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ERA, 31(1): 17–36.
DOI: 10.3934/era.2023002
Received: 16 August 2022
Revised: 30 September 2022
Accepted: 11 October 2022
Published: 19 October 2022

Research article

Energy equality for the multi-dimensional nonhomogeneous incompressible Hall-MHD equations in a bounded domain

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Abstract: This paper focuses on the energy equality for weak solutions of the nonhomogeneous incompressible Hall-magnetohydrodynamics equations in a bounded domain $\Omega \subset \mathbb{R}^n\,(n \geqslant 2)$. By exploiting the special structure of the nonlinear terms and using the coarea formula, we obtain some sufficient conditions for the regularity of weak solutions to ensure that the energy equality is valid. For the special case $n = 3,\ p = q = 2$, our results are consistent with the corresponding results obtained by Kang-Deng-Zhou in [Results Appl. Math. 12 : 100178, 2021]. Additionally, we establish the sufficient conditions concerning ∇u and ∇b, instead of u and b.

Keywords: Hall-MHD system; regularity; energy equality

1. Introduction

The goal of this paper is to study the energy equality for weak solutions of the nonhomogeneous incompressible Hall-magnetohydrodynamics (Hall-MHD) equations in $\Omega \times (0, T)$,

$$\partial_t (\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mu\Delta u + \nabla P - (\nabla \times b) \times b = 0, \tag{1.1}$$

$$\partial_t b + d_I \nabla \times \left(\frac{(\nabla \times b) \times b}{\rho}\right) - \nabla \times (u \times b) - \Delta b = 0, \tag{1.2}$$

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0, \tag{1.3}$$

$$\mathrm{div}\ u = 0,\ \ \mathrm{div}\ b = 0, \tag{1.4}$$

with initial data

$$\rho u(x, 0) = \rho_0 u_0(x),\ \ \rho(x, 0) = \rho_0(x),\ \ b(x, 0) = b_0(x), \tag{1.5}$$

and homogeneous Dirichlet boundary conditions

$$u = 0, \quad b = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ is bounded with $\partial\Omega \in C^2$. ρ, u, b and P are the density, velocity field, magnetic field and scalar pressure, respectively. The constant $\mu > 0$ represents the viscosity coefficient of the flow. The term $d_I \nabla \times \left(\frac{(\nabla\times b)\times b}{\rho}\right)$ denotes the Hall effect, and d_I is the Hall coefficient.

In 1960, Lighthill [1] said that the MHD equations cannot provide a precise description of physical phenomena and introduced first the Hall term into the MHD equations, which constitute the so-called Hall-MHD equations. Subsequently, Arichetogaray et al. [2] analyzed the Hall-MHD equations from either a two-fluid Euler-Maxwell system for electrons and ions or kinetic models, which are used for describing many physical phenomena in geophysics and astrophysics, such as the magnetic reconnection of space plasma, star formation and neutron stars. From the mathematical viewpoint, the Hall-MHD equations have been widely studied, involving the local and global well-posedness [2–5], blow-up criteria [4, 6], large time behavior [7–10], Liouville-type theorems [11–14] and energy conservation [15, 16]. It is worth pointing out that Kang et al. [16] recently studied the energy conservation for systems (1.1)–(1.4) in a bounded domain $\Omega \subset \mathbb{R}^3$ and obtained that if weak solutions (ρ, u, b, P) satisfy

$$\begin{aligned} &0 < c_1 \leqslant \rho(x,t) \leqslant c_2 < \infty, \;\; u, b \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \\ &P \in L^2((0,T)\times\Omega), \;\; u, b \in L^4((0,T)\times\Omega), \;\; \nabla\times b \in L^4((0,T)\times\Omega), \end{aligned} \tag{1.7}$$

then the following energy equality is valid:

$$\begin{aligned} \int_\Omega \left(\frac{1}{2}\rho\,|u|^2 + \frac{1}{2}\,|b|^2\right)dx + \int_0^t \int_\Omega \mu\,|\nabla u|^2\,dxds + \int_0^t \int_\Omega |\nabla b|^2\,dxds& \\ = \int_\Omega \left(\frac{1}{2}\rho_0\,|u_0|^2 + \frac{1}{2}\,|b_0|^2\right)dx, \quad \forall\, t \in (0,T).& \end{aligned} \tag{1.8}$$

Concerning the energy equality for the MHD equations, one can refer to the works [17–20], etc.

When $b = 0$, systems (1.1)–(1.4) reduce to the incompressible Navier-Stokes system. For the energy equality of the incompressible Navier-Stokes system, the Lions-Shinbrot type criterion on the velocity was obtained by Lions [21], Shinbrot [22], Da Veiga and Yang [23] and Yu [24]. Later, Yu [25] extended Shinbrot's result to the bounded domain, with an additional Besov regularity imposed on the velocity, which is essential to deal with the boundary effects, and Nguyen et al. [26] handled the boundary effects without requiring extra conditions of velocity field u near the boundary.

For the energy equality of the compressible Navier-Stokes equations, Yu [27] proved the energy equality holds true if $u \in L_t^p L_x^q$, ρ is bounded, and $\sqrt{\rho} \in L^\infty(0,T;H^1(\Omega))$. Later on, Chen et al. [28] extended the result of [27] to the bounded domain by performing global mollification. In addition, it is worth mentioning that Berselli and Chiodaroli [29], Liang [30] and Wang and Ye [31] derived the energy equality criteria in terms of the velocity and its gradient.

Inspired by the works [16, 26, 31], we provide sufficient conditions on the regularity of solutions for systems (1.1)–(1.4) to ensure the energy equality holds. Compared with the results of [16], we obtain the sufficient conditions concerning ∇u and ∇b, rather than u and b, to guarantee that the energy equality is valid.

Before stating our main results, we give the definition of weak solutions to systems (1.1)–(1.4).

Definition 1.1. A couple (ρ, u, b, P) is called a weak solution to systems (1.1)–(1.4) with initial data (1.5) if (ρ, u, b, P) satisfies the following,

1) Equations (1.1)–(1.4) hold in $D'(0, T; \Omega)$, and

$$|b|^2\,,\ \rho\,|u|^2 \in L^\infty(0, T; L^1(\Omega)), \quad \nabla u, \nabla b \in L^2(0, T; L^2(\Omega)). \tag{1.9}$$

2) $\rho(\cdot, t) \rightharpoonup \rho_0$ in $D'(\Omega)$ as $t \to 0$, i.e.,

$$\lim_{t\to 0} \int_\Omega \rho(x, t)\varphi(x)dx = \int_\Omega \rho_0(x)\varphi(x)dx, \tag{1.10}$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

3) $(\rho u)(\cdot, t) \rightharpoonup \rho_0 u_0$ in $D'(\Omega)$ as $t \to 0$, i.e.,

$$\lim_{t\to 0} \int_\Omega (\rho u)(x, t)\psi(x)dx = \int_\Omega (\rho_0 u_0)(x)\psi(x)dx, \tag{1.11}$$

for every test vector field $\psi \in C_0^\infty(\Omega)^n$.

4) The energy inequality holds, i.e.,

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\rho\,|u|^2 + \frac{1}{2}\,|b|^2\right) dx + \int_0^t \int_\Omega \mu\,|\nabla u|^2\, dxds + \int_0^t \int_\Omega |\nabla b|^2\, dxds \\ &\qquad \leqslant \int_\Omega \left(\frac{1}{2}\rho_0\,|u_0|^2 + \frac{1}{2}\,|b_0|^2\right) dx. \end{aligned} \tag{1.12}$$

Next, we state our main results as follows.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ be a bounded domain with C^2 boundary $\partial\Omega$. The energy equality (1.8) of weak solutions (ρ, u, b, P) to systems (1.1)–(1.4) with initial data (1.5) and Dirichlet boundary conditions (1.6) is valid provided

$$\begin{aligned} &0 < c_1 \leqslant \rho(x, t) \leqslant c_2 < \infty, \quad P \in L^2(0, T; L^2(\Omega)), \quad \nabla \times b \in L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(\Omega)), \\ &\nabla u\,,\ \nabla b \in L^p(0, T; L^q(\Omega)), \quad u, b \in L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(\Omega)) \cap L^p(0, T; L^q(\Omega)), \end{aligned} \tag{1.13}$$

where $1 < p, q < \infty$.

Remark 1.1. Thanks to the embeddings

$$\begin{cases} L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(\Omega)) \hookrightarrow L^p(0, T; L^q(\Omega)), & 1 < p, q \leqslant 3, \\ L^p(0, T; L^q(\Omega)) \hookrightarrow L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(\Omega)), & p, q > 3, \end{cases}$$

the conditions $u, b \in L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(\Omega)) \cap L^p(0, T; L^q(\Omega))$ in (1.13) could be replaced by the following:

$$u, b \in \begin{cases} L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(\Omega)), & 1 < p, q \leqslant 3, \\ L^p(0, T; L^q(\Omega)), & p, q > 3. \end{cases} \tag{1.14}$$

Remark 1.2. Setting $n = 3$, $p = q = 2$ in Theorem 1.1, we obtain the sufficient conditions in (1.7) due to the work [16].

On the other hand, when $n = 3$, we get the following corollary by exploiting the Gagliardo-Nirenberg inequality.

Corollary 1.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary $\partial\Omega$ and (ρ, u, b, P) be a weak solution of systems (1.1)–(1.4) with initial data (1.5) and Dirichlet boundary conditions (1.6). Assume that one of the following conditions is satisfied:

1) $0 < c_1 \leqslant \rho(x,t) \leqslant c_2 < \infty$, $P \in L^2(0,T;L^2(\Omega))$, $\nabla \times b \in L^4(0,T;L^4(\Omega))$,

$$u, b \in L^s(0,T;L^t(\Omega)) \quad \text{with} \quad \begin{cases} \frac{2}{s} + \frac{2}{t} = 1, & t \geqslant 4, \\ \frac{1}{s} + \frac{3}{t} = 1, & 3 < t < 4; \end{cases} \tag{1.15}$$

2) $0 < c_1 \leqslant \rho(x,t) \leqslant c_2 < \infty$, $P \in L^2(0,T;L^2(\Omega))$, $\nabla \times b \in L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))$,

$$\nabla u,\ \nabla b \in L^p(0,T;L^q(\Omega)) \quad \text{with} \quad \begin{cases} \frac{1}{p} + \frac{6}{5q} = 1, & \frac{9}{5} \leqslant q \leqslant 3, \\ \frac{1}{p} + \frac{3}{q} = 2, & \frac{3}{2} < q < \frac{9}{5}. \end{cases} \tag{1.16}$$

Then, the energy equality (1.8) holds.

Remark 1.3. The conditions (1.16) show that we can get the regularity involving ∇u and ∇b, rather than u and b, to ensure that the energy equality (1.8) is valid.

2. Preliminaries

Let $\eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be a standard mollifier, i.e., $\eta(x) = C_0 e^{-\frac{1}{1-|x|^2}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geqslant 1$, where C_0 is a constant such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$. For any $\varepsilon > 0$, we define the re-scaled mollifier $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$. For any function $f \in L^1_{loc}(\Omega)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} f(x-y)\eta_\varepsilon(y)dy, \quad x \in \Omega_\varepsilon,$$

where $\Omega_\varepsilon =: \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$.

Next, we recall the results involving the mollifier established in [26].

Lemma 2.1. ([26]) Let $2 \leqslant n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary $\partial\Omega$, $1 \leqslant p, q \leqslant \infty$ and $f : \Omega \times (0,T) \to \mathbb{R}$.

1) Suppose that $f \in L^p(0,T;L^q(\Omega))$. Then, for any $0 < \varepsilon < \delta$, there holds

$$\|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\Omega_\delta))} \leqslant C\varepsilon^{-1} \|f\|_{L^p(0,T;L^q(\Omega))}. \tag{2.1}$$

Moreover, if $p, q < \infty$, then

$$\limsup_{\varepsilon \to 0} \varepsilon \|\nabla f^\varepsilon\|_{L^p(0,T;L^q(\Omega_\delta))} = 0.$$

2) Assume $f \in L^p(0,T;L^q(\Omega))$ with $p,q < \infty$ and $g : \Omega \times (0,T) \to \mathbb{R}$ with $0 < c_1 \leqslant g \leqslant c_2 < \infty$. Then, for any $0 < \varepsilon < \delta$, there holds

$$\left\| \nabla \left(\frac{f^\varepsilon}{g^\varepsilon} \right) \right\|_{L^p(0,T;L^q(\Omega_\delta))} \leqslant C(c_1,c_2)\varepsilon^{-1} \|f\|_{L^p(0,T;L^q(\Omega))} . \tag{2.2}$$

Lemma 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary $\partial\Omega$, $1 \leqslant p, q, p_1, q_1, p_2, q_2 \leqslant \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Assume $f \in L^{p_1}(0,T;W^{1,q_1}(\Omega))$ and $g \in L^{p_2}(0,T;L^{q_2}(\Omega))$. Then, for any $0 < \varepsilon < \delta$ small, there holds

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))} \leqslant C\varepsilon \|f\|_{L^{p_1}(0,T;W^{1,q_1}(\Omega_\delta))} \|g\|_{L^{p_2}(0,T;L^{q_2}(\Omega_\delta))} . \tag{2.3}$$

Moreover, if $p_2, q_2 < \infty$, then

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1} \|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))} = 0. \tag{2.4}$$

Remark 2.1. The above lemma with $p = q$, $p_1 = q_1$ and $p_2 = q_2$ was proved in [26].

Proof. The proof is similar to that of [19, Lemma 2.2]. For any $(x,s) \in \Omega_{2\delta} \times (0,T)$, $\Omega_{2\delta} \subset \Omega_\delta \subset \Omega$, we know that

$$(fg)^\varepsilon - f^\varepsilon g^\varepsilon = R^\varepsilon - (f^\varepsilon - f)(g^\varepsilon - g), \tag{2.5}$$

and

$$\begin{aligned} R^\varepsilon &= \int_\Omega \Big(f(y,s) - f(x,s)\Big)\Big(g(y,s) - g(x,s)\Big)\eta_\varepsilon(x-y)dy \\ &= \int_\Omega \Big(f(y,s) - f(x,s)\Big)\Big(g(y,s) - g(x,s)\Big)\frac{1}{\varepsilon^n}\eta\left(\frac{x-y}{\varepsilon}\right)dy \\ &= \int_{B(x,\varepsilon)} \Big(f(y,s) - f(x,s)\Big)\Big(g(y,s) - g(x,s)\Big)\frac{1}{\varepsilon^n}\eta\left(\frac{x-y}{\varepsilon}\right)dy \\ &= \int_{B(0,1)} \Big(f(x+\varepsilon z,s) - f(x,s)\Big)\Big(g(x+\varepsilon z,s) - g(x,s)\Big)\eta(z)dz, \end{aligned}$$

where $z = \frac{y-x}{\varepsilon}$ and $B(x,\varepsilon)$ is an open ball centered at x with radius ε. The triangle inequality yields

$$\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))} \leqslant \|R^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))} + \|(f^\varepsilon - f)(g^\varepsilon - g)\|_{L^p(0,T;L^q(\Omega_{2\delta}))} . \tag{2.6}$$

Next, we will handle the term $\|R^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))}$. By utilizing the Minkowski inequality and Hölder inequality, we get

$$\begin{aligned} &\|R^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))} \\ =& \left\| \int_{B(0,1)} \Big(f(x+\varepsilon z,s) - f(x,s)\Big)\Big(g(x+\varepsilon z,s) - g(x,s)\Big)\eta(z)dz \right\|_{L^p(0,T;L^q(\Omega_{2\delta}))} \\ \leqslant& \int_{B(0,1)} \left\| \Big(f(x+\varepsilon z,s) - f(x,s)\Big)\Big(g(x+\varepsilon z,s) - g(x,s)\Big) \right\|_{L^p(0,T;L^q(\Omega_{2\delta}))} |\eta(z)|\, dz \\ \leqslant& \int_{B(0,1)} \|f(x+\varepsilon z,s) - f(x,s)\|_{L^{p_1}(0,T;L^{q_1}(\Omega_{2\delta}))} \\ &\quad \times \Big(\|g(x+\varepsilon z,s)\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{2\delta}))} + \|g(x,s)\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{2\delta}))} \Big) dz. \end{aligned} \tag{2.7}$$

Due to $|z| \leqslant 1$, we have

$$\begin{aligned}
&\|g(x+\varepsilon z, s)\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{2\delta}))}\\
=&\left\{\int_0^T \left(\int_{\Omega_{2\delta}} |g(x+\varepsilon z, s)|^{q_2}\, dx\right)^{\frac{p_2}{q_2}} ds\right\}^{\frac{1}{p_2}}\\
\leqslant&\left\{\int_0^T \left(\int_{\Omega_{\delta}} |g(x+\varepsilon z, s)|^{q_2}\, d(x+\varepsilon z)\right)^{\frac{p_2}{q_2}} ds\right\}^{\frac{1}{p_2}}\\
=&\,\|g(x, s)\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{\delta}))},
\end{aligned} \tag{2.8}$$

and

$$\|g(x, s)\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{2\delta}))} \leqslant \|g(x, s)\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{\delta}))}. \tag{2.9}$$

On the other hand, in view of Leibniz's formula, we conclude that

$$\begin{aligned}
f(x+\varepsilon z, s) - f(x, s) &= \int_0^1 \partial_\theta f(x+\theta\varepsilon z, s) d\theta\\
&= \int_0^1 \nabla f(x+\theta\varepsilon z, s) d\theta \cdot \varepsilon z.
\end{aligned}$$

Taking the norm for both sides of the above formula, we deduce

$$\begin{aligned}
&\left\|\Big(f(x+\varepsilon z, s) - f(x, s)\Big)\right\|_{L^{p_1}(0,T;L^{q_1}(\Omega_{2\delta}))}\\
=&\ \varepsilon\left\|\int_0^1 \nabla f(x+\theta\varepsilon z, s) d\theta \cdot z\right\|_{L^{p_1}(0,T;L^{q_1}(\Omega_{2\delta}))}\\
\leqslant&\ \varepsilon\int_0^1 \|\nabla f(x+\theta\varepsilon z, s)\|_{L^{p_1}(0,T;L^{q_1}(\Omega_{2\delta}))} d\theta\\
\leqslant&\ \varepsilon\int_0^1 \|\nabla f(x, s)\|_{L^{p_1}(0,T;L^{q_1}(\Omega_{\delta}))} d\theta\\
\leqslant&\ C\varepsilon\|\nabla f(x, s)\|_{L^{p_1}(0,T;L^{q_1}(\Omega_{\delta}))},
\end{aligned} \tag{2.10}$$

where we have used the fact that $|z| \leqslant 1$, $\theta \in [0, 1]$, and the constant $C \geqslant 1$ does not rely on θ, z, ε, f and g.

Plugging (2.8)–(2.10) into (2.7) yields

$$\|R^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))} \leqslant C\varepsilon \|f\|_{L^{p_1}(0,T;W^{1,q_1}(\Omega_{\delta}))} \|g\|_{L^{p_2}(0,T;L^{q_2}(\Omega_{\delta}))}\,. \tag{2.11}$$

For the second term of the right hand side of (2.6), one has

$$\begin{aligned}
|(f^\varepsilon - f)(g^\varepsilon - g)| =& \int_\Omega |f(y, s) - f(x, s)|\eta_\varepsilon(x-y) dy \int_\Omega |g(y, s) - g(x, s)|\eta_\varepsilon(x-y) dy\\
=& \int_\Omega \Big(f(y, s) - f(x, s)\Big)\frac{1}{\varepsilon^n}\eta\Big(\frac{x-y}{\varepsilon}\Big) dy \int_\Omega \Big(g(y, s) - g(x, s)\Big)\frac{1}{\varepsilon^n}\eta\Big(\frac{x-y}{\varepsilon}\Big) dy\\
=& \int_{B(x,\varepsilon)} \Big(f(y, s) - f(x, s)\Big)\Big(g(y, s) - g(x, s)\Big)\frac{1}{\varepsilon^n}\eta\Big(\frac{x-y}{\varepsilon}\Big) dy
\end{aligned} \tag{2.12}$$

$$=\int_{B(0,1)}\Big(f(x+\varepsilon z,s)-f(x,s)\Big)\Big(g(x+\varepsilon z,s)-g(x,s)\Big)\eta(z)dz,$$

where $z=\frac{y-x}{\varepsilon}$. Similar to (2.7), we arrive at

$$\|(f^\varepsilon-f)(g^\varepsilon-g)\|_{L^p(0,T;L^q(\Omega_{2\delta}))}\leqslant C\varepsilon\,\|f\|_{L^{p_1}(0,T;W^{1,q_1}(\Omega_\delta))}\,\|g\|_{L^{p_2}(0,T;L^{q_2}(\Omega_\delta))}\,. \tag{2.13}$$

Combining (2.11) and (2.13), we get

$$\|(fg)^\varepsilon-f^\varepsilon g^\varepsilon\|_{L^p(0,T;L^q(\Omega_{2\delta}))}\leqslant C\varepsilon\,\|f\|_{L^{p_1}(0,T;W^{1,q_1}(\Omega_\delta))}\,\|g\|_{L^{p_2}(0,T;L^{q_2}(\Omega_\delta))}\,, \tag{2.14}$$

which yields (2.3), while (2.4) follows from (2.3) by the density arguments.

Taking into account the boundary terms, we need to use the following coarea formula for $0<\kappa_1<\kappa_2$ established in [26]:

$$\int_{\kappa_1}^{\kappa_2}\int_{\partial\Omega_\kappa}g(\theta)dH^{n-1}(\theta)d\kappa=\int_{\Omega_{\kappa_1}\backslash\Omega_{\kappa_2}}g(x)dx. \tag{2.15}$$

Lemma 2.3. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with C^2 boundary. If $f\in L^p(0,T;W_0^{1,q}(\Omega))$, $p,q\in[1,\infty)$, then for $\varepsilon>0$ small,

$$\|f\|_{L^p(0,T;L^q(\Omega\backslash\Omega_\varepsilon))}\leqslant C\varepsilon\,\|\nabla f\|_{L^p(0,T;L^q(\Omega\backslash\Omega_{2\varepsilon}))}\,. \tag{2.16}$$

Proof. Let $\varepsilon>0$ be small. For any $x\in\Omega\backslash\Omega_\varepsilon$, there exists a unique $x_{\partial\Omega}$ such that $|x_{\partial\Omega}-x|=d(x,\partial\Omega)$. We define the projection mapping as

$$\mathcal{T}(x):=x_{\partial\Omega}.$$

Then, we know that

$$\|\nabla\mathcal{T}-\mathbf{id}\|_{L^\infty(\Omega\backslash\Omega_\delta)}=o(1)\ as\ \ \delta\to 0, \tag{2.17}$$

where **id** represents the identity matrix.

Thanks to $f=0$ on $\partial\Omega\times(0,T)$, from the coarea formula and Leibniz's rule, we deduce

$$\begin{aligned}\int_{\Omega\backslash\Omega_\varepsilon}|f(x,s)|^q\,dx&=\int_0^\varepsilon\int_{\partial\Omega_\kappa}|f(\theta,s)|^q\,dH^{n-1}(\theta)d\kappa\\&=\int_0^\varepsilon\int_{\partial\Omega_\kappa}|f(\theta,s)-f(\mathcal{T}(\theta),s)|^q\,dH^{n-1}(\theta)d\kappa\\&\leqslant\int_0^\varepsilon\int_{\partial\Omega_\kappa}\int_0^1|\nabla f(\theta+\rho(\mathcal{T}(\theta)-\theta),s)|^q\,|\mathcal{T}(\theta)-\theta|^q\,d\rho dH^{n-1}(\theta)d\kappa\\&\leqslant\varepsilon^q\int_0^\varepsilon\int_{\partial\Omega_\kappa}\int_0^1|\nabla f(\theta+\rho(\mathcal{T}(\theta)-\theta),s)|^q\,d\rho dH^{n-1}(\theta)d\kappa\\&=\varepsilon^q\int_0^1\int_{\Omega\backslash\Omega_\varepsilon}|\nabla f(x+\rho(\mathcal{T}(x)-x),s)|^q\,dxd\rho.\end{aligned} \tag{2.18}$$

Set $\mathcal{T}_\rho(x)=:x+\rho(\mathcal{T}(x)-x)$. For $\varepsilon>0$ small and $\rho\in(0,1)$, we conclude from (2.17) that

$$\frac{1}{2}\leqslant|\det(\nabla\mathcal{T}_\rho(x))|\leqslant\frac{3}{2}\ \text{ and }\ \mathcal{T}_\rho(\Omega\backslash\Omega_\varepsilon)\subset\Omega\backslash\Omega_{2\varepsilon}.$$

Therefore,

$$\int_0^1 \int_{\Omega\setminus\Omega_\varepsilon} |\nabla f(x+\rho(\mathcal{T}(x)-x), s)|^q \, dxd\rho = \int_0^1 \int_{\mathcal{T}_\rho(\Omega\setminus\Omega_\varepsilon)} |\nabla f(x,s)|^q \frac{dx}{|\det(\nabla\mathcal{T}_\rho)(\mathcal{T}_\rho^{-1}(x))|} d\rho$$
$$\leqslant 2\int_{\Omega\setminus\Omega_{2\varepsilon}} |\nabla f(x,s)|^q \, dx, \tag{2.19}$$

which combined with (2.18) yields (2.16).

3. Proof of Theorem 1.1 and Corollary 1.1

In this part, we first give the proof of Theorem 1.1. Unlike the periodic region, we are concerned with the boundary terms produced by using integration by parts. Then, making use of the Gagliardo-Nirenberg inequality, we prove Corollary 1.1 by the results of Theorem 1.1.

Proof of Theorem 1.1. First of all, we denote $m =: \nabla \times b$ and mollify the systems (1.1)–(1.4) in space to obtain

$$\partial_t (\rho u)^\varepsilon + \operatorname{div} (\rho u \otimes u)^\varepsilon - \mu\Delta u^\varepsilon + \nabla P^\varepsilon - (m\times b)^\varepsilon = 0, \tag{3.1}$$
$$\partial_t b^\varepsilon + d_I \nabla\times\left(\frac{m\times b}{\rho}\right)^\varepsilon - \nabla\times(u\times b)^\varepsilon - \Delta b^\varepsilon = 0, \tag{3.2}$$
$$\partial_t \rho^\varepsilon + \operatorname{div} (\rho u)^\varepsilon = 0, \tag{3.3}$$
$$\operatorname{div}\, u^\varepsilon = 0,\ \operatorname{div}\, b^\varepsilon = 0, \tag{3.4}$$

in $\Omega_\varepsilon \times (0,T)$. Next, selecting $0 < \varepsilon < \varepsilon_1/10 < \varepsilon_2/10 < r_0/100$, we multiply (3.1) by $(\rho u)^\varepsilon/\rho^\varepsilon$ and integrate on $(\tau,t)\times\Omega_{\varepsilon_2}$ with $0<\tau<t<T$. Choosing $\varepsilon_3 > 0$ small, by integrating with respect to ε_2 on $(\varepsilon_1, \varepsilon_1+\varepsilon_3)$, we have

$$\begin{aligned}
0 =& \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\partial_t(\rho u)^\varepsilon\, dxdsd\varepsilon_2 + \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\operatorname{div}(\rho u\otimes u)^\varepsilon\, dxdsd\varepsilon_2 \\
&- \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mu\Delta u^\varepsilon dxdsd\varepsilon_2 + \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\nabla P^\varepsilon dxdsd\varepsilon_2 \\
&- \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \frac{(\rho u)^\varepsilon}{\rho^\varepsilon}(m\times b)^\varepsilon dxdsd\varepsilon_2 \\
=:& F+G+H+K+L.
\end{aligned} \tag{3.5}$$

The terms F, G, H, K and L will be estimated, separately, as follows.

Estimate of F. By direct calculations and (3.3), we arrive at

$$\begin{aligned}
F =& \frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \partial_t\left(\frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon}\right) dxdsd\varepsilon_2 + \frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \partial_t\rho^\varepsilon \frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2} dxdsd\varepsilon_2 \\
=& \frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \partial_t\left(\frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon}\right) dxdsd\varepsilon_2 - \frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}} \operatorname{div}(\rho u)^\varepsilon \frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2} dxdsd\varepsilon_2 \\
=:& F_1 + F_2,
\end{aligned}$$

where F_1 is the desired term, while F_2 will be canceled with G_4 later.

Estimate of G. Taking advantage of integration by parts and free divergence condition (3.4) implies

$$\begin{aligned}
G = & \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}[(\rho u\otimes u)^\varepsilon-(\rho u)^\varepsilon\otimes u^\varepsilon]dxdsd\varepsilon_2\\
& +\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\mathrm{div}\,((\rho u)^\varepsilon\otimes u^\varepsilon)\,dxdsd\varepsilon_2\\
=: & G_1+\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla|(\rho u)^\varepsilon|^2\frac{u^\varepsilon}{\rho^\varepsilon}dxdsd\varepsilon_2\\
=: & G_1+\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}|(\rho u)^\varepsilon|^2\frac{u^\varepsilon}{\rho^\varepsilon}n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2\\
& -\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}|(\rho u)^\varepsilon|^2\mathrm{div}\left(\frac{u^\varepsilon}{\rho^\varepsilon}\right)dxdsd\varepsilon_2\\
=: & G_1+\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}|(\rho u)^\varepsilon|^2\frac{u^\varepsilon}{\rho^\varepsilon}n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2\\
& +\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\mathrm{div}\,(\rho^\varepsilon u^\varepsilon)\,\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}dxdsd\varepsilon_2\\
=: & G_1+\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}|(\rho u)^\varepsilon|^2\frac{u^\varepsilon}{\rho^\varepsilon}n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2\\
& +\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\mathrm{div}[(\rho^\varepsilon u^\varepsilon)-(\rho u)^\varepsilon]\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}dxdsd\varepsilon_2\\
& +\frac{1}{2\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\mathrm{div}\,(\rho u)^\varepsilon\,\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}dxdsd\varepsilon_2\\
=: & G_1+G_2^{bdr}+G_3+G_4,
\end{aligned}$$

where the superscript "bdr" in G_2^{bdr} indicates that the term includes a boundary layer, and it is clear that $G_4+F_2=0$. Thus, we only consider the terms G_1, G_2^{bdr} and G_3. First of all, for G_1 and G_3, by exploiting integration by parts, we have

$$\begin{aligned}
G_1 = & -\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)[(\rho u\otimes u)^\varepsilon-(\rho u)^\varepsilon\otimes u^\varepsilon]dxdsd\varepsilon_2\\
& +\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}[(\rho u\otimes u)^\varepsilon-(\rho u)^\varepsilon\otimes u^\varepsilon]n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2\\
=: & G_{11}+G_{12}^{bdr},
\end{aligned}$$

and

$$\begin{aligned}
G_3 = & -\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)\nabla\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)[(\rho^\varepsilon u^\varepsilon)-(\rho u)^\varepsilon]dxdsd\varepsilon_2\\
& +\frac{1}{2}\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}[(\rho^\varepsilon u^\varepsilon)-(\rho u)^\varepsilon]n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2
\end{aligned}$$

$$=: G_{31} + G_{32}^{bdr}.$$

For G_{11}, by Hölder's inequality, Lemma 2.1 (ii) and Lemma 2.2, we get

$$\begin{aligned}
|G_{11}| &= \left| \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \nabla \frac{(\rho u)^\varepsilon}{\rho^\varepsilon} [(\rho u \otimes u)^\varepsilon - (\rho u)^\varepsilon \otimes u^\varepsilon] dx ds d\varepsilon_2 \right| \\
&\leqslant \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \left\| \nabla \left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \\
&\qquad \times \| [(\rho u \otimes u)^\varepsilon - (\rho u)^\varepsilon \otimes u^\varepsilon] \|_{L^{\frac{2p}{p+1}}(0,T;L^{\frac{2q}{q+1}}(\Omega))} d\varepsilon_2 \\
&\leqslant C \left\| \nabla \left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \varepsilon \|u\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \|u\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\leqslant C \|u\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \|u\|_{L^p(0,T;W^{1,q}(\Omega))},
\end{aligned}$$

which implies

$$\limsup_{\varepsilon \to 0} \limsup_{\tau \to 0} |G_{11}| = 0.$$

Similarly, we deduce

$$\begin{aligned}
|G_{31}| &= \left| \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_2}} \left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right) \nabla \left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right) [(\rho^\varepsilon u^\varepsilon) - (\rho u)^\varepsilon] dx ds d\varepsilon_2 \right| \\
&\leqslant \frac{1}{\varepsilon_3} \int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3} \left\| \frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \left\| \nabla \left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \\
&\qquad \times \| [(\rho^\varepsilon u^\varepsilon) - (\rho u)^\varepsilon] \|_{L^p(0,T;L^q(\Omega))} d\varepsilon_2 \\
&\leqslant C \left\| \nabla \left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \|u\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \\
&\qquad \times \varepsilon \|\rho\|_{L^\infty(0,T;L^\infty(\Omega))} \|u\|_{L^p(0,T;W^{1,q}(\Omega))} \\
&\leqslant C \|u\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \|\rho\|_{L^\infty(0,T;L^\infty(\Omega))} \|u\|_{L^p(0,T;W^{1,q}(\Omega))}.
\end{aligned}$$

As $\rho \in L^\infty(0,T;L^\infty(\Omega))$, $u \in L^p(0,T;W^{1,q}(\Omega))$, we obtain

$$\limsup_{\varepsilon \to 0} \limsup_{\tau \to 0} |G_{31}| = 0.$$

Next, we estimate the boundary terms G_2^{bdr}, G_{12}^{bdr} and G_{32}^{bdr}. For G_2^{bdr}, we employ coarea Eq (2.15), Hölder's inequality and Lemma 2.3 to conclude

$$\begin{aligned}
\limsup_{\varepsilon_1, \varepsilon \to 0} \limsup_{\tau \to 0} |G_2^{bdr}| &= \limsup_{\varepsilon_1, \varepsilon \to 0} \limsup_{\tau \to 0} \left| \frac{1}{2\varepsilon_3} \int_\tau^t \int_{\Omega_{\varepsilon_1} \setminus \Omega_{\varepsilon_1+\varepsilon_3}} |(\rho u)^\varepsilon|^2 \frac{u^\varepsilon}{\rho^\varepsilon} n(x) dx ds \right| \\
&\leqslant \frac{C}{2\varepsilon_3} \|u\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega \setminus \Omega_{\varepsilon_3}))} \|u\|_{L^p(0,T;L^q(\Omega \setminus \Omega_{\varepsilon_3}))} \\
&\leqslant C \|u\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega \setminus \Omega_{\varepsilon_3}))} \|\nabla u\|_{L^p(0,T;L^q(\Omega \setminus \Omega_{2\varepsilon_3}))}.
\end{aligned}$$

Since $\nabla u \in L^p(0,T;L^q(\Omega))$ and $u \in L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))$, by letting $\varepsilon_3 \to 0$, we get

$$\limsup_{\varepsilon_3 \to 0} \limsup_{\varepsilon_1, \varepsilon \to 0} \limsup_{\tau \to 0} |G_2^{bdr}| = 0.$$

Likewise, for G_{12}^{bdr} and G_{32}^{bdr}, we have

$$\begin{aligned}
&\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|G_{12}^{bdr}|\\
&=\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}\left|\frac{1}{\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\setminus\Omega_{\varepsilon_1+\varepsilon_3}}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}[(\rho u\otimes u)^\varepsilon-(\rho u)^\varepsilon\otimes u^\varepsilon]n(x)dxds\right|\\
&\leqslant\frac{C}{\varepsilon_3}\left\|\left(\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right)\right\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\setminus\Omega_{\varepsilon_3}))}\|[(\rho u\otimes u)^\varepsilon-(\rho u)^\varepsilon\otimes u^\varepsilon]\|_{L^{\frac{2p}{p+1}}(0,T;L^{\frac{2q}{q+1}}(\Omega\setminus\Omega_{\varepsilon_3}))}\\
&\leqslant C\|u\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\setminus\Omega_{\varepsilon_3}))}\|u\|_{L^p(0,T;W^{1,q}(\Omega\setminus\Omega_{\varepsilon_3}))},
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|G_{32}^{bdr}|\\
&=\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}\left|\frac{1}{2\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\setminus\Omega_{\varepsilon_1+\varepsilon_3}}\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}[(\rho^\varepsilon u^\varepsilon)-(\rho u)^\varepsilon]n(x)dxds\right|\\
&\leqslant\frac{C}{2\varepsilon_3}\left\|\frac{|(\rho u)^\varepsilon|^2}{(\rho^\varepsilon)^2}\right\|_{L^{\frac{p}{p-1}}(0,T;L^{\frac{q}{q-1}}(\Omega\setminus\Omega_{\varepsilon_3}))}\|[(\rho^\varepsilon u^\varepsilon)-(\rho u)^\varepsilon]\|_{L^p(0,T;L^q(\Omega\setminus\Omega_{\varepsilon_3}))}\\
&\leqslant C\|u\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\setminus\Omega_{\varepsilon_3}))}\|u\|_{L^p(0,T;W^{1,q}(\Omega\setminus\Omega_{\varepsilon_3}))}\|\rho\|_{L^\infty(0,T;L^\infty(\Omega\setminus\Omega_{\varepsilon_3}))}.
\end{aligned}$$

Owing to the assumption (1.13), by letting $\varepsilon_3\to 0$, we derive

$$\limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|G_{12}^{bdr}|=0,\quad \limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|G_{32}^{bdr}|=0.$$

Estimate of H. We see that

$$\begin{aligned}
H=&-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\mu\Delta u^\varepsilon dxdsd\varepsilon_2-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\mu\Delta u^\varepsilon u^\varepsilon dxdsd\varepsilon_2\\
=:&H_1-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}\mu\nabla u^\varepsilon u^\varepsilon n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\mu|\nabla u^\varepsilon|^2dxdsd\varepsilon_2\\
=:&H_1+H_2^{bdr}+H_3.
\end{aligned}$$

As H_3 is the desired term, we only need to handle terms H_1 and H_2^{bdr}. For H_1, taking advantage of Hölder's inequality and Lemmas 2.1 (i) and 2.2, we deduce

$$\begin{aligned}
|H_1|&=\left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\mu\Delta u^\varepsilon dxdsd\varepsilon_2\right|\\
&\leqslant\mu\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\left\|\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\right\|_{L^2(0,T;L^2(\Omega))}\|\Delta u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}d\varepsilon_2\\
&\leqslant C\varepsilon\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;H^1(\Omega))}\|\Delta u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}\\
&\leqslant C\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;H^1(\Omega))}\|\nabla u\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

According to $u\in L^2(0,T;H^1(\Omega))$ and $\rho\in L^\infty(0,T;L^\infty(\Omega))$, we get

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|H_1|=0.$$

The boundary term H_2^{bdr} is estimated by using coarea Eq (2.15) and Lemma 2.3 as

$$\begin{aligned}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|H_2^{bdr}| &= \limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}\left|\frac{1}{\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}}\mu\nabla u^\varepsilon u^\varepsilon n(x)dxds\right|\\
&\leqslant \mu\frac{C}{\varepsilon_3}\|\nabla u\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))}\|u\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))}\\
&\leqslant C\|\nabla u\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))}\|\nabla u\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{2\varepsilon_3}))}.\end{aligned}$$

Since $u\in L^2(0,T;H^1(\Omega))$, by $\varepsilon_3\to 0$, we obtain

$$\limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|H_2^{bdr}| = 0.$$

Estimate of K. In light of the free divergence condition (3.4), we rewrite K as

$$\begin{aligned}K =&\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\nabla P^\varepsilon dxdsd\varepsilon_2+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\nabla P^\varepsilon u^\varepsilon dxdsd\varepsilon_2\\
=:&K_1+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}P^\varepsilon u^\varepsilon n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2\\
=:&K_1+K_2^{bdr}.\end{aligned}$$

For K_1, by making use of Hölder's inequality, we have

$$\begin{aligned}|K_1| &= \left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\nabla P^\varepsilon dxdsd\varepsilon_2\right|\\
&\leqslant \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\left\|\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\right\|_{L^2(0,T;L^2(\Omega))}\|\nabla P^\varepsilon\|_{L^2(0,T;L^2(\Omega))}d\varepsilon_2\\
&\leqslant C\varepsilon\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;H^1(\Omega))}\|\nabla P^\varepsilon\|_{L^2(0,T;L^2(\Omega))}\\
&\leqslant C\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^2(0,T;H^1(\Omega))}\|P\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Due to Lemmas 2.1 (i) and 2.2, one has

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|K_1| = 0.$$

Exploiting Hölder's inequality, coarea Eq (2.15) and Lemma 2.3 again, we deduce

$$\begin{aligned}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|H_2^{bdr}| &= \limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}\left|\frac{1}{\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}}P^\varepsilon u^\varepsilon n(x)dxds\right|\\
&\leqslant \frac{C}{\varepsilon_3}\|P\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))}\|u\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))}\\
&\leqslant C\|P\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))}\|\nabla u\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{2\varepsilon_3}))}.\end{aligned}$$

Owing to $u\in L^2(0,T;H^1(\Omega))$ and $P\in L^2(0,T;L^2(\Omega))$, by $\varepsilon_3\to 0$, we derive

$$\limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|K_2^{bdr}| = 0.$$

Estimate of L. First, it is clear that

$$L = -\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}(m\times b)^\varepsilon dxdsd\varepsilon_2$$

$$
\begin{aligned}
=&-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}[(m\times b)^\varepsilon-m^\varepsilon\times b^\varepsilon]dxdsd\varepsilon_2\\
&-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}(m^\varepsilon\times b^\varepsilon)dxdsd\varepsilon_2\\
&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}m^\varepsilon[(u^\varepsilon\times b^\varepsilon)-(u\times b)^\varepsilon]dxdsd\varepsilon_2 \qquad (3.6)\\
&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}b^\varepsilon\cdot\nabla\times(u\times b)^\varepsilon dxdsd\varepsilon_2\\
&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}(b^\varepsilon\times(u\times b)^\varepsilon)n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2\\
=&:L_1+L_2+L_3+L_4+L_5^{bdr},
\end{aligned}
$$

where $\operatorname{div}(b^\varepsilon\times(u\times b)^\varepsilon)=(u\times b)^\varepsilon\cdot\nabla\times b^\varepsilon-b^\varepsilon\cdot\nabla\times(u\times b)^\varepsilon$.

Next, multiplying (3.2) by b^ε yields

$$
\frac{1}{2}\partial_t|b^\varepsilon|^2+d_I\nabla\times\left(\frac{m\times b}{\rho}\right)^\varepsilon\cdot b^\varepsilon-\nabla\times(u\times b)^\varepsilon\cdot b^\varepsilon-\Delta b^\varepsilon\cdot b^\varepsilon=0. \qquad (3.7)
$$

Plugging (3.7) into (3.6), we infer that

$$
\begin{aligned}
L_4=&\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{1}{2}\partial_t|b^\varepsilon|^2dxdsd\varepsilon_2-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\Delta b^\varepsilon\cdot b^\varepsilon dxdsd\varepsilon_2\\
&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}d_I\nabla\times\left(\frac{m\times b}{\rho}\right)^\varepsilon\cdot b^\varepsilon dxdsd\varepsilon_2\\
=&:L_{41}-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}\nabla b^\varepsilon\cdot b^\varepsilon n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2 \qquad (3.8)\\
&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}|\nabla b^\varepsilon|^2dxdsd\varepsilon_2+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}d_Im^\varepsilon\cdot\left(\frac{m\times b}{\rho}\right)^\varepsilon dxdsd\varepsilon_2\\
&-\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\partial\Omega_{\varepsilon_2}}d_Ib^\varepsilon\times\left(\frac{m\times b}{\rho}\right)^\varepsilon n(\theta)dH^{n-1}(\theta)dsd\varepsilon_2\\
=&:L_{41}+L_{42}^{bdr}+L_{43}+L_{44}+L_{45}^{bdr},
\end{aligned}
$$

where L_{41}, L_{43} are our expected terms, and we just need to deal with the rest of the above items.

For L_1, applying Hölder's inequality and Lemma 2.2, we have

$$
\begin{aligned}
|L_1|=&\left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}[(m\times b)^\varepsilon-m^\varepsilon\times b^\varepsilon]dxdsd\varepsilon_2\right|\\
\leqslant&\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\left\|\frac{(\rho u)^\varepsilon}{\rho^\varepsilon}\right\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\|[(m\times b)^\varepsilon-m^\varepsilon\times b^\varepsilon]\|_{L^{\frac{2p}{p+1}}(0,T;L^{\frac{2q}{q+1}}(\Omega))}d\varepsilon_2\\
\leqslant&C\|u\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\varepsilon\|m\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\|b\|_{L^p(0,T;W^{1,q}(\Omega))}.
\end{aligned}
$$

Under the hypothesis (1.13), by $\varepsilon \to 0, \tau \to 0$, we obtain

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|L_1| = 0.$$

The term L_2 is estimated by utilizing Hölder's inequality and Lemma 2.2 as

$$\begin{aligned}|L_2| &= \left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}(m^\varepsilon\times b^\varepsilon)dxdsd\varepsilon_2\right|\\ &\leqslant \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\left\|\frac{(\rho u)^\varepsilon-\rho^\varepsilon u^\varepsilon}{\rho^\varepsilon}\right\|_{L^p(0,T;L^q(\Omega))}\|(m^\varepsilon\times b^\varepsilon)\|_{L^{\frac{p}{p-1}}(0,T;L^{\frac{q}{q-1}}(\Omega))}d\varepsilon_2\\ &\leqslant C\varepsilon\|\rho\|_{L^\infty(0,T;L^\infty(\Omega))}\|u\|_{L^p(0,T;W^{1,q}(\Omega))}\|m\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\|b\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}.\end{aligned}$$

As $\rho \in L^\infty(0,T;L^\infty(\Omega))$, $u \in L^p(0,T;W^{1,q}(\Omega))$, we get

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|L_2| = 0.$$

Similarly, we deal with L_3 and L_{44} as

$$\begin{aligned}|L_3| &= \left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}m^\varepsilon[(u^\varepsilon\times b^\varepsilon)-(u\times b)^\varepsilon]dxdsd\varepsilon_2\right|\\ &\leqslant \frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\|m^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\|[(u^\varepsilon\times b^\varepsilon)-(u\times b)^\varepsilon]\|_{L^{\frac{2p}{p+1}}(0,T;L^{\frac{2q}{q+1}}(\Omega))}d\varepsilon_2\\ &\leqslant C\|m\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\varepsilon\|u\|_{L^p(0,T;W^{1,q}(\Omega))}\|b\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))},\end{aligned}$$

and

$$\begin{aligned}|L_{44}| &= \left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}d_I m^\varepsilon\cdot\left(\frac{m\times b}{\rho}\right)^\varepsilon dxdsd\varepsilon_2\right|\\ &\leqslant C\left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}m^\varepsilon\cdot(m\times b)^\varepsilon dxdsd\varepsilon_2\right|\\ &=C\left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}m^\varepsilon\cdot[(m\times b)^\varepsilon-m^\varepsilon\times b^\varepsilon]dxdsd\varepsilon_2\right|\\ &\leqslant C\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\|m^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\|(m\times b)^\varepsilon-m^\varepsilon\times b^\varepsilon\|_{L^{\frac{2p}{p+1}}(0,T;L^{\frac{2q}{q+1}}(\Omega))}d\varepsilon_2\\ &\leqslant C\varepsilon\|m\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))}\|b\|_{L^p(0,T;W^{1,q}(\Omega))}.\end{aligned}$$

Then, we deduce that

$$\limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|L_3| = 0, \limsup_{\varepsilon\to 0}\limsup_{\tau\to 0}|L_{44}| = 0.$$

Next, we treat the terms about the boundary L_{42}^{bdr}, L_{45}^{bdr} and L_5^{bdr}, separately. For L_{42}^{bdr}, by using coarea Eq (2.15) and Lemma 2.3, we have

$$\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}|L_{42}^{bdr}| = \limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0}\left|\frac{1}{\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}}\nabla b^\varepsilon\cdot b^\varepsilon n(x)dxds\right|$$

$$\leqslant \frac{C}{\varepsilon_3} \|\nabla b\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))} \|b\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))}$$
$$\leqslant C \|\nabla b\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{\varepsilon_3}))} \|\nabla b\|_{L^2(0,T;L^2(\Omega\backslash\Omega_{2\varepsilon_3}))},$$

which, together with $b \in L^2(0,T;H^1(\Omega))$, implies

$$\limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} |L_{42}^{bdr}| = 0.$$

Similarly, for L_{45}^{bdr} and L_5^{bdr}, we get

$$\begin{aligned}
&\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} |L_{45}^{bdr}| \\
&= \limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} \left| \frac{1}{\varepsilon_3}\int_\tau^t \int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}} d_I b^\varepsilon \times \left(\frac{m\times b}{\rho}\right)^\varepsilon n(x)dxds \right| \\
&\leqslant C\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} \left| \frac{1}{\varepsilon_3}\int_\tau^t \int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}} \{b^\varepsilon \times [(m\times b)^\varepsilon - m^\varepsilon \times b^\varepsilon] + b^\varepsilon \times m^\varepsilon \times b^\varepsilon\} n(x)dxds \right| \\
&\leqslant \frac{C}{\varepsilon_3} \|b^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|(m\times b)^\varepsilon - m^\varepsilon\times b^\varepsilon\|_{L^{\frac{2p}{p+1}}(0,T;L^{\frac{2q}{q+1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \\
&\quad + \frac{C}{\varepsilon_3} \|b\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|m\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|b\|_{L^p(0,T;L^q(\Omega\backslash\Omega_{\varepsilon_3}))} \\
&\leqslant C \|b\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|m\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|b\|_{L^p(0,T;W^{1,q}(\Omega\backslash\Omega_{\varepsilon_3}))} \\
&\quad + C \|b\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|m\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|\nabla b\|_{L^p(0,T;L^q(\Omega\backslash\Omega_{2\varepsilon_3}))},
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} |L_5^{bdr}| \\
&= \limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} \left| \frac{1}{\varepsilon_3}\int_\tau^t \int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}} b^\varepsilon \times (u\times b)^\varepsilon n(x)dxds \right| \\
&= \limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} \left| \frac{1}{\varepsilon_3}\int_\tau^t \int_{\Omega_{\varepsilon_1}\backslash\Omega_{\varepsilon_1+\varepsilon_3}} \left(b^\varepsilon \times [(u\times b)^\varepsilon - u^\varepsilon\times b^\varepsilon] + b^\varepsilon\times u^\varepsilon\times b^\varepsilon\right) n(x)dxds \right| \\
&\leqslant \frac{C}{\varepsilon_3} \|b^\varepsilon\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|(u\times b)^\varepsilon - u^\varepsilon\times b^\varepsilon\|_{L^{\frac{2p}{p+1}}(0,T;L^{\frac{2q}{q+1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \\
&\quad + \frac{C}{\varepsilon_3} \|b\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|u\|_{L^p(0,T;L^q(\Omega\backslash\Omega_{\varepsilon_3}))} \\
&\leqslant C \|b\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|u\|_{L^p(0,T;W^{1,q}(\Omega\backslash\Omega_{\varepsilon_3}))} \\
&\quad + C \|b\|^2_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega\backslash\Omega_{\varepsilon_3}))} \|\nabla u\|_{L^p(0,T;L^q(\Omega\backslash\Omega_{2\varepsilon_3}))}.
\end{aligned}$$

Under the assumption (1.13), by letting $\varepsilon_3 \to 0$, we obtain

$$\limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} |L_{45}^{bdr}| = 0, \quad \limsup_{\varepsilon_3\to 0}\limsup_{\varepsilon_1,\varepsilon\to 0}\limsup_{\tau\to 0} |L_5^{bdr}| = 0.$$

Then, we collect all the above estimates F_1, H_3, L_{41}, L_{43} and put them into the right side of equation (3.5) to conclude

$$\begin{aligned}&\limsup_{\varepsilon_3\to0}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}\Bigg|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\partial_t\left(\frac12\frac{|(\rho u)^\varepsilon|^2}{\rho^\varepsilon}+\frac12|b^\varepsilon|^2\right)dxdsd\varepsilon_2\\&+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\mu|\nabla u^\varepsilon|^2dxdsd\varepsilon_2+\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}|\nabla b^\varepsilon|^2dxdsd\varepsilon_2\Bigg|=0.\end{aligned}$$

Using the weak continuity of ρ, ρu in (1.10) and (1.11) and limits

$$\limsup_{\varepsilon_3\to0}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}\left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\mu|\nabla u^\varepsilon|^2dxdsd\varepsilon_2-\int_0^t\int_\Omega\mu|\nabla u|^2dxds\right|=0,$$

$$\limsup_{\varepsilon_3\to0}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}\left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}|\nabla b^\varepsilon|^2dxdsd\varepsilon_2-\int_0^t\int_\Omega|\nabla b|^2dxds\right|=0,$$

and

$$\limsup_{\varepsilon_3\to0}\limsup_{\varepsilon_1,\varepsilon\to0}\limsup_{\tau\to0}\left|\frac{1}{\varepsilon_3}\int_{\varepsilon_1}^{\varepsilon_1+\varepsilon_3}\int_\tau^t\int_{\Omega_{\varepsilon_2}}\frac12\partial_t|b^\varepsilon|^2dxdsd\varepsilon_2-\int_0^t\int_\Omega\frac12\partial_t|b|^2dxds\right|=0,$$

we finish the proof of Theorem 1.1.

In what follows, we will prove Corollary 1.1 based on Theorem 1.1, where $n=3$.

Proof of Corollary 1.1. First, we prove the first case of (1.15). The basic regularity of weak solutions gives $u, b\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$. Choosing $p=q=2$ in (1.13), we know from Theorem 1.1 that the conditions $u,b\in L^4(0,T;L^4(\Omega))$ and $\nabla\times b\in L^4(0,T;L^4(\Omega))$ could guarantee the energy equality (1.8). Thus, to prove the first case of (1.15), it suffices to prove that $u,b\in L^s(0,T;L^t(\Omega))$ with $\frac2s+\frac2t=1, t\geqslant4$ could yield $u,b\in L^4(0,T;L^4(\Omega))$. To this end, with the help of the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{L^4(0,T;L^4(\Omega))}\leqslant C\|u\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{t-4}{2t-4}}\|u\|_{L^s(0,T;L^t(\Omega))}^{\frac{t}{2t-4}}\leqslant C,$$

$$\|b\|_{L^4(0,T;L^4(\Omega))}\leqslant C\|b\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{t-4}{2t-4}}\|b\|_{L^s(0,T;L^t(\Omega))}^{\frac{t}{2t-4}}\leqslant C,$$

which finishes the proof of the first case of (1.15).

Next, we consider the second case of (1.15) with $\frac1s+\frac3t=1$, $3<t<4$. Using the Gagliardo-Nirenberg inequality, we deduce

$$\begin{aligned}\|u\|_{L^4(0,T;L^4(\Omega))}&\leqslant C\|u\|_{L^2(0,T;L^6(\Omega))}^{\frac{3(4-t)}{2(6-t)}}\|u\|_{L^s(0,T;L^t(\Omega))}^{\frac{t}{2(6-t)}}\\&\leqslant C(\|\nabla u\|_{L^2(0,T;L^2(\Omega))}+\|u\|_{L^\infty(0,T;L^2(\Omega))})^{\frac{3(4-t)}{2(6-t)}}\|u\|_{L^s(0,T;L^t(\Omega))}^{\frac{t}{2(6-t)}}\leqslant C,\\\|b\|_{L^4(0,T;L^4(\Omega))}&\leqslant C\|b\|_{L^2(0,T;L^6(\Omega))}^{\frac{3(4-t)}{2(6-t)}}\|b\|_{L^s(0,T;L^t(\Omega))}^{\frac{t}{2(6-t)}}\\&\leqslant C(\|\nabla b\|_{L^2(0,T;L^2(\Omega))}+\|b\|_{L^\infty(0,T;L^2(\Omega))})^{\frac{3(4-t)}{2(6-t)}}\|b\|_{L^s(0,T;L^t(\Omega))}^{\frac{t}{2(6-t)}}\leqslant C.\end{aligned}$$

Thus, by Theorem 1.1, we get that (1.15) could ensure the energy equality (1.8).

In what follows, we give the proof of the first case of (1.16). According to Theorem 1.1, for $1 < p, q \leqslant 3$, one knows that the conditions $\nabla u, \nabla b \in L^p(0,T; L^q(\Omega))$ and $u, b \in L^{\frac{2p}{p-1}}(0,T; L^{\frac{2q}{q-1}}(\Omega))$ can ensure that energy equality (1.8) is valid. Therefore, to prove the first case of (1.16), we need to show that the conditions $\nabla u, \nabla b \in L^p(0,T; L^q(\Omega))$ can yield $u, b \in L^{\frac{2p}{p-1}}(0,T; L^{\frac{2q}{q-1}}(\Omega))$. To this end, for $\frac{9}{5} \leqslant q \leqslant 3$, by virtue of the Gagliardo-Nirenberg inequality, we know that

$$\|u\|_{L^{\frac{2q}{q-1}}(\Omega)} \leqslant C \|u\|_{L^2(\Omega)}^{\frac{5q-9}{5q-6}} \|\nabla u\|_{L^q(\Omega)}^{\frac{3}{5q-6}},$$

$$\|b\|_{L^{\frac{2q}{q-1}}(\Omega)} \leqslant C \|b\|_{L^2(\Omega)}^{\frac{5q-9}{5q-6}} \|\nabla b\|_{L^q(\Omega)}^{\frac{3}{5q-6}}.$$

Furthermore, in view of $\frac{1}{p} + \frac{6}{5q} = 1$, we deduce that

$$\|u\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \leqslant C \|u\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{5q-9}{5q-6}} \|\nabla u\|_{L^p(0,T;L^q(\Omega))}^{\frac{3}{5q-6}} \leqslant C,$$

$$\|b\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \leqslant C \|b\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{5q-9}{5q-6}} \|\nabla b\|_{L^p(0,T;L^q(\Omega))}^{\frac{3}{5q-6}} \leqslant C.$$

Then, we complete the proof of the first case of (1.16).

Next, we treat the remaining case of (1.16). For $\frac{3}{2} < q < \frac{9}{5}$, it follows from the Gagliardo-Nirenberg inequality that

$$\|u\|_{L^{\frac{2q}{q-1}}(\Omega)} \leqslant C \|u\|_{L^6(\Omega)}^{\frac{9-5q}{6-3q}} \|\nabla u\|_{L^q(\Omega)}^{\frac{2q-3}{6-3q}},$$

$$\|b\|_{L^{\frac{2q}{q-1}}(\Omega)} \leqslant C \|b\|_{L^6(\Omega)}^{\frac{9-5q}{6-3q}} \|\nabla b\|_{L^q(\Omega)}^{\frac{2q-3}{6-3q}}.$$

Thanks to $\frac{1}{p} + \frac{3}{q} = 2$, we further infer that

$$\|u\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \leqslant C \|u\|_{L^2(0,T;L^6(\Omega))}^{\frac{9-5q}{6-3q}} \|\nabla u\|_{L^p(0,T;L^q(\Omega))}^{\frac{2q-3}{6-3q}} \leqslant C,$$

$$\|b\|_{L^{\frac{2p}{p-1}}(0,T;L^{\frac{2q}{q-1}}(\Omega))} \leqslant C \|b\|_{L^2(0,T;L^6(\Omega))}^{\frac{9-5q}{6-3q}} \|\nabla b\|_{L^p(0,T;L^q(\Omega))}^{\frac{2q-3}{6-3q}} \leqslant C.$$

Then, from Theorem 1.1, we know that (1.16) could guarantee the energy equality (1.8), and the proof of Corollary 1.1 is finished.

4. Conclusions

This paper is dedicated to the energy equality of nonhomogeneous incompressible Hall-MHD equations in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$. Through the special structure of the nonlinear terms, and using the coarea formula, we get some types of regularity conditions to guarantee that the energy equality is valid. It is worth noting that among them are the regularity conditions concerning ∇u and ∇b, rather than u and b.

Acknowledgments

This work was sponsored by the NNSF of China (No. 11871305). The authors thank the referees for careful reading and valuable suggestions.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. M. J. Lighthill, Studies on magneto-hydrodynamic waves and other anisotropic wave motions, *Philos. Trans. R. Soc. Lond. Ser. A.*, **252** (1960), 397–430. https://doi.org/10.1098/rsta.1960.0010
2. M. Arichetogaray, P. Degond, Y. Frouvelle, J. G. Liu, Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system, *Kinet. Relat. Models*, **4** (2011), 901–918. http://dx.doi.org/10.3934/krm.2011.4.901
3. D. Chae, R. Wan, J. Wu, Local well-posedness for the Hall-MHD equations with fractional magnetic diffusion, *J. Math. Fluid Mech.*, **17** (2015), 627–638. https://doi.org/10.1007/s00021-015-0222-9
4. R. Wan, Y. Zhou, Global well-posedness, BKM blow-up criteria and zero h limit for the 3D incompressible Hall-MHD equations, *J. Differ. Equations*, **267** (2019), 3724–3747. https://doi.org/10.1016/j.jde.2019.04.020
5. Z. Ye, Well-posedness results for the 3D incompressible Hall-MHD equations, *J. Differ. Equations*, **321** (2022), 130–216. https://doi.org/10.1016/j.jde.2022.03.012
6. J. Fan, B. Ahmad, T. Hadyat, Y. Zhou, On blow-up criteria for a new Hall-MHD system, *Appl. Math. Comput.*, **274** (2016), 20–24. https://doi.org/10.1016/j.amc.2015.10.073
7. D. Chae, M. Schonbek, On the temporal decay for the Hall-magneto hydrodynamic equations, *J. Differ. Equations*, **255** (2013), 3971–3982. https://doi.org/10.1016/j.jde.2013.07.059
8. M. Dai, H. Liu, Long time behavior of solutions to the 3D Hall-magnetohydrodynamics system with one diffusion, *J. Differ. Equations*, **266** (2019), 7658–7677. https://doi.org/10.1016/j.jde.2018.12.008
9. S. Weng, Space-time decay estimates for the incompressible viscous resistive MHD and Hall-MHD equations, *J. Funct. Anal.*, **270** (2016), 2168–2187. https://doi.org/10.1016/j.jfa.2016.01.021
10. X. Zhai, Global wellposedness and large time behavior of solutions to the Hall-magnetohydrodynamics equations, *Z. Anal. Anwend.*, **39** (2020), 395–419. https://doi.org/10.4171/zaa/1665
11. Y. Zhou, Q. Bie, Q. Wang, Z. Yao, On Liouville type theorems for three-dimensional stationary MHD and Hall-MHD equations, *Sci. Sin. Math.*, **52** (2022), 1–10. https://doi.org/10.1360/SSM-2022-0059

12. D. Chae, P. Degond, J. G. Liu, Well-posedness for Hall-magnetohydrodynamics, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **31** (2014), 555–565. https://doi.org/10.1016/j.anihpc.2013.04.006

13. D. Chae, S. Weng, Liouville type theorems for the steady axially symmetric Navier-Stokes and magnetohydrodynamic equations, *Discrete Contin. Dyn. Syst.*, **36** (2016), 5267–5285. https://doi.org/10.48550/arXiv.1512.03491

14. Z. Zhang, X. Yang, S. Qiu, Remarks on Liouville type result for the 3D Hall-MHD system, *J. Partial Differ. Equations*, **3** (2015), 286–290. http://dx.doi.org/10.4208/jpde.v28.n3.7

15. L. Kang, X. Deng, Q. Bie, Energy conservation for the nonhomogeneous incompressible Hall-MHD equations, *J. Math. Phys.*, **62** (2021), 031506. https://doi.org/10.1063/5.0042696

16. L. Kang, X. Deng, Y. Zhou, Energy conservation for the nonhomogeneous incompressible Hall-MHD equations in a bounded domain, *Results Appl. Math.*, **12** (2021), 100178. https://doi.org/10.1016/j.rinam.2021.100178

17. Q. Bie, L. Kang, Q. Wang, Z. Yao, Regularity and energy conservation for the compressible MHD equations (in Chinese), *Sci. Sin. Math.*, **52** (2022), 741–756. https://doi.org/10.1360/SSM-2020-0339

18. Y. Wang, B. J. Zuo, Energy and cross-helicity conservation for the three-dimensional ideal MHD equations in a bounded domain, *J. Differ. Equations*, **268** (2020), 4079–4101. https://doi.org/10.1016/j.jde.2019.10.045

19. X. Wang, S. Liu, Energy conservation for the weak solutions to the 3D compressible magnetohydrodynamic equations of viscous non-resistive fluids in a bounded domain, *Nonlinear Anal. RWA*, **62** (2021), 103359. https://doi.org/10.1016/j.nonrwa.2021.103359

20. T. Wang, X. Zhao, Y. Chen, M. Zhang, Energy conservation for the weak solutions to the equations of compressible magnetohydrodynamic flows in three dimensions, *J. Math. Anal. Appl.*, **480** (2019), 123373. https://doi.org/10.1016/j.jmaa.2019.07.063

21. J. L. Lions, Sur la régularité et l'unicité des solutions turbulentes des équations de Navier Stokes, *Rend. Semin. Mat. Univ. Padova*, **30** (1960), 16–23.

22. M. Shinbrot, The energy equation for the Navier-Stokes system, *SIAM J. Math. Anal.*, **5** (1974), 948–954. https://doi.org/10.1137/0505092

23. H. B. da Veiga, J. Yang, On the Shinbrot's criteria for energy equality to Newtonian fluids: a simplified proof, and an extension of the range of application, *Nonlinear Anal.*, **196** (2020), 111809. https://doi.org/10.1016/j.na.2020.111809

24. C. Yu, A new proof to the energy conservation for the Navier-Stokes equations, preprint, arXiv:1604.05697.

25. C. Yu, The energy equality for the Navier-Stokes equations in bounded domains, preprint, arXiv:1802.07661.

26. Q. Nguyen, P. Nguyen, B. Tang, Energy equalities for compressible Navier-Stokes equations, *Nonlinearity*, **32** (2019), 4206–4231. https://doi.org/10.1088/1361-6544/ab28ae

27. C. Yu, Energy conservation for the weak solutions of the compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.*, **225** (2017), 1073–1087. https://doi.org/10.1007/s00205-017-1121-4

28. R. M. Chen, Z. L. Liang, D. H. Wang, R. Z. Xu, Energy equality in compressible fluids with physical boundaries, *SIAM J. Math. Anal.*, **52** (2020), 1363–1385. https://doi.org/10.1137/19M1287213

29. L. C. Berselli, E. Chiodaroli, On the energy equality for the 3D Navier-Stokes equations, *Nonlinear Anal.*, **192** (2020), 111704. https://doi.org/10.1016/j.na.2019.111704

30. Z. Liang, Regularity criterion on the energy conservation for the compressible Navier-Stokes equations, *Proc. Roy. Soc. Edinburgh Sect. A.*, **151** (2021), 1954–1971. https://doi.org/10.1017/prm.2020.87

31. Y. Wang, Y. Ye, Energy conservation via a combination of velocity and its gradient in the Navier-Stokes system, preprint, arXiv:2106.01233.