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http://www.aimspress.com/journal/era

ERA, 30(9): 3422–3434.
DOI: 10.3934/era.2022174
Received: 13 May 2022
Revised: 23 June 2022
Accepted: 28 June 2022
Published: 20 July 2022

Research article

Stability analysis of Boolean networks with Markov jump disturbances and their application in apoptosis networks

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Abstract: In this paper, the finite-time stability (FTS) of switched Boolean networks (SBNs) with Markov jump disturbances under the conditions of arbitrary switching signals is studied. By using the tool of the semi-tensor product, the equivalent linear-like form of SBNs with Markov jump disturbances is first established. Next, to facilitate investigation, we convert the addressed system into an augmented Markov jump Boolean network (MJBN), and propose the definition of the switching set reachability of MJBNs. A necessary and sufficient criterion is developed for the FTS of SBNs with Markov jump disturbances under the conditions of arbitrary switching signals. Finally, we give two examples to illustrate the effectiveness of our work.

Keywords: Switched Boolean network; semi-tensor product; Markov jump disturbance; finite-time stability

1. Introduction

Boolean networks (BNs) have been a common model to describe a number of complex systems such as gene networks [1–3], social neural networks [4] and others [5, 6]. Unfortunately, due to the lack of suitable mathematical tools, the previous research on BNs has limitations. Now, according to the semi-tensor product (STP) of the matrix proposed by Cheng [7, 8], a BN can be transformed into its quasi-linear form, which promotes the research of BNs. At present, a series of breakthrough results on BNs have been obtained.

Recently, switched BNs(SBNs) have become more important and attracted many researchers' attention. It is a powerful model to describe the hybrid phenomena in gene regulatory networks [1, 2, 9, 10]. Many fundamental and critical problems regardig SBNs have been studied in recent years, including point controllability [11–14], set controllability [15–19], reachability [20, 21], point stability [22–24], set stability [25–27], observability [18, 28, 29] and robustness [15, 30]. However, in the process of system modeling, some random factors, including random external interference and internal uncertainty, often exists. These factors often have some random properties, such as the Markov property, which will increase the difficulty of the research on original systems to some extent. As we all know, some excellent results have been obtained on BNs with Markov jump disturbances [14, 21, 31, 32]. Unfortunately, there are few results on SBNs with Markov jump disturbances.

An SBN with Markov jump disturbances is not a typical SBN. Generally, an SBN with Markov jump disturbances contains both a switching signal and disturbances with the Markov property, making it more complex and more difficult to analyze than BNs with Markov jump disturbances. Stability analysis is one of the most important issues in gene regulatory networks since it can effectively explain some living phenomena and offer good theoretical basis to treat various human cancers. There are some excellent results on the stability of Markovian jump Boolean networks (MJBNs) [14, 21] and SBNs [22–25]. For an MJBN, its stability, including point stability and set stability, represents the ability of the initial state to reach a given target state. More specifically, [21] focused on the point stability of MJBNs while it also illustrated the application of it. In [14], the author studied the set stability by performing induction, and by dividing the state space into several pieces, they consequently proposed some necessary and sufficient conditions.

As for SBNs, [12] introduced the basic concept of switched systems. The study in [22] concentrated on the stability of switched linear systems. The studies in [23, 25] expanded it to normal SBNs. By the way, [24] proposed a new concept named switching point reachability. Based on it, [24] discussed the pointwise stabilizability and global stability of SBNs. The authors of [23] developed an algorithm to calculate the largest invariant subset (LIS) of a given subset for SBNs. By using this tool, a criterion for the set stability of SBNs was established and the application of robustness was also mentioned.

According to the references listed above, many good results have been achieved in these areas. However, the stability of SBNs with Markov jump disturbances is rarely studied, even though it is more common and useful in practice. This problem needs to be solved and new theories are required urgently.

In this paper, our purpose is to discuss the finite-time stability (FTS) of SBNs with Markov jump disturbances. The main contributions are two-fold. On the one hand, the SBN with Markov jump disturbances is converted into a typical SBN. On the other hand, on the basis of the switching set reachability, an easily verifiable criterion for the stability of SBNs with Markov jump disturbances is proposed.

After that, the next three sections are as follows. Several essential notations and the problem considered in this paper are cited in Section 2; Section 3 contains the main results; In Section 4, two easily verifiable examples are proposed to illustrate the obtained results, and this is followed by a brief conclusion.

2. Preliminaries

2.1. Notations

- $D = \{0, 1\}$.
- $D^n = \underbrace{D \times D \times \cdots \times D}_{n}$.
- I_n represents the $n \times n$ identity matrix.

- $L_{m\times n}$ represents the set of all $m \times n$ logical matrices.
- $\text{Row}_i(A)$ ($\text{Col}_i(A)$) denotes the i-th row (column) of the matrix A.
- $\Delta_n = \left\{\delta_n^k \mid 1 \leq k \leq n\right\}$, and $\delta_n^k = Col_k(I_n)$.
- $|M|$ represents the cardinal number of the set M.
- $\otimes$ represents the Kronecker product.
- $\delta_n\left[j_1, \cdots, j_m\right] = \left[\delta_n^{j_1}, \cdots, \delta_n^{j_m}\right]$.
- The power-reducing matrix $M_{r,n}$ is defined as $\delta_{n^2}\left[1, n+2, 2n+3, \cdots, n^2\right]$.
- A $p \times q$ stochastic matrix is denoted as:

$$\Upsilon^{p\times q} = \left\{ Q \in \mathbb{R}^{p\times q} \mid Q_{a,b} \geq 0, \sum_{a=1}^{p} Q_{a,b} = 1, \forall b = 1, 2, \cdots, q \right\}.$$

- A vector π is said to be $\pi > 0$ if every element of π is more than 0.

2.2. Properties of STP and structure matrices

Definition 1. *[8] The STP of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is*

$$A \ltimes B = \left(A \otimes I_{\frac{\alpha}{n}}\right)\left(B \otimes I_{\frac{\alpha}{p}}\right)$$

where $\alpha = lcm(n, p)$ is the least common multiple of n and p.

The following lemmas are the properties of the STP.

Lemma 1. *[8] (Pseudo-Commutativity) Given $U \in \mathbb{R}^t$ and $C \in \mathbb{R}^{m\times n}$, then*

$$UC = (I_t \otimes C)U.$$

Lemma 2. *[8] For any logical function $f : D^n \mapsto D$, there exists a unique structure matrix $M_f \in L_{2\times 2^n}$ of f satisfying $f(X(t)) \sim M_f\, x(t)$, where $x(t) \in \Delta_{2^n}$ is the vector form of $X(t) \in D^n$.*

2.3. Problem setting

Consider an SBN with n nodes and ω models:

$$\begin{cases} X_1(t+1) = f_1^{\sigma(t)}(X(t), \tau(t)); \\ X_2(t+1) = f_2^{\sigma(t)}(X(t), \tau(t)); \\ \qquad\qquad \vdots \\ X_n(t+1) = f_n^{\sigma(t)}(X(t), \tau(t)), \end{cases} \tag{2.1}$$

where $X(t) = (X_1(t), ..., X_n(t)) \in D^n$ and $X_i \in D$ is the state of the node $i \in \{1, 2, \cdots, n\}$; $\sigma : \mathbb{N} \mapsto \boldsymbol{A} = \{1, 2, \cdots, \omega\}$ is the switching signal; $f_i^j : D^{q+n} \mapsto D, i = 1, \cdots, n, j = 1 \cdots, \omega$ are logical functions; $\tau(t) = (\varepsilon_1(t), \cdots, \varepsilon_q(t)) \in D^q$ represents the disturbances.

In this paper, we consider the external disturbances $\varepsilon_i(t), i = 1, 2, \cdots, q$ with the Markov property. Specifically, $\{\varepsilon_i(t) : t \in \mathbb{Z}_{\geq 0}\}$ is the homogeneous Markov chain and it satisfies

$$P\left\{\varepsilon_i(t+1) = b_i \mid \varepsilon_i(t) = a_i\right\} = P_{b_i,a_i} \in (0, 1) \tag{2.2}$$

$b_i, a_i \in D$ and $\sum_{b_i=0}^{1} P_{b_i,a_i} = 1$.

To promote the investigation, based on the STP and the vector forms of $X(t)$ and $\varepsilon(t)$, we further convert the system (2.1) into the following algebraic form:

$$x(t+1) = F\sigma(t)\varepsilon(t)x(t) \tag{2.3}$$

where $x(t) \in \Delta_{2^n}$, $\sigma(t) \in \Delta_{\omega}$, $\varepsilon(t) \in \Delta_{2^q}$ and $F \in L_{2^n \times \omega \cdot 2^{n+q}}$

For (2.2), we identify $\varepsilon_i(t+1) = b_i$ and $\varepsilon_i(t) = a_i$ as the vector forms $\delta_2^{2-b_i}$ and $\delta_2^{2-a_i}$, respectively. Setting $\varepsilon(t) = \ltimes_{i=1}^{q} \varepsilon_i(t) = \delta_{2^q}^{a}$ and $\varepsilon(t+1) = \ltimes_{i=1}^{q} \varepsilon_i(t+1) = \delta_{2^q}^{b}$, from the unique factorization of $\varepsilon(t)$ and $\varepsilon(t+1)$, we have

$$\begin{aligned} & P\left\{\varepsilon(t+1) = \delta_{2^q}^{b} \mid \varepsilon(t) = \delta_{2^q}^{a}\right\} \\ =& \prod_{i=1}^{q} P\left\{\varepsilon_i(t+1) = \delta_2^{2-b_i} \mid \varepsilon_i(t) = \delta_2^{2-a_i}\right\} \\ =& \prod_{i=1}^{q} P_{b_i,a_i} \\ :=& P_{b,a} \in (0,1), \end{aligned} \tag{2.4}$$

where $a, b \in \{1, \cdots, 2^q\}$, $a_i, b_i \in D$, $\sum_{b=1}^{2^q} P_{b,a} = 1$, and the discrete time homogeneous Markov chain $\{\varepsilon(t) : t \in \mathbb{Z}_{\geq 0}\}$ is ergodic. From [33], the Markov chain $\{\varepsilon(t) : t \in \mathbb{Z}_{\geq 0}\}$ has a unique stationary distribution, denoted as π.

According to (2.4), we have

$$\varepsilon(t+1) = P\varepsilon(t), \tag{2.5}$$

and

$$\pi = P\pi,$$

where $P \in \Upsilon^{2^q \times 2^q}$ with $(P)_{b,a} = P_{b,a}$ is the transition probability matrix.

For convenience, we abbreviate finite-time stable with probability one as FTSPO, and the following assumption is necessary.

Assumption 1. *The Markov chain* $\{\varepsilon(t) : t \in \mathbb{Z}_{\geq 0}\}$ *can realize the stationary distribution* π *in finite time, which means that* $\exists\, T \in \mathbb{Z}_{\geq 0}$*, when* $t \geq T$, $P^{(t)} = \pi$.

Definition 2. *The system given by (2.3) is said to be finite-time stable at* $x_e \in \Delta_{2^n}$ *with probability one (*x_e*-FTSPO), if for* $\forall \sigma(0) \in \Delta_{\omega}$*,* $x(0) \in \Delta_{2^n}$ *and* $\varepsilon(0) \in \Delta_{2^q}$*, there exists* $T \in \mathbb{Z}_+$ *such that for* $\forall t \geq T, t \in \mathbb{Z}_+$

$$P\left[x(t; x(0), \sigma(0), \varepsilon(0)) = x_e\right] = 1. \tag{2.6}$$

holds, where $x(t; x(0), \sigma(0), \varepsilon(0)$ *denotes the trajectory of the system (2.3) from* $x(0), \sigma(0)$ *and* $\varepsilon(0)$.

3. Main results

3.1. System's transformation

In this part, we transform the system (2.3) to make analyzing more convenient.

First, we divide F into several parts:

$$F = \left[F^1, F^2, \cdots, F^\omega\right],$$

where $F^i \in L_{2^n \times 2^{n+q}}$, and when $\sigma(t) = \delta_\omega^i, F\sigma(t) = F^i, i = 1, 2, \cdots, \omega$. At this moment, $x(t+1) = F^i \varepsilon(t) x(t)$.

Next, we set $\delta(t) = \varepsilon(t) \ltimes x(t)$.

Then, according to (2.5) and Lemma 1, we have

$$\begin{aligned}\delta(t+1) &= \varepsilon(t+1)x(t+1) \\ &= P\varepsilon(t)F^i\varepsilon(t)x(t) \\ &= \left(I_{2^q} \otimes F^i\right) PM_{r,2^q}\varepsilon(t)x(t) \\ &= Q^i\delta(t)\end{aligned}$$

where $Q^i = \left(I_{2^q} \otimes F^i\right) PM_{r,2^q} \in \mathbb{R}^{2^{n+q} \times 2^{n+q}}, i = 1, 2, \cdots, \omega$. Although Q is not a logical matrix, this system can still be regarded as an SBN, because every subsystem of it is a probabilistic BN(PBN), which makes Q a random matrix instead of a logical matrix.

What we want to research is the point stability of x(t); from Lemma 3, it is easy to know that when x(t) is stable, which means that x(t) is fixed, $\delta(t)$ may have different values because $\delta(t)$ may have different values. So the point stability problem is transformed into the set stability problem.

Now, the system given by (2.3) has been transformed into a new SBN:

$$\delta(t+1) = Q\sigma(t)\delta(t), \tag{3.1}$$

where $Q = \left[\begin{array}{cccc} Q^1 & Q^2 & \cdots & Q^\omega \end{array}\right] \in \Upsilon^{2^{n+q} \times \omega 2^{n+q}}$.

It is easy to find that when $x(t) = x_e$, the value of $\delta(t)$ can be determined by $\varepsilon(t)$.

To find out the relationship between $\delta(t)$ and $\varepsilon(t)$, the following lemma is necessary.

Lemma 3.

$$\delta_{2^m}^i \ltimes \delta_{2^n}^j = \delta_{2^{m+n}}^{(i-1)2^n + j}.$$

Proof. From Definition 1, we have

$$\delta_{2^m}^i \ltimes \delta_{2^n}^j = \left(\delta_{2^m}^i \otimes I_{2^n}\right)\delta_{2^n}^j.$$

Set

$$A = \left(\delta_{2^m}^i \otimes I_{2^n}\right) = \begin{pmatrix} 0_{2^n \times 2^n} \\ \vdots \\ 0_{2^n \times 2^n} \\ I_{2^n} \\ 0_{2^n \times 2^n} \\ \vdots \\ 0_{2^n \times 2^n} \end{pmatrix}$$

where $0_{2^n \times 2^n}$ denotes a $2^n \times 2^n$ null matrix. And we set $B = \delta_{2^n}^j$.

Then we set $C = \delta_{2^m}^i \ltimes \delta_{2^n}^j = AB \in L_{2^{n+m} \times 1}$, and for any element of C, $C_p = \sum_{a=1}^{2^n} A_{p,a}B_a, p = 1, 2, \cdots, 2^{m+n}$. So $C_p = 1$ if and only if (iff) $p = (i-1)2^n + j$, which corresponds the j-th row of I_{2^n} in A. The proof is completed.

Now we set $x_e = \delta_{2^n}^e$. Based on lemma 3, defining $M = \left\{\delta_{2^{n+q}}^{(i-1)2^n+e} \mid i \in [1:2^q]\right\}$, the equivalent definition of Definition 2 is obtained.

Definition 3. *Consider a subset $M \subseteq \Delta_{2^{n+q}}$. The system given by (3.1) is said to be M-FTSPO, if for $\forall \sigma(0) \in \Delta_\omega, \delta(0) \in \Delta_{2^{n+q}}, \exists T \in \mathbb{Z}_+$ such that*

$$P[\delta(t, \sigma(0), \delta(0)) \in M] = 1$$

holds for $\forall t \geq T, t \in \mathbb{Z}_+$.

Remark 1. *The authors of [34] have shown that the system (2.3) is M-FTSPO iff the system is I_M-stable, where I_M is the LIS of M. However, since the Markov chain $\{\varepsilon(t) : t \in \mathbb{Z}_{\geq 0}\}$ is ergodic, the stationary distribution π is greater than 0. In other words, for $\forall i, j \in [1:2^q]$, $t \in \mathbb{N}$ and $\left[P^{(t)}\right]_{ij} > 0$. From the definition of LIS, we have $I_M = M$.*

3.2. Switching set reachability

In this part, we first introduce the switching set reachability of the SBN (3.1).

Definition 4. *(Switching set reachability) Consider the system (3.1) for (no punctuation) a given initial state $\delta(0) \in \Delta_{2^{n+q}}$ and set $M \subseteq \Delta_{2^{n+q}}$. The system given by (3.1) is said to be switching reachable to M from $\delta(0)$, if $\exists t > 0$ with $t \in \mathbb{N}$, $\exists \sigma$ such that $P(\delta(t;\sigma,\delta(0)) \in M) > 0$.*

Set $M = \left\{\delta_{2^{n+q}}^{\alpha_i} \mid i \in [1:2^q]\right\}$ and $S = \sum_{i=1}^{\omega} Q_i$, where $(S)_{i,j}$ denotes that the one-step transition probability from $\delta_{2^{n+q}}^{\alpha_j}$ to $\delta_{2^{n+q}}^{\alpha_i}$ of the system (3.1). The following result can be derived.

Theorem 1. *Let $\delta(0) = \delta_{2^{n+q}}^{\beta}$. The system (3.1) is switching reachable to M from $\delta(0)$ at time Z, iff*

$$\sum_{i=1}^{2^q} \left(S^Z\right)_{\alpha_i,\beta} > 0 \tag{3.2}$$

Proof. We can use induction to prove the results for $Z \geq 1$.

When $Z = 1$, M is switching reachable from $\delta(0) = \delta_{2^{n+q}}^{\beta}$, iff $\exists \theta \in [1:\omega]$ such that $P(\delta(1;\sigma = \theta, \delta(0)) \in M) > 0$, that is

$$\sum_{i=1}^{2^q} (Q_\theta)_{\alpha_i,\beta} > 0.$$

Then, we have

$$\sum_{i=1}^{2^q} (S)_{\alpha_i,\beta} = \sum_{i=1}^{2^q} [\sum_{j=1}^{\omega} \left(Q_j\right)]_{\alpha_i,\beta} = \sum_{i=1}^{2^q} \left(\sum_{j=1, j\neq\theta}^{\omega} Q_j + Q_\theta\right)_{\alpha_i,\beta} > 0$$

which can conclude that (3.2) holds for $Z = 1$.

Theorem 1 holds for $Z = s$, i.e. $\sum_{i=1}^{2^q} (S^s)_{\alpha_i,\beta} > 0$ is assumed.

We consider that $Z = s + 1$ and M is switching reachable from $\delta(0) = \delta_{2^{n+q}}^{\beta}$ at time $s+1$ iff $\exists\,\sigma \in \{\sigma(t) \in [1:\omega] : t \in [0:s]\}$ such that

$$\begin{aligned}
&\mathbb{P}(\delta(s+1;\sigma,\delta(0)) \in M)\\
=&\sum_{c=1}^{2^{n+q}} P\left(\delta(s+1;\sigma,\delta(0)) \in M \mid \delta(s) = \delta_{2^{n+q}}^{c}\right)\\
&\times P\left(\delta(s;\sigma,\delta(0)) = \delta_{2^{n+q}}^{c} \mid \delta(0) = \delta_{2^{n+q}}^{\beta}\right) > 0,
\end{aligned}$$

which implies that

$$\sum_{c=1}^{2^{n+q}} \sum_{i=1}^{2^q} (S)_{\alpha_i,c}(S^s)_{c,\beta} = \sum_{i=1}^{2^q} S^{s+1}{}_{\alpha_i,\beta} > 0.$$

Additionally, (3.2) is true for $Z = s + 1$.

In conclusion, Theorem 1 holds for $\forall Z \in N$. The proof is completed.

3.3. FTS under an arbitrary switching signal

Lemma 4. *For* $\forall y = 1, 2\cdots, 2^{n+q}, k \in \mathbb{Z}_+$, *we have*

$$\sum_{\upsilon=1}^{2^{n+q}} \left(S^k\right)_{\upsilon,y} = \omega^k \tag{3.3}$$

Proof. It can be proven by induction.

When $k = 1$, for $\forall y = 1, 2\cdots, 2^{n+q}$, since P is a stochastic matrix, $\left(I_{2^q} \bigotimes F^{\upsilon}\right)$ and $M_{r,2^q}$ are logical matrices, for $\forall s = 1, 2, \cdots, \omega$, we have $\sum_{\upsilon=1}^{2^{n+q}} (Q_s)_{\upsilon,y} = 1$. Thus

$$\sum_{\upsilon=1}^{2^{n+q}} (S)_{\upsilon,y} = \sum_{\upsilon=1}^{2^{n+q}} \sum_{s=1}^{\omega} (Q_s)_{\upsilon,y} = \sum_{s=1}^{\omega} \sum_{\upsilon=1}^{2^{n+q}} (Q_s)_{\upsilon,y} = \omega,$$

which implies that (3.3) holds for $k = 1$.

Assume that for $\forall j = 1, 2\cdots, 2^{n+q}$, (3.3) is true for $k = t$. When $k = t + 1$, for $\forall j = 1, 2\cdots, 2^{n+q}$, it can be concluded that

$$\begin{aligned}
&\sum_{\upsilon=1}^{2^{n+q}} (S^{t+1})_{\upsilon,y}\\
=&\sum_{\upsilon=1}^{2^{n+q}} \sum_{c=1}^{2^{n+q}} (S^t)_{\upsilon,c}(S)_{c,y}\\
=&\sum_{\upsilon=1}^{2^{n+q}} (S^t)_{\upsilon,c} \sum_{c=1}^{2^{n+q}} (S)_{c,y}\\
=&\omega^t \sum_{c=1}^{2^{n+q}} (S)_{c,y} = \omega^{t+1}.
\end{aligned}$$

So, (3.3) holds for $t + 1$.

By induction, for $\forall j = 1, 2\cdots, 2^{n+q}$, (3.3) holds for $\forall k \in N$. The proof is completed.

In fact, denoting the set consisting of all admissible switching signals during $[0, k]$ as $\Omega_a^k := \{\sigma(t) \in \boldsymbol{A}, t \in [0, k]\}$, gives $\left|\Omega_a^k\right| = \omega^k$.

Then, based on Lemma 4, the following result can be derived.

Theorem 2. *The system given by (3.1) is M-FTSPO under the conditions of arbitrary switching signals iff* $\exists\, T < 2^{n+q} - 2^q$ *with* $T \in N$ *such that*

$$\sum_{i=1}^{2^q} Row_{a_i}\left(S^T\right) = [\underbrace{\omega^T \cdots \omega^T}_{2^{n+q}}] \tag{3.4}$$

holds.

Proof. **(Necessity)** Assume that the system (3.1) is M-FTSPO under the conditions of any switching signal above. Then, according to Definition 3, we conclude that for $\forall \sigma(0) \in \Delta_\omega, \delta(0) \in \Delta_{2^{n+q}}, \exists\, T \in \mathbb{Z}_+$ such that

$$P[\delta(t, \sigma(0), \delta(0)) \in M] = \sum_{i=1}^{2^q} P\left[\delta(t, \sigma(0), \delta(0)) = \delta_{2^{n+q}}^{a_i}\right] = 1$$

holds for $\forall t \geq T, t \in \mathbb{Z}_+$.

By Theorem 1 and Lemma 4, one can see that for any integer $t \geq T, (S^t)_{a,b} = 0, \forall a \notin \{a_1, \cdots, a_{2^q}\}$, and $\sum_{i=1}^{q} (S^t)_{a_i,b} = \omega^t$; it implies that (3.4) holds.

Next, we prove by contradiction that $T < 2^{n+q} - 2^q$ when (3.4) holds. If $T \geq 2^{n+q} - 2^q$, then $\exists \mu$, $\vartheta \in [1, 2^{n+q}]$ with $\mu \notin \{a_1, \cdots, a_{2^q}\}$ such that $\left(S^T\right)_{\mu,\vartheta} > 0$. Then, $\exists\, \sigma = \{\sigma(0), \cdots, \sigma(2^{n+q} - 2^q \ -1)\}$ such that $P(\delta(2^{n+q} - 2^q; \sigma, \delta(0)) = \delta_{2^{n+q}}^{\mu}) > 0$. Since the number of elements in $\Delta_{2^{n+q}} \backslash M$ is $2^{n+q} - 2^q$, we can find $t_1, t_2 \in \mathbb{N}$ with $0 \leq t_1 < t_2 \leq 2^{n+q} - 2^q - 1$ such that $\delta(t_1) = \delta(t_2)$.

Now, for $\bar{\delta}(0) = \delta(t_1)$, we set $\bar{\sigma}(t) = \sigma(t_1 + t), t = 0, 1, \cdots, t_2 - t_1 - 1$. Then, given $\bar{\sigma}(t)$, it is easy to see that $P(\delta(t_2 - t_1; \bar{\sigma}(t), \delta(0) = \delta(t_1)) = \delta(t_1)) > 0$. Generally, for $s \in \mathbb{Z}_+$, we construct the following signal:

$$\bar{\sigma}(t) = \begin{cases} \sigma(t_1), t = s(t_2 - t_1) \\ \sigma(t_1 + 1), t = s(t_2 - t_1) + 1 \\ \vdots \\ \sigma(t_2 - 1), t = (s+1)(t_2 - t_1) - 1 \end{cases}$$

which is a periodic switching signal. Then, given $\bar{\sigma}(t)$, $P(\delta(2^{n+q} - 2^q + s(t_2 - t_1); \bar{\sigma}(t), \delta(0)) = \delta_{2^{n+q}}^{\mu} > 0$ which contradicts the minimality of T. Hence, $T < 2^{n+q} - 2^q$.

(Sufficiency) Assuming that (3.4) holds, we show that the system (3.1) is M-FTSPO under the conditions of an arbitrary switching signal.

Actually, according to (3.4), we can see that for any $\forall j \in [1 : 2^q]$,

$$\left(S^T\right)_{a,j} = 0, \forall a \notin \{a_1, \cdots, a_{2^q}\},$$

and

$$\left(S^T\right)_{a,j} = \omega^t, a \in \{a_1, \cdots, a_{2^q}\}.$$

Thus, for any $t \geq T$, we have

$$(S^t)_{a,j} = 0, \forall a \notin \{a_1, \cdots, a_{2^q}\}$$

and

$$(S^t)_{a,j} = \omega^t, a \in \{a_1, \cdots, a_{2^q}\}.$$

Hence, by Theorem 1 and Lemma 4, it can be concluded that for $\forall\delta(0) = \delta_{2^{n+q}}^j$, $P(\delta(T;\sigma(t),\delta(0)) \in M) = 1$ at time $T \leq 2^{n+q}$ under the conditions of an arbitrary switching signal. Therefore, system (3.1) is M-FTSPO under the conditions of arbitrary switching signaling.

4. Illustrate examples

Example 1. *The reduced model of an apoptosis network is described [4]:*

$$\begin{cases} x_1(t+1) = \neg x_2(t) \wedge (u(t) \wedge \varepsilon(t)), \\ x_2(t+1) = -x_1(t) \wedge (x_3(t) \wedge \varepsilon(t)), \\ x_3(t+1) = x_2(t) \vee (u(t) \wedge \varepsilon(t)), \end{cases} \tag{4.1}$$

where the biological meaning of each varible can be found in [28]. $\varepsilon(t)$ is a Markov process, which has the following transition probability matrix:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

We need to verify whether the system given by (4.1) is finite-time stable at $(1,0,1)$ (cell survival) with probability one for any switching signal.

It is easy to know that P has invariant measure:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

The algebraic form of (4.1) can be derived:

$$x(t+1) = F\sigma(t)\varepsilon(t)x(t)$$

where $F^1 = \delta_8[7\ 7\ 3\ 3\ 5\ 7\ 1\ 3 \quad 7\ 7\ 8\ 8\ 7\ 7\ 8\ 8]$,
$F^2 = \delta_8[7\ 7\ 8\ 8\ 5\ 7\ 6\ 8 \quad 7\ 7\ 8\ 8\ 7\ 7\ 8\ 8]$. Moreover, $x_e = \delta_8^3$ and $M = \left\{\delta_{16}^3, \delta_{16}^{11}\right\}$.

By calculation, we find that $\mathrm{Row}_3\left(S^{14}\right) + \mathrm{Row}_{11}\left(S^{14}\right) \neq 2^{14}$. So as stated by Theorem 2, this system is not finite-time stable at $(1,0,1)$ with probability one.

Example 2. *We consider the SBN (2.1) with $n = 2$, $q = 1$ and $\sigma = 2$, where we have the following:* $f_1^1 = (\neg x_1 \wedge x_2) \wedge \varepsilon(t)$, $f_2^1 = 0$; $f_1^2 = (x_1 \bar{\vee} x_2) \wedge \varepsilon(t)$, $f_2^2 = [\neg(x_1 \leftrightarrow x_2)] \wedge \varepsilon(t)$; $f_1^3 = (x_1 \bar{\vee} x_2) \wedge \varepsilon(t)$, $f_2^3 = (x_1 \bar{\vee} x_2) \wedge \varepsilon(t)$. *$\varepsilon(t)$ is a Markov process with*

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

Given the desired state $x_e = (0,0) \sim \delta_4^4 := x_e$, we need to check whether this system is x_e-FTSPO under the conditions of arbitrary switching signaling.

Then, (2.3) can be obtained with $L_1 = \delta_4[4\ 4\ 2\ 4\quad 4\ 4\ 4\ 4]$, $L_2 = \delta_4[4\ 1\ 2\ 4\quad 4\ 4\ 4\ 4]$ and $L_3 = \delta_4[4\ 1\ 1\ 4\quad 4\ 4\ 4\ 4]$. Moreover, $x_e = \delta_4^4$ and $M = \left\{\delta_8^4, \delta_8^8\right\}$.

$$S = Q_1 + Q_2 + Q_3 = \begin{bmatrix} 0 & 1 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.5 & 0.5 & 0 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 \\ 0 & 1 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.5 & 0.5 & 0 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 \end{bmatrix}$$

By calculation,

$$S^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13.5 & 13.5 & 13.5 & 13.5 & 13.5 & 13.5 & 13.5 & 13.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13.5 & 13.5 & 13.5 & 13.5 & 13.5 & 13.5 & 13.5 & 13.5 \end{bmatrix}$$

$\text{Row}_4\left(S^3\right) + \text{Row}_8\left(S^3\right) = [27\ 27\ 27\ 27\ 27\ 27\ 27\ 27]$. From Theorem 2, this system is δ_4^4-FTSPO under the conditions of arbitrary switching signaling.

5. Conclusions

In this paper, we investigated the FTSPO of SBNs with Markov jump disturbances under the conditions of arbitrary switching signaling. First, the target system was an SBN with Markov jump disturbances, which means that we cannot handle it like a typical SBN. We transformed it into SBN-form so that we can handle it in the manner of an SBN. We transformed the algebraic expression using the STP. Next, we proposed an easily verifiable, necessary and sufficient criterion for the FTSPO of SBNs under the conditions of arbitrary switching signaling. Finally, we illustrated the obtained results by two examples.

Conflict of interest

The authors declare that there is no conflicts of interest.

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