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Research article

Oscillatory solutions and smoothing of a higher-order p-Laplacian operator

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Abstract: The goal of this paper was to provide a general analysis of the solutions to a higher-order p-Laplacian operator with nonlinear advection. Generally speaking, it is well known that any solution to a higher-order operator exhibits oscillations . In the present study, an advection term is introduced. This will allow us to analyze smoothing conditions in the solutions. The study of existence and uniqueness is based on a variational approach. Solutions are analyzed with an energy formulation initially discussed by Saint-Venant and extended in the works by Tikhonov and Täklind. This variational principle is supported by the definition of generalized norms under Hilbert-Sobolev spaces, enabling focus on the oscillating properties of solutions. Afterward, the paper introduces an analysis to characterize the traveling wave kind of solutions together with their characterization to understand the oscillations. Finally, a numerical exploration focuses on the smoothing conditions by the action of the nonlinear advection term. As a main finding to report: There exist a traveling wave speed (λ) and an advection coefficient (c^*) for which the profile‘s first minimum is almost positive, and such positivity holds beyond the first minimum.

Keywords: higher order p-Laplacian operator; homotopy; traveling waves; nonlinear advection

1. Problem description and objectives

Several forms of diffusion have been considered to model phenomena in applied sciences. Some of the most frequent operators involving diffusion can be mentioned as follows: linear order two ($\Delta\, v$), higher-order ($-(-\Delta)^m\, v$, $m \geq 2$), p-Laplacian ($\nabla\cdot(|\nabla v|^{p-2}\, \nabla v)$, $p > 1$) and thin-film ($-\nabla\cdot(|v|^n \nabla\Delta v)$, $n > 0$).

In several occasions, a combination of the cited operators is given (e.g., the higher order thin film operator). The particular selection of a diffusion operator is typically done by considering the involved diffusive principles. Some approaches to diffusion have been derived from statistical interpretations based on the concept of random walk (see [1] together with the studies cited therein).

Diffusion may be characterized by energy approaches as well. In [2] and [3], the free energy concept was employed. This was previously introduced by Landau and Ginzburg. Particularly, in [2] the authors proposed a free-energy formulation of the form $\frac{1}{2}k(\nabla u)^2$ (u is the particles concentration). Afterward, they used the chemical potential to obtain a fourth order diffusion. It is relevant to highlight that this kind of diffusion does not allow us to define a maximum principle in general (refer to [4], [5] and [6]).

Other approaches to diffusion have been focused on the observation of the phenomena using a model and the subsequent proposal of a newly defined operator, e.g., the Keller and Segel model to predict chemotaxis in cells [7] (see as well the mathematical treatment under smooth regularity conditions in [8], [9] and [10]). Other nonlinear diffusion formulations can be found in [11] to model complex geometric effects, and in [12] for the simulation of peristaltic processes in a Jeffrey fluid type. Further, regarding applications to chemistry, physics and engineering, diffusion resulting in a p-Laplacian operator has been explored as well (see [13] for an application to fluids, [14] for the Emden-Fowler equation formulated with a p-Laplacian, [15] for the Einstein-Yang-Mills equation and [16] for a heterogeneous reaction with a p-Laplacian formulation). In the search of positive solutions to general operators, it is worth considering the reference [17], wherein the author studied several analysis about the existence of positive solutions to a fractional differential equation based on a fixed-point argument.

As an alternative to the mentioned studies, our intention is to introduce a p-Laplacian operator together with a higher order spatial derivative (up to fourth order) in combination with a nonlinear advection term. The proposed problem can be formulated as:

$$\begin{gathered} v_t = -\Delta(|\Delta v|^m \Delta v) + (c \cdot \nabla v)\ |v|^{p-1} \\ m > 2, \quad p \geq 1, \quad v_0(x),\ v_{0,x}^p(x)\ \in H_0^n(\mathbb{R}^d), \quad n > 1,\ d \geq 1. \end{gathered} \tag{1.1}$$

Finite mass solutions to a p-Laplacian operator exhibit finite propagation in their support while keeping a maximum principle (refer to [18]). Note that when a reaction term of the super-linear type is introduced, solutions may blow-up [19]. In our analysis, no reaction is considered; on the contrary, the intention is to study the smoothing effect of an advection term.

The fourth order p-Laplacian operator was previously introduced in [19]) wherein several paramount properties are mentioned, namely, the fourth order p-Laplacian is potential in L^2, and it keeps the finite speed of propagation, typical of the basic p-Laplacian (see [20]).

Solutions to (1.1) are described based on a variational approach as introduced in [21]. The problem is then discussed under the conditions of the traveling wave domain where oscillatory and smoothing conditions are characterized.

The theory of traveling waves was first introduced by Fisher to explore the dynamics of genes in biological applications [22] and by Kolmogorov, Petrovskii and Piskunov (KPP) for combustion theory [23]. Since its formulation, the Fisher-KPP model has been employed in different applied sciences (refer to [24], [25] and [26] for biological applications and [27] for fluid applications). Furthermore, the Fisher-KPP problem has been employed to model bistable systems (see [28]) together with degenerate diffusion to extend already known equations. For instance, in the studies presented in [29, 30] and [31] the extended Fisher-Kolmogorov equation was analyzed. In addition, the authors of [32] proposed a p-Laplacian kind of diffusion for a Fisher-KPP reaction. Some other modelling exercises in different areas of sciences can be cited; see [33], [34] and [35] as representative examples. Furthermore, the traveling waves approach has been used to explore solutions for higher order operators (refer to [36], [37], [38] and [40]).

In this analysis, it was the objective to study the properties of traveling wave solutions. To this end, a variational approach is introduced and we referred to [21] for the formulation of the general Cauchy problem. To support a generalization exercise, we work within the Sobolev spaces scope. This approach permits the study of solution properties as the oscillations, mollification and compact support evolution.

It is important to mention that the term oscillation is employed to state that any solution is not monotone (decreasing or increasing).

In addition and in [19], the authors obtained blow-up patterns for a higher order p-Laplacian operator with a superlinear reaction with no advection. In our case, it is shown that the null critical state in (1.1) acts as an attractor hindering the formation of blow-up phenomena. In addition, the introduced advection acts as a smoother for the oscillations induced by the higher order p-Laplacian operator. Such a smoothing process is of relevance to define a region in which the solutions are positive.

2. Preliminary definitions

As introduced, the analysis of solutions is given within the scope of a variational formulation. To this end, the following definition holds (refer to [21] for further details on variational principles to general diffusion problems):

Definition 1. *$v(x,t)$ is an energy solution to (1.1) in $\mathbb{R}^d \times (0,t)$ if for any τ satisfying $0 < \tau < t$, the following equality holds:*

$$\int_0^\tau \int_{\mathbb{R}^d} v\, \zeta_t \, dxdt - \int_0^\tau \int_{\mathbb{R}^d} |\Delta v|^m \, \Delta v \, \Delta \zeta \, dxdt - \int_0^\tau \int_{\mathbb{R}^d} v \; |v|^{p-1} \, c \cdot \nabla \zeta \, dxdt = 0, \tag{2.1}$$

where ζ is an arbitrarily supporting function, such that $\zeta \in C^1(0,\tau\,; H_0^2(\mathbb{R}^d))$.

Note that the space $H_0^2(\mathbb{R}^d)$ is given in Definition 5

Now, consider the coming proposition formulated based on known results in [42] and [43]:

Proposition 1. *Assume the functions $J, K, M \in C_0^q\left(\mathbb{R}^d\right)$, where $(q \geq 1)$; then, the following anisotropic Sobolev inequality reads as follows:*

$$\int_{\mathbb{R}^d} |J\, K\, M| \, dx \leq \; C\, \|J\|_{L^q}^{\frac{\beta-1}{\beta}} \, \|\nabla J\|_{L^r}^{\frac{1}{\beta}} \, \|K\|_{L^2}^{\frac{\beta-2}{\beta}} \, \|\nabla K\|_{L^2}^{\frac{1}{\beta}} \, \|M\|_{L^2} \,, \tag{2.2}$$

where $C > 0$, $\frac{\beta-1}{q} + \frac{1}{r} = 1$, $\beta \geq 2$ and $1 \leq q, r < \infty$.

Note that in [21], the authors proposed estimates for blow-up profiles. In the p-Laplacian case, such profiles are close enough to the linear case for a higher order operator, i.e., the case of $u_t = -\Delta^2 u$ (see the S and HS-regimes in [21]). After rescaling, the kernel of the linear case is given by $e^{-x^{4/3}}$. In addition, the problem (1.1) is formulated with a nonlinear advection term. Based on these mentioned ideas, a weighted norm is defined accordingly:

Definition 2. *Assume the following norm in a weighted space $H_\Omega(\mathbb{R}^d)$:*

$$\|g\|_\Omega^2 = \int_{\mathbb{R}^d} \Omega(x) \sum_{j=0}^{4} |D^j g(x)|^2 dx, \tag{2.3}$$

where $D = \frac{d}{dx}$ and $h \in H_\Omega(\mathbb{R}^d) \subset L^2_\Omega(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. The weight $\Omega(x)$ is defined to balance for the higher-order operator and the advection ([21, 41] and [4]):

$$\Omega(x) = e^{c_0|x|^{\frac{4}{3}} - \frac{1}{v^p}\frac{1}{t^\alpha}\int_0^t (\|v_{0,x}(\tau)\|^p + 1) d\tau}, \tag{2.4}$$

where v_0 is the initial condition for the problem (1.1)*, $c_0 > 0$ is a small constant (refer to [21] and [4]) and $\alpha > p + 1$.*

Definition 3. *Consider the following homogeneous equation:*

$$g_t = \Delta_{\beta,2} g, \tag{2.5}$$

where $\Delta_{\beta,2} = -\Delta(|\Delta \cdot|^\beta \Delta)$ according to the nomenclature in [44].

Note that the function g is required to belong to a Sobolev space $H_\Omega(\mathbb{R}^d)$ and the spaces coming in Definitions 4 and 5.

Definition 4. *The following norm in a mollifying space of functions is considered:*

$$\|g\|_{H_v^m}^2 = \int_{-\infty}^{\infty} e^{m\omega^2} |g(\omega, t)|^2 d\omega, \tag{2.6}$$

complying with the mollification A_p-condition, $p = 1$ (refer to [45]) and the weight $v(\omega) = e^{m\omega^2}$, where $m > 2$.

Note that, if solutions are sufficiently smooth because of the action of the advection term, the use of a mollifying norm can be replaced by the usual Sobolev space given in the following definition.

Definition 5. *The following classical order n Sobolev space is given as:*

$$H^n\left(\mathbb{R}^d\right) = \left\{g \in L^2\left(\mathbb{R}^d\right) ; \nabla^n(g) \in L^2\left(\mathbb{R}^d\right)\right\},$$

where $g(x, t)$ is a function with generalized derivatives up to order $n > 1$. In addition, the following norm is defined accordingly:

$$\|g\|_{H^n} = \|g\|_{L^2} + \|\nabla^n g\|_{L^2}. \tag{2.7}$$

It shall be noted that the previous definition also applies to functions with compact support. To this end, the space $H_0^n\left(\mathbb{R}^d\right)$ is defined with the same norm as in (2.7) that applies to Lebesgue functions with compact support up to the order n generalized derivative.

Now, assume a sequence of open bounded intervals $B(0, \eta) \subset \mathbb{R}^d$, $\eta \in \mathbb{N}$, $\eta = 1, 2, 3 ...$

Proposition 2. *Consider the Sobolev space $W^{n,p}(B(0, \eta))$. Recall that a function in $W^{n,p}(B(0, \eta))$ has weak derivatives up to the order n with a finite L^p norm in $B(0, \eta)$. Define $k = int\{n - \frac{d}{p}\}$; then, the following continuous (in the sense of Hölder) inclusion holds (see [46], page 79):*

$$W^{n,p}(B(0, \eta)) \hookrightarrow C^k(B(0, \eta)). \tag{2.8}$$

Note that in our case, the solution may be differentiable up to fourth order, i.e., $n = 4$ with $p = 2$; then, $k = int\{4 - \frac{d}{2}\}$.

3. First assessments

The following Theorem aims to provide bounds and embedding properties.

Theorem 3.1. *Given $g_0 \in L^2(\mathbb{R}^d)$, the following embedding properties hold:*

$$||g||_{L^2} \leq ||g_0||_{L^2}, \quad ||g||_{H_0^n} \leq ||g_0||_{H_0^n}.$$

$$||g||_{H_\nu^m} \leq A_1 ||g_0||_{L^2}, \quad A_1^2 = e^{m f_M^2 - t f_M^4 \frac{\Gamma(\beta+1)}{\pi (i f_M)^{\beta+1}}}, \quad f_M = \left(\frac{\pi i^{\beta+1}\, 2\, m}{t \Gamma(\beta+1)(3-\beta)} \right)^{\frac{1}{1-\beta}},$$

$$||g||_{H_\nu^m} \leq ||g_0||_{H_\nu^m}, \quad ||g||_{H_\nu^m} \leq A_1 ||g_0||_{H_0^n}.$$

$$||g||_{\Omega} \leq \Upsilon ||g_0||_{H_\nu^m}, \quad ||g||_{\Omega} \leq B_1 \Upsilon ||g_0||_{H_0^n},$$

where

$$B_1^2 = \sup_{\forall x \in B_R} \{ e^{c_0 |x|^{\frac{4}{3}}} \}, \qquad \Upsilon^2 = 25 \ \max\{g, D^1 g, D^2 g, D^3 g, D^4 g\},$$

such that n takes values of $1, 2, 3$ *and* 4 *and* $B_R \in \mathbb{R}^d$ *is a closed ball considered to be sufficiently large for our purposes (i.e.,* $R \to \infty$*).*

The norm in H_Ω *is bounded by the norm in* H_ν^m *(mollifying) and the compacting norm in* H_0^n *multiplied by the scaling terms* Υ *and* ΥB_1.

Proof. Consider the homogeneous problem $g_t = \Delta_{\beta,2} g$; then, an abstract evolution is given by $g(x,t) = e^{t \Delta_{\beta,2}} g_0(x)$. After making the Fourier transformation (the domain of which is given by the f variable) by convolution of the terms in $(|\Delta \cdot|^\beta \Delta)$ and weighted by the term $-\Delta$, the following holds:

$$\hat{g}(f,t) \leq e^{-t f^4 \frac{\Gamma(\beta+1)}{2\pi (i f)^{\beta+1}}} \hat{g}_0(f), \tag{3.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

First of all, and based on the isometry in the Fourier transformation under the condition of the L^2 norm, the following inequality is shown:

$$||g||_{L^2}^2 \leq \int |e^{-t f^4 \frac{\Gamma(\beta+1)}{\pi (i f)^{\beta+1}}}| |\hat{g}_0(f)|^2 df \leq \sup_{\forall f \in \mathbb{R}^d} \{ |e^{-t f^4 \frac{\Gamma(\beta+1)}{\pi (i f)^{\beta+1}}}| \} \int |\hat{g}_0(f)|^2 df = ||g_0||_{L^2}^2. \tag{3.2}$$

Then $||g||_{L^2} \leq ||g_0||_{L^2}$.

Note that considering the definition in (2.7), the following holds:

$$||g||_{H_0^n} \leq ||g_0||_{H_0^n}. \tag{3.3}$$

Now, consider $g_0 \in L^2(\mathbb{R}^d)$:

$$||g||_{H_\nu^m}^2 = \int e^{m f^2} |\hat{g}(f,t)|^2 df \leq \sup_{\forall f \in \mathbb{R}^d} \{ e^{m f^2} e^{-t f^4 \frac{\Gamma(\beta+1)}{\pi (i f)^{\beta+1}}} \} \int |\hat{g}_0(f)|^2 df. \tag{3.4}$$

After standard assessments, it holds that:

$$||g||^2_{H^m_\nu} \le e^{mf_M^2 - tf_M^4 \frac{\Gamma(\beta+1)}{\pi(if_M)^{\beta+1}}} ||g_0||^2_{L^2}, \tag{3.5}$$

where

$$f_M = \left(\frac{\pi i^{\beta+1}\, 2\, m}{t\Gamma(\beta+1)(3-\beta)} \right)^{\frac{1}{1-\beta}}. \tag{3.6}$$

A function in H^m_ν satisfies an internal embedding and is internally supported by the compact space H^n_0:

$$||g||^2_{H^m_\nu} = \int e^{mf^2} |\hat{g}(f,t)|^2 df \le \sup_{\forall f \in \mathbb{R}^d} \{ |e^{-tf^4 \frac{\Gamma(\beta+1)}{\pi(if)^{\beta+1}}}| \} \int e^{mf^2} |\hat{g}_0(f)|^2 df \le ||g_0||^2_{H^m_\nu}. \tag{3.7}$$

If g and g_0 have compact support, the following holds:

$$\begin{aligned} ||g||^2_{H^m_\nu} &= \int e^{mf^2} |\hat{g}(f,t)|^2 df \le \sup_{\forall f \in \mathbb{R}^d} \{ |e^{mf^2 - tf^4 \frac{\Gamma(\beta+1)}{\pi(if)^{\beta+1}}}| \} \int |\hat{g}_0(f)|^2 df \\ &\le \sup_{\forall f \in \mathbb{R}^d} \{ |e^{mf^2 - tf^4 \frac{\Gamma(\beta+1)}{\pi(if)^{\beta+1}}}| \} \int \left(|\hat{g}_0(f)|^2 + |\nabla^n g_0|^2 \right) df = e^{mf_M^2 - tf_M^4 \frac{\Gamma(\beta+1)}{\pi(if_M)^{\beta+1}}} ||g_0||^2_{H^n_0}, \end{aligned} \tag{3.8}$$

where f_M is given in (3.6).

The objective now is to explore embedding properties for the norm in (2.3). First, consider the mollifying norm given in (2.6) :

$$\begin{aligned} ||g||^2_\Omega &= \int \Omega(x) \sum_{j=0}^{4} |D^j g(x)|^2 dx \le \int e^{mx^2} \sum_{j=0}^{4} |D^j g(x)|^2 dx \le \Upsilon^2 \int e^{mx^2} |g(x)|^2 dx = \Upsilon^2 ||g||_{H^m_\nu} \\ &\le \Upsilon^2 ||g_0||^2_{H^m_\nu}. \end{aligned} \tag{3.9}$$

For the last inequality, the continuity Proposition 2 has been considered so that, the following scaling variable can be defined upon: $\Upsilon^2 = 25 \sup_{\zeta \in U_{n\to\infty}} \{g, D^1 g, D^2 g, D^3 g, D^4 g\}$. Note that Proposition 2 allows us to account for the regularity of each of the involved derivatives in Υ.

The weighted space H_Ω is internally closed by the space of compact support functions satisfying the norm (2.7):

$$\begin{aligned} ||g||^2_\Omega &= \int \Omega(x) \sum_{j=0}^{4} |D^j g(x)|^2 dx \le \sup_{\forall x \in B_R} \{ e^{c_0 |x|^{\frac{4}{3}}} \} \Upsilon^2 \int (|g(x)|^2 + |\nabla^n g(x)|^2) dx \\ &\le \sup_{\forall x \in B_R} \{ e^{c_0 |x|^{\frac{4}{3}}} \} \Upsilon^2 ||g||^2_{H^n_0} \le \sup_{\forall x \in B_R} \{ e^{c_0 |x|^{\frac{4}{3}}} \} \Upsilon^2 ||g_0||^2_{H^n_0}, \end{aligned} \tag{3.10}$$

such that n takes values of $1, 2, 3$ and 4 and $B_R \in \mathbb{R}^d$ is a closed ball considered to be sufficiently large for our purposes (i.e., $R \to \infty$).

The next intention is to prove the local bound of any compactly supported function. To this purpose, consider the following a priori conditions:

$$\zeta \in C^1(0,t\,;H_0^3 \cap C_0^q(\mathbb{R}^d)), \quad v_0(x),\ \frac{\partial v_0(x)}{\partial x} \ \in H_0^n(\mathbb{R}^d) \cap C_0^q\left(\mathbb{R}^d\right), \tag{3.11}$$

where $n > 1$, $q \geq 1$. This gives the following lemma.

Lemma 3.1. *Given the energy solution presented in* (2.1)*, it is shown that such a solution is bounded in* $H_0^4(\mathbb{R}^d)$ *(refer to* (2.7) *with n* = 4*), i.e., compactly supported functions preserve the support locally.*

Proof. First, the equality (2.1) is considered for any $0 < \tau \leq t$:

$$\int_0^\tau \int_{\mathbb{R}^d} v\,\zeta_t\,dxdt = \int_0^\tau \int_{\mathbb{R}^d} v\ |v|^{p-1}\,c \cdot \nabla\zeta\,dxdt + \int_0^\tau \int_{\mathbb{R}^d} |\Delta v|^m\,\Delta v\,\Delta\zeta\,dxdt. \tag{3.12}$$

Under the conditions defined in (3.11) and in lieu of Proposition 1, the involved integrals can be further analyzed. For this purpose, consider $q = 2$, $r = 2$ and $\beta = 2$ in (2.2):

$$\int_{\mathbb{R}^d} \Delta\zeta\,\Delta v\,|\Delta v|^m\,dx \leq C\|\Delta\zeta\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta\zeta\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\beta+1}. \tag{3.13}$$

According to Definition 5 and Theorem 3.1, the following holds:

$$\begin{aligned} C\|\Delta\zeta\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta\zeta\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\beta+1} &\leq C\|\zeta\|_{H_0^2}^{\frac{1}{2}}\|\zeta\|_{H_0^3}^{\frac{1}{2}}\|v\|_{H_0^3}^{\frac{1}{2}}\|v\|_{H_0^2}^{\beta+1} \\ &\leq C\|\zeta\|_{H_0^2}^{\frac{1}{2}}\|\zeta\|_{H_0^3}^{\frac{1}{2}}\|v_0\|_{H_0^3}^{\frac{1}{2}}\|v_0\|_{H_0^2}^{\beta+1}. \end{aligned} \tag{3.14}$$

The next integral involved is assessed as:

$$\begin{aligned} \int_{\mathbb{R}^d} v\ |v|^{p-1}\,c \cdot \nabla\zeta\,dx &\leq k\|\nabla\zeta\|_{L^2}^{\frac{1}{2}}\|\nabla(\nabla\cdot\zeta)\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}^{p-1} \\ &\leq k\|\zeta\|_{H_0^1}^{\frac{1}{2}}\|\zeta\|_{H_0^2}^{\frac{1}{2}}\|v\|_{H_0^1}^{\frac{1}{2}}\|v\|_{L^2}^{p-\frac{1}{2}} \leq k\|\zeta\|_{H_0^2}^{\frac{1}{2}}\|v_0\|_{H_0^1}^{\frac{1}{2}}\|v_0\|_{L^2}^{p-\frac{1}{2}}, \end{aligned} \tag{3.15}$$

where for the last inequality, we have considered the natural embedding $H_0^1 \subset H_0^2$.

Given the left-hand side term in (3.12) and considering Gronwall inequality, the following holds:

$$\int_0^\tau v\,\zeta_t dt \leq \int_0^\tau f(t)\,\zeta\,v dt, \tag{3.16}$$

where the continuous $f(t)$ is the required function as per the Gronwall theorem conditions. Recall that the supporting function ζ is $C^1(0,t)$ in accordance with the conditions of (3.11). Then, ζ is selected such that $\zeta\,f(t) = 1$:

$$\int_0^\tau v\,\zeta_t dt \leq \int_0^\tau v dt. \tag{3.17}$$

The constant C in (3.14) and the constant k in (3.15) are considered to be sufficiently large so that, given (3.12), the following holds:

$$\int_0^\tau ||v||_{H^4} dt \le k \int_0^\tau ||\zeta||^{\frac{1}{2}}_{H_0^2} ||v_0||^{p+\frac{1}{2}}_{H_0^1} dt + C \int_0^\tau ||\zeta||_{H_0^3} ||v_0||^{\beta+\frac{3}{2}}_{H_0^3} dt. \tag{3.18}$$

The functions ζ and v_0 comply with the conditions in (3.11); particularly, they have been considered as having compact support. As a consequence and for any local time, the compact support is kept as no local blow up is expected. Based on this, it is possible to conclude that:

$$||v||_{H_0^4} \le k\, ||\zeta||^{\frac{1}{2}}_{H_0^2} ||v_0||^{p+\frac{1}{2}}_{H_0^1} + C ||\zeta||_{H_0^3} ||v_0||^{\beta+\frac{3}{2}}_{H_0^3}, \tag{3.19}$$

as expressed initially in the Lemma.

The next intention is to obtain bounds for the oscillating solutions. To this end, the mollifying norm (2.6) is used while the support is considered as open and oscillating. We establish the following conditions:

$$\zeta \in C^1(0, t\,; H_\nu^m \cap C^n(\mathbb{R}^d)), \quad v_0(x),\ \frac{\partial v_0(x)}{\partial x} \ \in H_\nu^m(\mathbb{R}^d) \cap C^n\left(\mathbb{R}^d\right), \tag{3.20}$$

$n > 1$, $q \ge 1$.

Lemma 3.2. *Any unstable (in the sense of oscillating) energy solution in accordance with* (2.1) *is bounded by the mollifying norm defined in* (2.6).

Proof. Consider (3.12) together with the inequality (2.2) ($q = 2$, $r = 2$, $\beta = 2$) and the norm given in (2.3):

$$\begin{aligned}\int_{\mathbb{R}^d} \Delta\zeta\, \Delta v\, |\Delta v|^m\, dx &\le C ||\Delta\zeta||^{\frac{1}{2}}_{L^2} ||\nabla\Delta\zeta||^{\frac{1}{2}}_{L^2} ||\nabla\Delta v||^{\frac{1}{2}}_{L^2} ||\Delta v||^{\beta+1}_{L^2} \\ &\le K ||\zeta||^{\frac{1}{2}}_{\Omega} ||\zeta||^{\frac{1}{2}}_{\Omega} ||v||^{\frac{1}{2}}_{\Omega} ||v||^{\beta+1}_{\Omega}.\end{aligned} \tag{3.21}$$

Considering the results obtained in Theorem 3.1,

$$C ||\zeta||^{\frac{1}{2}}_{\Omega} ||\zeta||^{\frac{1}{2}}_{\Omega} ||v||^{\frac{1}{2}}_{\Omega} ||v||^{\beta+1}_{\Omega} \le C\, \Upsilon ||\zeta||_{H_\nu^m} ||v_0||^{m+\frac{3}{2}}_{H_\nu^m}. \tag{3.22}$$

Operating in a similar way, we have:

$$\begin{aligned}\int_{\mathbb{R}^d} v\, |v|^{p-1}\, c \cdot \nabla\zeta\, dx &\le k ||\nabla\zeta||^{\frac{1}{2}}_{L^2} ||\nabla(\nabla\cdot\zeta)||^{\frac{1}{2}}_{L^2} ||v||^{\frac{1}{2}}_{L^2} ||\nabla v||^{\frac{1}{2}}_{L^2} ||v||^{p-1}_{L^2} \\ &\le k ||\zeta||_{\Omega} ||v||^p_{\Omega} \le k\Upsilon ||\zeta||_{H_\nu^m} ||v_0||^p_{H_\nu^m}.\end{aligned} \tag{3.23}$$

Again, consider (3.12), apply Gronwall inequality as in (3.17) and assume C and k are sufficiently large, then, For $0 < \tau \le t$,

$$\int_0^\tau ||v||_{\Omega} dt \le k\, \Upsilon \int_0^\tau ||\zeta||_{H_\nu^m} ||v_0||^p_{H_\nu^m} dt + C\, \Upsilon \int_0^\tau ||\zeta||_{H_\nu^m} ||v_0||^{m+\frac{3}{2}}_{H_\nu^m} dt. \tag{3.24}$$

The supporting function ζ and the initial condition v_0 satisfy (3.20); as a consequence the boundedness of any oscillating solution in $(0,\tau)$ in the norm (2.3) is smoothed by the mollifying norm (2.6), which gives

$$\|v\|_\Omega \le k\,\Upsilon\|\zeta\|_{H^m_\nu}\|v_0\|^p_{H^m_\nu} + C\,\Upsilon\|\zeta\|_{H^m_\nu}\|v_0\|^{m+\frac{3}{2}}_{H^m_\nu}\,, \tag{3.25}$$

as intended to show.

3.1. Uniqueness

The uniqueness is shown considering that solutions are sufficiently small under the conditions of the norm (2.6). As it will be shown in Section 4.2, the oscillating behavior is observed in the proximity of the null solution, which acts as an attractor in the phase space. Such a lack of regularity may lead to the loss of uniqueness. As a consequence, the analysis focuses on the proximity of such a null solution. Further, the initial conditions v_0 and w_0 are sufficiently small for our purposes under the conditions of the norm (2.6).

Consider two solutions $v(x,t)$ and $w(x,t)$ satisfying the variational formulation (2.1) and such that $v, w \in H_\Omega(\mathbb{R}^d) \subset L^2_\Omega(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. In addition, assume $v \ge w$ with $v_0(x) = w_0(x)$, and that it is sufficiently small under the conditions of the norm (2.6). The uniqueness analysis is performed for the norm (2.3) to account for the oscillations induced by the higher order operator and the effect of the advection term. Both v and w satisfy the formulation (3.12), then

$$\int_0^\tau\int_{\mathbb{R}^d}(v-w)\,\zeta_t\,dxdt = \int_0^\tau\int_{\mathbb{R}^d}(\,v\,|v|^{p-1} - w\,|w|^{p-1}\,)c\cdot\nabla\zeta\,dxdt + \int_0^\tau\int_{\mathbb{R}^d}(|\Delta v|^m\,\Delta v - |\Delta w|^m\,\Delta w\,)\Delta\zeta\,dxdt. \tag{3.26}$$

Performing similar operations as previously done for the expression (2.2) (again $q = 2,\ s = 2,\ \alpha = 2$) gives

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^d}(|\Delta v|^m\,\Delta v - |\Delta w|^m\,\Delta w\,)\Delta\zeta\,dx \le C_v\|\zeta\|_\Omega\|v\|^{m+\frac{3}{2}}_\Omega + C_w\|\zeta\|_\Omega\|w\|^{m+\frac{3}{2}}_\Omega \\ &\le 2C\,\Upsilon\|\zeta\|_{H^m_\nu}\|v_0\|^{m+\frac{3}{2}}_{H^m_\nu}, \end{aligned} \tag{3.27}$$

where $C = \max\{C_v, C_w\}$.

Considering the following integral:

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^d}(\,v\,|v|^{p-1} - w\,|w|^{p-1}\,)c\cdot\nabla\zeta\,dx \le k_v\|\zeta\|_\Omega\|v\|^p_\Omega + k_w\|\zeta\|_\Omega\|w\|^p_\Omega \\ &\le 2k\Upsilon\|\zeta\|_{H^m_\nu}\|v_0\|^p_{H^m_\nu}, \end{aligned} \tag{3.28}$$

where $k = \max\{k_v, k_w\}$.

After employing Gronwall inequality to the left-hand term in (3.26) gives

$$0 \le \int_0^\tau\int_{\mathbb{R}^d}(v-w)\,\zeta_t\,dxdt \le \int_0^\tau\int_{\mathbb{R}^d}(\,(v_0 - w_0) + F(t)\,\zeta\,(v-w)\,)dxdt, \tag{3.29}$$

where $F(t)$ is a continuous function as required by the Gronwall theorem.

Returning to (3.26) and considering the assessments done for a C and a k sufficiently large, we have

$$0 \leq \int_0^{\tau} \int_{\mathbb{R}^d} (\,(v_0 - w_0) + F(t)\,\zeta\,(v - w)\,)dxdt \;\leq M, \tag{3.30}$$

where $M = \max\{2C\,\Upsilon \|\zeta\|_{H^m_\nu} \|v_0\|^{m+\frac{3}{2}}_{H^m_\nu},\; 2k\Upsilon\|\zeta\|_{H^m_\nu}\|v_0\|^{p}_{H^m_\nu}\}$.

Note that $v_0 = w_0$ and that they have been requested to be sufficiently small under the conditions of the norm (2.6) to have a small M. In addition, $F(t)\,\zeta$ is continuous in $(0, \tau]$; consequently, it is possible to conclude in the space of the proximity of both $v(x, t)$ and $w(x, t)$, leading to the proof of the solution‘s asymptotic uniqueness in $(0, \tau]$.

4. Traveling waves

The mathematical analysis under traveling wave formulation has been performed in consideration of a step function as initial data. It should be noted that a step function is not compactly supported in $\mathbb{R}^d$, as initially required for initial data in (1.1). However, such a step function as initial data allows for analysis of the evolution of a null and positive state (together with the interaction between them). This is particularly interesting as a measure to determine the stability of solutions when departing from a positive mass and ending in the null critical state. The involved step function is given by the classical Heaviside function. Note that the higher-order p-Laplacian operator is a potential in the L^2 space; then, upon evolution, any solution is transformed into L^2 and, in accordance with the embeddings shown in Theorem 3.1, the solutions are transformed into the Hilbert-Sobolev Spaces defined in Section 2.

Traveling Wave profiles are obtained with the change $v(x, t) = \rho(\omega)$ where $\omega = x \cdot n_d - \lambda t$. Note that $\omega \in \mathbb{R}$ and the vector $n_d \in \mathbb{R}^d$ gives the traveling wave direction of propagation. Furthermore, λ is the traveling wave speed and the traveling wave profile is given by

$$\rho : \mathbb{R} \to (0, \infty), \text{ with } \rho \in H_\Omega(\mathbb{R}) \subset L^2_\Omega(\mathbb{R}) \subset L^2(\mathbb{R}). \tag{4.1}$$

Consider that the Traveling Wave moves in the direction $n_d = (1, 0, 0, ..., 0)$; then, $\omega = x - \lambda t$ and $v(x, t) = \rho(\omega) \in \mathbb{R}$. In addition, assume that the advection vector c acts in the Traveling Wave direction. Then, the problem (1.1) reads as

$$-\lambda\rho' = -(|\rho''|^m \rho'')'' + c\,\rho'|\rho|^{p-1}\,. \tag{4.2}$$

4.1. Traveling waves oscillatory behavior

According to the problem defined as (4.2), there exists one null solution $\rho = 0$. The objective in this section is to characterize the oscillatory behavior of profiles in the proximity of the mentioned null solution by making use of the norm given in (2.3). In addition, the null state represents a centered attractor to the closing manifolds acting in the proximity.

The analysis of oscillatory patterns in the proximity of the null state is inspired by a series of lemmas that were first introduced to analyze the Kuramoto-Sivashinsky equation [36]. Afterward, a similar methodology was pursued for a Cahn-Hilliard equation (see [37]) and for a general equation with a sixth-order operator [38]. The mentioned results are split into four lemmas for the sake of simplicity.

The first of the proposed lemmas aims to introduce bounds for the oscillating behavior of the solutions. To this end, the defined norms in (2.3) and (2.7) are employed.

Lemma 4.1. *Any oscillating solution $v(x,t) \in L^2(\mathbb{R}^d)$ is bounded by the norms* (2.3) *and* (2.7) *i.e..:*

$$\|v\|_{L^2} \leq C_1 \|v\|_{H^4}, \quad \|v\|_{L^2} \leq C_2 \|v\|_{\Omega}. \tag{4.3}$$

Proof. The first inequality is trivially proved considering the definition of the norm in the space H^n, $n = 4$, (2.7):

$$\|v\|_{H^4} = \|v\|_{L^2} + \left\|\nabla^4 v\right\|_{L^2} \geq \|v\|_{L^2}. \tag{4.4}$$

Any norm is positive; then, $\|v\|_{L^2} \leq \|v\|_{H^4}$, i.e., $C_1 = 1$.

The second expression on the right is proved in consideration of the norm (2.3):

$$\|v\|_{L^2}^2 \leq \int_{\mathbb{R}^d} \sum_{j=0}^{4} |D^j v(x)|^2 dx \leq \int_{\mathbb{R}^d} \Omega(x) \sum_{j=0}^{4} |D^j v(x)|^2 dx = \|v\|_{\Omega}^2, \tag{4.5}$$

almost everywhere in $\mathbb{R}^d$. Consequently, $C_2 = 1$.

Now, aiming to introduce the convergence analysis to the traveling waves, the following function is defined: $V(x,t) = v(x,t) - \psi(x,t)$, where $\psi(x,t)$ is a random perturbation that shall be sufficiently small to ensure convergence between the traveling wave pattern and the solution, in particular the null one. In our case, in the proximity of the null equilibrium, $\psi(x,t)$ is considered to satisfy the following condition (refer to Lemma 3.1 together with the inequality (3.9)):

$$\|\psi\|_{L^2} \leq \|\psi\|_{\Omega} \leq \Upsilon \|\psi\|_{H^m_\nu}. \tag{4.6}$$

To ensure convergence of traveling wave profiles, $\Upsilon \to 0$, i.e., a mollification is required on the pattern ψ. The Eq (1.1) is expressed in the new functions $V(x,t)$ and $\psi(x,t)$ as:

$$V_t + \psi_t = -\Delta\left(\sum_{k=0}^{\infty} \binom{m}{k} (\Delta V)^{m-k} (\Delta\psi)^k \, \Delta(V+\psi)\right) + c \cdot \nabla(V+\psi) \sum_{j=0}^{\infty} \binom{p-1}{j} V^{p-1-j} \psi^j. \tag{4.7}$$

Assume that any perturbation can be bounded as $0 < \|\psi\|_{\Omega} \leq K$. Further, let us define:

$$V_t = F(V), \tag{4.8}$$

where $F(V) = -\Delta\left(\sum_{k=0}^{\infty} \binom{m}{k} (\Delta V)^{m-k} (\Delta\psi)^k \, \Delta(V+\psi)\right) + c \cdot \nabla(V+\psi) \sum_{j=0}^{\infty} \binom{p-1}{j} V^{p-1-j} \psi^j$.

Based on the definitions exposed, the following lemma holds:

Lemma 4.2. *The mapping $F : H^m_\nu \to L^2$ is bounded and continuous. Furthermore, there exist $C_3 > 0$, $\alpha_0 > 0$ and $\alpha_1 > 1$ such that $\|F(V)\|_{L^2} \leq C_3 \|V\|_{H^m_\nu}^{\alpha_1}$, for $0 < \|V\|_{H^m_\nu} < \alpha_0$.*

Proof.

$$\|F(V)\|_{L^2} \leq \|F(V)\|_{\Omega} \leq \sum_{k=0}^{\infty} \binom{m}{k} \|V\|_{\Omega}^{m-k} \|\psi\|_{\Omega}^k (\|V\|_{\Omega} + \|\psi\|_{\Omega}) + c \sum_{j=0}^{\infty} \binom{p-1}{j} \|V\|_{\Omega}^{p-j} \|\psi\|_{\Omega}^{j+1}. \tag{4.9}$$

Consider $0 < \|\psi\|_\Omega \leq K$ together with the inequality (3.9):

$$\|F(V)\|_{L^2} \leq \|F(V)\|_\Omega \leq \sum_{k=0}^{\infty} \binom{m}{k} 2\|V\|_{H_\nu^m}^{m-k+1} K^{k+1} + \sum_{j=0}^{\infty} \binom{p-1}{j} \|V\|_{H_\nu^m}^{p-j} K^{j+1} . \tag{4.10}$$

Assume than $m - k + 1 > p - j$; then,

$$\|F(V)\|_{L^2} \leq \|F(V)\|_\Omega \leq \sum_{k=0}^{\infty} \binom{m}{k} 3\|V\|_{H_\nu^m}^{m-k+1} K^{k+1} = C_3 \|V\|_{H_\nu^m}^{\alpha_1} \tag{4.11}$$

Then, $C_3 = \sum_{k=0}^{\infty} \binom{m}{k} 3\, K^{k+1}$ and $\alpha_1 = \max\{m - k + 1\} > 1$.

Assume that $p - j > m - k + 1$:

$$\|F(V)\|_{L^2} \leq \|F(V)\|_\Omega \leq \sum_{j=0}^{\infty} \binom{p-1}{j} 3\|V\|_{H_\nu^m}^{p-j} K^{j+2} = C_3 \|V\|_{H_\nu^m}^{\alpha_1}. \tag{4.12}$$

Then, $C_3 = \sum_{j=0}^{\infty} \binom{p-1}{j} 3\, K^{j+2}$ and $\alpha_1 = \max\{p - j\} > 1$.

The continuity of the mapping F can be proved based on the inequalities shown. To this end, it suffices to define a pair of close sequences converging uniformly. Such process follows from standard means.

Now, assume the following problem for $|\psi| \to 0$:

$$V_t = -\Delta \left(\sum_{k=0}^{\infty} \binom{m}{k} (\Delta V)^{m-k} (\Delta \psi)^k \, \Delta(V + \psi) \right) + c \cdot \nabla(V + \psi) \sum_{j=0}^{\infty} \binom{p-1}{j} V^{p-1-j} \, \psi^j = L_0 V + G(V), \tag{4.13}$$

where $L_0 V = -\Delta \left(\sum_{k=0}^{\infty} \binom{m}{k} (\Delta V)^{m-k} (\Delta \psi)^k \, \Delta(V + \psi) \right)$ and $G(V) = c \cdot \nabla(V + \psi) \sum_{j=0}^{\infty} \binom{p-1}{j} V^{p-1-j} \, \psi^j$. Based on the provided definition, consider the abstract form: $V(x,t) = e^{t L_0}\, V_0(x)$, where $V_0 = v_0$. This gives the following lemma.

Lemma 4.3. *L_0 is the infinitesimal representation of the continuous semi-group e^{tL_0} that complies with the following inequalities:*

$$\int_0^1 \|e^{tL_0}\|_{L^2 \to H_\nu^m} = C_5 < \infty \ , \qquad \int_0^1 \|e^{tL_0}\|_{L^2 \to H_\Omega} = C_6 < \infty \tag{4.14}$$

Proof. Showing that $E(t) = e^{tL_0}$ is a continuous semigroup follows from standard theory (see [39]); nonetheless, some basic ideas are introduced. To this end, consider the family $\{E(t)\}_{t \in \mathbb{R}^+}$ with the following terms:

$$e^{L_0 t} := \sum_{j=1}^{\infty} \frac{(L_0)^j \, t^j}{j!}, \tag{4.15}$$

with $(L_0)^0 = I$. Furthermore,

$$\|E(t)\|_{L^2} = \|e^{L_0 t}\|_{L^2} \leq \sum_{j=1}^{\infty} \frac{(\|L_0\|_{L^2}\, t)^j}{j!} = e^{\|L_0\|_{L^2}\, t}, \quad t > 0. \tag{4.16}$$

The family $\{E(t)\}_{t\in\mathbb{R}^+}$ is well defined satisfying $E(0) = I$, where I refers to the the identity map in L^2. The uniform continuity is shown as follows:

$$\|\frac{E(t) - I}{t} - (L_0)\|_{L^2} \leq \frac{1}{t} \sum_{j=2}^{\infty} \frac{(\|L_0\|_{L^2}\, t)^j}{j!} = \frac{1}{t}(e^{\|L_0\|_2 t} - I - t\|L_0\|_2), \tag{4.17}$$

which converges to zero for $t \to 0^+$. Therefore, L_0 can be regarded as the infinitesimal generator of the uniformly continuous semigroup family $\{E(t)\}_{t\in\mathbb{R}^+}$ in L^2.

Now, according to Theorem 3.1,

$$\|V\|_{H^m_\nu} \leq \|V_0\|_{H^m_\nu} \leq A_1 \|V_0\|_{L^2}. \tag{4.18}$$

Now, consider the abstract representation $V(x,t) = e^{t\,L_0}\, V_0(x)$ such that:

$$\|V\|_{H^m_\nu} \leq \|e^{tL_0}\|_{L^2\to H^m_\nu} \|V_0\|_{L^2}. \tag{4.19}$$

Based on the above inequalities:

$$\int_0^1 \|e^{tL_0}\|_{L^2\to H^m_\nu} = \int_0^1 A_1 = C_5 < \infty. \tag{4.20}$$

It can be easily checked that C_5 is finite upon integration of A_1 (see Theorem 3.1) in the interval $(0,1]$.

Performing similar assessments, we conclude that there is boundedness of the abstract representation in the space H_Ω. To show this, consider Theorem 3.1 and the inequality (3.9) as follows:

$$\|V\|_\Omega \leq \Upsilon\, \|V\|_{H^m_\nu} \leq \Upsilon \|V_0\|_{H^m_\nu} \leq \Upsilon\, A_1 \|V_0\|_{L^2}. \tag{4.21}$$

Again,

$$\|V\|_{H_\Omega} \leq \|e^{tL_0}\|_{L^2\to H_\Omega} \|V_0\|_{L^2}, \tag{4.22}$$

such that:

$$\int_0^1 \|e^{tL_0}\|_{L^2\to H_\Omega} = \int_0^1 A_1\, \Upsilon = C_6 < \infty. \tag{4.23}$$

The integral above is finite in the interval $(0,1]$.

The next objective is to determine the spectrum of L_0 (as defined in (4.13)) in the space H_Ω. For this purpose, the following Lemma is provided:

Lemma 4.4. *Consider the expression* (4.6) *for the perturbed term. Assume that* $0 < \|\psi\|_\Omega \leq C$. *Then, the spectrum of* L_0 (4.13) *in* H_Ω *has, at least, one eigenvalue* μ *with* $Re(\mu) > 0$ *in the proximity of the null equilibrium attractor.*

Proof. This lemma can be proved by making use of Evans functions because their roots are the same as those of the characteristic polynomial (refer to [47]).

To show the proposed lemma, the characteristic polynomial is employed. Firstly, it shall be considered that the eigenvalues are obtained via homotopy in the proximity of the null critical point $\rho = 0$. The homotopy graphs allow us to consider the parametric study with the traveling wave velocity λ. For this purpose, a computational procedure entailing the use of the Matlab function Charpoly was performed. First, the following first integral holds for the sake of simplicity:

$$-\lambda\rho' = -(|\rho''|^m \rho'')'' + c(\rho^p)' \to -\lambda\rho = -(|\rho''|^m \rho'')' + c\,\rho^p + a_1, \tag{4.24}$$

where $a_1 = 0$ for simplicity and there is no impact on the ending results. In addition, note that $p \geq 1$. For the case $p = 1$, the analysis has already been covered by the traveling wave velocity λ. To this end it suffices to define an equivalent traveling wave speed given by $\lambda^* = \lambda + c$. In the case of $p > 1$, and in the proximity of null critical, the following problem is analyzed involving the operator L_0:

$$\lambda\rho = -(|\rho''|^m \rho'')'. \tag{4.25}$$

Making use of the classical matrix representation, we have

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\lambda}{m\rho_3^{m-1}+\rho_3^m} & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \tag{4.26}$$

Note that ρ_3 is a the second derivative of ρ. For our purposes, this value has been kept free, i.e., as a parameter. In addition, it is in the proximity of the null critical point $|\rho_3| \ll 1$ so that the variable $\varepsilon \gg 1$ can be introduced as follows

$$R(\mu) = -\mu^3 + \frac{\lambda}{m\,\rho_3^{m-1} + \rho_3^m} = 0 \quad \to -\mu^3 + \varepsilon\,\lambda = 0. \tag{4.27}$$

After standard resolution of the last equation, it is possible to confirm that there exists, at least, a positive eigenvalue.

To further assess the impact of the traveling wave velocity on the given spectral properties, several graphs were constructed (Figures 1 and 2 for positive traveling wave velocities and Figures 3 and 4 for negative travelling wave velocities).

4.2. Positivity and smootheness of the traveling wave profiles

Once the oscillatory behavior of traveling wave profiles was observed, it became our intention to determine conditions in the advection term to provide smoothness to such oscillating profiles. For this purpose, a numerical scheme has been explored by using the Matlab solver bvp4c. This solver is based on a Runge-Kutta with interpolating extensions [48]. The bvp4c solver employs a collocation method for which conditions at $\omega \to -\infty$ and $\omega \to \infty$ shall be given (see [49] for further insights into the Jacobi-Gauss collocation method to numerically solve a fractional singular delay integro-differential equation). In our analysis, the conditions at the border were set as follows: at $-\infty$ solutions are

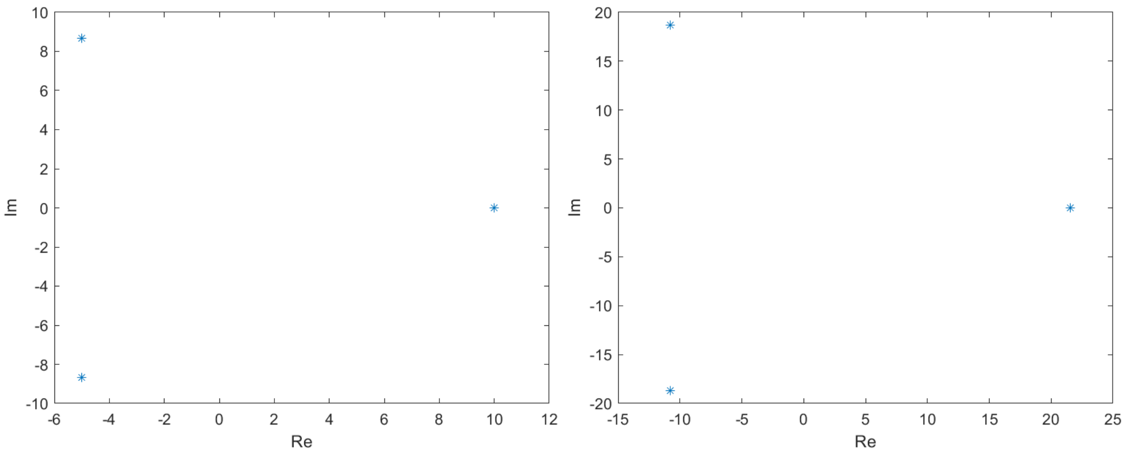

Figure 1. $\lambda = 1$ (on the left) and $\lambda = 10$ (on the right). Representation of $R(\mu)$ roots. The value of ε has been considered as arbitrarily large. Note that ε makes an impact on the scaling of the homotopy while keeping the same graph structure (for our purposes, one eigenvalue has been kept positive independently of the value considered for ε).

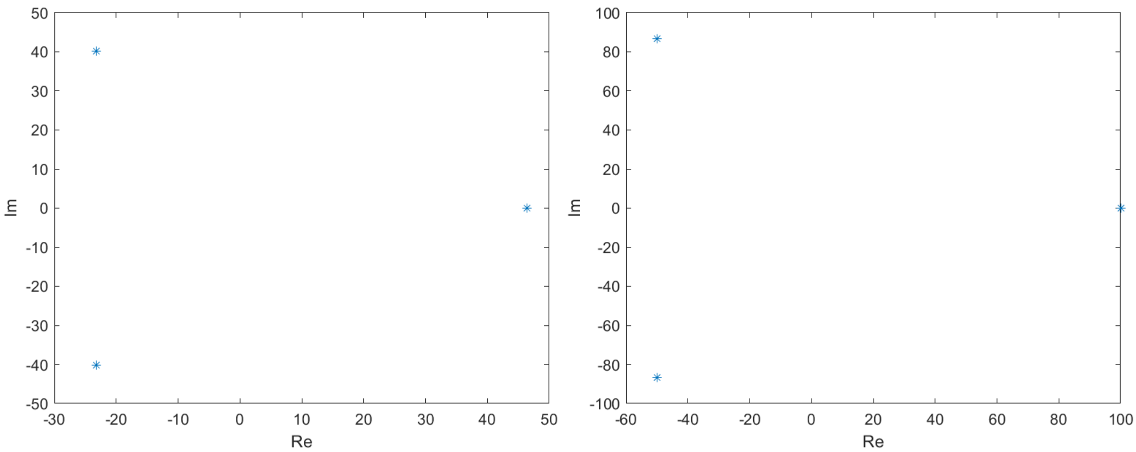

Figure 2. $\lambda = 100$ (on the left) and $\lambda = 1000$ (on the right). Representation of $R(\mu)$ roots.

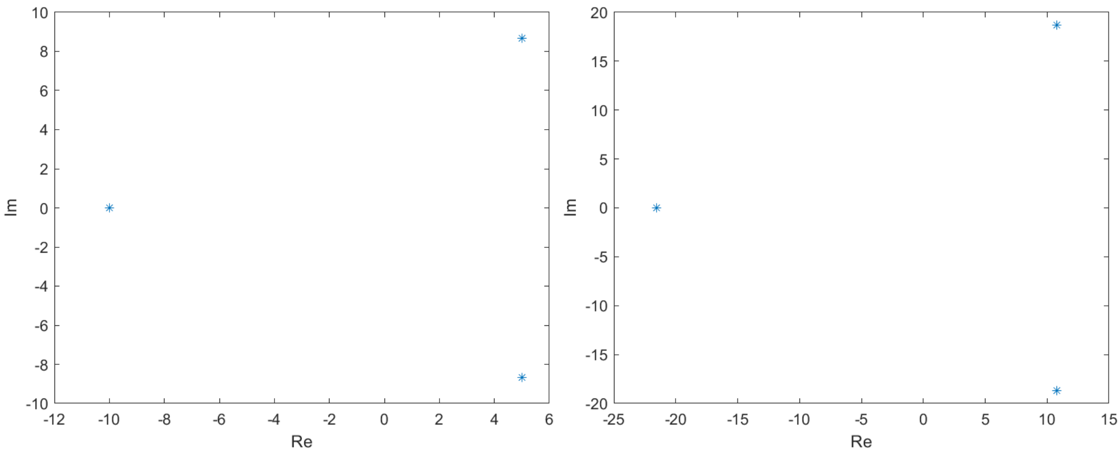

Figure 3. $\lambda = -1$ (on the left) and $\lambda = -10$ (on the right). Representation of $R(\mu)$ roots. The value of ε has been considered as arbitrarily large. Note that ε makes an impact on the scaling of the homotopy while keeping the same graph structure (for our purposes, one eigenvalue has been kept positive independently of the value considered for ε).

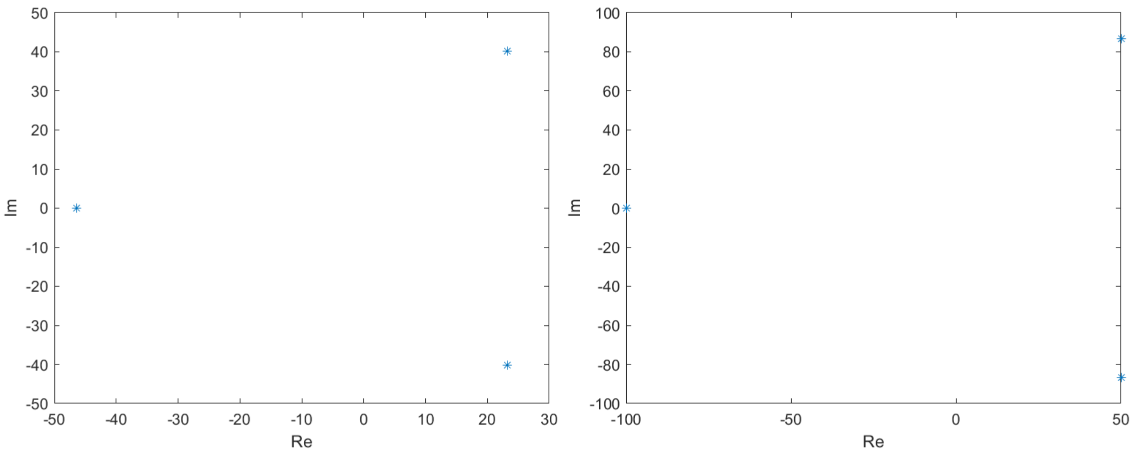

Figure 4. $\lambda = -100$ (on the left) and $\lambda = -1000$ (on the right). Representation of $R(\mu)$ roots.

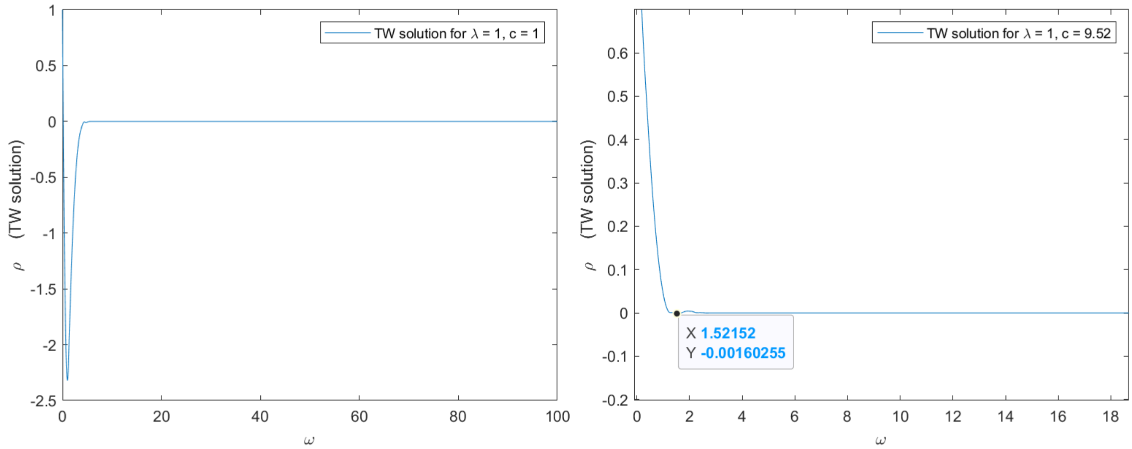

Figure 5. m = 3, p=2. Profiles of traveling wave solutions for $\lambda = 1$ and increasing values of the advection coefficient c. Note that, for increasing values of c, the first minimum is closer to zero. For the combination $\lambda = 1$ and $c = 9.52$ the profile is almost positive.

considered to be positive (i.e., $\rho(-\infty) = 1$) while at ∞, no conditions are imposed as the attractive properties of the null critical state shall lead solutions to the proximity.

The numerical scheme was run over a large domain $\omega \in [-1000, 1000]$ so that the possible impact of the collocation method at the pseudo-infinity borders would be minimized. The considered absolute error was 10^{-5}.

To account for a tractable problem, specific values for the involved parameters in (1.1) have been provided. In particular, $p = 2$ and $m = 3$. For this specific set of values and with no loss of generality, the solutions were studied for different combinations of traveling wave speeds (λ) and advection coefficients (c). It is possible to check that, for a given traveling wave speed, an increase of the advection magnitude has an smoothing effect on the profiles (see Figure 5), reducing the oscillations paths and increasing the region wherein positivity of the solutions holds. Nonetheless, when the advection coefficient is beyond a critical value, the oscillations emerge again (see Figure 6). Such a critical value has been precisely assessed as $c^* \in [9.52\,, 10]$ for $\lambda = 1$. Note that even when the numerical exercise has been done for the pair $m = 3$ and $p = 2$, the conclusions obtained in terms of the existence of a critical c^* holds for any other combination of $m > 2$ and $p \geq 1$.

5. Conclusions

The analysis provided has introduced results to characterize a problem with a higher order p-Laplacian operator and a nonlinear advection term. The problem was first formulated by applying a variational approach to study the local preservation of the support, uniqueness and bounds of the solutions. The oscillations induced by the p-Laplacian operator have been shown to hold based on the introduction of generalized Hilbert-Sobolev spaces. Afterward, a numerical algorithm was introduced to further characterize the smoothing effect of the introduced nonlinear advection term. It has been shown that for specific values of the traveling wave speed (λ) and advection coefficient (c^*), the first minimum in the traveling profile is almost positive and positivity holds beyond such a first minimum. Although the numerical assessment was done for particular values of m and p, the same conclusion

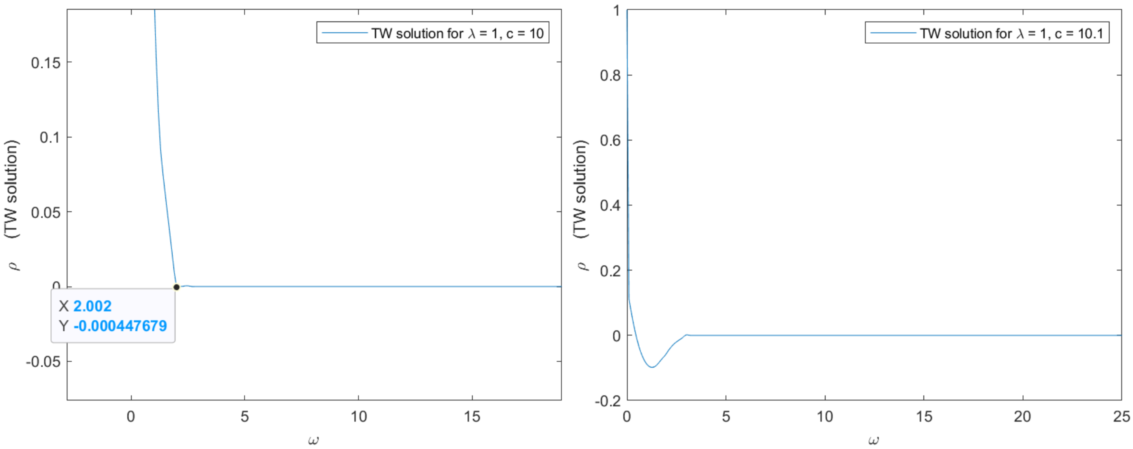

Figure 6. m = 3, p=2. For $c = 10$, the first minimum is almost positive and the evolution beyond such minimum is positive. For increasing values of c, as close as $c = 10.1$, the oscillating behavior appears again and positivity does not hold in general.

holds, i.e., there exists a combination of λ and c^* for which the first minimum is almost positive and positivity holds beyond such a minimum.

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Conflict of interests

The authors declare that there are not any conflicts of interest.

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