Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(9): 3435–3451.
DOI: 10.3934/era.2022175
Received: 23 May 2022
Revised: 14 July 2022
Accepted: 17 July 2022
Published: 21 July 2022

Research article

Zavadskij modules over cluster-tilted algebras of type $\mathbb{A}$*

Agustín Moreno Cañadas[1], Robinson-Julian Serna[2] and Isaías David Marín Gaviria[2,*]

[1] Departamento de Matemáticas, Universidad Nacional de Colombia, Edificio Yu Takeuchi 404, Kra 30 No 45-03, Bogotá 11001000, Colombia

[2] Escuela de Matemáticas y Estadística, Universidad Pedagógica y Tecnológica de Colombia, Avenida Central del Norte 39-115, Tunja 150003, Colombia

* **Correspondence:** Email: isaias.marin@uptc.edu.co.

Abstract: Zavadskij modules are uniserial tame modules. They arose from interactions between the poset representation theory and the classification of general orders. The main problem is to characterize Zavadskij modules over a finite-dimensional algebra A. In this setting, we prove that the indecomposable uniserial A-modules with a mast of multiplicity one in each vertex are Zavadskij modules. Since the Zavadskij property carries over to direct summands, but it is not invariant under the formation of direct sums, we give a criterion to determine when the direct sum of indecomposable Zavadskij modules is again a Zavadskij module. In addition, we use the triangulations of the $n + 3$-gon associated with the cluster-tilted algebra of type $\mathbb{A}_n$ to give a formula for the number of indecomposable Zavadskij modules over any cluster-tilted algebra of type $\mathbb{A}_n$. In this case, the formula gives the dimension of the cluster-tilted algebra. As an application, we discuss some integer sequences in the OEIS (The On-Line Encyclopedia of Integer Sequences) that allow us to enumerate indecomposable Zavadskij modules.

Keywords: algorithm of differentiation; categorification; cluster-tilted algebra of type $\mathbb{A}$; integer sequence; OEIS; quiver representation; uniserial module; Zavadskij module

---

1. Introduction

W. Rump introduced Zavadskij modules in 2000 [1]. They arose from a generalization of the two-point differentiation due to A. G. Zavdaskij [2]. Such an algorithm has been used to classify tiled orders of finite representation type, abelian groups of finite rank and different types of posets [3, 4, 5]. The Zavadskij property was introduced for semimaximal rings to establish a Zavadskij-type reduction for orders in a finite-dimensional algebra over a complete field with respect to a discrete valuation. Thus, the known versions of Zavadskij's differentiation algorithm (for tiled orders [6], representations

*In memory of Professor Daniel Simson (1942-2022).

of posets, and vector space categories [5]) were unified and extended in this way to a part of the representation theory of general orders. In the context of quiver representations, the Zavadskij property has not been characterized; thus, we discuss this property in the general case of finite-dimensional algebras, and particularly in the special case of the cluster-tilted algebras of type $\mathbb{A}$. These algebras are closely to cluster algebras and its categorification through cluster categories [7] (see also [8] for the case $\mathbb{A}$). A crucial role is played by the cluster tilting objects in the cluster category, which models the clusters in the cluster algebra. The endomorphism algebras of these cluster tilting objects are called *cluster-tilted algebras*. Such algebras have been studied, particularly if the quiver underlying the cluster algebra, and hence the cluster category, is of Dynkin type. Actually, cluster-tilted algebras of Dynkin type can be described as quivers with relations where the possible quivers are precisely the quivers in the mutation class of the Dynkin quiver, and the quiver uniquely determines the relations in an explicit way [9]. Moreover, the quivers in the mutation classes of Dynkin quivers are explicitly known; for instance, in the case $\mathbb{A}$ they can be found in [10]. Moreover, the cluster-tilted algebras of type $\mathbb{A}_n$ are naturally associated with triangulations of an $n + 3$-gon [8]. Furthermore, in [11], it was shown that there is a bijection between isomorphism classes of cluster-tilted algebras of type $\mathbb{A}_n$ and triangulations of the $(n+3)$-gon up to rotation. The combinatorics associated with this kind of algebras is well-understood.

Contributions

Indecomposable Zavadskij modules are uniserial modules, but the converse is not true. However, we use the mast associated with the uniserial modules to characterize in a combinatorial way the Zavadskij modules. We show that the indecomposable Zavadskij modules over a finite-dimensional algebra A are precisely the indecomposable uniserial modules whose mast is a path with multiplicity one in each vertex (Theorem 7). Furthermore, in the case of path algebras associated a tree quivers, we give a bijection between paths in the quiver and the indecomposable Zavadskij modules (Corollary 8). Thus, we establish that a direct sum of indecomposable Zavadskij modules is a Zavadskij module if and only if the support of each pairwise of direct summands is disjoint (Theorem 9). On the other hand, we use the geometric model given in [8] to establish a formula for the number of indecomposable Zavadskij modules over any cluster-tilted algebra of type $\mathbb{A}$ (Theorem 14). This formula is also described for the particular cases of Dynkin algebras of type $\mathbb{A}$ (Corollaries 16 and 17). Following [12], this formula allows us to give a categorification of the integer sequence A000217 (Theorem 18) via the indecomposable modules in the category $\mathrm{inj}^{lf}(Q_{\mathbb{N}})$ generated by the locally finite-dimensional representations of finite-length of the form

$$ {}_m\mathbb{I}_s:\ 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} \mathbb{k}_m \xrightarrow{1} \cdots \xrightarrow{1} \mathbb{k}_s \xrightarrow{0} 0 \xrightarrow{0} \cdots $$

where $\mathbb{k}_j = \mathbb{k}$ for all $m \le j \le s$ and $1 \le m \le s < \infty$, over the infinite-dimensional path algebra associated with the quiver

$$ Q_{\mathbb{N}}:\ 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n \longrightarrow \cdots $$

In this setting, the representations ${}_m\mathbb{I}_s$ are indecomposable Zavadskij modules, and direct sums of them are Zavadskij modules if and only if each pair of different direct summands have disjunct support. Moreover, we discuss the relationship between indecomposable Zavadskij modules in certain Dynkin algebras of type $\mathbb{A}$ and the integer sequences A005563, A002370, and A152948 in the OEIS [13] (Proposition 19).

Figure 1 shows a map of the main results presented, and the connections between the different topics dealt with in this paper.

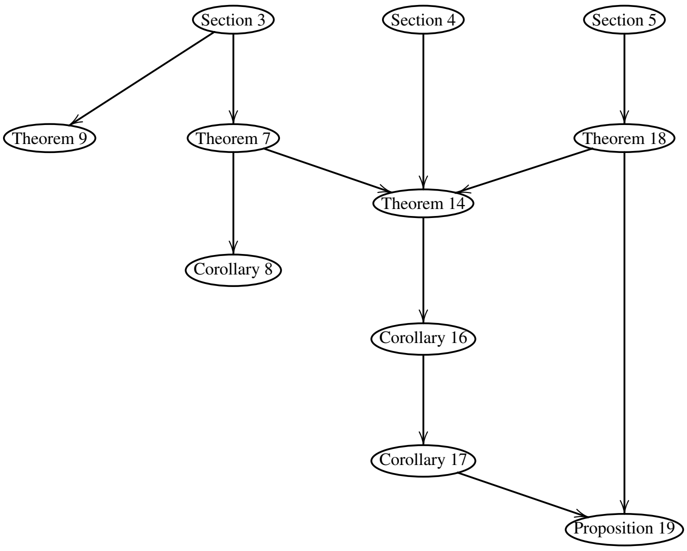


Figure 1. Main results of this paper. Theorems 7 and 9 establish a characterization of indecomposable modules and direct sums of them that are Zavadskij modules over any finite-dimensional algebra. Corollary 8 gives a bijection between paths in the quiver and the indecomposable Zavadskij modules when the algebra is associated with a tree quiver. Theorem 14 gives a formula for the number of indecomposable Zavadskij modules over any cluster-tilted algebra of type $\mathbb{A}$, and Corollaries 16 and 17 describe two particular cases. Finally, Theorem 18 categorifies the integer sequence A000217, where numbers in the sequence are invariants of indecomposable Zavadskij modules; moreover, Proposition 19 shows some integer sequences that count Zavadskij modules.

The paper is organized as follows. Section 2 deals with the modules over finite-dimensional $\mathbb{k}$-algebras in the setting of the representations of quivers. As a particular case, it introduces the cluster-tilted algebras of type $\mathbb{A}$. Moreover, following [1], Rump's characterization of the Zavadskij modules is described. In Section 3, we describe the results (Theorems 7 and 9, Corollary 8) about Zavadskij modules over finite-dimensional algebras. Finally, Section 4 is dedicated to giving a formula for the number of Zavadskij modules over cluster-tilted algebras of type $\mathbb{A}$ (Theorem 14), and Section 5 describes some integer sequences arising from Zavadskij modules.

2. Preliminaries

Through this work, $\mathbb{k}$ denotes an algebraically closed field. In this section, we recall some aspects important about modules over finite-dimensional $\mathbb{k}$-algebras in the language of quiver representations, cluster-tilted algebras of type $\mathbb{A}$, and the characterization due Rump of Zavadskij modules.

2.1. Modules over finite-dimensional $\mathbb{k}$-algebras

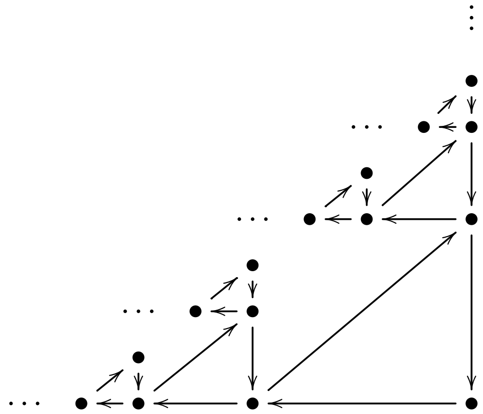
The use of quivers and their representations gives us the possibility to visualize very concretely the finitely generated modules over a finite-dimensional $\mathbb{k}$-algebra A. We write Q_0 for the set of vertices of a quiver Q, and Q_1 for its set of arrows, while $t\alpha$ and $h\alpha$ denote the tail and head of an arrow $t\alpha \xrightarrow{\alpha} h\alpha$ in Q_1. A vertex $x \in Q_0$ is said to be a *sink vertex* (resp. *source vertex*) if there is no arrow α in Q_1 such that $t\alpha = x$ (resp. $h\alpha = x$). A *representation* V of Q assigns a finite-dimensional vector space $V(x)$ to each $x \in Q_0$, and to each $\alpha \in Q_1$ a choice to linear map $V(\alpha) : V(t\alpha) \to V(h\alpha)$. A *subrepresentation* $W \subseteq V$ is a collection of subspaces $(W(x) \subseteq V(x))_{x \in Q_0}$ such that $V(\alpha)(W(t\alpha)) \subseteq W(h\alpha)$ for all $\alpha \in Q_1$. The *support* of a representation V, written Supp V, is the set of vertices $x \in Q_0$ such that $V(x) \neq 0$. Definitions of standard notions such as *morphisms*, *direct sum*, and *indescomposability* can be found in [14]. We denote by rep Q the category of representations of Q over $\mathbb{k}$.

There is a $\mathbb{k}$-algebra defined for a quiver Q in the following way: A *path of length l* in Q is a sequence of arrows $(\alpha_1, \dots, \alpha_l)$ of Q, of length l, such that $t\alpha_{i+1} = h\alpha_i$. A path of length 0, from a vertex x to x is denoted by e_x and it is called *stationary path*. A path $(\alpha_1, \dots, \alpha_l)$ of length $l \geq 1$ such that $t\alpha_1 = h\alpha_l$ is called an *oriented cycle*. The *path algebra* $\mathbb{k}Q$ of Q is defined to be the $\mathbb{k}$-algebra whose underlying $\mathbb{k}$-vector space has as a basis, the set of all paths $(x|\alpha_1, \dots, \alpha_l|y)$ of length ≥ 0. The product of two basis elements is given by the concatenation of paths. The path algebras $\mathbb{k}Q$ are *hereditary $\mathbb{k}$-algebras*, that is, $\mathbb{k}$-algebras such that the submodules of projective modules are projective. Now, the importance of quiver representations is that if Q has no oriented cycles, is equivalent to the category mod $\mathbb{k}Q$ of finite generated $\mathbb{k}Q$-modules; moreover, each finite-dimensional hereditary $\mathbb{k}$-algebra can be viewed as a path algebra ([14, Lemma 1.4 and Theorem 1.6]).

In a more general setting, a *bound quiver algebra* is the quotient $\mathbb{k}Q/I$ of a path algebra $\mathbb{k}Q$ by an *admissible ideal* I, that is, a two-sided ideal I of $\mathbb{k}Q$ that satisfies $R_Q^m \subset I \subset R_Q^2$ for some integer $m \geq 2$, where R_Q is two-sided ideal generated by all arrows in Q and for all $l \geq 1$, R_Q^l denotes the ideal of $\mathbb{k}Q$ generated, as a $\mathbb{k}$-vector space, by the set of all paths of length $\geq l$. Also, the pair (Q, I) is called a *bound quiver*. Let Q be a finite quiver and V be a representation of Q. For any nontrivial path $w = \alpha_1\alpha_2 \dots \alpha_l$ from x to y in Q, the *evaluation* of V on the path w is the $\mathbb{k}$-linear map from $V(x)$ to $V(y)$ defined by $V(w) = V(\alpha_l) \circ V(\alpha_{l-1}) \circ \cdots \circ V(\alpha_1)$. The definition of evaluation extends to $\mathbb{k}$-linear combinations of paths with a common tail and a common head which are called *relations*. A representation V of Q is said to be *bound by I*, if we have that $V(\rho) = 0$, for all relations $\rho \in I$. We denote by rep (Q, I) the full subcategory of rep Q, consisting of all representations of Q bounded by I. Now, the importance of bound quivers is that if A is a finite-dimensional $\mathbb{k}$-algebra then the category mod A of finitely generated A-modules is equivalent to the category of representations rep (Q_A, I_A) of an associated bound quiver (Q_A, I_A), where A and $\mathbb{k}Q_A/I_A$ are isomorphic as $\mathbb{k}$-algebras ([14, Theorem 1.6]). In the sequel, finite-dimensional algebras A are identified with bound quiver algebras $\mathbb{k}Q/I$ and the finitely generated A-modules are identified with the representations of Q bound by I.

2.2. Cluster-tilted algebras of type $\mathbb{A}$

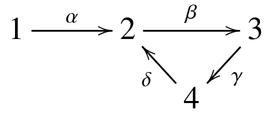
cluster-tilted algebras of type $\mathbb{A}$ are a particular case of bound quiver algebras. Although these were defined as the endomorphism algebras of the well-known cluster tilting objects (see [15, 16]) we recall a definition observed in [17], a *cluster-tilted algebra of type* $\mathbb{A}$ is the bound quiver algebra given by a finite connected full subquiver of the following infinite quiver (an infinite array of oriented 3-cycles attached at their vertices) modulo the relation that the composition of any two arrows in any oriented 3-cycle is zero.



(2.1)

Observe that a full subquiver Q containing two arrows in an oriented 3-cycle will contain the entire 3-cycle. Moreover, all 3-cycles in Q are oriented 3-cycles since the infinite quiver contains no unoriented 3-cycles.

Example 1. *A cluster-tilted algebra of type $\mathbb{A}$ is the bound quiver algebra (Q, I), where Q is the quiver*



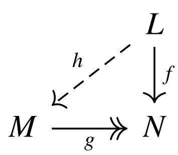
and I is the admissible ideal defined by two length paths in the 3-cycle $\beta\gamma\delta$, that is, $I = \{\beta\gamma, \gamma\delta, \delta\beta\}$.

2.3. Zavadskij modules and the Rump's characterization

Let M be a module over a ring A. Following [1], we recall that a module L over A is said to be M-*projective* if for any epimorphism $g : M \twoheadrightarrow N$ of A-modules the corresponding induced homomorphism

$$\mathrm{Hom}_A(L, M) \xrightarrow{g_*} \mathrm{Hom}_A(L, N)$$

is surjective. In other words, if $f : L \longrightarrow N$ is any morphism, then there exists a morphism $h : L \dashrightarrow M$ such that the diagram



commutes, that is, $f = g \circ h = g_*(h)$.

Similarly, a module L over A is said to be M-*injective* if for any monomorphism $g : N \hookrightarrow M$ the induced homomorphism

$$g^* : \mathrm{Hom}_A(M, L) \longrightarrow \mathrm{Hom}_A(N, L)$$

is surjective. In other words, if $f : N \longrightarrow L$ is any morphism, then there exists a morphism $h : M \longrightarrow L$ such that the diagram

$$\begin{array}{ccc} N & \overset{g}{\hookrightarrow} & M \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle h} & \\ L & & \end{array}$$

commutes, that is, $f = h \circ g = g^*(h)$. Moreover, if M itself is M-injective then M is said to be *quasi-injective*. For example, a projective (resp. injective) A-module is M-projective (resp. M-injective) for each A-module M, whereas any injective module is quasi-injective.

Definition 2. *An A-module M is a Zavadskij module if all of its submodules are M-projectives and each factor module is M-injective.*

In other words, this says that for each homomorphism $\overline{f} : U \longrightarrow W$ from a submodule U to a factor module W of M, the diagram

$$\begin{array}{ccc} M & \overset{f}{\dashrightarrow} & M \\ \uparrow & & \downarrow \\ U & \xrightarrow{\ \overline{f}\ } & W \end{array}$$

can be completed by an endomorphism f of M. Thus, for M to be a Zavadskij module, it suffices to see that each submodule is M-projective and that M itself is quasi-injective.

Example 3. *Let Q be the quiver $1 \longrightarrow 2$ then the path algebra $A = \Bbbk Q$ is the matrix ring $\left(\begin{smallmatrix}\Bbbk & 0 \\ \Bbbk & \Bbbk\end{smallmatrix}\right)$ and the indecomposable projective A-modules are Zavadskij modules, whereas the module ${}_AA$ is not a Zavadskij module because it is not a quasi-injective module. Thus, a direct sum of Zavadskij modules is not a Zavadskij module.*

We recall that a nonzero module M over a finite-dimensional $\Bbbk$-algebra A is *uniserial* if the lattice of its submodules forms a chain, i.e., if every two submodules of M are comparable. Clearly, all subfactors of a uniserial module are again uniserial. The lattice of proper nonzero submodules of a length-finite A-module M is then a finite chain with maximal element $\operatorname{rad} M$ and minimal element $\operatorname{soc} M$, where these are the notations for Jacobson radical and socle of M, respectively. Note that, if $J = \operatorname{rad} A$ then the submodules of M are given by $J^l M$, with $l = 0, \dots, \operatorname{length}(M)$ (see [14]).

If M is a uniserial A-module with submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ and composition factors $U_i = M_{i+1}/M_i$ then there are ring homomorphisms

$$\operatorname{End}_A M \longrightarrow \operatorname{End}_A U_i \tag{2.2}$$

which are all injective if the U_i are pairwise nonisomorphic. In this case, the maps (2.2) are monomorphisms between skew fields. In this setting, we recall that a uniserial A-module M is *tame* if the U_i are mutually nonisomorphic and the maps (2.2) are isomorphisms.

Concerning uniserial modules, a characterization of Zavadskij modules was achieved by W. Rump. In the indecomposable case, we have the following Theorem 4.

Theorem 4. *[1, Proposition 3] Let M be a module over a finite-dimensional $\Bbbk$-algebra. M is an indecomposable Zavadskij module if and only if M is a tame uniserial module.*

In the general case, we have the following Theorem 5.

Theorem 5. *[1, Proposition 4 and Theorem 1] An A-module M is a Zavadskij module if and only if it satisfies one of the following conditions.*

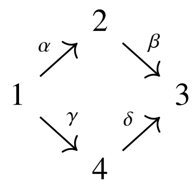
(a) The reduced part $M_1 \oplus \cdots \oplus M_r$ of $M = M_1^{n_1} \oplus \cdots \oplus M_r^{n_r}$ is a Zavadskij module, where all of M_i are indecomposable pairwise non isomorphic.
(b) M can be decomposed into tame uniserial modules and any two indecomposable direct summands of M with a common composition factor are isomorphic.

3. A combinatorial characterization of Zavadskij modules over finite-dimensional algebras

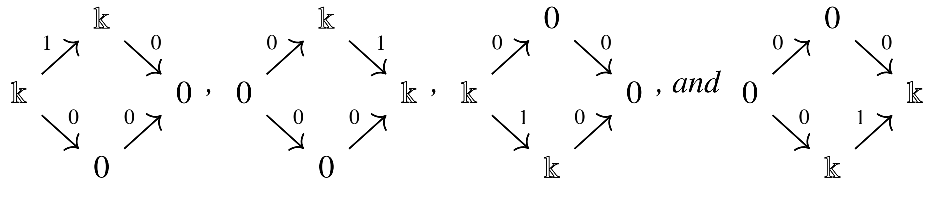
In this section, we prove Theorem 7, Theorem 9 and Corollary 8 to characterize finitely generated Zavadskij modules over finite-dimensional algebras using the mast associated with each indecomposable uniserial module. In the sequel of this section, A denotes a finite-dimensional $\Bbbk$-algebra; thus, A can be identified with a bound quiver algebra $A = \Bbbk Q/I$, where Q is a connected quiver, and I is an admissible ideal of the path algebra $\Bbbk Q$.

Following [19, 18], we recall that a path p in the quiver Q is said to be a *mast* of the uniserial A-module M if $\text{length}(p) = \text{length}(M) - 1$ and $pM \neq 0$. We let $\mu(x)$ denotes the *multiplicity* of a vertex x in a mast p which is the number of times that the vertex x occurs in p.

Example 6. *Let $A = \Bbbk Q/I$ be a bound quiver algebra, where Q is the quiver*



and $I = \langle \beta\alpha - \delta\gamma \rangle$. Then, the paths of length one α, β, γ, and δ are masts of the uniserial modules



respectively. Clearly, the simple modules $S_1 = \langle e_1 \rangle$, $S_2 = \langle e_2 \rangle$, $S_3 = \langle e_3 \rangle$, and $S_4 = \langle e_4 \rangle$ are uniserial whose masts are the stationary paths e_1, e_2, e_3, and e_4, respectively. However, neither $\beta\alpha$ nor $\delta\gamma$ is a mast of a uniserial A-module.

According to [19, 18], we have the following elementary observations:

1) Not every path in $\Bbbk Q$ with a nonzero image in A needs to occur as a mast of a uniserial module.

2) If the quiver Q has no double arrows, each uniserial A-module has a unique mast. Conversely, the uniqueness of masts in all uniserial modules implies an absence of double arrows.

3) For each uniserial A-module M with sequence $(S_1, \ldots, S_{l+1})$ of consecutive composition factors, there exists at least one path p of length l in $\Bbbk Q$ such that $pM \neq 0$ (see [18]); necessarily p passes in order through the sequence $(1, \ldots, l+1)$ of those vertices in Q_0 which represent the simple modules S_i. Moreover, $0 \neq \operatorname{soc} M = J^l M$ and J^l is generated by the images of the paths of length l.

The following result gives a combinatorial characterization of indecomposable Zavadskij modules over any finite-dimensional $\Bbbk$-algebra.

Theorem 7. *An indecomposable A-module M is a Zavadskij module if and only if it is a uniserial module, and for any mast p of M it holds that $\mu(x) = 1$ for every vertex $x \in Q_0$ that p passes through. In particular, if the algebra A is hereditary, M is a Zavadskij module if and only if it is uniserial.*

Proof. First, we consider the case where A is a hereditary algebra. The necessary condition follows immediately from Theorem 4. On the other hand, we consider a uniserial representation M of the quiver Q. We suppose that the mast p has the form

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$$

thus, M is defined by the formulas

$$M(\alpha) = \begin{cases} 1_{\Bbbk}, & \text{if } \alpha \in \{\alpha_1, \ldots, \alpha_{n-1}\}, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad M(x) = \begin{cases} \Bbbk, & \text{if } x \in \{1, ..., n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since M is indecomposable and it has the above form, the ring of endomorphisms $\operatorname{End} M$ is isomorphic to $\Bbbk$. Moreover, the composition factors have the form M_{i+1}/M_i, where M_j is the subrepresentation of M such that $M_j(x) = M(x)$ if $x \in \{j, \ldots, n\}$ and zero otherwise. Thus, M^{i+1}/M^i is the simple representation S_i at vertex i, thus $\operatorname{End}(M_{i+1}/M_i)$ is isomorphic to $\Bbbk$. So, the maps (2.2) are isomorphisms. Moreover, the composition factors are mutually nonisomorphic because they are simple representations in different vertices. In effect, we suppose that $\dim_{\Bbbk}(\operatorname{Hom}_A(P_i, M)) > 1$ for some i, where P_i denotes the indecomposable projective representation at vertex i. Since M is finitely generated then there exists a projective resolution of M of the form $0 \longrightarrow P' \longrightarrow P \longrightarrow M \longrightarrow 0$, with $P = \bigoplus_{i \in Q_0} d_i P_i$ and $P' = \bigoplus_{\alpha \in Q_1} d_{t\alpha} P_{h\alpha}$, where d_i denotes the dimension of $M(i)$ and $d_i P_i$ stands for the direct sum of d_i copies of P_i. It is clear that the indecomposable projective module P_i appears once, thus $\dim_{\Bbbk}(\operatorname{Hom}_A(P_i, M)) = 1$. Therefore, the simple module S_i at vertex i appears only once as factor composition of M. These arguments prove that M is a tame module via Theorem 4.

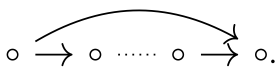
In the general case, let M be an indecomposable Zavadskij A-module over a bound quiver algebra $A = \Bbbk Q/I$. Theorem 4 implies that M is a tame uniserial module of length $l + 1 \geq 1$, therefore, indecomposable simple modules S_i, $1 \leq i \leq l+1$ in the sequence of composition factors $(S_1, \ldots, S_{l+1})$ are pairwise nonisomorphic. Additionally, if p is a mast of M then we note that p is a path in $\Bbbk Q$ of length l such that $p \notin I$ and $pM \neq 0$; necessarily p passes in order through the sequence $(1, \ldots, l+1)$ of those vertices in Q_0 which represent the simple modules S_i. Thus, $\mu(x) = 1$ for any $x \in p$. On the other hand, let us suppose that M is uniserial and that p is a mast of M with $\mu(x) = 1$ for any $x \in p_0$. Therefore, factors composition of M are precisely the indecomposable simple modules S_i corresponding bijectively to the set $p_0 = \{1, \ldots, l, l+1\}$ of vertices in p, therefore S_i is not isomorphic to S_j if $i \neq j$. Since $\operatorname{End} M \cong \Bbbk$ we have that M is a tame A-module. Thus, Theorem 4 allows to conclude that M is an indecomposable Zavadskij module. We are done.

We recall that a connected quiver with n vertices and $n-1$ arrows is called a *tree quiver* (this means that the underlying graph is a tree in the sense of graph theory). We have an easy way to determine indecomposable Zavadskij modules over path algebras of tree quivers.

Corollary 8. *If Q is tree quiver then the following bijection holds*

$$\{\textit{paths in } Q\} \leftrightarrows \{\textit{indecomposable Zavadskij } \Bbbk Q\textit{-modules}\}.$$

Proof. Note that each indecomposable uniserial module determines a unique mast in Q because there are no double arrows in Q. Furthermore, [20, Lemma 2.1.1] implies that each path in Q determines exactly one uniserial $\Bbbk Q$-module because there are only finitely many paths in Q and Q does not contain a subquiver of form



Let us consider the case when the direct sum of indecomposable Zavadskij modules over a finite-dimensional algebra A is a Zavadskij module via the masts associated.

Theorem 9. *A direct sum $M = M_1^{n_1} \oplus \cdots \oplus M_r^{n_r}$ of indecomposable pairwise non isomorphic Zavadskij A-modules is a Zavadskij module if and only if for each pair $i \neq j$ the masts p_i and p_j associated with M_i and M_j respectively do not have common vertices.*

Proof. Let $p_i : a_0 \longrightarrow a_1 \longrightarrow \cdots \longrightarrow a_s$ and $p_j : a'_0 \longrightarrow a'_1 \longrightarrow \cdots \longrightarrow a'_t$ be the masts associated with the indecomposable A-modules M_i and M_j respectively, where $i \neq j$. First, we suppose that p_i and p_j share the vertex $a'_k = a_k$. Since $M_i(a_j) = \Bbbk$ for all $0 \leq j \leq s$, then $M_j(a_k) = \Bbbk$. Thus, the simple S_{a_k} is a composition factor of M_i and of M_j. Then $M_i \cong M_j$ which is contradiction (see Theorem 5). On the other hand, we assume that p_i and p_j do not have a vertex in common. Then M is of finite length because the $\Bbbk$-algebra A is finite-dimensional, thus the reduced part $\overline{M} = \bigoplus_{i=1}^{n} M_i$ of M can be decomposed in tame uniserial A-modules provided that each module M_i is an indecomposable Zavadskij A-module. Now, if the simple S_a is a composition factor of M_i and M_j with $i \neq j$, then $M_i(a) \cong \Bbbk$ and $M_j(a) \cong \Bbbk$. Therefore a is a vertex in p_i and p_j, which contradicts the hypothesis, so M_i and M_j do not have a common composition factor. Since Theorem 5 establishes that $\overline{M}$ and M are Zavadskij modules. We are done.

Example 10. *Continuing Example 1, the indecomposable Zavadskij modules over the cluster-tilted algebra $\Bbbk Q/I$ correspond to: 1) the simple representations S_1 to S_4, whose masts associated are the stasionary paths e_1 to e_4 respectively. 2) the representations M^p such that $p \in \{\alpha, \beta, \gamma, \delta\}$, where $M^p(x) = \Bbbk$ if $x \in \{tp, hp\}$ and zero otherwise. Moreover, $M(p)$ is $1_\Bbbk$ whenever it is possible and zero otherwise. The masts associated with these modules are the arrows in Q. 3) the representation M such that $M(x) = \Bbbk$ if x is $1, 2,$ or 3 and zero otherwise. Moreover $M(\alpha) = M(\beta) = 1_\Bbbk$. Note that the mast of M is the path $\alpha\beta$ in Q. Direct sums as $M^\alpha \oplus M^\gamma$ and $M \oplus S_4$ are Zavadskij modules while the direct sum $M^\beta \oplus M^\delta$ is not a Zavadskij module. Observe that in this case, the number of indecomposable Zavadskij modules is 9, and it is equal to the dimension of the algebra.*

4. The number of indecomposable Zavadskij modules over cluster-tilted algebras ($\mathbb{A}_n$ case)

This section gives a formula for the number of indecomposable Zavadskij modules over a cluster-tilted algebra A. Note that such a number equals the dimension of the algebra A. To do that, we use the geometric model for these algebras given in [8]. Moreover, we discuss the role of some integer sequences in the enumeration of indecomposable Zavadskij modules over certain families of algebras.

In [8] P. Caldero, F. Chapoton, and R. Schiffler considered regular polygons with $n+3$ vertices and triangulations of such polygons. A *diagonal* is a straight line between two non-adjacent vertices on the border of the polygon, and a *triangulation* is a maximal set of diagonals that do not cross. In fact, a triangulation of an $(n+3)$-gon consists of exactly n diagonals. In [8] the category of diagonals of such polygons was defined, and it was shown to be equivalent to the cluster category of type $\mathbb{A}_n$; which, in greater generality, was defined simultaneously by A.B. Buan et al., [7]. They defined cluster categories as orbit categories of the bounded derived category of hereditary algebras. As an application in [8], the module category of a cluster-tilted algebra of type $\mathbb{A}_n$ is described by a category of diagonals C_T, where T is a triangulation of the $n+3$-gon. Indeed, the cluster-tilted quivers of type $\mathbb{A}_n$ are precisely those that are associated with an arbitrary triangulation of the $(n+3)$-gon.

For any triangulation of the regular $(n+3)$-gon, we can define a quiver with n vertices in the following way. The vertices are the midpoints of the diagonals. There is an arrow between i and j if the corresponding diagonals bound a common triangle. The orientation is $i \to j$ if the diagonal corresponding to j can be obtained from the diagonal corresponding to i by rotating anticlockwise about their common vertex (see Figure 2).

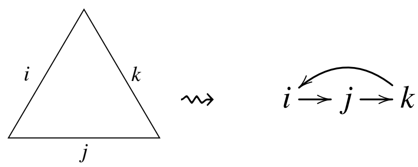


Figure 2. j lies counterclockwise of i.

It is also known from [8] that all quivers obtained in this way are quivers of cluster-tilted algebras of type $\mathbb{A}_n$. Furthermore, in [11], it was shown that there is a bijection between isomorphism classes of cluster-tilted algebras of type $\mathbb{A}_n$ and triangulations of the $(n+3)$-gon up to rotation.

Define the set of relations I to be the set of all paths $i \to j \to k$ such that there exists an arrow $k \to i$. It is important to recall that every triangulation T of the $(n+3)$-gon gives rise to a cluster-tilted algebra $\Bbbk Q_T/I$ of type $\mathbb{A}_n$ and every cluster-tilted algebra is of this form. In particular, every Dynkin quiver of type $\mathbb{A}_n$ corresponds to a triangulation without internal triangles; that is, each triangle has at least one side on the boundary of the polygon.

According [21], a *fan* in T is a maximal subset $\Sigma_v \subseteq T$ of at least two diagonals such that all the diagonals in Σ_v share the vertex v of Π_{n+3}. We denote by $\mathfrak{F}_T$ the set of fans in T.

Example 11. *Continuing with Example 1, we get the triangulation T (Figure 3).*
Observe that the fans determined by the triangulation are $\mathfrak{F}_T = \{\{2,4\}, \{4,3\}, \{1,2,3\}\}$.

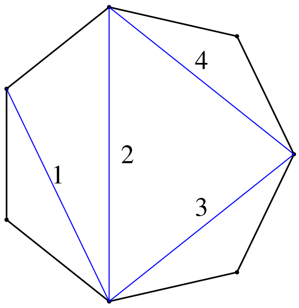


Figure 3. Triangulation associated with a cluster-tilted algebra of type $\mathbb{A}_4$.

Definition 12. *Let (Q, I) be a bound quiver. A path $p \notin I$ is said to be a maximal path in (Q, I) if there is no an arrow $\alpha \in Q_1$ such that $p\alpha \notin I$ or $\alpha p \notin I$.*

Lemma 13. *Given a triangulation T of the $(n+3)$-gon and its corresponding cluster-tilted algebra $\Bbbk Q_T/I$ of type $\mathbb{A}_n$, there is a bijection*

$$\{\textit{maximal paths in } (Q_T, I)\} \leftrightarrows \{\textit{fans in } T\}.$$

Proof. Since the quiver Q_T is connected, then a maximal path in (Q_T, I) has a length of at least one. For each arrow $i \longrightarrow j$ in Q_T, Figure 2 implies that the diagonals i and j belong to the same fan. Given a maximal path $p : i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_k$ in (Q_T, I), since $p \notin I$, p is an acycle path. If p belongs to a 3-cycle then the length of p is one because the length of two paths are zero, and we have a fan $\{i_1, i_2\}$, where the diagonals i_1 and i_2 are in a internal triangle in T. When p does not belong to a 3-cycle, we have a fan $F = \{i_1, i_2, \ldots, i_k\}$. Indeed, the maximality of p implies the property of maximality of F. It is clear that, different fans correspond to different maximal paths in (Q_T, I). Finally, note that a fan $F = \{i_1, \ldots, i_k\}$ in T, which has been drawn with blue color in Figure 4, defines a path

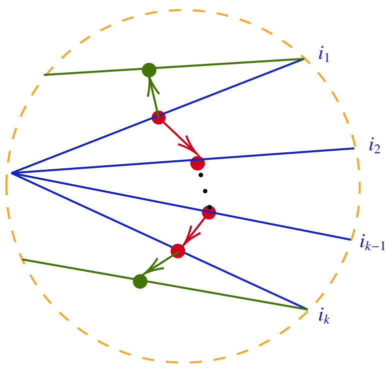


Figure 4. Maximal path defined by a fan.

$p : i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_k$ (red color), which is maximal in (Q_T, I), because although the green arrows in Figure 4 are adjacent to p, these do not satisfy Definition 12.

We denote by t_n the triangular number $t_n = \frac{n(n+1)}{2}$.

Theorem 14. *Let $\mathfrak{F}_T = \{F_1, \ldots, F_r\}$ be the set of fans in a triangulation T of the $(n+3)$-gon. The number of indecomposable Zavadskij modules over the cluster-tilted algebra $\Bbbk Q_T/I$ of type $\mathbb{A}_n$ is given* *by*

$$\sum_{i=1}^{r} t_{|F_i|} - \sum_{i \neq j} t_{|F_i \cap F_j|},$$

where t_j denotes the j-th triangular number.

Proof. Theorem 7 implies that the indecomposable Zavadskij modules over the algebra $\Bbbk Q_T/I$ are determined by all of the subpaths of maximal paths in (Q_T, I). Note that if p is a maximal path in (Q_T, I) and lenght$(p) = r$, there are exactly $t_r = \frac{r(r+1)}{2}$ subpaths in p, including the subpaths of length zero. Lemma 13 implies that each fan corresponds to a maximal path in (Q_T, I). Thus, the number of indecomposable Zavadskij modules over $\Bbbk Q_T/I$ is given by $\sum_{F \in \mathfrak{F}_T} t_{|F|} - h$, where the number h corresponds to the paths of length zero which have been counted twice. In other words, h is the number of diagonals in T that belong to exactly two different fans. Since if $F_i \cap F_j \neq \emptyset$, $|F_i \cap F_j| = 1$, we have that $h = \sum_{i \neq j} t_{|F_i \cap F_j|}$.

Example 15. *Continuing our running example, the number of indecomposable Zavadskij modules over the cluster-tilted algebra in Example 1 is $(t_3 + t_2 + t_2) - 3 = 9$. Note that, the cardinality of the fans $\{1, 2, 3\}$, $\{2, 4\}$, and $\{4, 3\}$, determine the summands $t_3 = 6$, $t_2 = 3$, and $t_2 = 3$ respectively. Moreover, observe that there are three diagonals (2, 3 and 4) that belong to exactly two fans in T.*

A particular case of cluster-tilted algebras of type $\mathbb{A}$ are the Dynkin algebras of type $\mathbb{A}_n$, that is, path algebras associated with Dynkin quivers Q of type $\mathbb{A}_n$, whose underlying graph $\overline{Q}$ has the form given in Figure 5. In this case, the vertices 1 and n are called *extreme vertices* of Q. Corollary 16 provides a

Figure 5. Dynkin diagram of type $\mathbb{A}_n$.

formula for the number of indecomposable Zavadskij modules over a Dynkin algebra of type $\mathbb{A}_n$.

Corollary 16. *Let $\Bbbk Q$ be a Dynkin algebra of type $\mathbb{A}_n$ and let $\mathfrak{F}_T$ be the set of fans in the triangulation T associated with Q. Then the number of indecomposable Zavadskij modules over $\Bbbk Q$ is given by*

$$\sum_{F \in \mathfrak{F}_T} t_{|F|} - (|\mathfrak{F}_T| - 1).$$

Proof. In the enumeration that we did in the proof of Theorem 14, h corresponds to the number of diagonals that belong to precisely two fans. In case (a), if $k \in F \cap F'$, with $F, F' \in \mathfrak{F}_T$, we have that k is either a non-extremal sink-vertex or a non-extremal source-vertex in Q and the number of these vertices in Q is equal to $|\mathfrak{F}_T| - 1$.

Recall that an algebra A is said to be *right serial* if every indecomposable projective right A-module is uniserial. An algebra A is called *left serial* if every indecomposable projective left A-module is uniserial. It is well known that a basic $\Bbbk$-algebra A is right serial if and only if, for every vertex x of its ordinary quiver Q_A, there exists at most one arrow of source x (see [14]). Moreover, note that the path algebra $A = \Bbbk Q$ of a tree quiver Q is a right serial algebra if and only if Q has only one sink vertex $i \in Q_0$.

Corollary 17. *Let $A = \Bbbk Q$ be the right serial Dynkin algebra of type $\mathbb{A}_n$ whose unique sink vertex is i. Then the number of indecomposable Zavadskij A-modules is given by $Z_i^{\mathbb{A}}(n) = t_i + t_{n-(i-1)} - 1$.*

Proof. Since the quiver Q has the shape $1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} i \xleftarrow{\alpha_i} \cdots \xleftarrow{\alpha_{n-1}} n$ then the triangulation T of the $(n+3)$-gon associated with Q has two fans with i and $n-(i-1)$ diagonals respectively. Moreover, there is exactly one diagonal (the diagonal i) that belong to the two fans. Theorem 14 implies the result.

5. Integer sequences arising from Zavadskij modules

As an application of the combinatorial approach studied in Sections 2 and 3, we found integer sequences arising from the number of indecomposable Zavadskij modules. In particular, a categorification of the sequence A000217 is given. In addition, we show some families of right serial path algebras whose number of indecomposable Zavadskij modules defines the integer sequences A005563, A002370, and A152948 encoded in OEIS [13].

Categorification of an integer sequence is a term coined by M. Ringel and P. Fahr [22, 23] to the process for which numbers in the sequence can be seen as invariants of objects in such a way that functional relations between objects of the category can be interpreted as identities between numbers in the sequence (see [24, 25] for more examples of categorifications).

Given the linearly oriented quiver Q_n

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n.$$

of type $\mathbb{A}_n$, all of the indecomposable modules in rep Q are uniserial and each path in Q is the mast of a unique uniserial module. Then the number of indecomposable Zavadskij modules is the n-th triangular number. Now, we consider the linearly oriented infinite Dynkin quiver of type $\mathbb{A}_\infty$, that is,

$$Q_{\mathbb{N}}:\ 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n \longrightarrow \cdots$$

A representation V of $Q_{\mathbb{N}}$ is said to be *of finite length* if $V(x) = 0$ for almost all vertices $x \in (Q_{\mathbb{N}})_0$. Following [12], we denote by $\mathrm{Rep}^{lf} Q_{\mathbb{N}}$ the category formed by *locally finite dimensional* representations, that is, directed unions of representations of finite length. It follows that the category $\mathrm{Rep}^{lf} Q_{\mathbb{N}}$ is locally finite and $\mathrm{rep}^{lf} Q_{\mathbb{N}}$ consist of all objects of $\mathrm{Rep}^{lf} Q_{\mathbb{N}}$ of finite length. In view of [12], each indecomposable object in $\mathrm{rep}^{lf} Q_{\mathbb{N}}$ is isomorphic to a *finite-interval representation* of the form

$${}_m\mathbb{I}_s:\ 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} \Bbbk_m \xrightarrow{1} \cdots \xrightarrow{1} \Bbbk_s \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

where $\Bbbk_j = \Bbbk$ for all $m \le j \le s$ and $1 \le m \le s < \infty$. Note that the finite-interval representations are indecomposable Zavadskij modules over the path algebra $\Bbbk Q_{\mathbb{N}}$ and direct sums of them are Zavadskij modules if and only if each pair of different direct summands have disjunct support.

Let $\mathrm{inj}^{lf}(Q_{\mathbb{N}})$ be the full subcategory of $\mathrm{rep}^{lf} Q_{\mathbb{N}}$ generated by the representations ${}_1\mathbb{I}_s$, for all $s \ge 1$. The Auslander-Reiten quiver of $\mathrm{inj}^{lf}(Q_{\mathbb{N}})$ has the form

$$\cdots \hookrightarrow {}_1\mathbb{I}_3 \hookrightarrow {}_1\mathbb{I}_2 \hookrightarrow {}_1\mathbb{I}_1.$$

Then, we give the following categorification.

Theorem 18. *The category* $inj^{lf}(Q_{\mathbb{N}})$ *give a categorification of the integer sequence A000217 in the OEIS.*

Proof. Let $\phi : \text{inj}^{lf}(Q_{\mathbb{N}}) \to \mathbb{N}$ be the function given by

$$\phi(M) = \sum_{i \in \text{Supp} M} i.$$

It associates to each indecomposable object ${}_1\mathbb{I}_s \in \text{inj}^{lf}(Q_{\mathbb{N}})$ the triangular number t_s. Note that this number is an invariant of the indecomposable objects in $\text{inj}^{lf}(Q_{\mathbb{N}})$. Hence, we obtain that

$$\cdots > \phi({}_1\mathbb{I}_3) > \phi({}_1\mathbb{I}_2) > \phi({}_1\mathbb{I}_1).$$

This order relation of the indecomposable objects in $\text{inj}^{lf}(Q_{\mathbb{N}})$ indicates that there are irreducible morphisms in $\text{inj}^{lf}(Q_{\mathbb{N}})$ from ${}_1\mathbb{I}_k$ to ${}_1\mathbb{I}_{k+1}$, for all $k \geq 1$.

Let $A = \Bbbk Q_{n,i}$ be a right serial Dynkin algebra of type $\mathbb{A}_n$, where the quiver $Q_{n,i}$ has the form

$$1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} i \xleftarrow{\alpha_i} \cdots \xleftarrow{\alpha_{n-1}} n,$$

that is, $Q_{n,i}$ has a unique sink vertex i. For any $i \geq 1$ fixed and $\mathbb{N}_{\geq i} = \{i, i+1, \dots\}$, we define the function

$$Z_i^{\mathbb{A}} : \mathbb{N}_{\geq i} \longrightarrow \mathbb{N},$$

where $Z_i^{\mathbb{A}}(n)$ is the number of indecomposable Zavadskij modules over the path algebra $\Bbbk Q_{n,i}$. Table 1 shows the values of the formula $Z_i^{\mathbb{A}}(n)$, for $3 \leq n \leq 10$ and a unique sink at vertex i (see Corollary 17).

$i \backslash n$	3	4	5	6	7	8	9	10
1	6	10	15	21	28	36	45	55
2	5	8	12	17	23	30	38	47
3			11	15	20	26	33	41
4					19	24	30	37
5							29	35

Table 1. Values of the formula $Z_i^{\mathbb{A}}(n)$, which count the number of indecomposable Zavadskij modules over the path algebra $\Bbbk Q_{n,i}$.

Proposition 19. *The integer sequences A152948, A005563, and A002370 in the OEIS count indecomposable Zavadskij modules over certain right serial Dynkin algebras of type* $\mathbb{A}_n$.

Proof. For $i = 2$, we have the sequence of quivers $Q_{2,2} = 1 \to 2$, $Q_{3,2} = 1 \to 2 \leftarrow 3$, $Q_{4,2} = 1 \to 2 \leftarrow 3 \leftarrow 4$, and so on. In this case, $Z_i^{\mathbb{A}}(2) = t_2$, $Z_i^{\mathbb{A}}(3) = t_2 + t_2 - t_1$, $Z_i^{\mathbb{A}}(4) = t_2 + t_3 - t_1$. In general, we have $Z_i^{\mathbb{A}}(k) = t_2 + t_{k-1} - t_1 = \frac{(k-1)(k)}{2} + 2 = \frac{k^2-k+4}{2}$. Replacing k by $2 - n$, we have the sequence $a(n) = \frac{n^2-3n+6}{2}$ for $n \geq 3$, which is encoded by A152948 in the OEIS. Let $\{Q_k\}_{k\geq 1}$ be the family of Dynkin quivers of type $\mathbb{A}_{2k}$ whose unique sink vertex is $k+1$, that is, $Q_1 = Q_{2,2} : 1 \to 2$, $Q_2 = Q_{4,3} : 1 \to 2 \to 3 \leftarrow 4$, $Q_3 = Q_{6,4} : 1 \to 2 \to 3 \to 4 \leftarrow 5 \leftarrow 6$, and so on. Note that the number of indecomposable Zavadskij modules over the algebra $\Bbbk Q_n$ is given by $t_{n+1} + t_n - 1 = (n+1)^2 - 1$, this formula corresponds to the integer sequence encoded by A005563 in OEIS. Finally, we consider the family $\{Q_k\}_{k\geq 1}$ of Dynkin quivers of type $\mathbb{A}_{2k+1}$ whose unique sink vertex is $k+2$. More precisely, the sequence of quivers is $Q_1 = Q_{3,3} : 1 \to 2 \to 3$, $Q_2 = Q_{5,4} : 1 \to 2 \to 3 \to 4 \leftarrow 5$, $Q_3 = Q_{7,5} : 1 \to 2 \to 3 \to 4 \to 5 \leftarrow 6 \leftarrow 7$, and so on. Note that the number of indecomposable Zavadskij modules over the algebra $\Bbbk Q_k$ is given by $t_{k+2} + t_k - t_1 = k^2 + 3k + 2 = (k+1)(k+2)$. If we replace $k+1$ by n, we obtain the sequence $n(n+1)$ for $n \geq 2$, which is encoded by A002370 in OEIS.

6. Concluding remarks

The indecomposable Zavadskij modules over a finite-dimensional algebra $\Bbbk Q/I$ are precisely the uniserial modules whose support is a path (called mast) in Q that does not belong to I, and this path does not pass by one same vertex twice or more. The direct sum of Zavadskij modules is a Zavadskij module when each pair of masts associated with the summands have disjoint support. The number of indecomposable Zavadskij modules is determined when the algebra is cluster-tilted of type $\mathbb{A}$, in this case, there are some integer sequences associated with this formula in some particular cases. An interesting case is given when the algebra is the path algebra associated with the natural numbers, where there is an arrow from n to $n+1$ for each $n \in \mathbb{N}$; in this case, a categorification of the integer sequence A000217 was obtained via Zavadskij modules that are finite-interval modules.

Acknowledgments

The first author has been supported by the Seminar Alexander Zavadskij on Representation of Algebras and Its Applications at Universidad Nacional de Colombia. The second and third author have been supported by Minciencias-Colombia Convocatoria fortalecimiento de vocaciones y formación en CTeI para la reactivación económica en el marco de la postpandemia 2020. No 891.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. W. Rump, Two-point differentiation for general orders, *J. Pure Appl. Algebra*, **153** (2000), 171–190. https://doi.org/10.1016/S0022-4049(99)00085-7
2. A. G. Zavadskij, Differentiation with respect to a pair of points, *Matrix probl.*, (1977), 115–121.
3. D. Arnold, *Abelian Groups and Representations of Finite Partially Ordered Sets*, Springer Science & Business Media, 2012.
4. W. Rump, Differentiation for orders and Artinian rings, *Algebras Represent. Theory*, **7** (2004), 395–417. https://doi.org/10.1023/B:ALGE.0000042182.98997.7c
5. D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra, Logic and Applications, Gordon and Breach Science Publishers, Montreux, 1992.

6. A. G. Zavadskiĭ, V. V. Kiričenko, Semimaximal rings of finite type, *Mat. Sb. (N.S.)*, **103** (1977), 323–345.

7. A. B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.*, **204** (2006), 572–618. https://doi.org/10.1016/j.aim.2005.06.003

8. P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Am. Math. Soc.*, **358** (2006), 1347–1364. https://doi.org/10.1090/S0002-9947-05-03753-0

9. A. B. Buan, R. J. Marsh, I. Reiten, Cluster-tilted algebras of finite representation type, *J. Algebra*, **306** (2006), 412–431. https://doi.org/10.1016/j.jalgebra.2006.08.005

10. A. B. Buan, D. F. Vatne, Derived equivalence classification for cluster-tilted algebras of type A_n, *J. Algebra*, **319** (2008), 2723–2738. https://doi.org/10.1016/j.jalgebra.2008.01.007

11. H. A. Torkildsen, Counting cluster-tilted algebras of type A_n, *Int. Electron. J. Algebra*, **4** (2008), 149–158.

12. S. Nowak, D. Simson, Locally Dynkin quivers and hereditary coalgebras whose left comodules are direct sums of finite dimensional comodules, *Comm. Algebra*, **30** (2002), 455–476. https://doi.org/10.1081/AGB-120006503

13. N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at http://oeis.org, 1964.

14. I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, London Mathematical Society Student Texts, 65, Cambridge University Press, Cambridge, 2006. https://doi.org/10.1017/CBO9780511614309

15. A. B. Buan, H. A. Torkildsen, The number of elements in the mutation class of a quiver of type D_n, *Electron. J. Combin.*, **16** (2009), 23. https://doi.org/10.37236/138

16. J. Bastian, T. Holm, S. Ladkani, Towards derived equivalence classification of the cluster-tilted algebras of Dynkin type D, *J. Algebra*, **410** (2014), 277–332. https://doi.org/10.1016/j.jalgebra.2014.03.034

17. I. Assem, S. Trepode, Homological methods, representation theory, and cluster algebras, in *Papers Based on Six Mini-Courses Delivered at the 2016 CIMPA School Held at the Universidad Nacional de Mar del Plata, Mar del Plata* (eds. Ibrahim Assem and Sonia Trepode) Springer, Cham, 2018. https://doi.org/10.1007/978-3-319-74585-5

18. A. Boldt, A. Mojiri, On uniserial modules in the Auslander-Reiten quiver, *J. Algebra*, **319** (2008), 1825–1850. https://doi.org/10.1016/j.jalgebra.2007.11.026

19. B. Huisgen-Zimmermann, The geometry of uniserial representations of finite-dimensional algebra. I, *J. Pure Appl. Algebra*, **127** (1998), 39–72. https://doi.org/10.1016/S0022-4049(96)00184-3

20. A. Boldt, *Two Aspects of Finite-Dimensional Algebras: Uniserial Modules and Coxeter Polynomials*, ProQuest LLC, Ann Arbor, MI, University of California, PhD Thesis, 1996.

21. R. Schiffler, R. J. Serna, A geometric realization of socle-projective categories for posets of type $\mathbb{A}$, *J. Pure Appl. Algebra*, **224** (2020), 23. https://doi.org/10.1016/j.jpaa.2020.106436

22. P. Fahr, C. M. Ringel, A partition formula for Fibonacci numbers, *J. Integer Seq.*, **11** (2008), 9.

23. P. Fahr, C. M. Ringel, Categorification of the Fibonacci numbers using representations of quivers, *J. Integer Seq.*, **15** (2012), 12. https://doi.org/10.48550/arXiv.1107.1858
24. A. M. Cañadas, P. F. F. Espinosa, I. D. M. Gaviria, Categorification of some integer sequences via Kronecker modules, *JP J. Algebra, Number Theory Appl.*, **38** (2016), 339–347. http://dx.doi.org/10.17654/NT038040339
25. A. M. Cañadas, I. D. M. Gaviria, H. Giraldo, Representation of equipped posets to generate Delannoy numbers, *Far East J. Math. Sci.*, **8** (2017), 1677–1695. https://doi.org/10.17654/MS102081677