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Research article

Exponential stability of Thermoelastic system with boundary time-varying delay

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Abstract: In this paper, we consider a Thermoelastic system with boundary time-varying delay. Under some appropriate assumptions, the global well-posedness and exponential stability are obtained by using the variable norm technique of Kato and the energy method respectively.

Keywords: Thermoelastic system; time-varying delay; exponential stability

1. Introduction

It is well known that the delay affects dynamical systems in many fields [1–4] and creates some instabilities [5–8]. Regarding the influence of the delay in hyperbolic equations, under some appropriate assumptions on the delay term, the authors of [8] proved that the delay term generates the stability mechanism in the following wave equation with a delayed velocity term and the mixed Dirichlet-Neumann boundary condition

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \ \text{ in } \ \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \ \text{ on } \ \Gamma_D \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau) \ \text{ on } \ \Gamma_N \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded set with a boundary Γ, which is divided into two parts Γ_D and Γ_N. The exponential stability of (1.1) was obtained by introducing the suitable energies and using some observability inequalities when $\mu_2 < \mu_1$. For the case that one of the assumptions has not been satisfied, some instability examples are also given in [8].

One has that if the boundary condition on Γ_N in (1.1) is replaced by

$$\frac{\partial u}{\partial \nu} = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau(t)) \ \text{ on } \ \Gamma_N \times \mathbb{R}^+, \tag{1.2}$$

where $\tau(t)$ is the time-varying delay. The balance between the delay term and the damping term is quite different from the constant delay for the time-varying delay in hyperbolic equations [9–11].

Under some suitable assumptions, the exponential stability result of (1.1) with the boundary time-varying delay (1.2) was obtained in [12] for $\mu_2 < \sqrt{1-d}\mu_1$ and $\tau'(t) \leq d$. The exponential stability of (1.1) was also extended to a nonlinear version in [12] and distributed version in [2].

For the heat equation with a time-varying delay, the authors of [13] investigated the method of the construction of Lyapunov functionals in order to get the stability of the following abstract equation with an internal time-varying delay

$$\frac{du(t)}{dt} = A(t, u(t)) + F(u(t - h(t))),$$

where $A(t, \cdot)$, $F(\cdot)$ are appropriate partial differential operators and $h(t)$ is the time-varying delay. The exponential stability result was proved. If we consider the heat equation with constant delay or distributed delay, the two-dimensional parabolic functional-differential equation with both fixed and distributed delays is as can be found in [14]

$$\frac{\partial u}{\partial t} = a(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + bu(t-\tau, x, y) + c\int_{-\tau}^{0} u(t+s, x, y)\, ds. \tag{1.3}$$

The asymptotic stability of the zero solution for (1.3) was proved and the asymptotic stability result for the versions with a missing constant or distributed delays are also available in [14]. Moreover, the heat equation with a non-constant delay and nonlinear weights can be seen in [15].

If we put the time-varying delay on the boundary for the heat equation, the interested system can be described by

$$\begin{cases} u_t(x,t) - au_{xx}(x,t) = 0 \ \text{ in } \ (0,\pi) \times \mathbb{R}^+, \\ u(0,t) = 0 \ \text{ on } \ \mathbb{R}^+, \\ u_x(\pi,t) = -\mu_0 u(\pi,t) - \mu_1 u(\pi, t-\tau(t)) \ \text{ on } \ \mathbb{R}^+, \end{cases} \tag{1.4}$$

where $a > 0$, μ_0 and μ_1 are fixed nonnegative constants. The energy of (1.4) decays exponentially when $\tau'(t) \leq d$ and $\mu_1^2 < (1-d)\mu_0^2$ (see [16]).

We are interested in the effect of a time-varying delay in the boundary stabilization of a thermoelastic system in one dimension. The uniform stability of a thermoelastic system equipped with the boundary time-varying delay on the heat equation has been proved in [17]. Let us put the boundary time-varying delay on the wave equation, that is to say, let us equip the temperature field with a Dirichlet boundary and the velocity field with a dynamic boundary with a time-varying delay. Our interested system can be specifically expressed by

$$\begin{cases} au_{tt}(x,t) - du_{xx}(x,t) + \gamma\theta_x(x,t) = 0 \ \text{ in } \ (0,L) \times \mathbb{R}^+, \\ b\theta_t(x,t) - k\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0 \ \text{ in } \ (0,L) \times \mathbb{R}^+, \\ u_x(0,t) = u(0,t) = 0, \ \ \theta(0,t) = \theta(L,t) = 0 \ \text{ in } \ \mathbb{R}^+, \\ u_x(L,t) = -\mu_1 u_t(L,t) - \mu_2 u_t(L, t-\rho(t)) \ \text{ in } \ \mathbb{R}^+, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \ \theta(x,0) = \theta_0(x) \ \text{ in } \ (0,L), \\ u_t(L, t-\rho(0)) = \sigma_0(L, t-\rho(0)) \ \text{ in } \ (0,\rho(0)), \end{cases} \tag{1.5}$$

where u_0, u_1, θ_0 and σ_0 represent the general initial conditions, μ_1 and μ_2 are positive constants. a is the density of the material medium, b is the heat capacity per unit volume at constant strain, k is the thermal conductivity, γ is the product of the volume coefficient of thermal expansion and the bulk modulus, d is the sum of Lamé modulus. In general, we assume that a, b, k, $|\gamma|$ and d are all positive constants, which is reasonable (see [18] for the coupled partial differential equations of standard thermoacoustic wave propagation.) The function $\rho(t) : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the nonlinear time-varying delay, u and θ are the displacement and the temperature difference relative to the reference value respectively. We study the asymptotic behavior of the solution of (1.5) and seek sufficient conditions on μ_2 which guarantee the exponential stability of (1.5).

From the physical point, we can see that the system (1.5) is fixed at one end ($x = 0$) and controlled by a boundary controller at the other end ($x = L$). We consider the stability of the closed-loop system with a combination of a no delay term and a variable delay in the feedback loop (see [19, 20]).

Inspired by the idea given in [12, 21–24], we introduce a new unknown variable, as follows, to represent the time-varying delay term:

$$z(L, \delta, t) = u_t(L, t - \delta\rho(t)) \ \text{ in } \ (0, 1) \times \mathbb{R}^+,$$

which results in

$$\begin{cases} \rho(t) z_t(L, \delta, t) + (1 - \delta\rho'(t)) z_\delta(L, \delta, t) = 0 \ \text{ in } \ (0, 1) \times \mathbb{R}^+, \\ z(L, 1, t) = u_t(L, t - \rho(t)) \ \text{ in } \ \mathbb{R}^+, \\ z(L, 0, t) = u_t(L, t) \ \text{ in } \ \mathbb{R}^+, \\ z(L, \delta, 0) = \sigma_0(L, -\delta\rho(0)) \ \text{ in } \ (0, 1). \end{cases} \tag{1.6}$$

It follows from (1.6) that System (1.5) can be written in the following equivalent form

$$\begin{cases} a u_{tt}(x, t) - d u_{xx}(x, t) + \gamma\theta_x(x, t) = 0 \ \text{ in } \ (0, L) \times \mathbb{R}^+, \\ b\theta_t(x, t) - k\theta_{xx}(x, t) + \gamma u_{xt}(x, t) = 0 \ \text{ in } \ (0, L) \times \mathbb{R}^+, \\ \rho(t) z_t(L, \delta, t) + (1 - \delta\rho'(t)) z_\delta(L, \delta, t) = 0 \ \text{ in } \ (0, 1) \times \mathbb{R}^+, \\ u_x(0, t) = u(0, t) = 0, \ \ \theta(0, t) = \theta(L, t) = 0 \ \text{ in } \ \mathbb{R}^+, \\ u_x(L, t) = -\mu_1 u_t(L, t) - \mu_2 z(L, 1, t) \ \text{ in } \ \mathbb{R}^+, \\ u(x, 0) = u_0(x), \ \ u_t(x, 0) = u_1(x), \ \ \theta(x, 0) = \theta_0(x) \ \text{ in } \ (0, L), \\ z(L, 0, t) = u_t(L, t) \ \text{ in } \ \mathbb{R}^+, \\ z(L, \delta, 0) = \sigma_0(L, -\delta\rho(0)) \ \text{ in } \ (0, 1). \end{cases} \tag{1.7}$$

In order to study the global well-posedness and stability of (1.7), we transform (1.7) into the abstract form given by (2.3), where the abstract non-autonomous operator $\mathcal{L}(t)$ can be seen in (2.4). The main results and features of our work are summarized as follows:

(i) For the non-autonomous operator $\mathcal{L}$, we choose Kato's variable norm technique to replace the commonly used Lumer-Phillips theorem to guarantee the closure of $\mathcal{L}$.

(ii) Suppose that $\rho(t)$ and $\rho'(t)$ are positive and limited; we can get the balance between the damping term and the time-varying delay term. We choose ξ with $\frac{\mu_2}{\sqrt{1-\tilde{\rho}_1}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-\tilde{\rho}_1}}$ as a weight for the energy space of the function z, where $\mu_2 < \sqrt{1 - \tilde{\rho}_1}\mu_1$ and $\rho'(t) \le \tilde{\rho}_1$. The balance is similar to that presented in [9–11, 25].

(iii) The exponential stability of (1.7) can be achieved by constructing a suitable Lyapunov functional, which needs some delicate estimates for the total energy and the time-varying delay term.

The rest of this paper is organized as follows. In Section 2, some preparations are presented, including some assumptions and the abstract form of (1.7). The global well-posedness of (1.7) is given in Section 3 and the exponential stability is done in the last section.

2. Preliminaries

We assume that the nonlinear time-varying delay $\rho \in W^{2,\infty}([0,T])$ for all $T > 0$; there exist four constants $\rho_0, \rho_1, \tilde{\rho_0}$ and $\tilde{\rho_1}$, for all $t > 0$,

$$0 < \rho_0 < \rho(t) \leq \rho_1, \;\; 0 < \tilde{\rho_0} < \rho'(t) \leq \tilde{\rho}_1 < 1. \tag{2.1}$$

Moreover, assume that

$$\mu_2 < \sqrt{1-\tilde{\rho}_1}\mu_1. \tag{2.2}$$

For simplicity, the norms of the Lebesgue spaces $L^p(\Omega)$ are noted as $\|\cdot\|_p$, $1 \leq p \leq \infty$. We abbreviate $z(L,\delta,t)$ as $z(L)$, $z(L,0,t)$ as $z(L,0)$ and $z(L,1,t)$ as $z(L,1)$ below.

If we denote

$$U = (u, v, \theta, z(L))^\top,$$

Problem (1.7) can be turned into the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}U = \mathcal{L}(t)U, \\ U(0) = (u_0, u_1, \theta_0, \sigma_0)^\top, \end{cases} \tag{2.3}$$

where $(\cdot)^\top$ means the transpose of the vector $(\cdot)$ and

$$\mathcal{L}(t)U = \begin{pmatrix} v \\ \frac{d}{a}u_{xx} - \frac{\gamma}{a}\theta_x \\ \frac{k}{b}\theta_{xx} - \frac{\gamma}{b}u_{xt} \\ -\frac{1-\delta\rho'}{\rho}z_\delta(L) \end{pmatrix}. \tag{2.4}$$

We consider the following Hilbert space

$$V = \{u \in H^1(0,L), u(0) = 0\}.$$

Assume that a constant $\xi > 0$ satisfies

$$\frac{\mu_2}{\sqrt{1-\tilde{\rho_1}}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-\tilde{\rho_1}}}. \tag{2.5}$$

Note that, such a constant ξ exists from (2.2). Then for all $t \in \mathbb{R}^+$, $z(L,\cdot,t)$, $z_1(L,\cdot,t)$ and $z_2(L,\cdot,t) \in L^2(0,1)$, the inner product and the corresponding norm can be defined as

$$(z_1, z_2)_\xi = \xi\rho(t)\int_0^1 z_1(L,\delta,t)z_2(L,\delta,t)\,d\delta$$

and

$$\|z\|_\xi^2 = \xi\rho(t)\int_0^1 |z(L,\delta,t)|^2\,d\delta$$

respectively.

The phase space of (2.3) can be defined as follows

$$\mathcal{H} = V \times L^2(0, L) \times L^2(0, L) \times L^2(0, 1)$$

endowed with the inner product

$$\left(U, \tilde{U}\right)_{\mathcal{H}} = \int_0^L (du_x\tilde{u}_x + av\tilde{v} + b\theta\tilde{\theta})\,dx + d\xi\rho(t)\int_0^1 z(L)\tilde{z}(L)\,d\delta$$

for $U = (u, v, \theta, z(L))^\top$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{z}(L))^\top \in \mathcal{H}$. The strong phase space $D(\mathcal{L}(t))$ can be defined as

$$\begin{aligned}\mathcal{D}(\mathcal{L}(t)) = \{&(u, v, \theta, z(L))^\top \in (E(\Delta, L^2(0, L)) \cap V) \times V \times (E(\Delta, L^2(0, L)) \cap H_0^1(0, L)) \\ &\times H^1(0, 1), u_t(L) = z(L, 0), u_x(L) = -\mu_1 v(L) - \mu_2 z(L, 1), u_x(0) = 0\},\end{aligned}$$

where

$$E(\Delta, L^2(0, L)) = \{u \in H^1(0, L), u_{xx} \in L^2(0, L)\}.$$

Note that the domain of the operator $\mathcal{L}(t)$ is independent of the time t, that is,

$$\mathcal{D}(\mathcal{L}(t)) = \mathcal{D}(\mathcal{L}(0)) \text{ for all } t > 0. \tag{2.6}$$

It is worth noting that $\mathcal{L}$ is a non-autonomous operator because the fourth term in $\mathcal{L}$ is dependent on t. Moreover, we always assume that (2.1), (2.2) and (2.5) hold without any additional clarification.

3. Global well-posedness

In this section, the well-posedness of Cauchy problem (2.3) is established by using the variable norm technique of Kato for the non-autonomous operators and estimating the total energy.

Theorem 3.1. *Assume the initial data* $U_0 = (u_0, u_1, \theta_0, \sigma_0)^\top \in \mathcal{D}(\mathcal{L}(0))$ *and fix* ξ *such that (2.5) holds. Then, Problem (2.3) possesses a unique solution*

$$U \in C([0, \infty), \mathcal{D}(\mathcal{L}(0))) \cap C^1([0, \infty), \mathcal{H}).$$

In order to show the global well-posedness of Problem (2.3), we first introduce the following proposition ([26]) which is a strong version of Theorem 1.9 by Kato in [27].

Proposition 3.1. *Let* $\mathcal{L}(t)$ *be an operator with the dense domain* $\mathcal{D}(\mathcal{L}(t))$ *in* $\mathcal{H}$ *and the domain be independent of t. Assume the following:*

1) For $T > 0$, $\mathcal{L} = \{\mathcal{L}(t)|t \in [0, T]\}$ *is a stable family of generators in* $\mathcal{H}$*, and the stability constants are independent of t.*

2) $\mathcal{L}_t$ *belongs to* $L_*^\infty([0, T], B(\mathcal{D}(\mathcal{L}(0), \mathcal{H}))$*. (*$B(X, Y)$ *is the set of all bounded linear operators on X to Y;* $L_*^\infty(I, Z)$ *denotes the equivalence classes of strongly measurable, essentially bounded functions on I to Z.)*

Then for all initial data in $\mathcal{D}(\mathcal{L}(0)$*, there exists a unique solution*

$$U \in C([0, T], \mathcal{D}(\mathcal{L}(0)) \cap C^1([0, T], \mathcal{H})$$

for Problem (2.3).

In this proposition, the triplet $\{\mathcal{L}, \mathcal{H}, \mathcal{D}(\mathcal{L}(0))\}$, with $\mathcal{L} = \{\mathcal{L}(t) \mid t \in [0, T]\}$, for some fixed $T > 0$, forms a constant domain system (see [27]). This proposition guarantees the existence and uniqueness of (2.3); the same application of Proposition 3.1 can be found in [12, 22]. In order to apply the above proposition, we next prove the density of $D(\mathcal{L}(0))$ in $\mathcal{H}$ in the following lemma.

Lemma 3.2. *$D(\mathcal{L}(0))$ is dense in $\mathcal{H}$.*

Proof. Establish that $\hat{U} = (\hat{u}, \hat{v}, \hat{\theta}, \hat{z}(L))^\top \in \mathcal{H}$ is orthogonal to all elements of $D(\mathcal{L}(0))$, that is, for all $U = (u, v, \theta, z(L))^\top \in D(\mathcal{L}(0))$, it holds that

$$(\hat{U}, U)_{\mathcal{H}} = \int_0^L (du_x\hat{u}_x + av\hat{v} + b\theta\hat{\theta})\, dx + d\xi\rho(t) \int_0^1 z(L)\hat{z}(L)\, d\delta = 0. \tag{3.1}$$

Let $u = v = \theta = 0$ and $z(L) \in C_0^\infty(0, 1)$; the following result holds by (3.1) for $U = (0, 0, 0, z(L))^\top \in D(\mathcal{L}(0))$:

$$\rho(t) \int_0^1 \hat{z}(L)z(L)\, d\delta = 0.$$

Due to the density of $C_0^\infty(0, 1)$ in $L^2(0, 1)$ and the fact that $\rho(t) > 0$, it follows that $\hat{z}(L) = 0$ for all $(\delta, t) \in (0, 1) \times \mathbb{R}^+$. In the same way, let $u = v = z = 0$; (3.1) implies that

$$\int_0^L \theta\hat{\theta}\, dx = 0,$$

where $\theta \in C_0^\infty(0, L)$. Since $C_0^\infty(0, L)$ is dense in $L^2(0, L)$, $\hat{\theta} = 0$ can be obtained. Similarly $\hat{v} = 0$ can be obtained. Finally, set $v = \theta = z(L) = 0$; there is

$$\int_0^L u_x\hat{u}_x\, dx = 0.$$

We find that $(u, 0, 0, 0)^\top \in D(\mathcal{L}(0))$ if and only if $u \in \mathcal{D}(\Delta) = \{E(\Delta, L^2(0, L)) \cap V, u_x(L) = 0\}$, where $\mathcal{D}(\Delta)$ is the domain of the Laplace operator with the mixed boundary conditions. Since $\mathcal{D}(\Delta)$ is dense in V, $\hat{u} = 0$ can be obtained. Thus, $D(\mathcal{L}(0))$ is dense in $\mathcal{H}$.

Based on the result described by (2.6) and Lemma 3.2, let us check the assumptions in Proposition 3.1 one by one. We first give a lemma which ensures that the operator $\widetilde{\mathcal{L}}$ generates a C_0-semigroup in $\mathcal{H}$.

Lemma 3.3. *Let $\widetilde{\mathcal{L}}(t) = \mathcal{L}(t) - \kappa(t)I$ ($\kappa(t)$ will be defined below). Then the operator $\widetilde{\mathcal{L}}(t)$ generates a C_0-semigroup in $\mathcal{H}$ for a fixed t.*

Proof. For $U = (u, v, \theta, z(L))^\top \in \mathcal{D}(\mathcal{L}(0))$, we calculate directly $(\mathcal{L}(t)U, U)_{\mathcal{H}}$ for a fixed t as follows

$$\begin{aligned}(\mathcal{L}(t)U, U)_{\mathcal{H}} = &\int_0^L [d(u_{xt}u_x + u_{xx}u_t) - \gamma(\theta_x u_t + u_{xt}\theta) + k\theta_{xx}\theta]\, dx \\ &- d\xi \int_0^1 (1 - \delta\rho'(t))z(L)z_\delta(L)\, d\delta.\end{aligned}$$

By Green's formula, for the boundary conditions of u and θ, the following holds:

$$\int_0^L u_{xt}u_x\,dx = -\int_0^L u_{xx}u_t\,dx - \mu_1 u_t^2(L) - \mu_2 u_t(L)z(L,1)$$

and

$$\int_0^L \theta_{xx}\theta\,dx = -\int_0^L \theta_x^2\,dx.$$

Moreover, we find that

$$\int_0^L (\theta_x u_t + u_{xt}\theta)\,dx = 0$$

and

$$\int_0^1 (1-\delta\rho'(t))z(L)z_\delta(L)\,d\delta = \frac{1}{2}(1-\rho'(t))z^2(L,1) - \frac{1}{2}u_t^2(L) + \frac{1}{2}\rho'(t)\int_0^1 z^2(L)\,d\delta. \tag{3.2}$$

From the above estimates, we have

$$\begin{aligned}(\mathcal{L}U,U)_{\mathcal{H}} =& -d(\mu_1-\frac{\xi}{2})u_t^2(L) - d\mu_2 u_t(L)z(L,1) - k\int_0^L \theta_x^2\,dx\\ &-\frac{d\xi}{2}(1-\rho'(t))z^2(L,1) - \frac{d\xi}{2}\rho'(t)\int_0^1 z^2(L)\,d\delta.\end{aligned}$$

Using Young's inequality, we get

$$\Big| -u_t(L)z(L,1)\Big| \le \frac{1}{2\sqrt{1-\tilde{\rho_1}}}u_t^2(L) + \frac{\sqrt{1-\tilde{\rho_1}}}{2}z^2(L,1).$$

Let $\kappa(t) = \frac{\sqrt{1+(\rho'(t))^2}}{2\rho(t)}$; there is

$$\Big| -\frac{d\xi}{2}\rho'(t)\int_0^1 z^2(L)\,d\delta\Big| \le \kappa(t)d\xi\rho(t)\int_0^1 z^2(L)\,d\delta \le \kappa(t)(U,U)_{\mathcal{H}}.$$

Therefore, it follows that

$$\begin{aligned}(\mathcal{L}(t)U,U)_{\mathcal{H}} \le& -d\Big(\mu_1-\frac{\xi}{2}-\frac{\mu_2}{2\sqrt{1-\tilde{\rho_1}}}\Big)u_t^2(L) - d\Big(\frac{\xi}{2}(1-\rho'(t)) - \frac{\mu_2}{2}\sqrt{1-\tilde{\rho_1}}\Big)z^2(L,1)\\ &+\kappa(t)(U,U)_{\mathcal{H}} - k\int_0^L \theta_x^2\,dx.\end{aligned}$$

Due to the range of ξ in (2.5), one can obtain

$$\Big((\mathcal{L}(t)-\kappa(t)I)U,U\Big)_{\mathcal{H}} \le 0.$$

Now we define a new operator

$$\widetilde{\mathcal{L}}(t) = \mathcal{L}(t) - \kappa(t)I;$$

then, $\widetilde{\mathcal{L}}(t)$ is dissipative.

Now, we shall show that $I-\mathcal{L}(t)$ is surjective for a fixed $t>0$. For all $F=(f_1,f_2,f_3,f_4(L))^\top\in\mathcal{H}$, we seek a solution $U=(u,v,\theta,z(L))^\top\in D(\mathcal{L}(t))$ such that

$$(I-\mathcal{L}(t))U=F.$$

Equivalently, we want to find a U that satisfies the following equations

$$\begin{cases} u-v=f_1, \\ av-du_{xx}+\gamma\theta_x=af_2, \\ b\theta-k\theta_{xx}+\gamma u_{xt}=bf_3, \\ z(L)+\frac{1-\delta\rho'(t)}{\rho(t)}z_\delta(L)=f_4(L). \end{cases} \tag{3.3}$$

Equation (3.3) implies $v=u-f_1$. The solution of the fourth equation in (3.3) with the initial data $z(L,0)=v(L)$ can be found as

$$z(L,\delta)=v(L)e^{\sigma(\delta,t)}+\rho(t)e^{\sigma(\delta,t)}\int_0^\sigma \frac{f_4(L)}{1-s\rho'(t)}e^{-\sigma(\delta,t)}\,ds \quad \text{for a fixed} \quad t,$$

where $\sigma(\delta,t)=\frac{\rho(t)}{\rho'(t)}\ln(1-\delta\rho'(t))$. Then we get

$$z(L,\delta)=u(L)e^{\sigma(\delta,t)}-f_1(L)e^{\sigma(\delta,t)}+\rho(t)e^{\sigma(\delta,t)}\int_0^\delta \frac{f_4(L)}{1-s\rho'(t)}e^{-\sigma(\delta,t)}\,ds. \tag{3.4}$$

In particular, we obtain that

$$z(L,1)=u(L)e^{\sigma(1,t)}+z_0(L),$$

where $z_0(L)=-f_1(L)e^{\sigma(1,t)}+\rho(t)e^{\sigma(1,t)}\int_0^1 \frac{f_4(L)}{1-s\rho'(t)}e^{-\sigma(\delta,t)}\,ds$. Thus, due to $v=u-f_1$, the Eqs $(3.3)_2$ and $(3.3)_3$ become

$$\begin{cases} au-du_{xx}+\gamma\theta_x=\ell_1, \\ b\theta-k\theta_{xx}+\gamma u_x=\ell_2, \end{cases} \tag{3.5}$$

where $\ell_1=a(f_1+f_2)$ and $\ell_2=bf_3+\gamma f_{1x}\in L^2(0,L)$. The bilinear and linear forms of the variational equations corresponding to (3.5) can be defined as

$$\mathcal{B}((u,\theta),(\phi,w))=\mathcal{G}((\phi,w)), \tag{3.6}$$

where the bilinear form $\mathcal{B}:(V\times H_0^1(0,L))^2\mapsto\mathbb{R}$ is given by

$$\begin{aligned}\mathcal{B}((u,\theta),(\phi,w))=&\int_0^L(du_x\phi_x+au\phi+\gamma\theta_x\phi+b\theta w+\gamma u_x w+k\theta_x w_x)\,dx\\ &+d(\mu_1+\mu_2e^{\sigma(1,t)})u(L)\phi(L),\end{aligned}$$

and the linear form $\mathcal{G}:V\times H_0^1(0,L)\mapsto\mathbb{R}$ is defined as

$$\mathcal{G}((\phi,w))=\int_0^L(\ell_1\phi+\ell_2 w)\,dx+d(\mu_1 f_1(L)-\mu_2 z_0(L))\phi(L).$$

For $(u,\theta) \in V \times H_0^1(0,L)$, the norm of $V \times H_0^1(0,L)$ is

$$\|(u,\theta)\|^2_{V\times H_0^1(0,L)} = \|u\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2.$$

Similar forms to (3.6) can be seen in (19) and (41) in [16]. Similar to [16], it is easy to show that $\mathcal{B}$ is bounded and coercive in $V \times H_0^1(0,L)$. Therefore, by the Lax-Milgram theorem, there is a unique solution $(u,\theta) \in V \times H_0^1(0,L)$ for (3.5). Then, we can obtain

$$v = u - f_1 \in V.$$

Given the boundedness of $\rho(t)$, $v(L)$ and $f_4(L)$, we find that $z(L,\delta) \in L^2(0,1)$ by (3.4). From the fourth equation in (3.3), we have $z_\delta(L,\delta) \in L^2(0,1)$ and $z(L,\delta) \in H^1(0,1)$.

Moreover, for all $(\phi, w) \in C_0^\infty(0,L) \times C_0^\infty(0,L) \subset V \times H_0^1(0,L)$, there is

$$\mathcal{B}((u,\theta),(\phi,w)) = \mathcal{G}((\phi,w)). \tag{3.7}$$

Then, θ_{xx} and $u_{xx} \in L^2(0,L)$; we have $u \in E(\Delta, L^2(0,L)) \cap V$ and $\theta \in E(\Delta, L^2(0,L)) \cap H_0^1(0,L)$.

Next, we verify the boundary conditions of u. From (3.7), we have

$$\int_0^L (du_x, k\theta_x)\cdot(\phi_x, w_x)\,dx = \int_0^L \Big((\ell_1 - au - \gamma\theta_x), (\ell_2 - b\theta - \gamma u_x)\Big)\cdot(\phi, w)\,dx. \tag{3.8}$$

Since $((\ell_1 - au - \gamma\theta_x), (\ell_2 - b\theta - \gamma u_x)) \in L^2(0,L) \times L^2(0,L)$, u_x and θ_x have weak derivations in $L^2(0,L)$ from (3.8). Then we have

$$d\int_0^L u_x\phi_x\,dx = \int_0^L (-au\phi + \gamma\theta_x\phi + \ell_1\phi)\,dx + du_x(L)\phi(L)$$

and

$$k\int_0^L \theta_x w_x\,dx = \int_0^L (-b\theta w - \gamma u_x w + \ell_2 w)\,dx.$$

Bringing the above two equations into (3.6), we find $u_x(L) = -\mu_1 v(L) - \mu_2 z(L,1)$.

Therefore, we find a solution $U = (u, v, \theta, z(L))^\top \in \mathcal{D}(\mathcal{L}(t))$ for (3.3). Again, given $\kappa(t) > 0$, $\kappa(t)I - \mathcal{L}(t)$ is surjective because of the boundedness of $\kappa(t)$, so $\widetilde{\mathcal{L}}(t)$ is maximal for a fixed $t > 0$. The proof of Lemma 3.3 is completed.

The stability of the operator $\widetilde{\mathcal{L}}$ in $\mathcal{H}$ can be obtained in the following lemma; a similar process can be seen in [12, 16, 22].

Lemma 3.4. $\widetilde{\mathcal{L}} = \{\widetilde{\mathcal{L}}(t),\ t \in [0,T]\}$ *is a stable family in* $\mathcal{H}$ *for all* $T > 0$.

Proof. For $U = (u, v, \theta, z(L))^\top \in \mathcal{H}$, let $\|U\|_t = \int_0^L (du_x^2 + av^2 + b\theta^2)\,dx + d\xi\rho(t)\int_0^1 z^2(L)\,d\delta$ for $0 \le s \le t \le T$. We claim that

$$\frac{\|U\|_t}{\|U\|_s} \le e^{\frac{\widetilde{\rho}_1}{2\rho_0}|t-s|} \quad \text{for all} \quad t,\ s \in [0,T]. \tag{3.9}$$

A direct computation deduces that

$$\begin{aligned}&\|U\|_t^2-\|U\|_s^2e^{\frac{\widetilde{\rho}_1}{\rho_0}(t-s)}\\ =&(1-e^{\frac{\widetilde{\rho}_1}{\rho_0}(t-s)})\int_0^L(du_x^2+av^2+b\theta^2)\,dx+d\xi(\rho(t)-\rho(s)e^{\frac{\widetilde{\rho}_1}{\rho_0}(t-s)})\int_0^1z^2(L)\,d\delta.\end{aligned}$$

Since $\rho\in W^{2,\infty}([0,T])\hookrightarrow C^1([0,T])$, there exists a constant $a\in(s,t)$, such that

$$\rho(t)=\rho(s)+\rho'(a)(t-s).$$

Then it follows from (2.1) that

$$\frac{\rho(t)}{\rho(s)}\le 1+\frac{\widetilde{\rho}_1}{\rho_0}(t-s)\le e^{\frac{\widetilde{\rho}_1}{\rho_0}(t-s)},$$

which results in $\rho(t)-\rho(s)e^{\frac{\widetilde{\rho}_1}{\rho_0}|t-s|}\le 0$. Combining $1-e^{\frac{\widetilde{\rho}_1}{\rho_0}|t-s|}\le 0$, we find that (3.9) holds.

Lemma 3.4 and (3.9) imply that the family $\mathcal{L}=\{\mathcal{L}(t),t\in[0,T]\}$ is a stable family of generators in $\mathcal{H}$ by Proposition 1.1 in [27]. Due to the boundedness of κ, we find that $\widetilde{\mathcal{L}}(t)=\mathcal{L}(t)-\kappa(t)I$ is stable in $\mathcal{H}$ from Proposition 1.2 in [27].

By applying Proposition 3.1, a property of $\widetilde{\mathcal{L}}$ can be verified in the following lemma, which is similar with Theorem 2.3 in [12].

Lemma 3.5. $\widetilde{\mathcal{L}}_t\in L_*^\infty([0,T],B(D(\mathcal{L}(0)),\mathcal{H}))$.

Proof. It follows from $\kappa(t)=\frac{\sqrt{1+(\rho'(t))^2}}{2\rho(t)}$ that

$$\kappa'(t)=\frac{\rho'(t)\rho''(t)}{2\rho(t)\sqrt{1+(\rho'(t))^2}}+\frac{\rho'(t)\sqrt{1+(\rho'(t))^2}}{2\rho^2(t)}.$$

The boundedness of $\kappa'(t)$ can be obtained from (2.1). By (2.4), we get

$$\mathcal{L}(t)=\begin{pmatrix}0&1&0&0\\ \frac{d}{a}\frac{\partial}{\partial xx}&0&-\frac{\gamma}{a}\frac{\partial}{\partial x}&0\\ 0&-\frac{\gamma}{b}\frac{\partial}{\partial x}&\frac{k}{b}\frac{\partial}{\partial xx}&0\\ 0&0&0&-\frac{1-\delta\rho'(t)}{\rho(t)}\frac{\partial}{\partial\delta}\big|_{(x=L)}\end{pmatrix}.$$

A direct computation and (2.1) yield that

$$\mathcal{L}_t(t)U=\begin{pmatrix}0\\0\\0\\ \frac{\delta\left(\rho(t)\rho''(t)-(\rho'(t))^2\right)+\rho'(t)}{\rho^2(t)}z_\delta(L)\end{pmatrix}$$

is also bounded for $t\in\mathbb{R}^+$. For $U\in\mathcal{D}(\mathcal{L}(0))$, $\widetilde{\mathcal{L}}_t(t)U=(\mathcal{L}_t(t)-\kappa'(t)I)U$ is bounded from the definition of the norm of the operator. Thus, we have proved this lemma.

We define the energy of Problem (2.3) as follows

$$E(t)=\frac{1}{2}\int_0^L(du_x^2+au_t^2+b\theta^2)\,dx+\frac{d\xi}{2}\rho(t)\int_0^1z^2(L)\,d\delta. \tag{3.10}$$

Some properties of the energy $E(t)$ can be illustrated in the following lemma.

Lemma 3.6. *The energy $E(t)$ is non-increasing. Moreover, there exists a constant $c_0 > 0$ such that*

$$E'(t) \leq -c_0\Big(u_t^2(L) + z^2(L,1) + \int_0^L \theta_x^2\, dx\Big), \quad \textit{for all} \quad t \geq 0. \tag{3.11}$$

Proof. Multiplying the first equation in (1.7) by u_t and the second one by θ, then adding the two results together, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}\int_0^L (du_x^2 + au_t^2 + b\theta^2)\, dx\Big] = -d\mu_1 u_t^2(L) - d\mu_2 u_t(L)z(L,1) - k\int_0^L \theta_x^2\, dx. \tag{3.12}$$

By the formula of derivation calculus, we have

$$\frac{d}{dt}\Big[\rho(t)\int_0^1 z^2(L)\, d\delta\Big] = \rho'(t)\int_0^1 z^2(L)\, d\delta + 2\rho(t)\int_0^1 z(L)z_t(L)\, d\delta. \tag{3.13}$$

It follows from $z_t = -\frac{1-\delta\rho(t)}{\rho(t)} z_\delta$ that

$$\int_0^1 z(L)z_t(L)\, d\delta = -\frac{\rho^{-1}(t)}{2}(1-\rho'(t))z^2(L,1) + \frac{\rho^{-1}(t)}{2}u_t^2(L) - \frac{\rho^{-1}(t)}{2}\rho'(t)\int_0^L z^2(L)\, d\delta. \tag{3.14}$$

Putting (3.14) into (3.13) and combing (3.10), (3.12) and (3.13), we get

$$E'(t) = -d\mu_1 u_t^2(L) - d\mu_2 u_t(L)z(L,1) - k\int_0^L \theta_x^2\, dx + \frac{d\xi}{2}u_t^2(L) - \frac{d\xi}{2}(1-\rho'(t))z^2(L,1).$$

Using Young's inequality, we find that

$$E'(t) \leq -d\Big(\mu_1 - \frac{\xi}{2} - \frac{\mu_2}{2\sqrt{1-\tilde{\rho_1}}}\Big)u_t^2(L) - d\Big(\frac{\xi}{2}(1-\rho'(t)) - \frac{\mu_2}{2}\sqrt{1-\tilde{\rho_1}}\Big)z^2(L,1) - k\int_0^L \theta_x^2\, dx.$$

Due to (2.1) and (2.5), there exists a positive constant c_0 which satisfies

$$c_0 = min\{d\Big(\mu_1 - \frac{\xi}{2} - \frac{\mu_2}{2\sqrt{1-\tilde{\rho_1}}}\Big), d\Big(\frac{\xi}{2}(1-\tilde{\rho_1}) - \frac{\mu_2}{2}\sqrt{1-\tilde{\rho_1}}\Big), k\}.$$

Then (3.11) holds.

Proof of Theorem 3.1. From Lemmas 3.2–3.5, (2.6) and Proposition 3.1, the problem

$$\begin{cases} \frac{d\widetilde{U}}{dt} = \widetilde{\mathcal{L}}(t)\widetilde{U} \\ \widetilde{U}(0) = U_0 \end{cases}$$

has a unique solution:

$$\widetilde{U} = e^{\int_0^t \widetilde{\mathcal{L}}(s)\, ds}U_0 \in C([0,T], D(\mathcal{L}(0))) \cap C^1([0,T], \mathcal{H}).$$

Then the solution of (2.3) can be given as

$$U(t) = e^{\int_0^t \kappa(s)\, ds}\widetilde{U}(t).$$

In fact, by a simple computation, we have

$$\begin{aligned}\frac{dU}{dt}(t) =&\kappa(t)e^{\int_0^t \kappa(s)\,ds}\widetilde{U}(t) + e^{\int_0^t \kappa(s)\,ds}\widetilde{U}_t(t)\\ =&e^{\int_0^t \kappa(s)\,ds}(\kappa(t)I + \widetilde{\mathcal{L}}(t))\widetilde{U}(t)\\ =&\mathcal{L}(t)e^{\int_0^t \kappa(s)\,ds}\widetilde{U}(t)\\ =&\mathcal{L}(t)U(t).\end{aligned}$$

Lemma 3.6 guarantees that the local solutions can not blow-up in finite time; therefore we have $t_{max} = \infty$. Then the proof of Theorem 3.1 is completed.

4. Exponential stability

In this section, the exponential stability of Problem (2.3) can be obtained by the energy method.

Theorem 4.1. *For all solutions of Problem (2.3), there exist two positive constants C_1 and C_2 such that*

$$E(t) \leq C_1 E(0)e^{-C_2 t} \text{ for all } t \geq 0.$$

The proof of Theorem 4.1 depends on the following three lemmas. First, a functional can be defined as

$$\varphi(t) = 2a\int_0^L mu_x(t)u_t(t)\,dx, \tag{4.1}$$

where we assume that there exists an $x_0 \in \mathbb{R}$ such that $m(x)$ is a standard multiplier:

$$m(x) = x - x_0.$$

The applications of $m(x)$ can be found in [12]. Moreover, it holds that $m(0) \leq 0$ and $m(L) \geq \alpha_0 > 0$. An estimate of φ can be obtained in the following lemma.

Lemma 4.1. *For all $\varepsilon > 0$, there exists a consant $C_\varepsilon > 0$ such that, for all $t > 0$,*

$$\varphi'(t) \leq m(L)(a + 2d\mu_1^2)u_t^2(L) + 2\mu_2^2 z^2(L,1) - a\int_0^L u_t^2\,dx - (d-\varepsilon)\int_0^L u_x^2\,dx + C_\varepsilon\int_0^L \theta_x^2\,dx. \tag{4.2}$$

Proof. We can write

$$\varphi'(t) = 2a\int_0^L mu_{xt}u_t\,dx + 2a\int_0^L mu_x u_{tt}\,dx = I_1 + I_2.$$

Using the boundary conditions of u and the first equation in (1.7), we have

$$I_1 = am(L)u_t^2(L) - a\int_0^L u_t^2\,dx \tag{4.3}$$

and

$$I_2 = 2d\int_0^L mu_x u_{xx}\,dx - 2\gamma\int_0^L mu_x\theta_x\,dx.$$

For all $\varepsilon > 0$, by Young's inequality, we obtain

$$2d\int_0^L mu_xu_{xx}\,dx = dm(L)u_x^2(L) - d\int_0^L u_x^2\,dx$$
$$\leq 2dm(L)\mu_1^2u_t^2(L) + 2dm(L)\mu_2^2z^2(L,1) - d\int_0^L u_x^2\,dx$$

and

$$\Big| -2\gamma\int_0^L mu_x\theta_x\,dx\Big| \leq \varepsilon\int_0^L u_x^2\,dx + C_\varepsilon\int_0^L \theta_x^2\,dx.$$

Then we arrive at

$$I_2 \leq 2dm(L)\mu_1^2u_t^2(L) + 2dm(L)\mu_2^2z^2(L,1) - (d-\varepsilon)\int_0^L u_x^2\,dx + C_\varepsilon\int_0^L \theta_x^2\,dx. \tag{4.4}$$

From (4.3) and (4.4), the desired result is given.

If we add a perturbation to the function z, the functional with respect to z can be expressed by

$$J(t) = d\xi\rho(t)\int_0^L e^{-2\rho(t)\delta}z^2(L)\,d\delta; \tag{4.5}$$

the same choice for $J(t)$ can be seen in [12]. The following lemma gives a property of the functional J.

Lemma 4.2. *For the above functional, for all $t > 0$, it holds that*

$$J'(t) = -d\xi e^{-2\rho(t)}(1-\rho'(t))z^2(L,1) + d\xi u_t^2(L) - 2d\xi\rho(t)\int_0^1 e^{-2\rho(t)\delta}z^2(L)\,d\delta. \tag{4.6}$$

Proof. By a direct computation, we obtain

$$J'(t) = d\xi\int_0^1 \frac{d}{dt}[\rho(t)e^{-2\rho(t)\delta}]z^2(L)\,d\delta + 2d\xi\rho(t)\int_0^1 e^{-2\rho(t)\delta}z(L)z_t(L)\,d\delta. \tag{4.7}$$

By a similar process in (3.2), we deal with the second item on the right of (4.7). To be specific, by the relation $z_t(L) = -\frac{1-\delta\rho'(t)}{\rho(t)}z_\delta(L)$, it holds that

$$2d\xi\rho(t)\int_0^1 e^{-2\rho(t)\delta}z(L)z_t(L)\,d\delta = -d\xi e^{-2\rho(t)}(1-\rho'(t))z^2(L,1)$$
$$+ d\xi u_t^2(L) + d\xi\int_0^1 \frac{d}{d\delta}[e^{-2\rho(t)\delta}(1-\rho'(t))]z^2(L)\,d\delta. \tag{4.8}$$

Taking (4.8) into (4.7) and combing the two integrations, we obtain

$$d\xi\int_0^1 \Big(\frac{d}{dt}[\rho(t)e^{-2\rho(t)\delta}] + \frac{d}{d\delta}[e^{-2\rho(t)\delta}(1-\rho'(t))]\Big)z^2(L)\,d\delta = -2d\xi\rho(t)\int_0^1 e^{-2\rho(t)\delta}z^2(L)\,d\delta.$$

Therefore, (4.6) can be obtained from the above three equalities.

Let

$$G(t) = ME(t) + \varphi(t) + J(t),$$

where the constant M large enough will be fixed later, ϕ and J can be seen in (4.1) and (4.5) respectively. We can find the equivalence of $E(t)$ and $G(t)$ from the following lemma.

Lemma 4.3. *For the constant M, there exist c_1 and $c_2 > 0$ such that*

$$c_1 E(t) \leq G(t) \leq c_2 E(t). \tag{4.9}$$

Proof. From Young's inequality and the boundedness of ρ by (2.1), there exists a positive constant c_0' such that

$$|\phi(t) + J(t)| \leq c_0' E(t).$$

Taking $c_1 = M - c_0'$ and $c_2 = M + c_0'$, we get this lemma with $M > c_0'$.

Proof of Theorem 4.1. From (3.11), (4.2) and (4.6), we have

$$\begin{aligned} G'(t) =& ME'(t) + \phi'(t) + \psi'(t) \\ \leq& -[Mc_0 - m(L)(a + 2d\mu_1^2) - d\xi]u_t^2(L) + [Mc_0 - 2\mu_2^2 + d\xi e^{-2\rho(t)}(1 - \rho'(t))]z^2(L,1) \\ &- (d - \varepsilon)\int_0^L u_x^2\,dx - a\int_0^L u_t^2\,dx - (Mc_0 - C_\varepsilon)\int_0^L \theta_x^2\,dx \\ &- 2d\xi\rho(t)\int_0^1 e^{-2\rho(t)\delta} z^2(L)\,d\delta. \end{aligned} \tag{4.10}$$

Now we choose $0 < \varepsilon < d$ and $M > 0$ such that

$$M \geq \frac{1}{c_0} max\Big\{m(L)(a + 2d\mu_1^2) + d\xi, 2\mu_2^2 - d\xi e^{-2\rho_0}(1 - \tilde{\rho}_1), C_\varepsilon, c_o c_o'\Big\}.$$

Due to Poincaré's inequality and (4.10), there exists $c_3 > 0$ such that

$$G'(t) \leq -c_3 E(t).$$

Consequently, it follows from (4.9) that

$$E(t) \leq \frac{c_2}{c_1} E(0) e^{-\frac{c_3}{c_1} t}.$$

Let $C_1 = \frac{c_2}{c_1}$ and $C_2 = \frac{c_3}{c_1}$; then, Theorem 4.1 is completed.

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Conflict of interest

The authors declare that there is no conflicts of interest.

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