f(u) \leq u(a - bu^s)$, when $g_1(u) = \alpha u^k$ and $g_2(u) = \gamma u^l$ with $a, b, \alpha, \gamma, k, l, s > 0$, Hong et al. [52] showed that if $k < \max\{l, s, \frac{2}{n}\}$, then the system (1.4) admits a globally bounded solution. Furthermore, when $k = \max\{l, s\} \geq \frac{2}{n}$, the system (1.4) also admits a globally bounded solution if one of the following assumptions holds: (a) $k = l = s, \frac{kn-2}{kn}(\alpha\chi - \gamma\xi) < b$; (b) $k = l > s, \alpha\chi - \gamma\xi < 0$; (c) $k = s > l, \frac{kn-2}{kn}\alpha\chi < b$. More recently, on the basis of [52], in high dimension ($n \geq 2$), Zhou et al. [53] have further studied the boundedness of globally classical solution for the critical cases: (a) $k = l = s, \frac{kn-2}{kn}(\alpha\chi - \gamma\xi) = b$; (b) $k = s > l, \frac{kn-2}{kn}\alpha\chi = b$; (c) $k = l > s, \alpha\chi - \gamma\xi = 0, nk(nk - 2) < 4, 0 < k = l \leq 1$. Apart from that, there are some interesting findings about blow-up behavior of solutions for system (1.4). When $f(u) = 0$, $g_1(u) = \alpha u$ and $g_2(u) = \gamma u$ with $\alpha, \gamma > 0$, Li et al. [54] proved that the nonradial solutions to system (1.4) would blow-up in finite time if either $\alpha\chi - \xi\gamma > 0, \beta \neq \delta$ and $\int_\Omega u_0 dx > \frac{8\pi}{\alpha\chi - \xi\gamma}$ or $\alpha\chi\delta - \xi\gamma\beta > 0, \delta < \beta$ and $\int_\Omega u_0 dx > \frac{8\pi}{\alpha\chi\delta - \xi\gamma\beta}$ in the case $n = 2$. Yu et al. [55] improved the above finite-time blowup result under the condition of $\alpha\chi - \xi\gamma > 0$ and $\int_\Omega u_0 dx > \frac{8\pi}{\alpha\chi - \xi\gamma}$. More recently, when $f(u) = 0$, the second and third equations are replaced by $0 = \Delta v - m_1(t) + g_1(u), m_1(t) = \frac{1}{|\Omega|}\int_\Omega g_1(u)$

and $0 = \Delta w - m_2(t) + g_2(u), m_2(t) = \frac{1}{|\Omega|}\int_\Omega g_1(u)$, respectively, for $g_1(u) \geq k_1 u^{\gamma_1}$ and $g_2(u) \leq k_2 u^{\gamma_2}$ for all $u \geq 0$ with $k_1, k_2, \gamma_1, \gamma_2 > 0$, Liu et al. [56] proved that if $\gamma_1 > \max\{\gamma_2, \frac{2}{n}\}$, the radial solutions to system (1.4) would blow-up in finite time, and if $\gamma_1 < \frac{2}{n}$, the classical solution would be globally bounded. Later on, Wang et al. [57] extended such blow-up results to a quasilinear system with logistic source. Similar to classical Keller-Segel system, when considering the effect of nonlinear diffusion and logistic source, Chiyo et al. [58] studied the following parabolic-elliptic-elliptic system

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w) + f(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \alpha v + \beta u, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - \gamma w + \delta u, & x \in \Omega,\ t > 0, \end{cases}$$

with $m, p, q \in \mathbb{R}$, where they classify boundedness and blow-up into the cases $p < q$ and $p > q$ without any condition for the sign of $\chi\alpha - \xi\gamma$ and the case $p = q$ with $\chi\alpha - \xi\gamma > 0$ or $\chi\alpha - \xi\gamma < 0$.

In contrast to the systems mentioned above, we find that there are very few results on the existence of globally bounded classical solutions to the attraction-repulsion system with nonlinear diffusion and logistic source as well as nonlinear signal production at the same time. On the basis of work [52], the purpose of the present paper is to continue to detect the effect among nonlinear diffusion and logistic source as well as nonlinear signal production on the boundedness of the solution to the following attraction-repulsion system

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m}\nabla u - \chi u(u+1)^{\theta-1}\nabla v + \xi u(u+1)^{l-1}\nabla w) + au - bu^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v + \alpha u^{\gamma_1} - \beta v, & x \in \Omega,\ t > 0, \\ 0 = \Delta w + \gamma u^{\gamma_2} - \delta w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, ν denotes the outward unit normal vector on $\partial\Omega$. Here, u, v and w represent the density of cell, chemical concentration of attractant and chemical concentration of repellent, respectively, and the parameters satisfy $m, \theta, l \in \mathbb{R}$, $\chi, \xi, a, b, \alpha, \beta, \gamma, \delta, \gamma_1, \gamma_2 > 0, \kappa > 1$.

We state our main results to system (1.6) as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters satisfy $m, \theta, l \in \mathbb{R}$ and $\chi, \xi, a, b, \alpha, \beta, \gamma, \delta, \gamma_1, \gamma_2 > 0, \kappa > 1$. For any nonnegative initial data $u_0(x) \in C^0(\overline{\Omega})$, if*

$$\theta + \gamma_1 < \max\left\{l + \gamma_2, \kappa, m + \frac{2}{n} + 1\right\},$$

then the system (1.6) *admits a globally bounded classical solution.*

Remark 1.2. *Theorem 1.1 implies that the behavior of solutions to system* (1.6) *is determined by the interactions among the six mechanisms, namely, self-diffusion $\nabla \cdot ((u+1)^m \nabla u)$, cross-diffusion $-\nabla \cdot (\chi u(u+1)^{\theta-1}\nabla v)$, cross-diffusion $\nabla \cdot (\xi u(u+1)^{l-1}\nabla w)$, attraction, repulsion and logistic source. If attraction and cross-diffusion $-\nabla \cdot (\chi u(u+1)^{\theta-1}\nabla v)$ are dominated by the other four mechanisms with*

$\theta+\gamma_1 < \max\left\{l+\gamma_2, \kappa, m+\frac{2}{n}+1\right\}$, *then the solutions would be globally bounded. From the previous work by Winkler [22], we know that if the nonlinear exponent of attractant production is controlled by* $\frac{2}{n}$, *the classical solution would be globally bounded. Thus the boundedness criteria established in Theorem 1.1 are consistent with [22] for the case* $w=0, m=0, \theta=1, a=b, \kappa=1$. *Moreover, if the parameters* $m=0, \theta=l=1$, *Theorem 1.1 covers the conclusions of Theorem 1(i) in [52].*

Theorem 1.3. *Let* $\Omega\subset\mathbb{R}^n(n\geq 1)$ *be a bounded domain with smooth boundary and the parameters satisfy* $m,\theta,l\in\mathbb{R}$ *and* $\chi,\xi,a,b,\alpha,\beta,\gamma,\delta,\gamma_1,\gamma_2>0,\kappa>1$. *For any nonnegative initial data* $u_0(x)\in C^0(\overline{\Omega})$, *assume that* $\theta+\gamma_1=\max\{l+\gamma_2,\kappa\}\geq m+\frac{2}{n}+1$. *If one of the following three conditions holds:*

$$\text{(a) when } \theta+\gamma_1=l+\gamma_2=\kappa, \text{ if } \theta\geq l\geq 1 \text{ and } \frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n}<b$$
$$\text{or if } l\geq\theta\geq 1 \text{ and } \frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}<b;$$
$$\text{(b) when } \theta+\gamma_1=l+\gamma_2>\kappa, \text{ if } \theta\geq l\geq 1 \text{ and } 2\alpha\chi\leq\gamma\xi;$$
$$\text{(c) when } \theta+\gamma_1=\kappa>l+\gamma_2, \text{ if } \theta\geq 1 \text{ and } \frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}<b,$$

then the system (1.6) *admits a globally bounded classical solution.*

Remark 1.4. *Under the above three balance situations, namely,* $\theta+\gamma_1=l+\gamma_2=\kappa, \theta+\gamma_1=l+\gamma_2>\kappa$ *or* $\theta+\gamma_1=\kappa>l+\gamma_2$, *the boundedness of solutions would be determined by the sizes of the coefficients involved. When* $m=0,\theta=l=1$, *Theorem 1.3 is consistent with Theorem 1(ii) in [52]. Here, it should be noted that we only prove the case of* $\min\{l,\theta\}\geq 1$, *and we will continue to study the other cases in future.*

Remark 1.5. *Theorem 1.1 and Theorem 1.3 also leave an interesting problem, i.e. it is still unknown whether the boundedness criteria obtained in Theorem 1.1 and Theorem 1.3 are optimal for system* (1.6). *We will also further study the finite-time blow-up criteria of the solution for system* (1.6) *in the future research.*

We carry out this paper as follows. In Section 2, we state a result on the existence local solutions and give some useful lemmas. In Section 3, we construct the L^p-estimates for component u and then use the Moser iteration to prove Theorem 1.1 and Theorem 1.3.

2. Preliminaries

In this section, we first state the existence of local solutions to system (1.6). The proof relies on Schauder fixed theorem. We omit it for brevity and refer the readers to [59, 60] for more details.

Lemma 2.1. *Let* $\Omega\subset\mathbb{R}^n(n\geq 1)$ *be a bounded domain with smooth boundary. For any nonnegative initial data* $u_0\in C^0(\overline{\Omega})$, *there exists* $T_{\max}\in(0,\infty]$ *such that the system* (1.6) *admits a unique nonnegative classical solution* (u,v,w) *belonging to* $C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max}))$ *in* $\Omega\times(0,T_{\max})$ *with*

$$u,v,w\geq 0 \ \text{ in } \ \overline{\Omega}\times(0,T_{\max}). \tag{2.1}$$

Furthermore,

$$\text{if } T_{\max} < \infty, \text{ then } \limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2}$$

The following two lemmas are useful in obtaining the estimate of $\int_\Omega u^p$. The proof of Lemma 2.2 can be found in [58, Lemma 2.3], we omit it here.

Lemma 2.2. *Let $\sigma > 1$. Then for all $\epsilon > 0$,*

$$(x+1)^\sigma \le (1+\epsilon)x^\sigma + C_\epsilon, \ x \ge 0, \tag{2.3}$$

where $C_\epsilon := (1+\epsilon)((1+\epsilon)^{\frac{1}{\sigma-1}} - 1)^{1-\sigma}$.

Lemma 2.3. *Let (u, v, w) be a solution of system (1.6), then for any $\gamma_2, \eta > 0$ and $\tau > 1$, there exists $c_0 > 0$ depending only on γ_2, η and τ such that*

$$\int_\Omega w^\tau \le \eta \int_\Omega u^{\gamma_2 \tau} + c_0 \ \textit{for all } t \in (0, T_{\max}). \tag{2.4}$$

Proof. Integrating the first equation of system (1.6) over Ω, we find

$$\frac{d}{dt}\int_\Omega u dx = \int_\Omega au - bu^\kappa \le a\int_\Omega u - \frac{b}{|\Omega|^{\kappa-1}}\left(\int_\Omega u\right)^\kappa \ \text{ for all } t \in (0, T_{\max}), \tag{2.5}$$

where we have used Hölder's inequality $(\int_\Omega u)^\kappa \le |\Omega|^{\kappa-1}\int_\Omega u^\kappa$. Thus, using a standard ODI comparison theory, we can obtain

$$\int_\Omega u \le \max\left\{\int_\Omega u_0, \left(\frac{a}{b}\right)^{\frac{1}{\kappa-1}}|\Omega|\right\} \ \text{ for all } t \in (0, T_{\max}). \tag{2.6}$$

Moreover, we can derive directly by integrating the third equation over Ω,

$$\|w\|_{L^1(\Omega)} = \frac{\gamma}{\delta}\|u^{\gamma_2}\|_{L^1(\Omega)}. \tag{2.7}$$

Multiplying the third equation of system (1.6) with $w^{\tau-1}$ and integrating over Ω, we can get

$$\frac{4(\tau-1)}{\tau^2}\int_\Omega |\nabla w^{\frac{\tau}{2}}|^2 + \delta\int_\Omega w^\tau = \gamma\int_\Omega u^{\gamma_2}w^{\tau-1} \le \frac{\tau-1}{\tau}\delta\int_\Omega w^\tau + \frac{\gamma^\tau}{\tau\delta^{\tau-1}}\int_\Omega u^{\gamma_2\tau} \tag{2.8}$$

by Young's inequality and thus

$$\|w\|_{L^\tau(\Omega)} \le \frac{\gamma}{\delta}\|u^{\gamma_2}\|_{L^\tau(\Omega)} \ \text{ for all } t \in (0, T_{\max}) \tag{2.9}$$

and

$$\frac{4(\tau-1)}{\tau}\int_\Omega |\nabla w^{\frac{\tau}{2}}|^2 \le \frac{\gamma^\tau}{\delta^{\tau-1}}\int_\Omega u^{\gamma_2\tau} \text{ for all } t \in (0, T_{\max}). \tag{2.10}$$

By Ehrling's lemma, for any $\eta > 0$ and $\tau > 1$, there exists $C_1 = C_1(\eta, \tau) > 0$ such that

$$\|\phi\|^2_{L^2(\Omega)} \le \eta \|\phi\|^2_{W^{1,2}(\Omega)} + C_1 \|\phi\|^{\tau}_{L^{\frac{2}{\tau}}(\Omega)} \text{ for all } \phi \in W^{1,2}(\Omega). \tag{2.11}$$

Let $\phi = w^{\frac{\tau}{2}}$, from (2.7), (2.9) and (2.10), there exists $C_1 = C_1(\eta, \tau) > 0$ such that

$$\int_\Omega w^\tau \le \eta \int_\Omega u^{\gamma_2 \tau} + C_1 \|u^{\gamma_2}\|^{\tau}_{L^1(\Omega)}. \tag{2.12}$$

For $\gamma_2 \in (0, 1]$, using Hölder's inequality, from (2.6) one may obtain

$$\|u^{\gamma_2}\|^{\tau}_{L^1(\Omega)} \le C_2 \tag{2.13}$$

with $C_2 = C_2(\eta, \tau, \gamma_2) > 0$. For $\gamma_2 \in (1, \infty)$, using interpolation inequality and Young's inequality, from (2.6) we have

$$\|u^{\gamma_2}\|^{\tau}_{L^1(\Omega)} \le \|u^{\gamma_2}\|^{\tau\vartheta}_{L^\tau(\Omega)} \|u^{\gamma_2}\|^{\tau(1-\vartheta)}_{L^{\frac{1}{\gamma_2}}(\Omega)} \le \eta \int_\Omega u^{\gamma_2 \tau} + C_3, \tag{2.14}$$

where $\vartheta = \frac{\gamma_2 - 1}{\gamma_2 - \frac{1}{\tau}} \in (0, 1)$ and $C_3 = C_3(\eta, \tau, \gamma_2) > 0$. Thus (2.4) is the direct result of combining (2.12)–(2.14). □

3. Global existence and boundedness

In this section, we construct the L^p−estimate for component u under different conditions to prove Theorem 1.1 and Theorem 1.3.

Lemma 3.1. *Let $p > \max\{1, 1 - \theta, 2 - \theta - \gamma_1, 3 - l\}$. Then there exists a constant $C > 0$ such that the solution of system* (1.6) *satisfies*

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le& -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m}{2}}|^2 + \frac{2\alpha\chi(p-1)}{p+\theta-1}\int_\Omega u^{p+\theta+\gamma_1-1} \\ &+ \frac{\xi\delta(p-1)}{p+l-1}\int_\Omega (u+1)^{p+l-1} w - \frac{\gamma\xi(p-1)}{p+l-1}\int_\Omega u^{p+l+\gamma_2-1} \\ &+ a\int_\Omega (u+1)^p - b\int_\Omega u^{p+\kappa-1} + C \ \textit{for all } t \in (0, T_{\max}).\end{aligned} \tag{3.1}$$

Proof. Multiplying the first equation of system (1.6) by $(u + 1)^{p-1}$ and integrating by parts over Ω, we derive

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p =& \int_\Omega (u+1)^{p-1}\nabla\cdot((u+1)^m \nabla u) - \chi\int_\Omega (u+1)^{p-1}\nabla\cdot(u(u+1)^{\theta-1}\nabla v) \\ &+ \xi\int_\Omega (u+1)^{p-1}\nabla\cdot(u(u+1)^{l-1}\nabla w) + a\int_\Omega u(u+1)^{p-1} - b\int_\Omega u^\kappa (u+1)^{p-1} \\ =& -(p-1)\int_\Omega (u+1)^{p+m-2}|\nabla u|^2 + \chi(p-1)\int_\Omega u(u+1)^{p+\theta-3}\nabla u\cdot\nabla v \\ &- \xi(p-1)\int_\Omega u(u+1)^{p+l-3}\nabla u\cdot\nabla w + a\int_\Omega u(u+1)^{p-1} - b\int_\Omega u^\kappa (u+1)^{p-1}\end{aligned}$$

$$:= I_1 + I_2 + I_3 + I_4 \tag{3.2}$$

for all $t \in (0, T_{\max})$.

As to the first term I_1, it can be rewritten as

$$I_1 = -(p-1)\int_\Omega (u+1)^{p+m-2}|\nabla u|^2 = -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m}{2}}|^2. \tag{3.3}$$

For the second term I_2, by integrating by parts we can obtain from the second equation of system (1.6)

$$\begin{aligned}
I_2 =& \chi(p-1)\int_\Omega u(u+1)^{p+\theta-3}\nabla u \cdot \nabla v \\
=& \chi(p-1)\int_\Omega \nabla\left[\int_0^u s(s+1)^{p+\theta-3}ds\right]\cdot\nabla v \\
=& \chi(p-1)\int_\Omega \left[\int_0^u s(s+1)^{p+\theta-3}ds\right](-\Delta v) \\
=& \chi(p-1)\int_\Omega \left[\int_0^u s(s+1)^{p+\theta-3}ds\right](\alpha u^{\gamma_1} - \beta v) \\
\le& \alpha\chi(p-1)\int_\Omega \left[\int_0^u s(s+1)^{p+\theta-3}ds\right]u^{\gamma_1}.
\end{aligned} \tag{3.4}$$

For $p > 1-\theta$, we can infer that

$$\begin{aligned}
\left[\int_0^u s(s+1)^{p+\theta-3}ds\right]u^{\gamma_1} \le& \left[\int_0^u (s+1)^{p+\theta-2}ds\right]u^{\gamma_1} \\
\le& \frac{1}{p+\theta-1}(u+1)^{p+\theta-1}u^{\gamma_1} \\
\le& \frac{1}{p+\theta-1}(u+1)^{p+\theta+\gamma_1-1}.
\end{aligned} \tag{3.5}$$

Using Lemma 2.2 with $\epsilon = 1$ and $p > 2-\theta-\gamma_1$, we can obtain from (3.4) and (3.5)

$$I_2 \le \frac{2\alpha\chi(p-1)}{p+\theta-1}\int_\Omega u^{p+\theta+\gamma_1-1} + c_1 \tag{3.6}$$

with some $c_1 > 0$.

Similarly, for the third term I_3, we deduce

$$\begin{aligned}
I_3 =& -\xi(p-1)\int_\Omega u(u+1)^{p+l-3}\nabla u \cdot \nabla w \\
=& -\xi(p-1)\int_\Omega \nabla\left[\int_0^u s(s+1)^{p+l-3}ds\right]\cdot\nabla w \\
=& \xi(p-1)\int_\Omega \left[\int_0^u s(s+1)^{p+l-3}ds\right]\Delta w \\
=& \xi(p-1)\int_\Omega \left[\int_0^u s(s+1)^{p+l-3}ds\right](\delta w - \gamma u^{\gamma_2}).
\end{aligned} \tag{3.7}$$

Noticing that $s^{p+l-2} \le s(s+1)^{p+l-3} \le (s+1)^{p+l-2}$ for $p > 3-l$, we derive that

$$\frac{1}{p+l-1}u^{p+l-1} \le \int_0^u s(s+1)^{p+l-3}ds \le \frac{1}{p+l-1}(u+1)^{p+l-1}. \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$I_3 \le \frac{\xi\delta(p-1)}{p+l-1}\int_\Omega (u+1)^{p+l-1}w - \frac{\gamma\xi(p-1)}{p+l-1}\int_\Omega u^{p+l+\gamma_2-1}. \tag{3.9}$$

For the term I_4, we can obtain

$$\begin{aligned} I_4 =& a\int_\Omega u(u+1)^{p-1} - b\int_\Omega u^\kappa(u+1)^{p-1} \\ \le& a\int_\Omega (u+1)^p - b\int_\Omega u^{p+\kappa-1}. \end{aligned} \tag{3.10}$$

Thus the inequality (3.1) is the direct result of combining (3.2), (3.3), (3.6), (3.9) and (3.10). □

Lemma 3.2. *Let* (u, v, w) *be a solution of system* (1.6). *If* $\theta + \gamma_1 < \max\{l + \gamma_2, \kappa, m + \frac{2}{n} + 1\}$, *then for any* $p > \max\{1, 1-\theta, 2-\theta-\gamma_1, 3-l\}$ *there exists a constant* $C > 0$ *such that*

$$\int_\Omega (u+1)^p \le C \ \textit{for all } t \in (0, T_{\max}). \tag{3.11}$$

Proof. Using Lemma 2.2 once more with $\epsilon = 1$ and $p > 2-l$, one may infer that

$$\delta\left[\int_0^u s(s+1)^{p+l-3}ds\right]w \le \frac{2\delta}{p+l-1}u^{p+l-1}w + c_2 w \tag{3.12}$$

with some $c_2 > 0$. Moreover, by Young's inequality and Lemma 2.3 we have

$$\begin{aligned} \frac{2\delta\xi(p-1)}{p+l-1}\int_\Omega u^{p+l-1}w \le& \frac{\varepsilon}{2}\int_\Omega u^{p+l+\gamma_2-1} + c_3(\varepsilon)\int_\Omega w^{\frac{p+l+\gamma_2-1}{\gamma_2}} \\ \le& \varepsilon\int_\Omega u^{p+l+\gamma_2-1} + c_3(\varepsilon) \end{aligned} \tag{3.13}$$

and

$$c_2\int_\Omega w = \frac{c_2\gamma}{\delta}\int_\Omega u^{\gamma_2} \le \varepsilon\int_\Omega u^{p+l+\gamma_2-1} + c_3(\varepsilon) \tag{3.14}$$

with any $\varepsilon > 0$.

Thus, from (3.12)–(3.14), we can get

$$\frac{\xi\delta(p-1)}{p+l-1}\int_\Omega (u+1)^{p+l-1}w - \frac{\gamma\xi(p-1)}{p+l-1}\int_\Omega u^{p+l+\gamma_2-1} \le \left(2\varepsilon - \frac{\gamma\xi(p-1)}{p+l-1}\right)\int_\Omega u^{p+l+\gamma_2-1} + 2c_3(\varepsilon). \tag{3.15}$$

Substitute (3.15) into (3.1) to get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(u+1)^{\frac{p+m}{2}}|^2 + \frac{2\alpha\chi(p-1)}{p+\theta-1}\int_\Omega u^{p+\theta+\gamma_1-1}$$

$$+\left(2\varepsilon-\frac{\gamma\xi(p-1)}{p+l-1}\right)\int_\Omega u^{p+l+\gamma_2-1}+a\int_\Omega(u+1)^p-b\int_\Omega u^{p+\kappa-1}+c_4 \tag{3.16}$$

for all $t\in(0,T_{\max})$.

Let $\theta+\gamma_1<\kappa$. In view of Young's inequality, we get

$$\frac{2\alpha\chi(p-1)}{p+\theta-1}\int_\Omega u^{p+\theta+\gamma_1-1}\le\frac{b}{2}\int_\Omega u^{p+\kappa-1}+c_5\ \text{ for all } t\in(0,T_{\max}). \tag{3.17}$$

Choosing $\varepsilon=\frac{\gamma\xi(p-1)}{2(p+l-1)}>0$ in (3.16), one can derive

$$\frac{1}{p}\frac{d}{dt}\int_\Omega(u+1)^p\le a\int_\Omega(u+1)^p-\frac{b}{2}\int_\Omega u^{p+\kappa-1}+c_6. \tag{3.18}$$

This proves (3.11).

Let $\theta+\gamma_1<l+\gamma_2$. We know from Young's inequality

$$\frac{2\alpha\chi(p-1)}{p+\theta-1}\int_\Omega u^{p+\theta+\gamma_1-1}\le\frac{\gamma\xi(p-1)}{2(p+l-1)}\int_\Omega u^{p+l+\gamma_2-1}+c_7\ \text{ for all } t\in(0,T_{\max}). \tag{3.19}$$

Substituting (3.19) into (3.16) and taking $\varepsilon:=\frac{\gamma\xi(p-1)}{4(p+l-1)}>0$, we can obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega(u+1)^p\le a\int_\Omega(u+1)^p-b\int_\Omega u^{p+\kappa-1}+c_8. \tag{3.20}$$

Hence we complete the proof of (3.11) under this case.

Let $\theta+\gamma_1<m+\frac{2}{n}+1$. Without loss of generality, suppose $\theta+\gamma_1\ge\max\{l+\gamma_2,\kappa\}$. Taking $\varepsilon:=\frac{\gamma\xi(p-1)}{2(p+l-1)}>0$ in (3.16), we can obtain by Young's inequality that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega(u+1)^p\le&-\frac{4(p-1)}{(p+m)^2}\int_\Omega|\nabla(u+1)^{\frac{p+m}{2}}|^2+\frac{2\alpha\chi(p-1)}{p+\theta-1}\int_\Omega(u+1)^{p+\theta+\gamma_1-1}\\&+a\int_\Omega(u+1)^p+c_4\\\le&-\frac{4(p-1)}{(p+m)^2}\int_\Omega|\nabla(u+1)^{\frac{p+m}{2}}|^2+\left(\frac{2\alpha\chi(p-1)}{p+\theta-1}+c_9\right)\int_\Omega(u+1)^{p+\theta+\gamma_1-1}+c_{10}.\end{aligned} \tag{3.21}$$

Using the Gagliardo-Nirenberg inequality and (2.6), we have

$$\begin{aligned}\int_\Omega(u+1)^{p+\theta+\gamma_1-1}=&\|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2(p+\theta+\gamma_1-1)}{p+m}}(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}}\\\le&c_{11}\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\theta^*}\|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2}{p+m}}(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}(1-\theta^*)}\\&+c_{11}\|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2}{p+m}}(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}}\\\le&c_{12}\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\theta^*}+c_{12}\ \text{ for all } t\in(0,T_{\max}),\end{aligned} \tag{3.22}$$

with $\theta^* = \frac{\frac{p+m}{2}-\frac{p+m}{2(p+\theta+\gamma_1-1)}}{\frac{p+m}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$, where $\frac{2(p+\theta+\gamma_1-1)}{p+m}\theta^* < 2$ due to $\theta+\gamma_1 < m+\frac{2}{n}+1$. This yields

$$\left(\frac{2\alpha\chi(p-1)}{p+\theta-1}+c_9+1\right)\int_\Omega (u+1)^{p+\theta+\gamma_1-1} \le \frac{4(p-1)}{(p+m)^2}\|\nabla(u+1)^{\frac{m+p}{2}}\|^2_{L^2(\Omega)}+c_{13} \tag{3.23}$$

with $c_{13}>0$ by Young's inequality. Now combining (3.21) and (3.23), we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -\int_\Omega (u+1)^{p+\theta+\gamma_1-1}+c_{13} \ \text{ for all } t\in(0,T_{\max}). \tag{3.24}$$

Thus there exist $c_{14}, c_{15}>0$ such that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + c_{14}\int_\Omega (u+1)^p &\le \frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p+\int_\Omega (u+1)^{p+\theta+\gamma_1-1}\\ &\le c_{15} \ \text{ for all } t\in(0,T_{\max}).\end{aligned} \tag{3.25}$$

This completes the proof of (3.11). □

Now we are in a position to prove Theorem 1.1.

The proof of Theorem 1.1 Let $m,\theta,l\in\mathbb{R},\chi,\xi,a,b,\alpha,\beta,\gamma,\delta,\gamma_1,\gamma_2>0,\kappa>1$ and $p>\max\{1,1-\theta,2-\theta-\gamma_1,3-l,n\gamma_1,n\gamma_2\}$. If $\theta+\gamma_1 < \max\left\{l+\gamma_2,\kappa,m+\frac{2}{n}+1\right\}$, there exists $C>0$ such that $\int_\Omega u^p \le C$ for all $t\in(0,T_{\max})$ from Lemma 3.2. By the elliptic L^p–estimate applied to the second and third equations in system (1.6), we have

$$\|v(\cdot,t)\|_{W^{2,p/\gamma_1}(\overline{\Omega})},\ \|w(\cdot,t)\|_{W^{2,p/\gamma_2}(\overline{\Omega})} \le C \ \text{ for all } t\in(0,T_{\max}), \tag{3.26}$$

and hence

$$\|v(\cdot,t)\|_{C^1(\overline{\Omega})},\ \|w(\cdot,t)\|_{C^1(\overline{\Omega})} \le C \ \text{ for all } t\in(0,T_{\max}) \tag{3.27}$$

by the Sobolev imbedding theorem. Using the technique of Moser iteration [45,61], we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \tag{3.28}$$

for all $t\in(0,T_{\max})$. Thus it follows from Lemma 2.1 that $T_{\max}=\infty$. This concludes the Theorem 1.1. □

Lemma 3.3. *Let* (u,v,w) *be a solution of system* (1.6). *Assume that* $\theta+\gamma_1=\max\{l+\gamma_2,\kappa\}\ge m+\frac{2}{n}+1$. *If one of the following three conditions holds:*

(a) when $\theta+\gamma_1=l+\gamma_2=\kappa$, *if* $\theta\ge l\ge 1$ *and* $\dfrac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n}<b$
or if $l\ge\theta\ge 1$ *and* $\dfrac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}<b$;

(b) when $\theta+\gamma_1=l+\gamma_2>\kappa$, *if* $\theta\ge l\ge 1$ *and* $2\alpha\chi\le\gamma\xi$;

(c) when $\theta+\gamma_1=\kappa>l+\gamma_2$, *if* $\theta\ge 1$ *and* $\dfrac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}<b$,

then for any $p>1$ *there exists a constant* $C>0$ *such that*

$$\int_\Omega (u+1)^p \le C \ \textit{for all } t\in(0,T_{\max}). \tag{3.29}$$

Proof. For $\theta + \gamma_1 = l + \gamma_2$, we can get from (3.16) that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m}{2}}|^2 + \left(2\varepsilon - \frac{\gamma\xi(p-1)}{p+l-1} + \frac{2\alpha\chi(p-1)}{p+\theta-1}\right)\int_\Omega u^{p+\theta+\gamma_1-1} + a\int_\Omega (u+1)^p - b\int_\Omega u^{p+\kappa-1} + c_4 \text{ for all } t\in(0,T_{\max}). \tag{3.30}$$

(a) Let $\theta + \gamma_1 = l + \gamma_2 = \kappa$. Then the inequality (3.30) can be rewritten as

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m}{2}}|^2 + \left(2\varepsilon - \frac{\gamma\xi(p-1)}{p+l-1} + \frac{2\alpha\chi(p-1)}{p+\theta-1} - b\right)\int_\Omega u^{p+\kappa-1} + a\int_\Omega (u+1)^p + c_4 \text{ for all } t\in(0,T_{\max}). \tag{3.31}$$

If $\theta \ge l \ge 1$, then we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le& -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m}{2}}|^2 + \left(2\varepsilon - \frac{\gamma\xi(p-1)}{p+l-1} + \frac{2\alpha\chi(p-1)}{p+\theta-1} - b\right)\int_\Omega u^{p+\kappa-1}\\ &+ a\int_\Omega (u+1)^p + c_4,\\ \le& -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m}{2}}|^2 + \left[2\varepsilon - \frac{(\gamma\xi-2\alpha\chi)(p-1)}{p+l-1} - b\right]\int_\Omega u^{p+\kappa-1}\\ &+ a\int_\Omega (u+1)^p + c_4 \text{ for all } t\in(0,T_{\max}).\end{aligned} \tag{3.32}$$

When $2\alpha\chi \le \gamma\xi$, by taking $\varepsilon := \frac{1}{4}\left[\frac{(\gamma\xi-2\alpha\chi)(p-1)}{p+l-1} + b\right] > 0$, we can get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -\frac{1}{2}\left[\frac{(\gamma\xi-2\alpha\chi)(p-1)}{p+l-1} + b\right]\int_\Omega u^{p+\kappa-1} + a\int_\Omega (u+1)^p + c_4 \text{ for all } t\in(0,T_{\max}). \tag{3.33}$$

This yields (3.29).

When $2\alpha\chi > \gamma\xi$, take $p \in (1, \frac{2\alpha\chi-\gamma\xi+b(l-1)}{(2\alpha\chi-\gamma\xi-b)_+})$ to ensure $\varepsilon = \frac{1}{4}\left[b - \frac{(2\alpha\chi-\gamma\xi)(p-1)}{p+l-1}\right] > 0$ in (3.30). We deduce

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -\frac{1}{2}\left[b - \frac{(2\alpha\chi-\gamma\xi)(p-1)}{p+l-1}\right]\int_\Omega u^{p+\kappa-1} + a\int_\Omega (u+1)^p + c_4 \tag{3.34}$$

for all $t \in (0, T_{\max})$. This concludes (3.29) with $p \in (1, \frac{2\alpha\chi-\gamma\xi+b(l-1)}{(2\alpha\chi-\gamma\xi-b)_+})$. So it suffices to deal with the case of $2\alpha\chi - \gamma\xi - b > 0$. Since $\frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n} < b$, we can take $p_0 \in (\frac{(\kappa-1-m)n}{2}, \frac{2\alpha\chi-\gamma\xi+b(l-1)}{(2\alpha\chi-\gamma\xi-b)})$.

In view of Gagliardo-Nirenberg inequality , we have

$$
\begin{aligned}
\int_\Omega (u+1)^{p+\kappa-1} =& \|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2(p+\kappa-1)}{p+m}}(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}} \\
\leq & c_{15}\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}\theta_0}\|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2p_0}{p+m}}(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}(1-\theta_0)} \\
& + c_{15}\|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2p_0}{p+m}}(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}} \\
\leq & c_{16}\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}\theta_0} + c_{16} \text{ for all } t \in (0, T_{\max}),
\end{aligned}
\tag{3.35}
$$

with $\theta_0 = \frac{\frac{m+p}{2p_0} - \frac{m+p}{2(p+\kappa-1)}}{\frac{m+p}{2p_0} + \frac{1}{n} - \frac{1}{2}} \in (0,1)$ when $p > p_0$. Due to $p_0 > \frac{(\kappa-1-m)n}{2}$, we have $\frac{2(p+\kappa-1)}{p+m}\theta_0 < 2$. By Young's inequality, for any $\epsilon > 0$ we have

$$
\int_\Omega (u+1)^{p+\kappa-1} \leq \epsilon\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^2 + c_{17}(\epsilon) \tag{3.36}
$$

with some $c_{17}(\epsilon) > 0$. Choosing $\varepsilon = \frac{b}{2}$ in (3.31), we can take ϵ small enough to obtain

$$
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \leq -c_{18}\int_\Omega (u+1)^{p+\kappa-1} + a\int_\Omega (u+1)^p + c_{19} \text{ for all } t \in (0, T_{\max}). \tag{3.37}
$$

This yields (3.29). If $l \geq \theta \geq 1$, taking $2\varepsilon = \frac{\gamma\xi(p-1)}{p+l-1}$, we have

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \leq & -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(u+1)^{\frac{p+m}{2}}|^2 + \left(2\varepsilon - \frac{\gamma\xi(p-1)}{p+l-1} + \frac{2\alpha\chi(p-1)}{p+\theta-1} - b\right)\int_\Omega u^{p+\kappa-1} \\
& + a\int_\Omega (u+1)^p + c_4, \\
= & -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(u+1)^{\frac{p+m}{2}}|^2 + \left[\frac{2\alpha\chi(p-1)}{p+\theta-1} - b\right]\int_\Omega u^{p+\kappa-1} \\
& + a\int_\Omega (u+1)^p + c_4 \text{ for all } t \in (0, T_{\max}).
\end{aligned}
\tag{3.38}
$$

Taking $p \in (1, \frac{2\alpha\chi + b(\theta-1)}{(2\alpha\chi - b)_+})$ to ensure $b - \frac{2\alpha\chi(p-1)}{p+\theta-1} > 0$, we can get

$$
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \leq a\int_\Omega (u+1)^p + c_4 \text{ for all } t \in (0, T_{\max}). \tag{3.39}
$$

This proves (3.29) with $p \in (1, \frac{2\alpha\chi + b(\theta-1)}{(2\alpha\chi - b)_+})$. So it suffices to deal with the case of $2\alpha\chi - b > 0$. Since $\frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n} < b$, we can take $p_1 \in (\frac{(\kappa-1-m)n}{2}, \frac{2\alpha\chi + b(\theta-1)}{(2\alpha\chi - b)})$. Using the Gagliardo-Nirenberg inequality once more, we have

$$\begin{aligned}\int_\Omega (u+1)^{p+\kappa-1} =& \|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2(p+\kappa-1)}{p+m}}(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}}\\ \le& c_{20}\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}\theta_1}\|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2p_1}{p+m}}(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}(1-\theta_1)}\\ &+c_{20}\|(u+1)^{\frac{m+p}{2}}\|_{L^{\frac{2p_1}{p+m}}(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}}\\ \le& c_{21}\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\kappa-1)}{p+m}\theta_1}+c_{21} \text{ for all } t\in(0,T_{\max}),\end{aligned} \tag{3.40}$$

with $\theta_1 = \frac{\frac{m+p}{2p_1}-\frac{m+p}{2(p+\kappa-1)}}{\frac{m+p}{2p_1}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$ when $p > p_1$. Due to $p_1 > \frac{(\kappa-1-m)n}{2}$, we have $\frac{2(p+\kappa-1)}{p+m}\theta_1 < 2$. By Young's inequality, for any $\epsilon_1 > 0$ we have

$$\int_\Omega (u+1)^{p+\kappa-1} \le \epsilon_1\|\nabla(u+1)^{\frac{m+p}{2}}\|_{L^2(\Omega)}^2 + c_{22}(\epsilon_1) \tag{3.41}$$

with some $c_{22}(\epsilon_1) > 0$. Choosing $2\varepsilon = \frac{\gamma\xi(p-1)}{p+l-1}$, in (3.38), we can take ϵ_1 small enough to obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -c_{24}\int_\Omega (u+1)^{p+\kappa-1} + a\int_\Omega (u+1)^p + c_{25} \text{ for all } t\in(0,T_{\max}), \tag{3.42}$$

with $c_{24}, c_{25} > 0$. This yields (3.29).

(b) Let $\theta+\gamma_1 = l+\gamma_2 > \kappa$. Since $\theta \ge l \ge 1$, we can obtain from (3.30)

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le& -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(u+1)^{\frac{p+m}{2}}|^2 + \left(2\varepsilon - \frac{\gamma\xi(p-1)}{p+l-1} + \frac{2\alpha\chi(p-1)}{p+\theta-1}\right)\int_\Omega u^{p+\theta+\gamma_1-1}\\ &+ a\int_\Omega (u+1)^p - b\int_\Omega u^{p+\kappa-1} + c_4\\ \le& -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(u+1)^{\frac{p+m}{2}}|^2 + \left[2\varepsilon - \frac{(\gamma\xi-2\alpha\chi)(p-1)}{p+l-1}\right]\int_\Omega u^{p+\theta+\gamma_1-1}\\ &+ a\int_\Omega (u+1)^p - b\int_\Omega u^{p+\kappa-1} + c_4 \text{ for all } t\in(0,T_{\max}).\end{aligned} \tag{3.43}$$

For $2\alpha\chi \le \gamma\xi$, by letting $\varepsilon = \frac{(\gamma\xi-2\alpha\chi)(p-1)}{4(p+l-1)}$, we know

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le -\frac{(\gamma\xi-2\alpha\chi)(p-1)}{2(p+l-1)}\int_\Omega u^{p+\theta+\gamma_1-1} + a\int_\Omega (u+1)^p + c_4 \tag{3.44}$$

for all $t \in (0, T_{\max})$. This implies (3.29).

(c) Let $\theta+\gamma_1 = \kappa > l+\gamma_2$. Taking $\varepsilon = \frac{\gamma\xi(p-1)}{2(p+l-1)}$ in (3.16), we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \le& -\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(u+1)^{\frac{p+m}{2}}|^2 - \left(b - \frac{2\alpha\chi(p-1)}{p+\theta-1}\right)\int_\Omega u^{p+\kappa-1}\\ &+ a\int_\Omega (u+1)^p + c_4 \text{ for all } t\in(0,T_{\max}).\end{aligned} \tag{3.45}$$

The process of proof is same as the case (a) with $\theta+\gamma_1 = \kappa+l+\gamma_2$ and $l \ge \theta \ge 1$. Thus we omit them here. □

The proof of Theorem 1.3 With the aid of Lemma 3.3 and Moser iteration in [45, 61], we can obtain the boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ for all $t \in (0, T_{\max})$. And we can also get the boundedness of $\|v(\cdot,t)\|_{C^1(\overline{\Omega})}$ and $\|w(\cdot,t)\|_{C^1(\overline{\Omega})}$ by (3.27) for all $t \in (0, T_{\max})$. Hence we deduce from Lemma from Lemma 2.1 that $T_{\max} = \infty$. This completes the proof of Theorem 1.3. □

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Conflict of interest

The authors declare that there is no conflicts of interest regarding the publication of this paper.

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