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Research article

Stability analysis and backward bifurcation on an SEIQR epidemic model with nonlinear innate immunity

Xueyong Zhou*and Xiangyun Shi

School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China

* **Correspondence:** Email: xueyongzhou@xynu.edu.cn; Tel: +8615039743856.

Abstract: Infectious diseases have a great impact on the economy and society. Dynamic models of infectious diseases are an effective tool for revealing the laws of disease transmission. Quarantine and nonlinear innate immunity are the crucial factors in the control of infectious diseases. Currently, there no mathematical models that comprehensively study the effect of both innate immunity and quarantine. In this paper, we propose and analyze an SEIQR epidemic model with nonlinear innate immunity. The boundedness and positivity of the solutions are discussed. Employing the next-generation matrix, we compute the expression of the basic reproduction number. Under certain conditions, the phenomenon of backward bifurcation may occur. That is to say, the stable disease-free equilibrium point and the stable endemic equilibrium point coexist when the basic reproduction ratio is less than one. And the basic reproduction number is no longer the threshold value to determine whether the disease breaks out. We investigate the globally asymptotical stability of the disease-free equilibrium point for the system by constructing Lyapunov function. Also, we research the global stability of the endemic equilibrium by using geometric approach. Numerical simulations are carried out to reveal the theoretical results and find some complex dynamics (for example, the existence of Hopf bifurcation) of the system. Both theoretical and numerical results indicate that the nonlinear innate immunity may cause backward bifurcation and Hopf bifurcation, which makes more difficult to eliminate the disease.

Keywords: innate immunity; SEIQR epidemic model; backward bifurcation; stability

1. Introduction

Infectious diseases can spread and turn into epidemics, taking thousands of lives within a matter of just a few days. Many scholars have studied the epidemiological mechanism of infectious diseases through various methods and given preventive strategies. Mathematical models can help us to gain insights into the dynamics of diseases and their control strategies [1–3]. Epidemic models have been developed by many scholars since the first epidemiological model created by Daniel Bernoulli [4].

In [5], considering both individual behavioral responses and governmental actions, Lin et al. proposed a conceptual model for the COVID-19 outbreak in Wuhan, China. In order to forecast the evolution of the COVID-19 outbreak in Mexico, Avila-Ponce de León et al. proposed an SEIARD mathematical model which incorporated the asymptomatic infected individuals [6]. In [7], a bacterial meningitis transmission dynamics was considered by Asamoah et al. The existence of backward bifurcation was discussed and optimal control problem was solved. In [8], an ordinary differential model of malaria was established. In this paper, stability of disease-free and endemic equilibria, bifurcation phenomena were investigated. In [9], a delay Ebola epidemic model was studied by Al-Darabsah et al. In [10], He et al. built an SEIR model for COVID-19 incorporating some general control strategies. In [11], based on the COVID-19 data in Ghana and Egypt, Asamoah et al. formulated a COVID-19 infection model to present the sensitivity assessment and optimal economic evaluation. In [12], Zhao et al. studied a stochastic switched SIRS epidemic model. The stationary distribution and extinction of the disease were discussed. In [13], Omame et al. considered a co-infection model for SARS-CoV-2 and ZIKV, which exhibited backward bifurcation. In [14], considering the infectious force in latent period and infected period, Zhao et al. lucubrated an SEIR epidemic model with discontinuous treatment strategy.

For the sake of the effective strategies to disease control and prevention, quarantine is the most effective way to reduce the transmission of the infected to the susceptible. For example, many countries have taken quarantine measures in the fight against COVID-19. Many scholars introduced the quarantine class into the epidemic models. In [15], Herbert et al. showed six epidemic models with quarantine and different forms of the incidence. In [16], a compartmental model incorporating asymptomatic class, quarantine and isolation was presented by Ali et al. The strategies for effective control of the epidemic were proposed by analyzing the model. In [17], Tulu et al. built a fractional-order model for Ebola with the strategies to vaccination and quarantine. They gained that vaccination and quarantine are effective control measures for Ebola.

Innate immunity is the body's natural immune defense function formed during germline development and evolution [18]. It is a series of defense mechanisms formed by organisms in the process of long-term evolution. In [19], Kabir et al. built an epidemic model for natural and artificial immunity with durability and imperfectness of protection. In [20, 21], the authors considered the mathematical models with the nonlinear innate immunity rate. However, there currently no models that comprehensively consider the effect of the nonlinear innate immunity and quarantine.

This paper is organized as follows. In Section 2, we built an SEIQR epidemic model and describe it. Preliminaries, such as boundedness, positivity of system (2.2) are discussed in Section 3. In Section 4, we present the expressions of equilibria and the basic reproduction number. The global stability result of the disease-free equilibrium and the existence result of backward bifurcation for system (2.2) are derived in Section 5. In Section 6, the global stability of endemic equilibrium is shown via using geometric approach. We present some numerical examples to verify the theoretical results obtained in previous sections. Finally, we conclude this article with a brief discussion.

2. Model formulation

In this work, therefore, we will investigate the effects of quarantine and nonlinear innate immunity. And we shall consider an SEIR model with quarantine and nonlinear innate immunity. Suppose that $N(t)$ denotes the number of total population. The total population is subdivided into five different classes, namely, the susceptible ($S(t)$) individuals, the exposed individuals ($E(t)$), the infected individuals ($I(t)$), the quarantined individuals ($Q(t)$) and the recovered individuals ($R(t)$). Hence, $N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$. In the present paper, we assume that the recovered individuals confer permanent immunity and they do not revert to the susceptible individuals, like varicella, measles, rubella [22,23]. The flowchart of the epidemiological SEIQR model is illustrated in Figure 1.

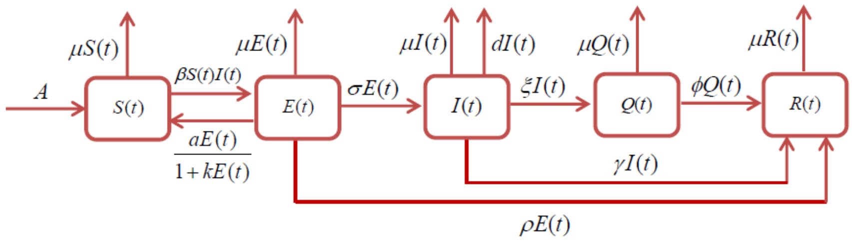


Figure 1. The flowchart of the proposed SEIQR model.

From the flowchart, we will establish the following model:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + \frac{aE(t)}{1+kE(t)}, \\ \frac{dE(t)}{dt} = \beta S(t)I(t) - \sigma E(t) - \rho E(t) - \mu E(t) - \frac{aE(t)}{1+kE(t)}, \\ \frac{dI(t)}{dt} = \sigma E(t) - \xi I(t) - \gamma I(t) - \mu I(t) - dI(t), \\ \frac{dQ(t)}{dt} = \xi I(t) - \phi Q(t) - \mu Q(t), \\ \frac{dR(t)}{dt} = \rho E(t) + \gamma I(t) + \phi Q(t) - \mu R(t). \end{cases} \tag{2.1}$$

In model (2.1), the parameters A, μ, d, σ, ξ, ρ, γ, ϕ, β, a and k are nonnegative. A is the replenishment rate of susceptibles; β is the infection rate; μ is the natural death rate of the population; d denotes the disease-related mortality rate; σ denotes the transfer rate between the exposed and the infectious; ξ is quarantine rate of the infected; ρ is the recovery rate of the infectious individuals; ϕ is the recovery rate of the quarantine individuals. We assume that the exposed transformed into the susceptible with the nonlinear innate immunity rate $\frac{aE(t)}{1+kE(t)}$ [20,21]. For biological significance, we postulate that the initial conditions of system (2.1) satisfy: $S(0) \geq 0$, $E(0) \geq 0$, $I(0) \geq 0$, $Q(0) \geq 0$ and $R(0) \geq 0$.

Since $R(t)$, the recovered population, is independent of the first four equations of system (2.1), the rest of the paper will consider only the following four-dimensional system:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + \frac{aE(t)}{1+kE(t)}, \\ \frac{dE(t)}{dt} = \beta S(t)I(t) - \sigma E(t) - \rho E(t) - \mu E(t) - \frac{aE(t)}{1+kE(t)}, \\ \frac{dI(t)}{dt} = \sigma E(t) - \xi I(t) - \gamma I(t) - \mu I(t) - dI(t), \\ \frac{dQ(t)}{dt} = \xi I(t) - \phi Q(t) - \mu Q(t). \end{cases} \tag{2.2}$$

The initial conditions associated with the system (2.2) are as follows

$$S(0) \geq 0, \ E(0) \geq 0, \ I(0) \geq 0, \ Q(0) \geq 0. \tag{2.3}$$

3. Basic properties

In this section, we shall present some basic properties such as boundedness, positivity of system (2.2).

Theorem 3.1. For all $t > 0$, the solutions $(S(t), E(t), I(t), Q(t))$ of system (2.2) with initial condition (2.3) are positive.

Proof. Set $t_1 = \sup\{t > 0 : S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, Q(0) \geq 0\}$. The following inequality is given by the first equation of system (2.2)

$$\frac{dS(t)}{dt} \geq A - \beta S(t)I(t) - \mu S(t).$$

The above inequality can be rewritten as:

$$\frac{dS(t)}{dt}\{S(t)\exp[\mu t + \int_0^t \beta I(\varsigma)d\varsigma]\} \geq A\{\exp[\mu t + \int_0^t \beta I(\varsigma)d\varsigma]\}.$$

Thus

$$S(t_1)\exp[\mu t_1 + \int_0^{t_1} \beta I(\varsigma)d\varsigma] - S(0) \geq \int_0^{t_1} A\exp[\mu z + \int_0^z \beta I(\varsigma)d\varsigma]dz,$$

so that

$$\begin{aligned} S(t_1) \geq\ & S(0)\exp[-\mu t_1 - \int_0^{t_1} \beta I(\varsigma)d\varsigma] + \\ & \exp[-\mu t_1 - \int_0^{t_1} \beta I(\varsigma)d\varsigma]\int_0^{t_1} A\exp[\mu z + \int_0^z \beta I(\varsigma)d\varsigma]dz \\ \geq\ & 0. \end{aligned}$$

Similar to the above method, we can obtain $E(t) \geq 0$, $I(t) \geq 0$, $Q(t) \geq 0$ for all time $t > 0$. Hence, for all $t > 0$, the solutions of system (2.2) satisfying the initial value condition (2.3) are positive. This completes the proof of Theorem 3.1.

Define

$$\mathcal{D} = \{(S, E, I, Q) \in \mathbb{R}_4^+ : 0 \leq S + E + I + Q \leq \frac{A}{\mu}\}. \tag{3.1}$$

Theorem 3.2. The region $\mathcal{D}$ is invariant, which indicates that all solutions of system (2.2) with initial condition (2.3) in $\mathcal{D}$ remain in $\mathcal{D}$ for all $t > 0$.

Proof. Adding the two sides of system (2.1) respectively, we have

$$\frac{dN(t)}{dt} = A - \mu N(t) - dI(t).$$

Hence

$$\frac{dN(t)}{dt} \leq A - \mu N(t).$$

From above we obtain that

$$0 \leq N(t) \leq \frac{A}{\mu} + (S(0) + E(0) + I(0) + Q(0) + R(0))\exp(-\mu t).$$

Therefore, if $N(0) < \frac{A}{\mu}$, then

$$\limsup_{t\to+\infty}(S(t) + E(t) + I(t) + Q(t) + R(t)) \leq \frac{A}{\mu}.$$

Hence, for all $t > 0$,

$$[S(t) + E(t) + I(t) + Q(t)] \le \frac{A}{\mu}.$$

That is, all orbits of system (2.2) with initial conditions $S(0) \ge 0, E(0) \ge 0, I(0) \ge 0, Q(0) \ge 0$ in $\mathcal{D}$ remain in $\mathcal{D}$ for all $t > 0$. Thus, the region $\mathcal{D}$ is positively-invariant. Furthermore, if $N(0) > \frac{A}{\mu}$, then either $N(t)$ approaches $\frac{A}{\mu}$ as $t \to \infty$ or the solution enters $\mathcal{D}$ in finite time. Hence, the region $\mathcal{D}$ attracts all solutions in $\mathbb{R}^4_+$.

Throughout this paper, we shall consider the dynamical behaviors of system (2.2) on the region $\mathcal{D}$.

4. Equilibria

Setting the right-hand sides of system (2.2) to 0, we can get that system (2.2) has only one disease-free equilibrium, denoted by $P_0(\frac{A}{\mu}, 0, 0, 0)$.

Next, we shall calculate the basic reproduction number, denoted by $\mathcal{R}_0$, of system (2.2) by applying the next generation matrix method in [24] offered by van den Driessche et al.

Let $x = (I, E)^\top$. We can re-express system (2.2) as follows

$$\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$\mathcal{F}(x) = \begin{pmatrix} \beta S I \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V}(x) = \begin{pmatrix} (\sigma + \rho + \mu)E + \frac{aE}{1+kE} \\ -\sigma E + (\xi + \gamma + \mu + d)I \\ -\xi I + (\phi + \mu)Q \\ -A + \beta S I + \mu S - \frac{aE}{1+kE} \end{pmatrix}.$$

We can obtain

$$F = \begin{pmatrix} \frac{\beta A}{\mu} & 0 \\ 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \sigma + \rho + \mu + a & 0 \\ -\sigma & \xi + \gamma + \mu + d \end{pmatrix}.$$

Then the next generation matrix for system (2.2) is

$$FV^{-1} = \begin{pmatrix} \frac{A\sigma\beta}{\mu(\xi+\gamma+\mu+d)(\mu+\sigma+\rho+a)} & \frac{A\beta}{\mu(\xi+\gamma+\mu+d)} \\ 0 & 0 \end{pmatrix}.$$

The spectral radius of matrix FV^{-1}, denoted by $\rho(FV^{-1})$, equals to $\dfrac{A\sigma\beta}{\mu(\xi + \gamma + \mu + d)(\mu + \sigma + \rho + a)}$. According to Theorem 2 in literature [24], the basic reproductive rate of system (2.2) is spectral radius $\rho(FV^{-1})$, i.e.,

$$\mathcal{R}_0 = \frac{A\sigma\beta}{\mu(\xi+\gamma+\mu+d)(\mu+\sigma+\rho+a)}.$$

From [24], the local stability of the disease free equilibrium $P_0(\frac{A}{\mu},0,0,0)$ for system (2.2) can directly established. The result is listed as following.

Theorem 4.1. The disease-free equilibrium $P_0(\frac{A}{\mu},0,0,0)$ of system (2.2) is locally asymptotically stable when $\mathcal{R}_0 < 1$; and $P_0(\frac{A}{\mu},0,0,0)$ is unstable when $\mathcal{R}_0 > 1$.

In the following, we shall discuss the existence of the endemic equilibrium. Let $P^*(S^*,E^*,I^*,Q^*)$ be an arbitrary endemic equilibrium of system (2.2). Setting the right-hand sides of (2.2) to 0, we can get $E^* = \frac{\xi+\gamma+\mu+d}{\sigma}I^*$, $S^* = \frac{A(1+kE^*)+aE^*}{\beta I^*+\mu}$, $Q^* = \frac{\xi}{\phi+\mu}I^*$. Here I^* satisfies the following quadratic equation

$$A_1(I^*)^2 + A_2 I^* + A_3 = 0, \tag{4.1}$$

where

$$\begin{cases} A_1 = k\beta(\sigma+\rho+\mu), \\ A_2 = (\sigma+\rho+\mu)(\beta\sigma+k\gamma\mu+k\mu d+k\mu^2+k\xi\mu) - Ak\beta\sigma, \\ A_3 = \sigma\mu(\xi+\gamma+\mu+d)(\mu+\sigma+\rho+a)(1-\mathcal{R}_0). \end{cases} \tag{4.2}$$

Denote

$$\begin{aligned}\Delta &= A_2^2 - 4A_1A_3 \\ &= A_2^2 - 4A_1\sigma\mu(\xi+\gamma+\mu+d)(\mu+\sigma+\rho+a)(1-\mathcal{R}_0).\end{aligned}$$

Solving the equation $\Delta = 0$, we can obtain that $\mathcal{R}_0 = \mathcal{R}^*$, where

$$\mathcal{R}^* = 1 - \frac{A_2^2}{4A_1\sigma\mu(\xi+\gamma+\mu+d)(\mu+\sigma+\rho+a)}.$$

The following equivalent relations are true:

$$\Delta < 0 \Leftrightarrow \mathcal{R}^* < \mathcal{R}_0;\ \ \Delta = 0 \Leftrightarrow \mathcal{R}^* = \mathcal{R}_0;\ \ \Delta > 0 \Leftrightarrow \mathcal{R}^* > \mathcal{R}_0.$$

Hence, for the existence of equilibria of system (2.2), the following conclusions are correct.

Theorem 4.2. System (2.2) always exists a disease free equilibrium P_0 and

(i) if $\mathcal{R}^* < \mathcal{R}_0$ or $\mathcal{R}^* = \mathcal{R}_0$ or $\mathcal{R}^* < \mathcal{R}_0 < 1$ and $A_2 > 0$, system (2.2) has no endemic equilibrium;

(ii) if $\mathcal{R}^* < \mathcal{R}_0 = 1$ and $A_2 < 0$ or $\mathcal{R}_0 > 1$ or $\mathcal{R}^* = \mathcal{R}_0 < 1$, system (2.2) has only one endemic equilibrium $P^*(S^*,E^*,I^*,Q^*)$;

(iii) if $\mathcal{R}^* < \mathcal{R}_0 < 1$ and $A_2 < 0$, system (2.2) has two unequal endemic equilibrium points denoted by $P_*(S_*,E_*,I_*,Q_*)$ and $P^*(S^*,E^*,I^*,Q^*)$, where

$$I_* = \frac{-A_2 - \sqrt{\Delta}}{2A_1},\ \ I^* = \frac{-A_2 + \sqrt{\Delta}}{2A_1}.$$

5. Stability of disease-free equilibrium P_0

In this section we shall investigate the locally and globally asymptotical stability of disease-free equilibrium $P_0(\frac{A}{\mu}, 0, 0, 0)$ and endemic equilibrium P^*.

Theorem 5.1. The disease free equilibrium $P_0(\frac{A}{\mu}, 0, 0, 0)$ is globally asymptotically stable when $\mathcal{R}_1 < 1$, where $\mathcal{R}_1 = \dfrac{A\sigma\beta}{\mu(\xi+\gamma+\mu+d)(\sigma+\rho+\mu+\frac{\mu a}{\mu+kA})}$.

Proof. Consider Lyapunov function $V(t) = a_1E(t) + a_2I(t) + a_3Q(t)$, where a_1, a_2, a_3 are undetermined non-negative real numbers. Then the derivative of $V(t)$ along the solution curves of (2.2) has the following form,

$$\begin{aligned}\frac{dV(t)}{dt} &= a_1\frac{dE(t)}{dt} + a_2\frac{dI(t)}{dt} + a_3\frac{dQ(t)}{dt}\\ &= a_1(\beta SI - \sigma E - \rho E - \mu E - \frac{aE}{1+kE}) + a_2(\sigma E - \xi I - \gamma I - \mu I - dI)\\ &\quad + a_3(\xi I - \phi Q - \mu Q)\\ &\le a_1[\frac{\beta A}{\mu}I - (\sigma+\rho+\mu+\frac{\mu a}{\mu+kA})E] + a_2[\sigma E - (\xi+\gamma+\mu+d)I]\\ &\quad + a_3[\xi I - (\phi+\mu)Q].\end{aligned}$$

Now we select the coefficients a_1, a_2 and a_3, with the zero coefficients of E and I. Hence we obtain

$$a_1 = \sigma,\ \ a_2 = \sigma+\rho+\mu+\frac{\mu a}{\mu+kA},\ \ a_3 = \frac{1}{\xi}[(\sigma+\rho+\mu+\frac{\mu a}{\mu+kA})(\xi+\gamma+\mu+d) - \frac{\sigma\beta A}{\mu}].$$

Substituting the values of a_1, a_2 and a_3 to $V(t)$, the derivative of $V(t)$ can be expressed as

$$\frac{dV(t)}{dt} \le \frac{1}{\xi}[\frac{\sigma\beta A}{\mu} - (\sigma+\rho+\mu+\frac{\mu a}{\mu+kA})(\xi+\gamma+\mu+d)](\phi+\mu)Q.$$

Clearly, $\frac{dV(t)}{dt} \le 0$ when $\mathcal{R}_1 = \dfrac{A\sigma\beta}{\mu(\xi+r+\mu+d)(\sigma+\rho+\mu+\frac{\mu a}{\mu+kA})} < 1$. Furthermore, $\frac{dV(t)}{dt} = 0$ if and only if $E = I = Q = 0$.

Next, we shall examine the possibility of backward bifurcation for system (2.2). To do this, we firstly introduce the approach presented by Castillo-Chaves et al. (see [25]), which is based on the use of the general center manifold theory [26]. Considering a general system of ODEs with a parameter ς:

$$\frac{d\chi}{dt} = F(\chi, \varsigma);\ \ F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \text{ and } F \in C^2(\mathbb{R}^n \times \mathbb{R}). \tag{5.1}$$

Assume that $\chi = 0$ is an equilibrium for system (5.1) for all values of the parameter ς, that is $F(0, \varsigma) \equiv 0$, for all ς. Let $Q = D_\chi F(0, 0) = (\frac{\partial F_i}{\partial \chi_i}(0, 0))$ be the Jacobian matrix of $F(\chi, \varsigma)$ at point $(0, 0)$.

Lemma 5.1. (Castillo-Chavez and Song [25]) Assume:

(A1) 0 is a simple eigenvalue of Q and all other eigenvalues of Q have negative real parts;

(A2) Matrix Q has a (non-negative) right eigenvector $\mathbf{w} = (w_1, w_2, \cdots, w_n)^\top$ and a left eigenvector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ corresponding to the zero eigenvalue.

Let F_k denotes the k^{th} component of F and,

$$\mathbf{a} = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 F_k}{\partial \chi_i \partial \chi_j}(0, 0),$$

$$\mathbf{b} = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 F_k}{\partial \chi_i \partial \varsigma}(0,0).$$

Then the local dynamics of system (5.1) around $\chi = 0$ are totally determined by **a** and **b**.

1) $\mathbf{a} > 0$, $\mathbf{b} > 0$. When $\varsigma < 0$, with $|\varsigma| \ll 1$, $\chi = 0$ is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 < \varsigma \ll 1$, $\chi = 0$ is unstable and there exists a negative and locally asymptotically stable equilibrium;

2) $\mathbf{a} < 0$, $\mathbf{b} < 0$. When $\varsigma < 0$, with $|\varsigma| \ll 1$, $\chi = 0$ is unstable; when $0 < \varsigma \ll 1$, $\chi = 0$ is locally asymptotically stable and there exists a positive unstable equilibrium;

3) $\mathbf{a} > 0$, $\mathbf{b} < 0$. When $\varsigma < 0$, with $|\varsigma| \ll 1$, $\chi = 0$ is unstable and there exists a locally asymptotically stable negative equilibrium; when $0 < \varsigma \ll 1$, $\chi = 0$ is stable and a positive unstable equilibrium appears;

4) $\mathbf{a} < 0$, $\mathbf{b} > 0$. When ς changes from negative to positive, $\chi = 0$ changes its stability from stable to unstable. Correspondently, a negative unstable equilibrium becomes positive and locally asymptotically stable.

Remark 5.1. The requirement that **w** is non-negative is unnecessary (see [25]).

Introducing $S = \chi_1$, $E = \chi_2$, $I = \chi_3$, $Q = \chi_3$, we rewrite system (2.2) as

$$\begin{cases} \frac{d\chi_1(t)}{dt} = A - \beta\chi_1\chi_3 - \mu\chi_1 + \frac{a\chi_2}{1+k\chi_2} := F_1, \\ \frac{d\chi_2(t)}{dt} = \beta\chi_1\chi_3 - \sigma\chi_2 - \rho\chi_2 - \mu\chi_2 - \frac{a\chi_2}{1+k\chi_2} := F_2, \\ \frac{d\chi_3(t)}{dt} = \sigma\chi_2 - \xi\chi_3 - \gamma\chi_3 - \mu\chi_3 - d\chi_3 := F_3, \\ \frac{d\chi_4(t)}{dt} = \xi\chi_3 - \phi\chi_4 - \mu\chi_4 := F_4. \end{cases} \tag{5.2}$$

We will apply Lemma 5.1 to show that system (5.2) may exhibit a backward bifurcation when $\mathcal{R}_0 = 1$. We consider the parameter β as bifurcation parameter. Corresponding to $\mathcal{R}_0 = 1$, we can get $\beta = \beta^* = \frac{\mu(\xi+r+\mu+d)(\mu+\sigma+\rho+a)}{A\sigma}$.

The Jacobi matrix of system (5.2) is

$$J(P_0,\beta^*) = \begin{pmatrix} -\mu & a & -\frac{\beta A}{\mu} & 0 \\ 0 & -(\sigma+\rho+\mu+a) & \frac{\beta A}{\mu} & 0 \\ 0 & \sigma & -(\mu+d+\gamma+\xi) & 0 \\ 0 & 0 & \xi & -(\phi+\mu) \end{pmatrix}.$$

And the eigenvalues of $J(P_0,\beta^*)$ are given by $\lambda_1 = -\mu$, $\lambda_2 = 0$, $\lambda_3 = -(\mu+d+\gamma+\xi)$ and $\lambda_4 = -(\phi+\mu)$.

Obviously, the matrix $J(P_0,\beta^*)$ has a simple zero eigenvalue $\lambda_2 = 0$. And the other eigenvalues of $J(P_0,\beta^*)$ are negative real numbers. Therefore, we can use the center manifold theory to discuss the dynamics of system (2.2) when $\mathcal{R}_0 = 1$. Hence, the disease free equilibrium P_0 is a nonhyperbolic equilibrium.when $\beta = \beta^*$ (or equivalently when $\mathcal{R}_0 = 1$). Therefore, the assumption (A1) of Lemma 5.1 is then verified.

Now we will calculate a right eigenvector of the matrix $J(P_0,\beta^*)$ associated with the zero eigenvalue $\lambda_2 = 0$, denoted by $\mathbf{w} = (w_1, w_2, w_3, w_4)^\top$. It is found by

$$\begin{pmatrix} -\mu & a & -\frac{\beta A}{\mu} & 0 \\ 0 & -(\sigma+\rho+\mu+a) & \frac{\beta A}{\mu} & 0 \\ 0 & \sigma & -(\mu+d+\gamma+\xi) & 0 \\ 0 & 0 & \xi & -(\phi+\mu) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = 0.$$

Thus, we can get

$$\begin{cases} -\mu w_1 + a w_2 - \frac{\beta A}{\mu} w_3 = 0, \\ -(\sigma+\rho+\mu+a) w_2 + \frac{\beta A}{\mu} w_3 = 0, \\ \sigma w_2 - (\mu+d+\gamma+\xi) w_3 = 0, \\ \xi w_3 - (\phi+\mu) w_4 = 0. \end{cases}$$

This implies $w_1 = \frac{\phi+\mu}{\mu}[\frac{a(\mu+d+\gamma+\xi)}{\sigma} - \frac{\beta A}{\mu}]$, $w_2 = \frac{(\phi+\mu)(\mu+d+\gamma+\xi)}{\sigma}$, $w_3 = \phi+\mu$, $w_4 = \xi$. Therefore, the right eigenvector is

$$\mathbf{w} = (\frac{\phi+\mu}{\mu}[\frac{a(\mu+d+\gamma+\xi)}{\sigma} - \frac{\beta A}{\mu}], \frac{(\phi+\mu)(\mu+d+\gamma+\xi)}{\sigma}, \phi+\mu, \xi)^\top. \tag{5.3}$$

Furthermore, the left eigenvector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ of the matrix $J(P_0,\beta^*)$ which satisfies $\mathbf{v}\cdot\mathbf{w} = 1$ is given by

$$\begin{cases} -\mu v_1 = 0, \\ a v_1 - (\sigma+\rho+\mu+a) v_2 + \sigma v_3 = 0, \\ -\frac{\beta A}{\mu} v_1 + \frac{\beta A}{\mu} v_2 - (\mu+d+\gamma+\xi) v_3 + \xi v_4 = 0, \\ -(\phi+\mu) v_4 = 0. \end{cases}$$

Then, the left eigenvector $\mathbf{v}$ turns out to be

$$\mathbf{v} = (0, \frac{\sigma}{(\phi+\mu)(2\mu+d+\gamma+\xi+\sigma+\rho+a)}, \frac{\sigma+\rho+\mu+a}{(\phi+\mu)(2\mu+d+\gamma+\xi+\sigma+\rho+a)}, 0). \tag{5.4}$$

Calculating all the partial derivatives of F_i $(i = 1,2,3,4)$ with respect to χ_i $(i = 1,2,3,4)$ and β at the disease-free equilibrium $P_0(\frac{A}{\mu}, 0, 0, 0)$, we get

$$\frac{\partial^2 F_1}{\partial\chi_1\partial\chi_3} = \frac{\partial^2 F_1}{\partial\chi_3\partial\chi_1} = -\beta,\ \frac{\partial^2 F_1}{\partial\chi_2^2} = -2ak,$$

$$\frac{\partial^2 F_2}{\partial\chi_1\partial\chi_3} = \frac{\partial^2 F_2}{\partial\chi_1\partial\chi_3} = \beta,\ \frac{\partial^2 F_2}{\partial\chi_2^2} = 2ak,$$

$$\frac{\partial^2 F_1}{\partial\chi_3\partial\beta} = -\frac{\beta A}{\mu},\ \frac{\partial^2 F_2}{\partial\chi_3\partial\beta} = \frac{\beta A}{\mu},$$

and all the other second-order partial derivatives are equal to 0.

Thus, we can calculate the coefficients $\mathbf{a}$ and $\mathbf{b}$ defined in Lemma 5.1, i.e.,

$$\mathbf{a} = \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 F_k}{\partial\chi_i\partial\chi_j}(P_0,\beta^*),$$

$$\mathbf{b} = \sum_{k,i=1}^{4} v_k w_i \frac{\partial^2 F_k}{\partial\chi_i\partial\beta}(P_0,\beta^*).$$

Considering system (5.2) and taking into account of $\mathbf{a}$ and $\mathbf{b}$ only the nonzero derivatives of the terms $\frac{\partial^2 F_k}{\partial\chi_i\partial\chi_j}(P_0,\beta^*)$ and $\frac{\partial^2 F_k}{\partial\chi_i\partial\beta}(P_0,\beta^*)$, it follows that

$$\begin{aligned}\mathbf{a} = {} & 2v_1 w_1 w_3 \tfrac{\partial^2 F_1}{\partial\chi_1\partial\chi_3}(P_0,\beta^*) + v_1 w_2^2 \tfrac{\partial^2 F_1}{\partial\chi_2^2}(P_0,\beta^*) \\ & +2v_2 w_1 w_3 \tfrac{\partial^2 F_2}{\partial\chi_1\partial\chi_3}(P_0,\beta^*) + v_2 w_2^2 \tfrac{\partial^2 F_2}{\partial\chi_2^2}(P_0,\beta^*),\end{aligned}$$

and

$$\mathbf{b} = v_1 w_3 \frac{\partial^2 F_1}{\partial \chi_3 \partial \beta}(P_0, \beta^*) + v_2 w_3 \frac{\partial^2 F_2}{\partial \chi_3 \partial \beta}(P_0, \beta^*).$$

From (5.3) and (5.4), we obtain

$$\mathbf{a} = \frac{2\sigma(\phi+\mu)^2}{(\phi+\mu)(2\mu+d+\gamma+\xi+\sigma+\rho+a)}[\frac{\beta}{\mu}(\frac{a(\mu+d+\gamma+\xi)}{\sigma} - \frac{\beta A}{\mu}) + \frac{ak(\mu+d+\gamma+\xi)^2}{\sigma^2}], \tag{5.5}$$

and

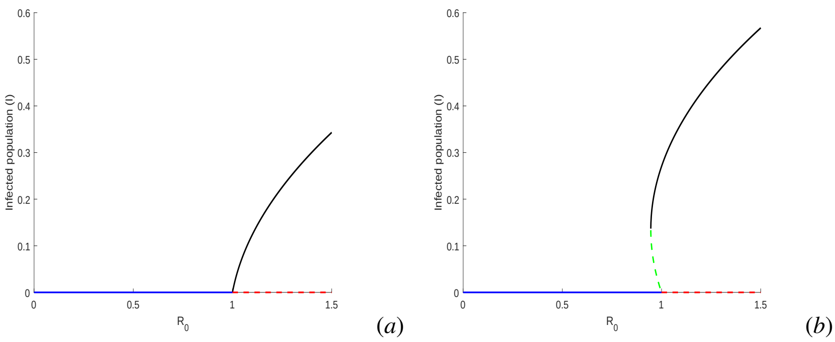
$$\mathbf{b} = \frac{\sigma}{2\mu + d + \gamma + \xi + \sigma + \rho + a}\frac{\beta A}{\mu}.$$

Obviously, the coefficient **b** is always positive. According to Lemma 5.1 we can conclude that the sign of **a** determines the local dynamics of the disease-free equilibrium P_0 when $\beta = \beta^*$.

Define $\mathcal{R}_1^* = \frac{a(\mu+d+\gamma+\xi)}{\beta A\sigma}(\frac{1}{\mu} + \frac{\mu^2 k}{\beta\sigma})$. Note that $\mathbf{a} < 0$ if $\mathcal{R}_1^* < 1$, and $\mathbf{a} > 0$ if $\mathcal{R}_1^* > 1$. Hence, from Lemma 5.1, we have the following results.

Theorem 5.2. (1) Assume that the basic reproductive rate $\mathcal{R}_0$ equals to 1. System (2.2) exhibits a backward bifurcation if $\mathcal{R}_1^* < 1$. Otherwise, system (2.2) exhibits a forward bifurcation if $\mathcal{R}_1^* > 1$.

(2) The endemic equilibrium is locally asymptotically stable when the basic reproductive rate $\mathcal{R}_0 > 1$ and close to one.



(*a*) Forward bifurcation. (*b*) Backward bifurcation.

Figure 2. Bifurcation diagram of system (2.2). The dash curve represents unstable equilibrium while the solid curve represent stable equilibrium. Here we set $A = 2$ for (a) and $A = 3$ for (b). The other parameter values are: $\beta = 0.0030, \mu = 0.006, a = 10.98, k = 15, \sigma = 0.84, \xi = 0.3, \gamma = 0.15, d = 0.009, \rho = 1.16, \phi = 0.32$.

6. Global stability of endemic equilibria

In the following, we will utilize the geometric approach to observe the global stability of the endemic equilibrium for system (2.2). Firstly, we present some preliminary results on the geometric approach to global dynamics, one can find them in [28–30]. Let $\mathcal{B}$ be the Euclidean unit ball in $\mathbb{R}^2$, and let $\partial\mathcal{B}$ and $\bar{\mathcal{B}}$ be its boundary and closure, respectively. Denote the set of Lipschitzian functions from X to Y by Lip$(X \to Y)$. We consider a function $\varphi \in$ Lip$(\bar{\mathcal{B}} \to \mathcal{D})$ as a simply connected rectifiable surface in $\mathcal{D} \subset \mathbb{R}^n$. A function $\psi \in$ Lip$(\partial\mathcal{B} \to \mathcal{D})$ is a closed rectifiable curve in $\mathcal{D}$ and is called simple if it is one-to-one. Define $\sum(\psi, \mathcal{D}) = \{\varphi \in \text{Lip}(\bar{\mathcal{B}} \to \mathcal{D}) | \varphi_{\partial\mathcal{B}} = \psi\}$. If $\mathcal{D}$ is an open, simply connected set, then $\sum(\psi, \mathcal{D})$ is nonempty for each simple closed rectifiable curve ψ in $\mathcal{D}$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^{\binom{n}{2}}$. A function $\mathcal{S}$ on surface in $\mathcal{D}$ is defined as follows:

$$\mathcal{S}\varphi = \int_{\mathcal{B}} \|P \cdot (\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2})\| du, \tag{6.1}$$

where $u = (u_1, u_2)$, $u \longmapsto \varphi(u)$ is Lipschitzian on $\bar{\mathcal{B}}$, P is an $\binom{n}{2} \times \binom{n}{2}$ matrix such that $\|P^{-1}\|$ is bounded on $\varphi(\bar{\mathcal{B}})$, and the wedge product $\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2}$ is a vector in $\mathbb{R}^{\binom{n}{2}}$. The following lemma develops on the results in [28] and [30].

Lemma 6.1. Suppose that ψ is an arbitrary simple, closed and rectitiable curve in $\mathbb{R}^n$. Then there exists $\delta > 0$ such that $\mathcal{S}\psi \geq \delta$ for all $\varphi \in \sum(\psi, \mathbb{R}^n)$.

Let $x \longmapsto f(x) \in \mathbb{R}^n$ be a C^1 function for x in a set $\mathcal{D} \subset \mathbb{R}^n$. We consider the autonomous equation in $\mathbb{R}^n$

$$\frac{dx}{dt} = f(x). \tag{6.2}$$

For any surface φ, the new surface φ_t is defined by $\varphi_t(u) = x(t, \varphi(u))$. If $\varphi_t(u)$ is reviewed as a function of u, $\varphi_t(u)$ is a time t map determined by system (6.2). If $\varphi_t(u)$ is viewed as a function of t, $\varphi_t(u)$ is the solution of (6.2) passing through the initial point $(0, \varphi(u))$. The right-hand derivative of $\mathcal{S}\varphi_t$, denoted $D_+\mathcal{S}\varphi_t$ is defined by

$$D_+\mathcal{S}\varphi_t \int_{\varphi_t} \lim_{h\to\infty} (\|\varrho + hQ(\varphi_t(u))\varrho\| - \|\varrho\|) du, \tag{6.3}$$

where the matrix $Q = P_f P^{-1} + P \frac{\partial f}{\partial x}^{[2]} P^{-1}$. Here P_f is the directional derivative of P in the direction of the vector field f, $\frac{\partial f}{\partial x}^{[2]}$ is the second additive compound matrix (we can find its definition in [31]) of $\frac{\partial f}{\partial x}$, and $\varrho = P \cdot (\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2})$ is a solution to the differential equation

$$\frac{d\varrho}{dt} = Q(\varphi_t(u))\varrho. \tag{6.4}$$

Then, the right-hand derivative $D_+\mathcal{S}\varphi_t$ is expressed as

$$D_+\mathcal{S}\varphi_t = \int_{\bar{B}} D_+\|\varrho\| du.$$

At a general point $P(S, E, I, Q)$, the Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{pmatrix} -\beta I - \mu & \frac{a}{(1+kE)^2} & -\beta S & 0 \\ \beta I & -\sigma - \rho - \mu - \frac{a}{(1+kE)^2} & \beta S & 0 \\ 0 & \sigma & -(\mu + d + \gamma + \xi) & 0 \\ 0 & 0 & \xi & -(\phi + \mu) \end{pmatrix}.$$

The second additive compound matrix of $\frac{\partial f}{\partial x}$ is the 6×6 matrix given by

$$\frac{\partial f^{[2]}}{\partial x} = \begin{pmatrix} M_{11} & \beta S & 0 & \beta S & 0 & 0 \\ \sigma & M_{22} & 0 & \frac{a}{(1+kE)^2} & 0 & 0 \\ 0 & \xi & M_{33} & 0 & \frac{a}{(1+kE)^2} & -\beta S \\ 0 & \beta I & 0 & M_{44} & 0 & 0 \\ 0 & 0 & \beta I & \xi & M_{55} & \beta S \\ 0 & 0 & 0 & 0 & \sigma & M_{66} \end{pmatrix},$$

where

$$M_{11} = -\beta I - \mu - \sigma - \rho - \mu - \frac{a}{(1+kE)^2},\ M_{22} = -\beta I - \mu - \xi - \gamma - \mu - d,$$

$$M_{33} = -\beta I - \mu - \rho - \mu,\ M_{44} = -\sigma - \rho - \mu - \frac{a}{(1+kE)^2} - (\mu + d + \gamma + \xi),$$

$$M_{55} = -\sigma - \rho - \mu - \frac{a}{(1+kE)^2} - (\phi + \mu),\ M_{66} = -(\mu + d + \gamma + \xi) - (\phi + \mu).$$

Let

$$P = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & E \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 1/I & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/I & 0 & 0 \\ 0 & 0 & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/E & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/E \end{pmatrix}.$$

The derivative of P in the direction of the vector field f is

$$P_f = \begin{pmatrix} -I'/I & 0 & 0 & 0 & 0 & 0 \\ 0 & -I'/I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I'/I & 0 & 0 \\ 0 & 0 & -E'/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -E'/E & 0 \\ 0 & 0 & 0 & 0 & 0 & -E'/E \end{pmatrix}.$$

Then

$$P_f P^{-1} = \begin{pmatrix} -I'/I & 0 & 0 & 0 & 0 & 0 \\ 0 & -I'/I & 0 & 0 & 0 & 0 \\ 0 & 0 & -I'/I & 0 & 0 & 0 \\ 0 & 0 & 0 & -E'/E & 0 & 0 \\ 0 & 0 & 0 & 0 & -E'/E & 0 \\ 0 & 0 & 0 & 0 & 0 & -E'/E \end{pmatrix}$$

and

$$\begin{aligned} Q &= P_f P^{-1} + P \frac{\partial f^{[2]}}{\partial x} P^{-1} \\ &= \begin{pmatrix} Q_{11} & \beta S & \beta S & 0 & 0 & 0 \\ \sigma & Q_{22} & \frac{a}{(1+kE)^2} & 0 & 0 & 0 \\ 0 & \beta I & Q_{33} & 0 & 0 & 0 \\ 0 & \frac{\xi I}{E} & 0 & Q_{44} & \frac{a}{(1+kE)^2} & -\beta S \\ 0 & 0 & \frac{\xi I}{E} & \beta I & Q_{55} & \beta S \\ 0 & 0 & 0 & 0 & \sigma & Q_{66} \end{pmatrix}, \end{aligned} \tag{6.5}$$

where

$$Q_{11} = -\frac{\beta SI}{E} - \mu - \beta I + \frac{akE}{(1+kE)^2},\ Q_{22} = -\frac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \frac{a}{1+kE},$$

$$Q_{33} = -\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \frac{akE}{(1+kE)^2},\ Q_{44} = -\frac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d,$$

$$Q_{55} = -\frac{\sigma E}{I} - \mu - \sigma - \rho - \phi - \frac{a}{(1+kE)^2} + \xi + \gamma + d,\ Q_{66} = -\frac{\sigma E}{I} - (\phi + \mu).$$

We consider the following norm introduced in [34, 35] for $\varrho = (\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6)^\top \in \mathbb{R}_6$, $\|\varrho\| = \max\{\mathcal{U}_1, \mathcal{U}_2\}$, where $\mathcal{U}_1(\varrho_1, \varrho_2, \varrho_3)$ has the following form

$$\mathcal{U}_1(\varrho_1, \varrho_2, \varrho_3) = \begin{cases} \max\{|\varrho_1|, |\varrho_2| + |\varrho_3|\} & \text{if } \operatorname{sgn}(\varrho_1) = \operatorname{sgn}(\varrho_2) = \operatorname{sgn}(\varrho_3), \\ \max\{|\varrho_2| + |\varrho_1| + |\varrho_3|\} & \text{if } \operatorname{sgn}(\varrho_1) = \operatorname{sgn}(\varrho_2) = -\operatorname{sgn}(\varrho_3), \\ \max\{|\varrho_1|, |\varrho_2|, |\varrho_3|\} & \text{if } \operatorname{sgn}(z_1) = -\operatorname{sgn}(\varrho_2) = \operatorname{sgn}(\varrho_3), \\ \max\{|\varrho_1| + |\varrho_3|, |\varrho_2| + |\varrho_3|\} & \text{if } -\operatorname{sgn}(\varrho_1) = \operatorname{sgn}(\varrho_2) = \operatorname{sgn}(\varrho_3) \end{cases} \tag{6.6}$$

and $\mathcal{U}_2(\varrho_4, \varrho_5, \varrho_6)$ has the following form

$$\mathcal{U}_2(\varrho_4, \varrho_5, \varrho_6) = \begin{cases} |\varrho_4| + |\varrho_5| + |\varrho_6| & \text{if } \operatorname{sgn}(\varrho_4) = \operatorname{sgn}(\varrho_5) = \operatorname{sgn}(\varrho_6), \\ \max\{|\varrho_4| + |\varrho_5|, |\varrho_4| + |\varrho_6|\} & \text{if } \operatorname{sgn}(\varrho_4) = \operatorname{sgn}(\varrho_5) = -\operatorname{sgn}(\varrho_6), \\ \max\{|\varrho_4|, |\varrho_4| + |\varrho_6|\} & \text{if } \operatorname{sgn}(\varrho_4) = -\operatorname{sgn}(\varrho_5) = \operatorname{sgn}(\varrho_5), \\ \max\{|\varrho_4| + |\varrho_6|, |\varrho_5| + |\varrho_6|\} & \text{if } -\operatorname{sgn}(\varrho_4) = \operatorname{sgn}(\varrho_5) = \operatorname{sgn}(\varrho_6). \end{cases} \tag{6.7}$$

Furthermore we use the following relations

$$|\varrho_2| < \mathcal{U}_1,\ |\varrho_3| < \mathcal{U}_1,\ |\varrho_2 + \varrho_3| < \mathcal{U}_1$$

and

$$|\varrho_i|,\ |\varrho_i + \varrho_j|,\ |\varrho_4 + \varrho_5 + \varrho_6| < \mathcal{U}_2(\varrho)\ (i, j = 4, 5, 6,\ i \neq j).$$

Lemma 6.2. There exists a positive constant η, such that $D_+\|\varrho\| \leq -\eta\|\varrho\|$ for all $\varrho \in \mathbb{R}^4$ and all $S, E, I, Q > 0$, provided that the following inequation

$$\begin{aligned} \max\{\tfrac{\beta A}{\mu} - \inf_{t\in(0,+\infty)} \tfrac{\beta SI}{E} - \mu + a,\ -\inf_{t\in(0,+\infty)} \tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + 2a, \\ \sup_{t\in(0,+\infty)} \tfrac{\xi I}{E} - \inf_{t\in(0,+\infty)} \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d + \tfrac{2\beta A}{\mu}\} < -\eta \end{aligned} \tag{6.8}$$

holds for some η. Here $D_+\|\varrho\|$ the right-hand derivative of $\|\varrho\|$, ϱ is the solution of $\frac{d\varrho}{dt} = Q\varrho$.

Proof. We show the existence of some $\eta > 0$ such that

$$D_+\|\varrho\| \leq -\eta\|\varrho\|$$

for all $\varrho \in \mathbb{R}^4$, where ϱ is a solution of Eq (6.4). By linearity, if the above inequality is true for some ϱ, then it also holds for $-\varrho$. Based on the different orthants and the definition of the norm $\|\cdot\|$ in (6.6) and (6.7) within each orthant, the full calculation to demonstrate the proof involves 16 separate cases and subcases.

Case 1. $\mathcal{U}_1 > \mathcal{U}_2$, $\varrho_1, \varrho_2, \varrho_3 > 0$, and $|\varrho_1| > |\varrho_2| + |\varrho_3|$. Then

$$||\varrho|| = |\varrho_1|,$$

so that

$$\begin{aligned} D_+||\varrho|| &= \varrho_1' \\ &= (-\frac{\beta SI}{E} - \mu - \beta I + \frac{akE}{(1+kE)^2})\varrho_1 + \beta S\varrho_2 + \beta S\varrho_3 \\ &\le (-\frac{\beta SI}{E} - \mu - \beta I + \frac{akE}{(1+kE)^2})|\varrho_1| + \beta S(|\varrho_2| + |\varrho_3|) \\ &\le (-\frac{\beta SI}{E} - \mu - \beta I + \frac{akE}{(1+kE)^2})|\varrho_1| + \beta\frac{A}{\mu}|\varrho_1| \\ &= (\frac{\beta A}{\mu} - \frac{\beta SI}{E} - \mu + \frac{akE}{(1+kE)^2})||\varrho|| \\ &= (\frac{\beta A}{\mu} - \frac{\beta SI}{E} - \mu + a)||\varrho||. \end{aligned}$$

Case 2. $\mathcal{U}_1 > \mathcal{U}_2$, $\varrho_1, \varrho_2, \varrho_3 > 0$, and $|\varrho_1| < |\varrho_2| + |\varrho_3|$. Then

$$||\varrho|| = |\varrho_2| + |\varrho_3|,$$

so that

$$\begin{aligned} D_+||\varrho|| &= \varrho_2' + \varrho_3' \\ &= \sigma\varrho_1 + (-\frac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \frac{a}{1+kE})\varrho_2 + \frac{a}{(1+kE)^2}\varrho_3 \\ &\quad +\beta I\varrho_2 + (-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \frac{akE}{(1+kE)^2})\varrho_3 \\ &\le \sigma|\varrho_1| + (-\frac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \frac{a}{1+kE})|\varrho_2| + \frac{a}{(1+kE)^2}|\varrho_3| \\ &\quad +\beta I|\varrho_2| + (-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \frac{akE}{(1+kE)^2})|\varrho_3| \\ &\le \sigma(|\varrho_2| + |\varrho_3|) + (-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \sigma + \rho + \frac{a}{1+kE})|\varrho_2| \\ &\quad +(-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \frac{a}{1+kE})|\varrho_3| \\ &\le (-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + \frac{a}{1+kE})||\varrho|| \\ &\le (-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + a)||\varrho||. \end{aligned}$$

Case 3. $\mathcal{U}_1 > \mathcal{U}_2$, $\varrho_1 < 0$, $\varrho_2, \varrho_3 > 0$, and $|\varrho_1| > |\varrho_2|$. Then

$$||\varrho|| = |\varrho_1| + |\varrho_3|,$$

so that

$$\begin{aligned} D_+||\varrho|| &= -\varrho_1' + \varrho_3' \\ &= -(-\frac{\beta SI}{E} - \mu - \beta I + \frac{akE}{(1+kE)^2})\varrho_1 - \beta S\varrho_2 - \beta S\varrho_3 \\ &\quad +\beta I\varrho_2 + (-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \frac{akE}{(1+kE)^2})\varrho_3 \\ &\le (-\frac{\beta SI}{E} - \mu - \beta I + \frac{akE}{(1+kE)^2})|\varrho_1| + \beta S|\varrho_2| + \beta S|\varrho_3| \\ &\quad +\beta I|\varrho_2| + (-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \frac{akE}{(1+kE)^2})|\varrho_3| \\ &\le (-\frac{\beta SI}{E} - \mu + \frac{akE}{(1+kE)^2} + \beta S)|\varrho_1| \\ &\quad +(-\frac{\beta SI}{E} - \mu - \xi - \gamma - d + \frac{akE}{(1+kE)^2} + \beta S)|\varrho_3| \\ &\le (\frac{\beta A}{\mu} - \frac{\beta SI}{E} - \mu + \frac{akE}{(1+kE)^2})||\varrho|| \\ &\le (\frac{\beta A}{\mu} - \frac{\beta SI}{E} - \mu + a)||\varrho||. \end{aligned}$$

Case 4. $\mathcal{U}_1 > \mathcal{U}_2$, $\varrho_1 < 0$, $\varrho_2, \varrho_3 > 0$, and $|\varrho_1| < |\varrho_2|$. Then

$$||\varrho|| = |\varrho_2| + |\varrho_3|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_2' + \varrho_3' \\ &= \sigma\varrho_1 + (-\tfrac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \tfrac{a}{1+kE})\varrho_2 + \tfrac{a}{(1+kE)^2}\varrho_3 \\ &\quad +\beta I\varrho_2 + (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{akE}{(1+kE)^2})\varrho_3 \\ &\le \sigma|\varrho_1| + (-\tfrac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \tfrac{a}{1+kE})|\varrho_2| + \tfrac{a}{(1+kE)^2}|\varrho_3| \\ &\quad +\beta I|\varrho_2| + (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{akE}{(1+kE)^2})|\varrho_3| \\ &\le \sigma(|\varrho_2| + |\varrho_3|) + (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \sigma + \rho + \tfrac{a}{1+kE})|\varrho_2| \\ &\quad +(-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{a}{1+kE})|\varrho_3| \\ &\le (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + \tfrac{a}{1+kE})\|\varrho\| \\ &\le (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + a)\|\varrho\|. \end{aligned}$$

Case 5. $\mathcal{U}_1 > \mathcal{U}_2, \varrho_1, \varrho_2 > 0, \varrho_3 < 0$, and $|\varrho_2| > |\varrho_1| + |\varrho_3|$. Then

$$\|\varrho\| = |\varrho_2|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_2' \\ &= \sigma\varrho_1 + (-\tfrac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \tfrac{a}{1+kE})\varrho_2 + \tfrac{a}{(1+kE)^2}\varrho_3 \\ &\le \sigma|\varrho_1| + (-\tfrac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \tfrac{a}{1+kE})|\varrho_2| + \tfrac{a}{(1+kE)^2}|\varrho_3| \\ &\le (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + \tfrac{a(2+kE)}{(1+kE)^2})\|\varrho\| \\ &\le (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + 2a)\|\varrho\|. \end{aligned}$$

Case 6. $\mathcal{U}_1 > \mathcal{U}_2, \varrho_1, \varrho_2 > 0, \varrho_3 < 0$, and $|\varrho_2| < |\varrho_1| + |\varrho_3|$. Then

$$\|\varrho\| = |\varrho_1| + |\varrho_3|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_1' - \varrho_3' \\ &= -(\tfrac{\beta SI}{E} + \mu + \beta I - \tfrac{akE}{(1+kE)^2})\varrho_1 + \beta S\varrho_2 + \beta S\varrho_3 \\ &\quad +\beta I\varrho_2 + (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{akE}{(1+kE)^2})\varrho_3 \\ &\le (-\tfrac{\beta SI}{E} - \mu - \beta I + \tfrac{akE}{(1+kE)^2})|\varrho_1| + \beta S|\varrho_2| + \beta S|\varrho_3| \\ &\quad +\beta I|\varrho_2| + (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{akE}{(1+kE)^2})|\varrho_3| \\ &\le (-\tfrac{\beta SI}{E} - \mu + \tfrac{akE}{(1+kE)^2} + \beta S)|\varrho_1| \\ &\quad +(-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{akE}{(1+kE)^2} + \beta S)|\varrho_3| \\ &\le (\tfrac{\beta A}{\mu} - \tfrac{\beta SI}{E} - \mu + \tfrac{akE}{(1+kE)^2})\|\varrho\| \\ &\le (\tfrac{\beta A}{\mu} - \tfrac{\beta SI}{E} - \mu + a)\|\varrho\|. \end{aligned}$$

Case 7. $\mathcal{U}_1 > \mathcal{U}_2, \varrho_2, \varrho_3 > 0, \varrho_2 < 0$, and $|\varrho_1| = \max\{|\varrho_2|, |\varrho_3|\}$. Then

$$\|\varrho\| = |\varrho_1|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_1' \\ &= (-\tfrac{\beta SI}{E} - \mu - \beta I + \tfrac{akE}{(1+kE)^2})\varrho_1 + \beta S\varrho_2 + \beta S\varrho_3 \\ &\le (-\tfrac{\beta SI}{E} - \mu - \beta I + \tfrac{akE}{(1+kE)^2})|\varrho_1| + \beta S(|\varrho_2| + |\varrho_3|) \\ &\le (-\tfrac{\beta SI}{E} - \mu - \beta I + \tfrac{akE}{(1+kE)^2})|\varrho_1| + \beta\tfrac{A}{\mu}|\varrho_1| \\ &\le (\tfrac{\beta A}{\mu} - \tfrac{\beta SI}{E} - \mu + \tfrac{akE}{(1+kE)^2})\|\varrho\| \\ &\le (\tfrac{\beta A}{\mu} - \tfrac{\beta SI}{E} - \mu + a)\|\varrho\|. \end{aligned}$$

Case 8. $\mathcal{U}_1 > \mathcal{U}_2, \varrho_2, \varrho_3 > 0, \varrho_2 < 0$, and $|\varrho_2| = \max\{|\varrho_1|, |\varrho_3|\}$. Then

$$\|\varrho\| = |\varrho_2|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= -\varrho_2' \\ &= \sigma\varrho_1 + (-\tfrac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \tfrac{a}{1+kE})\varrho_2 + \tfrac{a}{(1+kE)^2}\varrho_3 \\ &\leq \sigma|\varrho_1| + (-\tfrac{\beta SI}{E} - \beta I - \mu - \xi - \gamma - d + \sigma + \rho + \tfrac{a}{1+kE})|\varrho_2| + \tfrac{a}{(1+kE)^2}|\varrho_3| \\ &\leq (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + 2\sigma + \rho + 2a)\|\varrho\|. \end{aligned}$$

Case 9. $\mathcal{U}_1 > \mathcal{U}_2, \varrho_1, \varrho_3 > 0, \varrho_2 < 0$, and $|\varrho_3| > \max\{|\varrho_1|, |\varrho_2|\}$. Then

$$\|\varrho\| = |\varrho_3|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_3' \\ &= \beta I\varrho_2 + (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{akE}{(1+kE)^2})\varrho_3 \\ &\leq \beta I|\varrho_2| + (-\tfrac{\beta SI}{E} - \mu - \xi - \gamma - d + \tfrac{akE}{(1+kE)^2})|\varrho_3| \\ &\leq (\tfrac{\beta A}{\mu} - \tfrac{\beta SI}{E} - \mu + a)\|\varrho\| \end{aligned}$$

Case 10. $\mathcal{U}_1 < \mathcal{U}_2, \varrho_4, \varrho_5, \varrho_6 > 0$. Then

$$\|\varrho\| = |\varrho_4| + |\varrho_5| + |\varrho_6|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_4' + \varrho_5' + \varrho_6' \\ &= \tfrac{\xi I}{E}\varrho_2 + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)\varrho_4 + (\tfrac{a}{(1+kE)^2})\varrho_5 - \beta S\varrho_6 \\ &\quad + \tfrac{\xi I}{E}\varrho_3 + \beta I\varrho_4 + (-\tfrac{\sigma E}{I} - \mu - \sigma - \rho - \phi - \tfrac{a}{(1+kE)^2} + \xi + \gamma + d)\varrho_5 + \beta S\varrho_6 \\ &\quad + \sigma\varrho_5 + (-\tfrac{\sigma E}{I} - (\phi + \mu))\varrho_6 \\ &\leq \tfrac{\xi I}{E}(|\varrho_2 + \varrho_3|) + (-\tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d)|\varrho_4| \\ &\quad + (-\tfrac{\sigma E}{I} - \mu - \rho - \phi + \xi + \gamma + d)|\varrho_5| + (-\tfrac{\sigma E}{I} - (\phi + \mu))|\varrho_6|. \end{aligned}$$

Using $|\varrho_2 + \varrho_3| < \mathcal{U}_1 < |\varrho_4| + |\varrho_5| + |\varrho_6|$, it follows

$$D_+\|\varrho\| \leq (\tfrac{\xi I}{E} - \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d)\|\varrho\|.$$

Case 11. $\mathcal{U}_1 < \mathcal{U}_2, \varrho_4, \varrho_5 > 0, \varrho_6 < 0$ and $|\varrho_5| > |\varrho_6|$. Then

$$\|\varrho\| = |\varrho_4| + |\varrho_5|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_4' + \varrho_5' \\ &= \tfrac{\xi I}{E}\varrho_2 + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)\varrho_4 + \tfrac{a}{(1+kE)^2}\varrho_5 - \beta S\varrho_6 \\ &\quad + \tfrac{\xi I}{E}\varrho_3 + \beta I\varrho_4 + (-\tfrac{\sigma E}{I} - \mu - \sigma - \rho - \phi - \tfrac{a}{(1+kE)^2} + \xi + \gamma + d)\varrho_5 + \beta S\varrho_6 \\ &\leq \tfrac{\xi I}{E}(|\varrho_2 + \varrho_3|) + (-\tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d)|\varrho_4| \\ &\quad + (-\tfrac{\sigma E}{I} - \mu - \sigma - \rho - \phi + \xi + \gamma + d)|\varrho_5|. \end{aligned}$$

Using $|\varrho_2+\varrho_3| < \mathcal{U}_1 < |\varrho_4|+|\varrho_5|$, it follows

$$D_+\|\varrho\| \le (\tfrac{\xi I}{E} - \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d)\|\varrho\|.$$

Case 12. $\mathcal{U}_1 < \mathcal{U}_2, \varrho_4, \varrho_5 > 0, \varrho_6 < 0$ and $|\varrho_5| < |\varrho_6|$. Then

$$\|\varrho\| = |\varrho_4| + |\varrho_6|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_4' - \varrho_6' \\ &= \tfrac{\xi I}{E}\varrho_2 + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)\varrho_4 + \tfrac{a}{(1+kE)^2}\varrho_5 - \beta S\varrho_6 \\ &\quad +\sigma\varrho_5 + (-\tfrac{\sigma E}{I} - (\phi+\mu))\varrho_6 \\ &\le \tfrac{\xi I}{E}|\varrho_2| + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)|\varrho_4| + \tfrac{a}{(1+kE)^2}|\varrho_5| - \beta S|\varrho_6| \\ &\quad +\sigma|\varrho_5| + (-\tfrac{\sigma E}{I} - (\phi+\mu))|\varrho_6|. \end{aligned}$$

Using $|\varrho_2| < \mathcal{U}_1 < |\varrho_4|+|\varrho_5|$, it follows

$$D_+\|\varrho\| \le (\tfrac{\xi I}{E} - \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d + a + \sigma)\|\varrho\|.$$

Case 13. $\mathcal{U}_1 < \mathcal{U}_2, \varrho_4, \varrho_6 > 0, \varrho_5 < 0$ and $|\varrho_5| > |\varrho_4| + |\varrho_6|$. Then

$$\|\varrho\| = |\varrho_5|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= -\varrho_5' \\ &= \tfrac{\xi I}{E}\varrho_3 - \beta I\varrho_4 + (-\tfrac{\sigma E}{I} - \mu - \sigma - \rho - \phi - \tfrac{a}{(1+kE)^2} + \xi + \gamma + d)\varrho_5 - \beta S\varrho_6 \\ &\le \tfrac{\xi I}{E}|\varrho_3| + (-\tfrac{\sigma E}{I} - \mu - \sigma - \rho - \phi - \tfrac{a}{(1+kE)^2} + \xi + \gamma + d)|\varrho_5| \end{aligned}$$

Using $|\varrho_3| < \mathcal{U}_1 < |\varrho_5|$, it follows

$$D_+\|\varrho\| \le (\tfrac{\xi I}{E} - \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d)\|\varrho\|.$$

Case 14. $\mathcal{U}_1 < \mathcal{U}_2, \varrho_4, \varrho_6 > 0, \varrho_5 < 0$ and $|\varrho_5| < |\varrho_4| + |\varrho_6|$. Then

$$\|\varrho\| = |\varrho_4| + |\varrho_6|,$$

so that

$$\begin{aligned} D_+\|\varrho\| &= \varrho_4' + \varrho_6' \\ &= \tfrac{\xi I}{E}\varrho_2 + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)\varrho_4 + \tfrac{a}{(1+kE)^2}\varrho_5 - \beta S\varrho_6 \\ &\quad +\sigma\varrho_5 + (-\tfrac{\sigma E}{I} - (\phi+\mu))\varrho_6 \\ &\le \tfrac{\xi I}{E}|\varrho_2| + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)\varrho_4 + \tfrac{a}{(1+kE)^2}|\varrho_5| - \beta S|\varrho_6| \\ &\quad +\sigma|\varrho_5| + (-\tfrac{\sigma E}{I} - (\phi+\mu))|\varrho_6|. \end{aligned}$$

Using $|\varrho_2| < \mathcal{U}_1 < |\varrho_4|+|\varrho_6|$, it follows

$$D_+\|\varrho\| \le (\tfrac{\xi I}{E} - \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d + a + \sigma)\|\varrho\|.$$

Case 15. $\mathcal{U}_1 < \mathcal{U}_2, \varrho_5, \varrho_6 > 0, \varrho_4 < 0$ and $|\varrho_5| < |\varrho_4|$. Then

$$||\varrho|| = |\varrho_4| + |\varrho_6|,$$

so that

$$\begin{aligned} D_+||\varrho|| &= -\varrho_4' + \varrho_6' \\ &= -[\tfrac{\xi I}{E}\varrho_2 + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)\varrho_4 + \tfrac{a}{(1+kE)^2}\varrho_5 - \beta S\varrho_6] \\ &\quad +\sigma\varrho_5 + (-\tfrac{\sigma E}{I} - (\phi + \mu))\varrho_6 \\ &\leq \tfrac{\xi I}{E}|\varrho_2| + (-\tfrac{\sigma E}{I} - \mu - \beta I - \phi + \xi + \gamma + d)|\varrho_4| + \tfrac{a}{(1+kE)^2}|\varrho_5| - \beta S|\varrho_6| \\ &\quad +\sigma|\varrho_5| + (-\tfrac{\sigma E}{I} - (\phi + \mu))|\varrho_6|. \end{aligned}$$

Using $|\varrho_2| < \mathcal{U}_1 < |\varrho_4| + |\varrho_6|$, it follows

$$D_+||\varrho|| \leq (\tfrac{\xi I}{E} - \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d + a + \sigma)||\varrho||.$$

Case 16. $\mathcal{U}_1 < \mathcal{U}_2, \varrho_5, \varrho_6 > 0, \varrho_4 < 0$ and $|\varrho_5| > |\varrho_4|$. Then

$$||\varrho|| = |\varrho_5| + |\varrho_6|,$$

so that

$$\begin{aligned} D_+||\varrho|| &= \varrho_5' + \varrho_6' \\ &= \tfrac{\xi I}{E}\varrho_3 + \beta I\varrho_4 + (-\tfrac{\sigma E}{I} - \mu - \sigma - \rho - \phi - \tfrac{a}{(1+kE)^2} + \xi + \gamma + d)\varrho_5 + \beta S\varrho_6 \\ &\quad +\sigma\varrho_5 + (-\tfrac{\sigma E}{I} - (\phi + \mu))\varrho_6 \\ &\leq \tfrac{\xi I}{E}|\varrho_3| + \beta I|\varrho_4| + (-\tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d)|\varrho_5| + \beta S|\varrho_6| \\ &\quad +\sigma|\varrho_5| + (-\tfrac{\sigma E}{I} - (\phi + \mu))|\varrho_6|. \end{aligned}$$

Using $|\varrho_3| < \mathcal{U}_1 < |\varrho_5| + |\varrho_6|$, it follows

$$D_+||\varrho|| \leq (\tfrac{\xi I}{E} - \tfrac{\sigma E}{I} - \mu - \phi + \xi + \gamma + d + \tfrac{2\beta A}{\mu})||\varrho||.$$

From Case 1 to Case 16 and the condition of Lemma 6.2, we can obtain

$$D_+||\varrho|| \leq -\eta||\varrho||$$

for all $\varrho \in \mathbb{R}^4$.

From [32,33] we know that the geometric approach can be applied to prove the globally asymptotic stability when the epidemic model has a unique endemic equilibrium. In this situation, there exists a compact absorbing set $\mathcal{D}$, and a surface remains in $\mathcal{D}$ for all time. From Section 4, we find that system (2.2) will possibly exhibit bistability. For this case, system (2.2) does not exist an absorbing set. Hence, we shall consider the following sequence of surface $\{\varphi^k\}$ in the following lemma.

Lemma 6.3. Let ψ be a simple closed curve in $\mathcal{D}$. There exist a positive ϵ and a sequence of surface $\{\psi^k\}$ that minimize $\mathcal{S}$ given by (6.1) relative to $\sum(\psi, \mathcal{D})$ such that $\{\psi_t^k \subset \mathcal{D}\}$ for all $k = 1, 2, 3, \cdots$ and all $t \in [0, \epsilon]$.

Proof. Let $\zeta = \frac{1}{2}\min\{E, I : (S, E, I, Q) \in \psi\}$. It is easy to see $\zeta > 0$. From the second and third equations of (2.2), we can get the inequality $\frac{dE(t)}{dt} \geq -\sigma E(t) - \rho E(t) - \mu E(t) - \frac{aE(t)}{1+kE(t)}$, $\frac{dI(t)}{dt} \geq -(\xi+\gamma+\mu+d)I$ in $\mathcal{D}$. Hence we can conclude that there is an $\epsilon > 0$ such that, if a solution satisfied $E(0) \geq \zeta$, $I(0) \geq \zeta$, then it remains in $\mathcal{D}$ for $t \in [0, \epsilon]$. Thus we only require to prove there exists a sequence $\{\psi^k\}$, such that it minimizes $\mathcal{S}$ relative to $\Sigma(\psi, \tilde{\mathcal{D}})$, where $\tilde{\mathcal{D}} = \{(S, E, I, Q) \in \mathcal{D} : E \geq \zeta, I \geq \zeta\}$.

Let $\varphi(u) = (S(u), E(u), I(u), Q(u)) \in \Sigma(\psi, \mathcal{D})$. Define a new surface $\tilde{\varphi}(u) = (\tilde{S}(u), \tilde{E}(u), \tilde{I}(u), \tilde{Q}(u))$ by

$$\tilde{\varphi}(u) = \begin{cases} \varphi(u) & \text{if } E(u) \geq \zeta, I(u) \geq \zeta, \\ (S, \zeta, E, Q) & \text{if } E(u) < \zeta, I(u) \geq \zeta \text{ and } S(u) + \zeta + I(u) + Q(u) \leq \frac{A}{\mu}, \\ (\frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta), \zeta, \zeta, \frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta)) & \text{if } E(u) < \zeta, I(u) \geq \zeta \text{ and } S(u) + \zeta + I(u) + Q(u) > \frac{A}{\mu}, \\ (S, E, \zeta, Q) & \text{if } E(u) \geq \zeta, I(u) < \zeta \text{ and } S(u) + E(u) + \zeta + Q(u) \leq \frac{A}{\mu}, \\ (\frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta), \zeta, \zeta, \frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta)) & \text{if } E(u) \geq \zeta, I(u) < \zeta \text{ and } S(u) + E(u) + \zeta + Q(u) > \frac{A}{\mu}, \\ (S, \zeta, \zeta, Q) & \text{if } E(u) < \zeta, I(u) < \zeta \text{ and } S(u) + 2\zeta + Q(u) \leq \frac{A}{\mu}, \\ (\frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta), \zeta, \zeta, \frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta)) & \text{if } E(u) < \zeta, I(u) < \zeta \text{ and } S(u) + 2\zeta + Q(u) > \frac{A}{\mu}, \end{cases}$$

From the above definition of $\tilde{\varphi}(u)$, it is not difficult to know $\tilde{\varphi}(u) \in \Sigma(\psi, \tilde{\mathcal{D}})$. Also $\mathcal{S}\tilde{\varphi} = \int_{\bar{\mathcal{B}}} |\frac{1}{\tilde{E}}(\frac{\partial \tilde{\varphi}}{\partial u_1} \wedge \frac{\partial \tilde{\varphi}}{\partial u_2})| du$ and $\mathcal{S}\varphi = \int_{\bar{\mathcal{B}}} |\frac{1}{E}(\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2})| du$. In the following, we will prove $\mathcal{S}\tilde{\varphi} \leq \mathcal{S}\varphi$.

According to the definition of wedge product, we obtain that

$$\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2} = \begin{pmatrix} \frac{\partial S}{\partial u_1} \\ \frac{\partial E}{\partial u_1} \\ \frac{\partial I}{\partial u_1} \\ \frac{\partial Q}{\partial u_1} \end{pmatrix} \wedge \begin{pmatrix} \frac{\partial S}{\partial u_2} \\ \frac{\partial E}{\partial u_2} \\ \frac{\partial I}{\partial u_2} \\ \frac{\partial Q}{\partial u_2} \end{pmatrix} = \begin{pmatrix} \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial E}{\partial u_1} & \frac{\partial E}{\partial u_2} \end{pmatrix} \\ \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial I}{\partial u_1} & \frac{\partial I}{\partial u_2} \end{pmatrix} \\ \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \end{pmatrix} \\ \det\begin{pmatrix} \frac{\partial E}{\partial u_1} & \frac{\partial E}{\partial u_2} \\ \frac{\partial I}{\partial u_1} & \frac{\partial I}{\partial u_2} \end{pmatrix} \\ \det\begin{pmatrix} \frac{\partial E}{\partial u_1} & \frac{\partial E}{\partial u_2} \\ \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \end{pmatrix} \\ \det\begin{pmatrix} \frac{\partial I}{\partial u_1} & \frac{\partial I}{\partial u_2} \\ \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \end{pmatrix} \end{pmatrix}$$

is a vector in $\mathbb{R}^6$ for each $u \in \mathcal{B}$. Denote $\frac{\partial \tilde{\varphi}}{\partial u_1} \wedge \frac{\partial \tilde{\varphi}}{\partial u_2} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6)^\top$ and $\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2} = (x_1, x_2, x_3, x_4, x_5, x_6)^\top$. We will prove $|\tilde{x}_i| \leq |x_i|$ $(i = 1, 2, 3, 4, 5, 6)$ in the following seven cases.

Case 1. If $E(u) \geq \zeta$, $I(u) \geq \zeta$, then $\tilde{\varphi} = \varphi$ and therefore, $|\tilde{x}_i| \leq |x_i|$ $(i = 1, 2, 3, 4, 5, 6)$.

Case 2. If $E(u) < \zeta$, $I(u) \geq \zeta$ and $S(u) + E(u) + \zeta + Q(u) \leq \frac{A}{\mu}$, then $\tilde{\varphi}(u) = (S(u) + \zeta + I(u) + Q(u))$. Hence, we can obtain

$$\frac{\partial \tilde{\varphi}}{\partial u_1} \wedge \frac{\partial \tilde{\varphi}}{\partial u_2} = \begin{pmatrix} \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \end{pmatrix} \\ 0 \\ \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial I}{\partial u_1} & \frac{\partial I}{\partial u_2} \end{pmatrix} \\ 0 \\ \det\begin{pmatrix} \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \\ \frac{\partial I}{\partial u_1} & \frac{\partial I}{\partial u_2} \end{pmatrix} \\ 0 \end{pmatrix}.$$

Therefore, it follows $\tilde{x}_i = x_i$ $(i = 1, 3, 5)$ and $\tilde{x}_i = 0$ $(i = 2, 4, 6)$. Hence $|\tilde{x}_i| \leq |x_i|$ $(i = 1, 2, 3, 4, 5, 6)$.

Case 3. If $E(u) < \zeta$, $I(u) \geq \zeta$ and $S(u) + E(u) + \zeta + Q(u) > \frac{A}{\mu}$, then $\tilde{\varphi}(u) = (\frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta), \zeta, \zeta, \frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta))$. Therefore, $\frac{\partial\tilde{\varphi}}{\partial u_1} \wedge \frac{\partial\tilde{\varphi}}{\partial u_2} = 0$. Thus,

$$\frac{\partial\tilde{\varphi}}{\partial u_j} = (\frac{A}{\mu} - 2\zeta)\frac{S\frac{\partial S}{\partial u_j} - Q\frac{\partial Q}{\partial u_j}}{(S+Q)^2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

for $j = 1, 2$. Therefore, $\frac{\partial\tilde{\varphi}}{\partial u_1}$ and $\frac{\partial\tilde{\varphi}}{\partial u_2}$ have linear dependent. Hence, $\frac{\partial\tilde{\varphi}}{\partial u_1} \wedge \frac{\partial\tilde{\varphi}}{\partial u_2} = 0$. Thus $\tilde{x}_i = 0$ $(i = 1, 2, 3, 4, 5, 6)$. Therefore $|\tilde{x}_i| \leq |x_i|$ $(i = 1, 2, 3, 4, 5, 6)$.

Case 4. If $E(u) \geq \zeta$, $I(u) < \zeta$ and $S(u) + E(u) + \zeta + Q(u) \leq \frac{A}{\mu}$, then $\varphi(u) = (S(u), I(u), \zeta, Q(u))$. Hence,

$$\frac{\partial\varphi}{\partial u_1} \wedge \frac{\partial\varphi}{\partial u_2} = \begin{pmatrix} \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \end{pmatrix} \\ \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial E}{\partial u_1} & \frac{\partial E}{\partial u_2} \end{pmatrix} \\ 0 \\ \det\begin{pmatrix} \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \\ \frac{\partial E}{\partial u_1} & \frac{\partial E}{\partial u_2} \end{pmatrix} \\ 0 \\ 0 \end{pmatrix}$$

almost everywhere. Therefore, $\tilde{x}_i = x_i$ $(i = 1, 2, 4)$, $\tilde{x}_i = 0$ $(i = 3, 5, 6)$. Therefore $|\tilde{x}_i| \leq |x_i|$ $(i = 1, 2, 3, 4, 5, 6)$.

Case 5. If $E(u) \geq \zeta$, $I(u) > \zeta$ and $S(u) + E(u) + \xi + Q(u) > \frac{A}{\mu}$, then $\tilde{\varphi}(u) = (\frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta), \zeta, \zeta, \frac{S}{S+Q}(\frac{A}{\mu} - 2\zeta))$. Thus,

$$\frac{\partial\tilde{\varphi}}{\partial u_j} = (\frac{A}{\mu} - 2\zeta)\frac{S\frac{\partial S}{\partial u_j} - Q\frac{\partial Q}{\partial u_j}}{(S+Q)^2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

for $j = 1, 2$. Therefore, $\frac{\partial\tilde{\varphi}}{\partial u_1}$ and $\frac{\partial\tilde{\varphi}}{\partial u_2}$ have linear dependent. Hence, $\frac{\partial\tilde{\varphi}}{\partial u_1} \wedge \frac{\partial\tilde{\varphi}}{\partial u_2} = 0$. Thus $\tilde{x}_i = 0$ $(i = 1, 2, 3, 4, 5, 6)$. Therefore $|\tilde{x}_i| \leq |x_i|$ $(i = 1, 2, 3, 4, 5, 6)$.

Case 6. If $E(u) < \zeta$, $I(u) < \zeta$ and $S(u) + 2\zeta + Q(u) \leq \frac{A}{\mu}$, then $\tilde{\varphi}(u) = (S(u), \zeta, \zeta, Q(u))$, thus,

$$\frac{\partial\varphi}{\partial u_1} \wedge \frac{\partial\varphi}{\partial u_2} = \begin{pmatrix} \det\begin{pmatrix} \frac{\partial S}{\partial u_1} & \frac{\partial S}{\partial u_2} \\ \frac{\partial Q}{\partial u_1} & \frac{\partial Q}{\partial u_2} \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $\tilde{x}_i = x_i$ $(i = 1)$ and $\tilde{x}_i = 0$ $(i = 2, 3, 4, 5, 6)$. Hence $|\tilde{x}_i| \leq |x_i|$ $(i = 1, 2, 3, 4, 5, 6)$.

Case 7. If $E(u) < \zeta$, $I(u) < \zeta$ and $S(u)+E(u)+2\zeta > \frac{A}{\mu}$, then $\tilde{\varphi}(u) = (\frac{S}{S+Q}(\frac{A}{\mu}-2\zeta), \zeta, \zeta, \frac{S}{S+Q}(\frac{A}{\mu}-2\zeta))$. Thus,

$$\frac{\partial\tilde{\varphi}}{\partial u_j} = (\frac{A}{\mu} - 2\zeta)\frac{S\frac{\partial S}{\partial u_j} - Q\frac{\partial Q}{\partial u_j}}{(S+Q)^2}\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

for $j = 1,2$. Therefore, $\frac{\partial\tilde{\varphi}}{\partial u_1}$ and $\frac{\partial\tilde{\varphi}}{\partial u_2}$ have linear dependent. Hence, $\frac{\partial\tilde{\varphi}}{\partial u_1} \wedge \frac{\partial\tilde{\varphi}}{\partial u_2} = 0$. Thus $\tilde{x}_i = 0$ $(i = 1,2,3,4,5,6)$. Therefore $|\tilde{x}_i| \le |x_i|$ $(i = 1,2,3,4,5,6)$.

We also note that $\tilde{E}(u) = \max\{E(u),\zeta\}$ and $\tilde{I}(u) = \max\{I(u),\zeta\}$. Thus $\frac{1}{\tilde{E}(u)} \le \frac{1}{E(u)}$, $\frac{1}{\tilde{I}(u)} \le \frac{1}{I(u)}$. Let

$$\tilde{P} = \begin{pmatrix} 1/\tilde{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\tilde{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\tilde{I} & 0 & 0 \\ 0 & 0 & 1/\tilde{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\tilde{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\tilde{E} \end{pmatrix},$$

by comparing the vector $P \cdot (\frac{\partial\varphi}{\partial u_1} \wedge \frac{\partial\varphi}{\partial u_2})$ and the vector $\tilde{P} \cdot (\frac{\partial\tilde{\varphi}}{\partial u_1} \wedge \frac{\partial\tilde{\varphi}}{\partial u_2})$, we have $|\frac{1}{\tilde{I}}\tilde{x}_i| \le |\frac{1}{I}x_i|$ $(i = 1,2,4)$ and $|\frac{1}{\tilde{E}}\tilde{x}_i| \le |\frac{1}{E}x_i|$ $(i = 3,5,6)$ for their corresponding component.

From above seven cases, we can obtain $\|\frac{\partial\tilde{\varphi}}{\partial u_1} \wedge \frac{\partial\tilde{\varphi}}{\partial u_2}\| \le \|\frac{\partial\varphi}{\partial u_1} \wedge \frac{\partial\varphi}{\partial u_2}\|$ and $\mathcal{S}\tilde{\varphi} \le \mathcal{S}\varphi$.

From Lemma 6.2, we can choose $\varpi = \inf\{\mathcal{S}\varphi : \varphi \in \Sigma(\psi,\mathcal{D})\}$. Assume that $\{\varphi^k\}$ is a sequence of surfaces that minimizes $\mathcal{S}$ relative to $\Sigma(\psi,\mathcal{D})$, then $\lim_{t\to\infty} \mathcal{S}\varphi^k = \varpi$. Let $\{\tilde{\varphi}^k\}$ be a sequence of surfaces that minimizes $\mathcal{S}$ relative to $\Sigma(\psi,\tilde{\mathcal{D}})$ defined by the above construction. Hence, for each k, $\mathcal{S}\tilde{\varphi}^k \le \mathcal{S}\varphi^k$. On the other hand, since $\{\mathcal{S}\tilde{\varphi}^k\}$ is bounded, we assume that $\{\mathcal{S}\tilde{\varphi}^k\}$ is convergent without generality. From $\mathcal{S}\tilde{\varphi}^k \le \mathcal{S}\varphi^k$ (for each k), we can obtain that $\lim_{k\to\infty} \mathcal{S}\tilde{\varphi}^k \le \varpi$. From $\tilde{\varphi}^k \in \Sigma(\psi,\mathcal{D})$, we get $\tilde{\varphi}^k \in \Sigma(\psi,\tilde{\mathcal{D}})$. Then, for each k, $\lim_{k\to\infty} \mathcal{S}\tilde{\varphi}^k \ge \varpi$. Therefore, $\lim_{k\to\infty} \mathcal{S}\tilde{\varphi}^k = \varpi$. From $\mathcal{S}\tilde{\varphi}^k \le \mathcal{S}\varphi^k$ (for each k), we can obtain

$$\inf\{\mathcal{S}\tilde{\varphi} : \tilde{\varphi} \in \Sigma(\psi,\tilde{\mathcal{D}})\} \le \inf\{\mathcal{S}\varphi : \varphi \in \Sigma(\psi,\mathcal{D})\} = \varpi$$

From $\tilde{\varphi} \in \Sigma(\psi,\mathcal{D})$, we have $\inf\{\mathcal{S}\tilde{\varphi} : \tilde{\varphi} \in \Sigma(\psi,\tilde{\mathcal{D}})\} \ge \varpi$, which implies $\inf\{\mathcal{S}\tilde{\varphi} : \tilde{\varphi} \in \Sigma(\psi,\tilde{\mathcal{D}})\} = \varpi$. At last, we can show that $\lim_{k\to\infty} \mathcal{S}\tilde{\varphi}^k = \varpi = \inf\{\mathcal{S}\tilde{\varphi} : \tilde{\varphi} \in \Sigma(\psi,\tilde{\mathcal{D}})\}$. It follows that $\tilde{\varphi}^k$ minimizes $\mathcal{S}$ relative to $\Sigma(\psi,\tilde{\mathcal{D}})$.

From Lemmas 6.2 and 6.3, we can obtain the following theorem.

Theorem 6.1. If the inequality (6.8) holds true, then any $\omega-$limit point of system (2.2) in the interior of $\mathcal{D}$ is an equilibrium, and so each positive semitrajectory of system (2.2) in $\bar{\mathcal{D}}$ limits to a single equilibrium.

The proof of Theorem 6.1 is similar to the proof of Corollary 5.4 in [27], we omit it.

From Theorem 6.1, we have the following theorem.

Theorem 6.2. Suppose that inequality (6.8) is satisfies, then

(1) when system (2.2) has only one disease-free equilibrium, then all solutions of system (2.2) limit to P_0;

(2) when $\mathcal{R}_0 > 1$, then all solutions of system (2.2) converge to the the endemic equilibrium P^*;

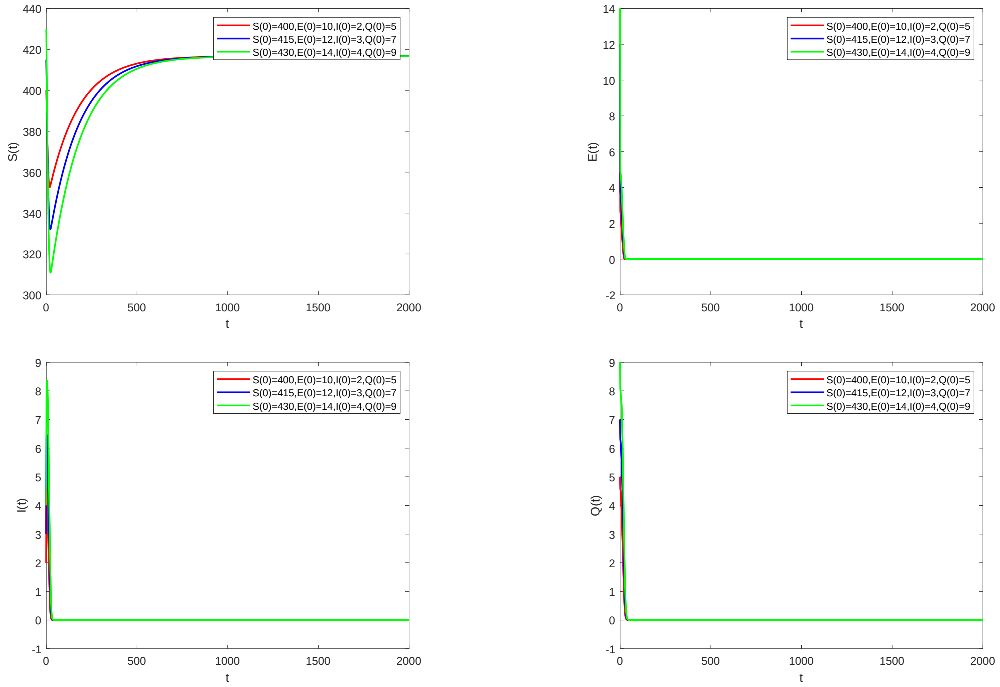


Figure 3. When $\mathcal{R}_0 < \mathcal{R}^* < 1$, the disease free equilibrium P_0 is locally stable.

(3) when system (2.2) has two endemic equilibrium point, i.e., $\mathcal{R}_0^* < \mathcal{R}_0 < 1$ and $A_2 < 0$, then solutions of system (2.2) either go to the disease-free equilibrium P_0, or tend to the upper equilibrium P^*.

7. Numerical simulation

In this section, we present some numerical simulations by using the method of Runge-Kutta through Matlab 2018a software to corroborate the theoretical results and find the complex dynamics of system (2.2).

Case i.. Let $A = 2.5, \beta = 0.003, \mu = 0.006$, $a = 10.98, k = 15, \sigma = 0.84, \xi = 0.3, \gamma = 0.15, d = 0.009, \rho = 1.16, \phi = 0.32$. We can get $A_2 = 0.00506 > 0$, $\mathcal{R}^* = 0.9973$, $\mathcal{R}_0 = 0.1739$ and $\mathcal{R}_0 < \mathcal{R}^* < 1$. It is easy to see that the conditions of Case (i) of Theorem 4.2 are satisfied. Hence, system (2.2) has only one disease free equilibrium $P_0(666.6667, 0, 0, 0)$. Furthermore, the condition of Theorem 4.1 is satisfied, $P_0(666.6667, 0, 0, 0)$ is locally asymptotically stable (See Figure 3).

Case ii. Set $A = 13, \beta = 0.003, \mu = 0.006$, $a = 10.98, k = 15, \sigma = 0.84, \xi = 0.3, \gamma = 0.15, d = 0.009, \rho = 1.16, \phi = 0.32$. We can get $\mathcal{R}^* = -13.7347$, $\mathcal{R}_0 = 0.9042$ and $\mathcal{R}^* < \mathcal{R}_0 < 1$. The conditions of Case (iii) of Theorem 4.2 are satisfied. System (2.2) has a disease free equilibrium $P_0(2666.6667, 0, 0, 0)$, two endemic equilibria $P_*(2163.778095, 0.0086, 0.0156, 0.0144)$ and $P^*(395.3322, 5.2981, 9.5708, 8.8075)$. P_* is unstable and P^* is locally asymptotically stable (See Figure 4).

Case iii. Suppose $A = 16, \beta = 0.003, \mu = 0.006$, $a = 10.98, k = 15, \sigma = 0.84, \xi = 0.3, \gamma = 0.15, d = 0.009, \rho = 1.16, \phi = 0.32$. We can get $\mathcal{R}_0 = 1.1129 > 1$. The conditions of Case (ii) of

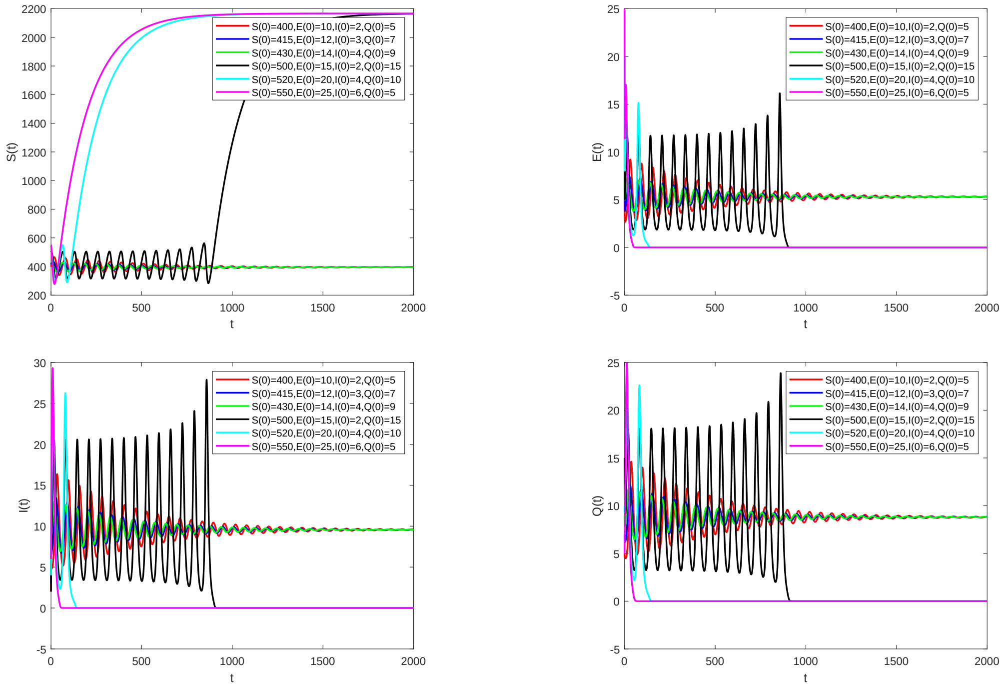


Figure 4. When $\mathcal{R}^* < \mathcal{R}_0 < 1$, the disease free equilibrium P_0 and endemic equilibrium P^* are locally stable.

Theorem 4.2 are satisfied. System (2.2) has a disease free equilibrium $P_0(2666.6667, 0, 0, 0)$ and an endemic equilibrium $P^*(389.7962, 6.8102, 12.3023, 11.3211)$. By the case (2) of Theorem 6.2, P^* is locally asymptotically stable (See Figure 5).

Case iv. Set $A = 3.2, \beta = 0.0072, \mu = 0.006, k = 15, \sigma = 0.84, \xi = 0.3, \gamma = 0.15, d = 0.009, \rho = 1.16, \phi = 0.32$. From Figure 6(a), we can find that Hopf bifurcation may occur around the endemic equilibrium when parameter a changes. Set $A = 3.2, \beta = 0.0072, \mu = 0.006, a = 1.2, \sigma = 0.84, \xi = 0.3, \gamma = 0.15, d = 0.009, \rho = 1.16, \phi = 0.32$. From Figure 6(b), we can find that system (2.2) exists Hopf bifurcation around the endemic equilibrium when parameter k changes, which shows that nonlinear innate immunity rate can cause system to produce periodic solutions.

8. Discussion

In this paper we considered an SEIQR epidemic model with nonlinear innate immunity. We found that system (2.2) might exist backward bifurcation under some conditions. The global stability of the disease free and endemic equilibria for system (2.2) was obtained. The global stability of the endemic equilibrium point for a four-dimensional nonlinear ordinary differential equation model is studied by using the method in [27, 34]. At present, this approach is rarely used in studying the global stability of the endemic equilibrium for the epidemic models. This is a good method to study the global stability of the endemic equilibrium when the models exist backward bifurcation and the Lyapunov function is not well constructed. We find that Hopf bifurcation might occur by numerical simulation. In theory, we can further study the direction of bifurcation and the stability of the bifurcating periodic solutions.

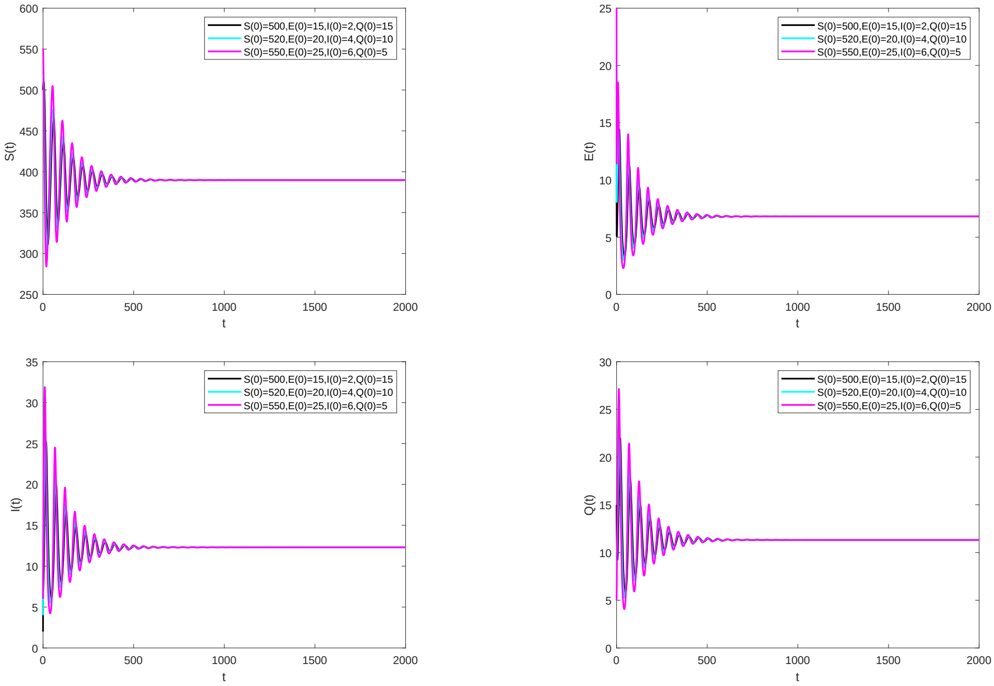


Figure 5. When $\mathcal{R}_0 > 1$, the endemic equilibrium P^* is locally stable.

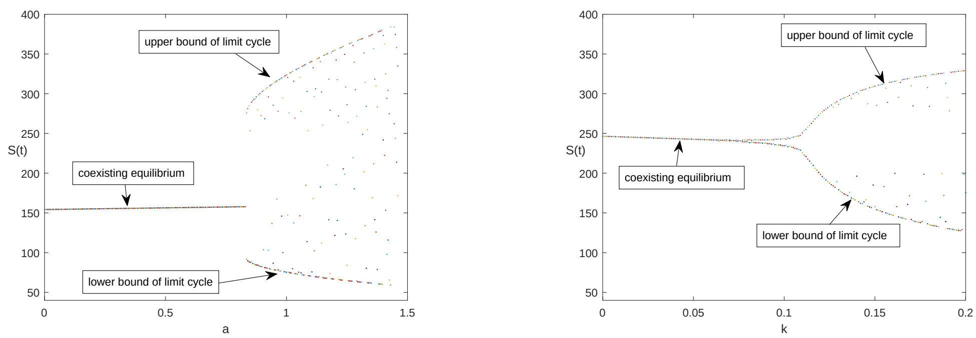
We'll leave the work for the future.

When the innate immunity is linear, i.e., $k = 0$, system (2.2) reduces to the following:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + aE(t), \\ \frac{dE(t)}{dt} = \beta S(t)I(t) - \sigma E(t) - \rho E(t) - \mu E(t) - aE(t), \\ \frac{dI(t)}{dt} = \sigma E(t) - \xi I(t) - \gamma I(t) - \mu I(t) - dI(t), \\ \frac{dQ(t)}{dt} = \xi I(t) - \phi Q(t) - \mu Q(t). \end{cases} \tag{8.1}$$

By calculating, we get that the basic reproduction number of (8.1) is

$$\mathcal{R}_0^l = \frac{A\sigma\beta}{\mu(\xi + \gamma + \mu + d)(\mu + \sigma + \rho + a)}.$$



(a) $S(t)$ vs. a. (b) $S(t)$ vs. k.

Figure 6. Hopf bifurcation diagram.

Hence $\mathcal{R}_0 = \mathcal{R}_0^l$. That is to say the parameter k does not effect the basic reproduction number. System (8.1) always exists a disease free equilibrium $P_0^l(\frac{A}{\mu}, 0, 0, 0)$. When $\mathcal{R}_0 > 1$, system (8.1) has a unique endemic equilibrium $P_l^*(S_l^*, E_l^*, I_l^*, Q_l^*)$, where

$$E_l^* = \frac{\xi + \gamma + \mu + d}{\sigma} I_l^*, Q_l^* = \frac{\xi}{\mu + \phi} I_l^*, S_l^* = \frac{A + aE_l^*}{\mu + \beta I_l^*},$$

$$I_l^* = \frac{A\sigma\beta - \mu(\xi + \gamma + \mu + d)(\sigma + \rho + \mu + a)}{\beta(\xi + \gamma + \mu + d)(\mu + \sigma + \rho)}.$$

By constructing the suitable Lyapunov functions we can conclude that (i) when $\mathcal{R}_0 < 1$, the disease free equilibrium $P_0^l(\frac{A}{\mu}, 0, 0, 0)$ of system (8.1) is globally asymptotically stable; (ii) when $\mathcal{R}_0 > 1$, the endemic equilibrium $P_l^*(S_l^*, E_l^*, I_l^*, Q_l^*)$ of system (8.1) is globally asymptotically stable. When the innate immunity rate is linear, backward bifurcation and Hopf bifurcation cannot exist. The dynamic properties of system (2.2) are more complex than those of system (8.1). Hence, the nonlinear innate immunity makes more difficult to eliminate the disease. In order to take appropriate prevention and control measures, we should pay attention to the impact of nonlinear innate immunity in disease control.

Some fascinating questions are well worth further investigation. For example, the phenomenon of stochastic disturbances is common in nature. The transmission coefficient is often randomly perturbed as the disease spreads. If we assume that the incidence rate in (2.1) is perturbed by white noise so that $\beta \to \beta + \nu\dot{B}(t)$, where $B(t)$ is a standard Brownian motion with intensity ν. System (2.1) becomes the following stochastic model

$$\begin{cases} dS(t) = [A - \beta S(t)I(t) - \mu S(t) + \frac{aE(t)}{1+kE(t)}]dt - \nu S(t)I(t)dB(t), \\ dE(t) = [\beta S(t)I(t) - \sigma E(t) - \rho E(t) - \mu E(t) - \frac{aE(t)}{1+kE(t)}]dt + \nu S(t)I(t)dB(t), \\ dI(t) = [\sigma E(t) - \xi I(t) - \gamma I(t) - \mu I(t) - dI(t)]dt, \\ dQ(t) = [\xi I(t) - \phi Q(t) - \mu Q(t)]dt, \\ dR(t) = [\rho E(t) + \gamma I(t) + \phi Q(t) - \mu R(t)]dt. \end{cases} \tag{8.2}$$

System (8.2) is more reasonable than model (2.1). And the dynamical behaviors of system (8.2) are more complicated than system (2.1). We leave it in the future.

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Conflict of interest

The authors declare there is no conflicts of interest.

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