



Research article

# Long-time dynamical behavior for a piezoelectric system with magnetic effect and nonlinear dampings

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**Abstract:** This paper is concerned with the long-time dynamical behavior of a piezoelectric system with magnetic effect, which has nonlinear damping terms and external forces with a parameter. At first, we use the nonlinear semigroup theory to prove the well-posedness of solutions. Then, we investigate the properties of global attractors and the existence of exponential attractors. Finally, the upper semicontinuity of global attractors has been investigated.

**Keywords:** piezoelectric system; the well posedness; global attractor; exponential attractors; upper semicontinuity of attractors

## 1. Introduction

In this paper, we investigate the piezoelectric beam system with magnetic effect:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + f_1(v, p) + g_1(v_t) = \varepsilon h_1, & (0, L) \times (0, T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + f_2(v, p) + g_2(p_t) = \varepsilon h_2, & (0, L) \times (0, T), \end{cases} \quad (1.1)$$

where  $\varepsilon \in [0, 1]$ ,  $\alpha = \alpha_1 + \gamma^2\beta$ , and  $\alpha, \rho, \gamma, \beta, \mu > 0$  mean elastic stiffness, the mass density per unit volume, piezoelectric coefficient, water resistance coefficient, magnetic permeability, respectively, which is supplemented by the following initial boundary conditions:

$$\begin{cases} v(0, t) = \alpha v_x(L, t) - \gamma\beta p_x(L, t) = 0, & t > 0, \\ p(0, t) = p_x(L, t) - \gamma v_x(L, t) = 0, & t > 0, \\ v(x, 0) = v_0, \quad v_t(x, 0) = v_1, & 0 < x < L, \\ p(x, 0) = p_0, \quad p_t(x, 0) = p_1, & 0 < x < L. \end{cases} \quad (1.2)$$

It is well known that the ability to convert mechanical energy and electrical energy into each other is a characteristic of piezoelectric materials. In addition, it is smaller, less expensive, and more efficient [1]. Therefore, it has been applied in automotive [2, 3], medical [4, 5], space structures [6, 7] and many other fields [8]. In the previous modeling stage, the magnetic effect was basically ignored due to its slight [9], but the latest results show that the magnetic effect may influence the performance of the system [10], so it is crucial to study piezoelectric beam models with magnetic effects. Nowadays, a growing number of scholars are beginning to do related research in this field.

The establishment of mathematical models of piezoelectric beams has a long history. In the early days, Tebou [11] and Haraux [12] established the model of a single beam due to Maxwell equation [13, 14] and the dynamic interaction between electromagnetism is ignored. Therefore, they obtained the following equation:

$$\begin{cases} \rho v_{tt} - \alpha_1 v_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ v(0, t) = \alpha_1 v_x(L, t) + \delta v_t(L, t) = 0, & t \in \mathbb{R}^+. \end{cases}$$

Based on the magnetic effects and variational method, the equations of a single beam were derived by Morris and Özer [15], as follows:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ v(0, t) = p(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, & t \in \mathbb{R}^+, \\ \beta p_x(L, t) + \frac{V(t)}{h} = 0, & t \in \mathbb{R}^+. \end{cases} \quad (1.3)$$

where  $V(t) = p_t(L, t)$  denotes electrical feedback controller. Then, they established the further results: the well-posedness of solution, strong stability of piezoelectric beam and so on [16, 17].

In [18], Ramos et al. inserted a dissipative term  $\delta v_t$  in the first equation of (1.3) and considered the following boundary condition

$$\begin{cases} v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, & 0 < t < T, \\ p(0, t) = p_x(L, t) - \gamma v_x(L, t) = 0, & 0 < t < T. \end{cases}$$

The authors applied energy method to prove the energy of the system and estimated that the energy is exponentially stable. After that, Ramos et al. changed the boundary condition in [19]:

$$\begin{cases} v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) + \xi_1 \frac{v_t(L, t)}{h} = 0, & 0 < t < T, \\ p(0, t) = p_x(L, t) - \gamma v_x(L, t) + \xi_2 \frac{p_t(L, t)}{h} = 0, & 0 < t < T, \end{cases}$$

and proved the exponential stability regardless of any relationship between system coefficients and there is equivalent to exact observability at the boundary.

Moreover, we know there are many ways to control piezoelectric vibration, among which time-delayed feedback control is a common way to improve stability of the system. Therefore, more and

more researchers studied the influence of time-delayed effect on the stability of hyperbolic systems, and also proved the so-called destabilizing effect.

Dakto et al. [20–23] has obtained a series of research results in this direction. Among them, Dakto studied the equation as follows,

$$u_{tt} - u_{xx} + 2u_t(x, t - \tau) = 0, \quad (x, t) \in (0, 1) \times (0, +\infty),$$

where  $\tau > 0$ , and proved that the time delay in the dissipative term charcatered by velocity term could make the beams lose stability.

Therefore, it is necessary to add some controlled term to stabilize the hyperbolic systems.

In 2021, Freitas and Ramos [24] considered the longitudinal vibration of the piezoelectric beams with thermal effect, magnetic effect and fraction damping:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \delta \theta_x + f_1(v, p) + v_t = h_1, & (0, L) \times (0, T), \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + A^\nu p_t + f_2(v, p) = h_2, & (0, L) \times (0, T), \\ c \theta_t - \kappa \theta_{xx} + \delta v_{tx} = 0, & (0, L) \times (0, T), \end{cases}$$

the authors applied the semigroup theory to prove the well-posedness of solutions, then they showed that the existence of global attractor, exponential attractor and the upper semi-continuity of global attractor when  $\nu \rightarrow 0^+$ .

Freitas et al. [25] studied nonlinear piezoelectric beam system with delay term:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + f_1(v, p) + v_t = h_1, \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + f_2(v, p) + \mu_1 p_t + \mu_2 p_t(x, t - \tau) = h_2, \end{cases}$$

and proved the system is asymptotically smooth gradient, and then they used stability estimation to obtain that the system is quasi-stable. Finally, they also proved the global attractor has finite fractal dimension.

Ma et al. [26] studied long-time dynamics of semilinear wave equation:

$$\partial_t^2 u - \Delta u + a(x)g(\partial_t u) + f(u) = \varepsilon h(x) \quad \text{in } \Omega \times \mathbb{R}^+,$$

where  $\Omega \subset \mathbb{R}^3$  is a bounded domain with a smooth boundary  $\partial\Omega$  and  $\varepsilon \in [0, 1]$ . In this paper, they mainly proved the existence and properties of global attractors and attractors are continuous under autonomous perturbations. We also mention the pioneering work [27] that is related to nonlinear damping term in the research of attractor. In [27], Lasiecka and Ruzmaikina considered the existence, regularity and finite dimensionality of attractors for wave equations with some assumptions on nonlinear interior damping in 2-D case.

Freitas et al. [28] investigated the following wave equations:

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\alpha_1} u_t + g_1(u_t) = f_1(u, v) + \varepsilon h_1, & \text{in } \Omega \times \mathbb{R}^+, \\ v_{tt} - \Delta v + (-\Delta)^{\alpha_2} v_t + g_2(v_t) = f_2(u, v) + \varepsilon h_2, & \text{in } \Omega \times \mathbb{R}^+, \end{cases}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain with smooth boundary  $\partial\Omega$ ,  $\alpha_1, \alpha_2 \in (0, 1)$  and  $\varepsilon \in [0, 1]$ . The authors showed that the dynamical system is quasi-stable and that the family of the global attractors is the continuous with respect to the parameter.

Recently, Freitas and Ramos [29] also considered a system modeling a mixture of three interacting continua with external forces with a parameter  $\varepsilon \in [0, 1]$ , and gave the smoothness of global attractors, the continuity of global attractors.

The main contributions of this paper are:

1) We consider the piezoelectric system with nonlinear damping and external forces with a parameter, we prove the existence of solutions by maximal monotone operator theory.

2) We establish a quasi-stability estimate to prove the system is quasi-stable and then we obtain the existence of exponential attractor and some properties of the global attractor.

3) We consider the upper semicontinuity of global attractors with respect to the parameter  $\varepsilon \in [0, 1]$ .

The present paper is organized as following. In Section 2, we mainly establish the well-posedness of the solution by nonlinear operator theory. In Section 3, we apply the infinite dynamical system theory to prove the existence of global attractors and exponential attractors. In Section 4, we mainly show the upper semi-continuity of the global attractors.

## 2. Existence of global solutions

In this section, we first give some assumptions and notations. Then, we are concerned with well-posedness of the solution of the systems (1.1) and (1.2) by the similar argument as in [30–32] and the references therein.

### 2.1. Assumption

Now, we give some assumptions which will be used hereinafter. In the following, we shall denote  $C, C_i$  be the positive generic constants, which may be different in various lines. We assume that

(1) The function  $F \in C^2(\mathbb{R}^2)$  satisfies

$$\nabla F = (f_1, f_2), \quad (2.1)$$

and there exists  $\beta_0, m_F \geq 0$  such that

$$F(v, p) \geq -\beta_0(|v|^2 + |p|^2) - m_F, \quad (2.2)$$

where

$$0 \leq \beta_0 < \frac{1}{2\beta_1}, \quad (2.3)$$

and  $\beta_1$  is the embedding constant defined in (2.13). Moreover, there exist constant  $r \geq 1, C_f > 0$  such that

$$|\nabla f_i(v, p)| \leq C_f(1 + |v|^{r-1} + |p|^{r-1}), \quad i = 1, 2. \quad (2.4)$$

Furthermore, for arbitrary  $v, p \in \mathbb{R}$ , we get

$$\nabla F(v, p) \cdot (v, p) - F(v, p) \geq -\beta_0(|v|^2 + |p|^2) - m_F. \quad (2.5)$$

(2) The external forces  $h_1, h_2 \in L^2(0, L)$ ,  $\varepsilon \in [0, 1]$ .

(3) For the nonlinear damping  $g_i (i = 1, 2)$ , we have

$$g_i \in C^1(\mathbb{R}), \quad g_i(0) = 0, \quad (2.6)$$

and there exists positive constants  $m, M_1$  and  $q \geq 1$  such that

$$m \leq g'_i(v) \leq M_1(1 + |v|^{q-1}), \quad \forall v \in \mathbb{R}. \quad (2.7)$$

If  $q \geq 3$ , there exist  $l > q - 1$ ,  $M_2 > 0$  such that

$$g_i(v)v \geq M_2|v|^l, \quad |v| \geq 1. \quad (2.8)$$

Furthermore, from (2.7), we obtain

$$(g_i(p) - g_i(v))(p - v) \geq m|p - v|^2, \quad p, v \in \mathbb{R}. \quad (2.9)$$

**Remark 2.1.** A simple example of the function  $F$  satisfying the assumption (1) can be

$$F(v, p) = |v + p|^4 - |v + p|^2 + c_1|vp|^2, \quad c_1 > 0,$$

which implies that

$$\begin{aligned} f_1(v, p) &= \frac{\partial F}{\partial v} = 4(v + p)^3 - 2(v + p) + 2c_1vp^2, \\ f_2(v, p) &= \frac{\partial F}{\partial p} = 4(v + p)^3 - 2(v + p) + 2c_1v^2p. \end{aligned}$$

It is easy to verify that

$$F(v, p) \geq \min_{\xi \in \mathbb{R}} \{\xi^4 - \xi^2\} = -\frac{1}{4},$$

which yields that the assumptions (2.2)–(2.4) hold with  $m_F = \frac{1}{4}$  and  $r = 4$ . Noticing that

$$\nabla F(v, p) \cdot (v, p) - F(v, p) \geq 3|v + p|^4 - |v + p|^2 \geq -\frac{1}{12} \geq -m_F,$$

we can derive that (2.5) also holds.

## 2.2. The Cauchy problem

In the rest of this paper, we denote

$$\|v\|_r = \|v\|_{L^r(0, L)}, \quad r \geq 1, \quad (v, p)_2 = (v, p)_{L^2(0, L)}.$$

We shall define the Sobolev Space

$$H_*^i(0, L) = \{v \in H^i(0, L) : v(0) = 0\}, \quad i = 1, 2.$$

Because of  $v(0) = 0$ , we obtain the Poincaré's inequality

$$\lambda_1 \|v\|_2^2 \leq \|v_x\|_2^2, \quad \forall v \in H_*^1(0, L), \quad (2.10)$$

where  $\lambda_1 > 0$ . Hence, we deduce an equivalent norm  $\|v\|_{H_*^1(0,L)} = \|v_x\|_2$  in  $H_*^1(0, L)$ .

The energy space  $\mathcal{H}$  is defined by

$$\mathcal{H} = V \times H = (H_*^1(0, L))^2 \times (L^2(0, L))^2.$$

The inner product on  $\mathcal{H}$  is

$$(z, \tilde{z})_{\mathcal{H}} = \alpha_1 \int_0^L v_x \tilde{v}_x dx + \beta \int_0^L (\gamma v_x - p_x)(\gamma \tilde{v}_x - \tilde{p}_x) dx + \rho \int_0^L v_t \tilde{v}_t dx + \mu \int_0^L p_t \tilde{p}_t dx,$$

where  $z = (v, p, v_t, p_t)^T$ ,  $\tilde{z} = (\tilde{v}, \tilde{p}, \tilde{v}_t, \tilde{p}_t)^T$ .

From the inner product, we can define the norm as

$$\|z(t)\|_{\mathcal{H}}^2 = \|(v, p, v_t, p_t)^T\|_{\mathcal{H}}^2 = \alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2 + \rho \|v_t\|_2^2 + \mu \|p_t\|_2^2. \quad (2.11)$$

Moreover, there exists a constant  $\kappa > 0$  such that

$$\|v\|_2^2 + \|p\|_2^2 \leq \frac{1}{\lambda_1} (\|v_x\|_2^2 + \|p_x\|_2^2) \leq \kappa (\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2). \quad (2.12)$$

In fact, observing that

$$\|p_x\|_2^2 = \|\gamma v_x - p_x - \gamma v_x\|_2^2 \leq 2\|\gamma v_x - p_x\|_2^2 + 2\gamma^2 \|v_x\|_2^2,$$

we have

$$\|v_x\|_2^2 + \|p_x\|_2^2 \leq (1 + 2\gamma^2) \|v_x\|_2^2 + 2\|\gamma v_x - p_x\|_2^2,$$

where  $\kappa = \max\{(2\gamma^2 + 1)\alpha_1^{-1}, 2\beta^{-1}\}$ , the inequality (Eq 2.12) holds. Combining the Poincaré's inequality (Eq 2.10) and the above formula, there exists  $\beta_1 = \frac{\kappa}{\lambda_1} > 0$  such that

$$\|v\|_2^2 + \|p\|_2^2 \leq \beta_1 (\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2). \quad (2.13)$$

Let us write the systems (1.1) and (1.2) as an equivalent Cauchy problem

$$\frac{d}{dt} z(t) + \mathcal{A}z(t) = \mathcal{F}z(t), \quad z(0) = z_0, \quad (2.14)$$

where  $z_0 = (v_0, p_0, v_1, p_1)^T \in \mathcal{H}$ , and  $\mathcal{A} : D(\mathcal{A}) = \{(v, p, \tilde{v}, \tilde{p})^T \in \mathcal{H} : v, p \in H^2(0, L), \tilde{v}, \tilde{p} \in H_*^1(0, L), v_x(L) = p_x(L) = 0\} \subset \mathcal{H} \hookrightarrow \mathcal{H}$  is defined by

$$\mathcal{A}z(t) = \begin{pmatrix} -\tilde{v} \\ -\tilde{p} \\ \frac{1}{\rho}(-\alpha v_{xx} + \gamma\beta p_{xx} + g_1(v_t)) \\ \frac{1}{\mu}(-\beta p_{xx} + \gamma\beta v_{xx} + g_2(p_t)) \end{pmatrix}.$$

The function  $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$  is defined by

$$\mathcal{F}(z) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\rho}(\varepsilon h_1 - f_1(v, p)) \\ \frac{1}{\mu}(\varepsilon h_2 - f_2(v, p)) \end{pmatrix}. \quad (2.15)$$

By a simple calculation, we have

$$(\mathcal{A}z, z)_{\mathcal{H}} = \int_0^L (g_1(\tilde{v})\tilde{v} + g_2(\tilde{p})\tilde{p}) dx \geq 0. \quad (2.16)$$

### 2.3. Energy identities

**Definition 2.1.** A function  $z(t) = (v, p, v_t, p_t)^T \in C([0, \infty), \mathcal{H})$  with  $z(0) = (v_0, p_0, v_1, p_1)^T$  is called a weak solution of the systems (1.1) and (1.2), if for any  $\varphi, \psi \in H_*^1(0, L)$  it satisfies

$$\begin{aligned} & \rho \frac{d}{dt}(v_t, \varphi)_2 + \mu \frac{d}{dt}(p_t, \psi)_2 + \alpha_1(v_x, p_x)_2 + \beta(\gamma v_x - p_x)_2(\gamma \varphi_x - \psi_x)_2 + (g_1(v_t), \varphi)_2 \\ & + (g_2(p_t), \psi)_2 + \int_0^L f_1(v, p)\varphi dx + \int_0^L f_2(v, p)\psi dx = (\varepsilon h_1, \varphi)_2 + (\varepsilon h_2, \psi)_2. \end{aligned}$$

Moreover, if

$$z \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); \mathcal{H}),$$

then  $z$  is called the strong solution.

The total energy of solutions of (1.1) and (1.2) is defined by

$$\mathcal{E}(t) = E(t) + \int_0^L F(v(t), p(t))dx - \varepsilon \int_0^L (h_1 v(t) + h_2 p(t))dx, \quad t \geq 0, \quad (2.17)$$

where  $E(t) = \frac{1}{2}\|z(t)\|_{\mathcal{H}}^2$ .

**Lemma 2.1.** If  $z = (v, p, v_t, p_t)^T$  is a strong solution of (1.1) and (1.2), the following conclusions will hold,

(1)

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= - \int (g_1(v_t)v_t + g_2(p_t)p_t)dx \\ &\leq -m(\|v_t(t)\|_2^2 + \|p_t(t)\|_2^2). \end{aligned} \quad (2.18)$$

(2) There exist constants  $\beta_2, C_F > 0$  such that

$$\beta_2\|z\|_{\mathcal{H}}^2 - C_F \leq \mathcal{E} \leq C_F(1 + \|z\|_{\mathcal{H}}^{r+1}), \quad t \geq 0. \quad (2.19)$$

*Proof.* Multiplying the equations in (1.1) by  $v_t, p_t$ , respectively, and using integration by parts, and applying the inequality (Eq 2.9), we can obtain (2.18).

Applying (2.2) and (2.14), we have

$$\begin{aligned} \int_0^L F(v, p)dx &\geq -\beta_0(\|v\|_2^2 + \|p\|_2^2) - Lm_F \\ &\geq -\beta_0\beta_1(\alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2) - Lm_F \\ &\geq -\beta_0\beta_1\|(v, p, v_t, p_t)^T\|_{\mathcal{H}}^2 - Lm_F. \end{aligned} \quad (2.20)$$

Using (2.3) and (2.20), we obtain

$$\begin{aligned} \mathcal{E}(t) &= \int_0^L F(v, p)dx - \varepsilon \int_0^L (h_1 v + h_2 p)dx + E(t) \\ &\geq \left(\frac{1}{2} - \beta_0\beta_1\right)\|z\|_{\mathcal{H}}^2 - Lm_F - \varepsilon \int_0^L (h_1 v + h_2 p)dx. \end{aligned}$$

Let

$$\beta_2 = \frac{1}{4}(1 - 2\beta_0\beta_1) > 0, \quad (2.21)$$

since  $\varepsilon \in [0, 1]$ , we have

$$\varepsilon \int_0^L (h_1 v + h_2 p) dx \leq \frac{\beta_2}{\beta_1} (\|v\|_2^2 + \|p\|_2^2) + \frac{\beta_1}{4\beta_2} (\|h_1\|_2^2 + \|h_2\|_2^2), \quad (2.22)$$

the first part of (2.19) is obtained with

$$C_F = Lm_F + \frac{\beta_1}{4\beta_2} (\|h_1\|_2^2 + \|h_2\|_2^2).$$

Moreover, by (2.4) we have

$$\int_0^L F(v, p) dx \leq C(\|v_x\|_2^{r+1} + \|p_x\|_2^{r+1} + 1). \quad (2.23)$$

By (2.13) and (2.25), we can deduce the second inequality in (2.19).

#### 2.4. Well-posedness

In this subsection, we will use the nonlinear operator theory to prove the well-posedness of solutions.

**Definition 2.2.** Let  $X$  be a reflexive Banach space, the operator  $A : X \rightarrow X'$  is called monotone if it satisfies

$$\langle Az^1 - Az^2, z^1 - z^2 \rangle \geq 0, \quad \forall z^1, z^2 \in D(A),$$

Furthermore, if  $(A + I) : X \rightarrow X'$  is onto, then  $A$  is maximal.

**Definition 2.3.** Let  $X$  be a reflexive Banach space, the operator  $B : X \rightarrow X'$  is called hemicontinuous, if

$$\lim_{\lambda \rightarrow 0} \langle B(u + \lambda v), w \rangle = \langle Bu, w \rangle, \quad \forall u, v, w \in X.$$

**Lemma 2.2.** The operator  $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$  defined in (2.15) is locally Lipschitz continuous.

*Proof.* Let the solutions  $z^1, z^2 \in \mathcal{H}$  such that  $\|z^1\|_{\mathcal{H}}, \|z^2\|_{\mathcal{H}} \leq \mathcal{R}$ , where  $\mathcal{R} > 0$ .

Then, we can deduce

$$\|\mathcal{F}(z^1) - \mathcal{F}(z^2)\|_{\mathcal{H}}^2 = \frac{1}{\rho} \int_0^L |f_1(v^1, p^1) - f_1(v^2, p^2)|^2 dx + \frac{1}{\mu} \int_0^L |f_2(v^1, p^1) - f_2(v^2, p^2)|^2 dx.$$

It follows from (2.4) that

$$|f_i(v^1, p^1) - f_i(v^2, p^2)|^2 \leq C_f (|v^1|^{r-1} + |p^1|^{r-1} + |v^2|^{r-1} + |p^2|^{r-1} + 1)^2 (|v^1 - v^2|^2 + |p^1 - p^2|^2). \quad (2.24)$$

It follows from (2.24) that there exists some constant  $C_{\mathcal{R}} > 0$  such that

$$\int_0^L |f_i(v^1, p^1) - f_i(v^2, p^2)|^2 dx \leq C_{\mathcal{R}} \|z^1 - z^2\|_{\mathcal{H}}^2, \quad i = 1, 2,$$

which implies that

$$\|\mathcal{F}(z^1) - \mathcal{F}(z^2)\|_{\mathcal{H}} \leq C'_{\mathcal{R}} \|z^1 - z^2\|_{\mathcal{H}},$$

for some constant  $C'_{\mathcal{R}} > 0$ . This finishes the proof of this lemma.



Now, we are in the position to give the existence of the solutions.

**Theorem 2.1.** *Suppose the assumptions (2.1)–(2.7) hold. If  $z_0 \in \mathcal{H}$ , the systems (1.1) and (1.2) have a unique weak solution  $z(t)$  satisfies  $z \in C([0, \infty); \mathcal{H})$ , and it depends continuously on the initial data  $z_0$ . In addition, if  $z_0 \in D(\mathcal{A})$ , the weak solution is a strong solution.*

*Proof.* Similar to (2.15) and (2.16), using the monotonicity of function  $g_i, i = 1, 2$ , we know  $\mathcal{A}$  is monotone. To obtain  $\mathcal{A}$  is a maximal monotone operator, we need to show that there exists  $z \in D(\mathcal{A})$ , for arbitrary  $w \in \mathcal{H}$  such that

$$\mathcal{A}z + z = w.$$

In fact, in the following, we decompose the operator  $\mathcal{A}$  as

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ B & G \end{pmatrix},$$

where  $B : D(B) = \{(v, p) \in H_*^2(0, L) \times H_*^2(0, L)\} \subset V \hookrightarrow H$ ,

$$B(v, p) = \begin{pmatrix} \frac{1}{\rho}(-\alpha v_{xx} + \gamma\beta p_{xx}) \\ \frac{1}{\mu}(-\beta p_{xx} + \gamma\beta v_{xx}) \end{pmatrix},$$

$G : H \rightarrow H$ ,

$$G(v_t, p_t) = \begin{pmatrix} \frac{1}{\rho}g_1(v_t) \\ \frac{1}{\mu}g_2(p_t) \end{pmatrix}.$$

Writing  $z = (v, p, v_t, p_t)^T = (\tau, \eta)^T \in V \times H$ ,  $w = (w_1, w_2)^T \in V \times H$ , so

$$\tau - \eta = w_1, \quad B(\tau) + G(\eta) + \eta = w_2,$$

then, we can analyze that  $\eta \in V$ . Hence, we need to prove  $K(\eta) = (B + I)\eta + G(\eta) : V \rightarrow V'$  is onto. By Corollary 2.2 of [33], we need only to prove  $K$  is maximal monotone and coercive. From (2.11) and the embedding  $V \hookrightarrow H = H' \hookrightarrow V'$  we have

$$\langle B\tau, \eta \rangle = (\tau, \eta)_V, \quad \tau, \eta \in V. \quad \langle \tau, \eta \rangle = (\tau, \eta)_H, \quad \tau \in V, \eta \in H.$$

Firstly, let  $\eta^1, \eta^2 \in V$ , then

$$\begin{aligned} \langle (B + I)(\eta^1 - \eta^2), \eta^1 - \eta^2 \rangle &= \langle B(\eta^1 - \eta^2), \eta^1 - \eta^2 \rangle + \langle \eta^1 - \eta^2, \eta^1 - \eta^2 \rangle \\ &= \|\eta^1 - \eta^2\|_V^2 + \|\eta^1 - \eta^2\|_H^2 \geq 0. \end{aligned} \quad (2.25)$$

By (2.9), we have

$$\langle G(\eta^1) - G(\eta^2), \eta^1 - \eta^2 \rangle = (G(\eta^1) - G(\eta^2), \eta^1 - \eta^2)_H \geq 0. \quad (2.26)$$

So we obtain that  $B + I$  and  $G$  are both monotone.

Secondly, let  $\tau, \eta, \phi \in V$ ,  $\lambda \in \mathbb{R}$ , we have

$$|\langle (B + I)(\tau + \lambda\eta), \phi \rangle - \langle (B + I)\tau, \phi \rangle| = |\lambda \langle (B + I)\eta, \phi \rangle| \leq |\lambda|(\langle B\eta, \phi \rangle + \langle \eta, \phi \rangle),$$

which implies that the continuity of  $\langle (B + I)(\tau + \lambda\eta), \phi \rangle$  at  $\lambda = 0$ . Moreover, we have

$$\langle G(\tau + \lambda_n\eta), \phi \rangle = \sum_{i=1}^2 \int_0^L g_i(\tau^i + \lambda_n\eta^i)\phi^i dx.$$

Clearly

$$g_i(\tau^i + \lambda_n\eta^i)\phi^i \rightarrow g_i(\tau^i)\phi^i \quad a.e \text{ in } (0, L) \text{ as } \lambda_n \rightarrow 0.$$

From (2.7) and Dominated Convergence Theorem, we deduce

$$\lim_{n \rightarrow \infty} \langle G(\tau + \lambda_n\eta), \phi \rangle = \langle G(\tau), \phi \rangle_H.$$

So we obtain that  $B + I$  and  $G$  are hemicontinuous.

In addition, we can deduce  $B + I$  and  $G$  are coercive from (2.25) and (2.26).

Therefore, according to Theorem 2.6 of [33], we know that  $K$  is coercive and maximal monotone, which implies that  $K$  is onto. That is,  $\mathcal{A}$  is maximal monotone in  $\mathcal{H}$ .

In conclusion, since  $\mathcal{A}$  is maximal monotone and  $\mathcal{F}$  is locally Lipschitz, by applying Theorem 7.2 of [34], we can obtain: when  $z_0 \in \mathcal{H}$ , the problem (2.14) has a unique weak solution  $z(t) \in C([0, t_{max}); \mathcal{H})$ . Moreover, if  $t_{max} < \infty$ , then  $\limsup_{t \rightarrow t_{max}^-} \|z\|_{\mathcal{H}} = \infty$ . When  $z_0 \in D(\mathcal{A})$ , the problem (2.14) has a unique strong solution  $z(t) \in C([0, t_{max}), D(\mathcal{A}))$ , ( $t_{max} \leq \infty$ ).

Now, we need to show the existence of global solutions, that is  $t_{max} = \infty$ . In fact, let  $z(t)$  be a strong solution defined in  $[0, t_{max})$ . From (2.18), we infer

$$\mathcal{E}(t) \leq \mathcal{E}(0). \quad (2.27)$$

It follows from (2.19) and (2.17) that

$$\|z\|_{\mathcal{H}}^2 \leq \frac{1}{\beta_2} (\mathcal{E}(0) + C_F).$$

By density argument, the conclusion also holds for weak solution. Therefore  $t_{max} = \infty$ .

Finally, let  $z^1, z^2$  be two weak solutions, by the standard arguments, for any  $T > 0$ , there exists constant  $C > 0$  such that

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}}^2 \leq e^{CT} \|z^1(0) - z^2(0)\|_{\mathcal{H}}^2, \quad t \in [0, T]. \quad (2.28)$$

The proof is complete.

### 3. The existence of global attractor

In this section, for the sake of completeness, we collect some known results in the theory of nonlinear dynamical systems (see [35–38]).

Let  $(v, p, v_t, p_t)^T$  be the unique solution for the systems (1.1) and (1.2). We can define the operator  $S(t) : \mathcal{H} \rightarrow \mathcal{H}$  by

$$S(t)(v_0, p_0, v_1, p_1)^T = (v(t), p(t), v_t(t), p_t(t))^T, \quad t \geq 0.$$

Hence,  $(\mathcal{H}, S(t))$  constitutes a dynamical system.

**Definition 3.1.** Let  $B \subset \mathcal{H}$  be a positively invariant set of a dynamical system  $(\mathcal{H}, S(t))$ .

1. A function  $\Phi(\eta)$  is said to be a Lyapunov function, if  $t \rightarrow \Phi(S(t)\eta)$  is a non-increasing function for any  $\eta \in B$ .

2. A Lyapunov function is called strict, if there exist  $t_0 > 0$ ,  $\eta \in B$  such that  $\Phi(S(t_0)\eta) = \Phi(\eta)$ , then  $\eta = S(t)\eta$ ,  $t > 0$ .

**Definition 3.2.** The dynamical system  $(\mathcal{H}, S_t)$  is quasi-stable on  $B \subset \mathcal{H}$ , if there exist a compact seminorm  $\chi_X(\cdot)$  on the space  $X$  and nonnegative scalar function  $a(t), b(t), c(t) \in \mathbb{R}^+$  satisfy:

- (1)  $a(t), c(t)$  are locally bounded on  $[0, \infty)$ ;
- (2)  $b(t) \in L^1(\mathbb{R}^+)$ , and  $\lim_{t \rightarrow \infty} b(t) = 0$ ;
- (3) for any  $\eta_1, \eta_2 \in B$  and  $t \geq 0$ , the estimates

$$\|S(t)\eta_1 - S(t)\eta_2\|_{\mathcal{H}}^2 \leq a(t)\|\eta_1 - \eta_2\|_{\mathcal{H}}^2,$$

and

$$\|S(t)\eta_1 - S(t)\eta_2\|_{\mathcal{H}}^2 \leq b(t)\|\eta_1 - \eta_2\|_{\mathcal{H}}^2 + c(t) \sup_{0 \leq s \leq t} [\chi_X(u^1(s) - u^2(s))]^2,$$

hold, where  $S(t) = (u^i(t), u_t^i(t))$ ,  $i = 1, 2$ .

**Lemma 3.1.** The dynamical system  $(\mathcal{H}, S(t))$  is gradient, that is, there exists a strict Lyapunov function  $\Phi \in \mathcal{H}$ . What's more,

$$\Phi(z) \rightarrow \infty \iff \|z\|_{\mathcal{H}} \rightarrow \infty.$$

*Proof.* Let  $\mathcal{E}(t)$  defined in (2.17) be a Lyapunov function  $\Phi$  and  $z_0 = (v_0, p_0, v_1, p_1)^T \in \mathcal{H}$ , we can infer that  $t \rightarrow \Phi(S(t)z_0)$  is a non-increasing function from (2.18).

Supposing that  $\Phi(S(t)z_0) = \Phi(z_0)$ , for  $t \geq 0$ , we have

$$\|v_t(t)\|_2^2 = \|p_t(t)\|_2^2 = 0, \quad t \geq 0.$$

We obtain

$$v_t = p_t = 0, \quad a.e \text{ in } (0, L), \quad t \geq 0.$$

Consequently,  $v_t(t) = v_0, p_t(t) = p_0$ , which implies  $S(t)z_0 = z(t) = (v_0, p_0, 0, 0)^T$  is a stationary solution of  $(\mathcal{H}, S(t))$ .

From (2.19), we obtain

$$\Phi(z) \leq C_F(1 + \|z\|_{\mathcal{H}}^{r+1}), \quad t \geq 0.$$

Let  $\Phi(z) \rightarrow \infty$ , we have  $\|z\|_{\mathcal{H}} \rightarrow \infty$ . Then, it follows from (2.19) that

$$\|z\|_{\mathcal{H}}^2 \leq \frac{\Phi(z) + C_F}{\beta_2}.$$

As a result, we infer from  $\|z\|_{\mathcal{H}} \rightarrow \infty$  that  $\Phi(z) \rightarrow \infty$ .

**Lemma 3.2.** The set of stationary points  $\mathcal{N}$  of  $S(t)$  is bounded in  $\mathcal{H}$ .

*Proof.* Let  $z = (v, p, 0, 0)^T \in \mathcal{N}$  be the stationary solution of systems (1.1) and (1.2). Then, we have the following elliptic equations

$$\begin{cases} -\alpha v_{xx} + \gamma\beta p_{xx} + f_1(v, p) = \varepsilon h_1, \\ -\beta p_{xx} + \gamma\beta v_{xx} + f_2(v, p) = \varepsilon h_2. \end{cases} \quad (3.1)$$

Multiplying the equations in (3.1) by  $v, p$ , respectively, and integrating the result over  $(0, L)$ , we have

$$\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2 = - \int_0^L (f_1(v, p)v + f_2(v, p)p) dx + \varepsilon \int_0^L (h_1 v + h_2 p) dx.$$

Hence, using (2.1), (2.2), and (2.5), we obtain

$$- \int_0^L (f_1(v, p)v + f_2(v, p)p) dx \leq 2\beta_0\beta_1(\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2) + 2Lm_F.$$

By (2.21), we have

$$4\beta_2(\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2) \leq 2Lm_F + \varepsilon \int_0^L (h_1 v + h_2 p) dx.$$

By Young's inequalities and (2.13), we infer

$$\begin{aligned} \int_0^L (h_1 v + h_2 p) dx &\leq \frac{\beta_2}{\beta_1} (\|v\|_2^2 + \|p\|_2^2) + \frac{\beta_1}{4\beta_2} (\|h_1\|_2^2 + \|h_2\|_2^2) \\ &\leq \beta_2(\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2) + \frac{\beta_1}{4\beta_2} (\|h_1\|_2^2 + \|h_2\|_2^2). \end{aligned}$$

Therefore, we conclude

$$3\beta_2 \|z\|_{\mathcal{H}}^2 \leq 2Lm_F + \frac{\beta_1}{4\beta_2} (\|h_1\|_2^2 + \|h_2\|_2^2). \quad (3.2)$$

The proof is complete.

**Lemma 3.3.** *Suppose that the Assumption 2.1 holds. Let  $B$  be a bounded forward invariant set in  $\mathcal{H}$  and  $S(t)z^i = (v^i, p^i, v_t^i, p_t^i)^T$  be a weak solution of the systems (1.1) and (1.2) with  $z_0^i \in B, i = 1, 2$ . Then there exist constant  $\sigma, \varsigma, C_B > 0$  independent of  $\varepsilon$  such that*

$$E(t) \leq \varsigma E(0) e^{-\sigma t} + C_B \sup_{s \in [0, t]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2), \quad \forall t \geq 0.$$

where  $E(t) = \frac{1}{2} \|z\|_{\mathcal{H}}^2, v = v^1 - v^2, p = p^1 - p^2, \theta \geq 2$ .

*Proof.* Let  $F_i(v, p) = f_i(v^1, p^1) - f_i(v^2, p^2), i = 1, 2, G_1(v_t) = g_1(v_t^1) - g_1(v_t^2), G_2(p_t) = g_2(p_t^1) - g_2(p_t^2)$ .

Then  $v = v^1 - v^2, p = p^1 - p^2$  satisfy

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + G_1(v_t) = -F_1(v, p), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + G_2(p_t) = -F_2(v, p), \\ (v(0), p(0), v_t(0), p_t(0)) = z^1 - z^2, \\ v(0) = p(0) = v_x(L) = p_x(L) = 0. \end{cases} \quad (3.3)$$

Multiplying the first equation of (3.3) by  $v$ , the second one by  $p$ , and integrating over  $[0, L] \times [0, T]$ , we have

$$\begin{aligned} \int_0^T E(t)dt &= -\frac{1}{2} \int_0^L (\rho v_t v + \mu p_t p) dx \Big|_0^T + \int_0^T (\rho \|v_t\|_2^2 + \mu \|p_t\|_2^2) dt \\ &\quad - \frac{1}{2} \int_0^T \int_0^L (G_1(v_t)v) + G_2(p_t)p dx dt - \frac{1}{2} \int_0^T \int_0^L (F_1(v, p)v + F_2(v, p)p) dx dt. \end{aligned}$$

Step 1. By Poincaré's and Hölder's inequalities, we have

$$\int_0^L (\rho v_t v + \mu p_t p) dx \leq CE(t).$$

Then

$$\int_0^L (\rho v_t v + \mu p_t p) dx \Big|_0^T \leq C(E(0) + E(T)),$$

where  $C$  is a positive constant. From (2.9), we conclude

$$\int_0^T (\rho \|v_t\|_2^2 + \mu \|p_t\|_2^2) dt \leq \frac{1}{m} \int_0^T \int_0^L (G_1(v_t)v) + G_2(p_t)p dx dt.$$

Step 2. According to Young's inequality and (2.9), we can deduce

$$\int_0^T \int_0^L G_1(v_t)v dx dt \leq \frac{1}{2} \int_0^T \int_0^L G_1(v_t)v_t dx dt + \frac{1}{2} \int_0^T \int_0^L G_1(v_t) \frac{|v|^2}{v_t} dx dt.$$

Applying (2.7), we infer

$$\int_0^L G_1(v_t)v dx \leq \frac{1}{2} \int_0^L G_1(v_t)v_t dx + \frac{M_1}{2} \int_0^L (1 + |v_t|^{q-1} + |v_t^{2q-1})|v|^2 dx.$$

For further estimation, we divide it into three cases.

Case a:  $q = 1$ . It is easy to get

$$\int_0^L (1 + |v_t|^{q-1} + |v_t^{2q-1})|v|^2 dx \leq 3\|v\|_2^2 (1 + \int_0^L (g_1(v_t^1)v_t^1 + g_1(v_t^2)v_t^2) dx).$$

Case b:  $1 < q < 3$ . Applying the assumption (2.9) and Hölder's inequality, we have

$$\begin{aligned} \int_0^L (1 + |v_t|^{q-1} + |v_t^{2q-1})|v|^2 dx &\leq C\|v\|_{2d_1}^2 \left( \int_0^L (1 + |v_t|^{d(q-1)} + |v_t^{2d(q-1)}) dx \right)^{\frac{1}{d}} \\ &\leq C\|v\|_{2d_1}^2 \left( 1 + \int_0^L (g_1(v_t^1)v_t^1 + g_1(v_t^2)v_t^2) dx \right)^{\frac{1}{d}} \\ &\leq C\|v\|_{2d_1}^2 \left( 1 + \int_0^L (g_1(v_t^1)v_t^1 + g_1(v_t^2)v_t^2) dx \right), \end{aligned}$$

where  $d = \frac{2}{q-1}$ , and  $\frac{1}{d} + \frac{1}{d_1} = 1$ .

Case c:  $q \geq 3$ . Using the assumption (2.8), by the similar argument as in Case b, let  $d = \frac{1}{q-1} > 1$ , we can obtain the same result.

Combining the estimates of the above three cases, we can deduce that there exist  $C > 0, \theta \geq 2$  such that

$$\int_0^L G_1(v_t)v dx \leq \frac{1}{2} \int_0^L G_1(v_t)v_t dx + C\|v\|_\theta^2(1 + \int_0^L (g_1(v_t^1)v_t^1 + g_1(v_t^2)v_t^2)dx). \quad (3.4)$$

Similarly,

$$\int_0^L G_2(p_t)p dx \leq \frac{1}{2} \int_0^L G_2(p_t)p_t dx + C\|p\|_\theta^2(1 + \int_0^L (g_2(p_t^1)p_t^1 + g_2(p_t^2)p_t^2)dx). \quad (3.5)$$

Furthermore, due to  $z^1, z^2 \in B$ , and by (2.18), (2.19), we conclude there exists  $C_B > 0$  such that

$$\int_0^L (g_1(v_t^1)v_t^1 + g_1(v_t^2)v_t^2)dx \leq C_B,$$

$$\int_0^L (g_2(p_t^1)p_t^1 + g_2(p_t^2)p_t^2)dx \leq C_B.$$

Combining with (2.8), (3.4), (3.5), and applying  $L^{2\theta}(0, L) \hookrightarrow L^\theta(0, L)$ , we deduce

$$\begin{aligned} -\frac{1}{2} \int_0^T \int_0^L (G_1(v_t)v + G_2(p_t)p)dxdt &\leq \frac{1}{2} \int_0^T \int_0^L (G_1(v_t)v_t + G_2(p_t)p_t)dxdt \\ &\quad + C_{B,T} \sup_{s \in [0,T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2). \end{aligned}$$

Step 3. Applying (2.4),  $H_*^1(0, L) \hookrightarrow L^r(0, L)$ ,  $r \in (1, \infty)$  and Hölder's inequality, we can infer

$$\begin{aligned} \int_0^L F_1(v, p)v dxdt &\leq C_f(\|v^1\|_{2\theta}^{r-1} + \|v^2\|_{2\theta}^{r-1} + \|p^1\|_{2\theta}^{r-1} + \|p^2\|_{2\theta}^{r-1})(\|v\|_{2\theta} + \|p\|_{2\theta})\|v\|_2 \\ &\leq C_B(\|v\|_{2\theta} + \|p\|_{2\theta})\|v\|_2 \\ &\leq C_B(\|v\|_{2\theta}^2 + \|p\|_{2\theta}^2). \end{aligned} \quad (3.6)$$

Similarly, we obtain

$$\int_0^L F_2(v, p)p dxdt \leq C_B(\|v\|_{2\theta}^2 + \|p\|_{2\theta}^2). \quad (3.7)$$

Combining (3.7) with (3.6), there exists  $C_T > 0$ , we have

$$-\frac{1}{2} \int_0^T \int_0^L (F_1(v, p)v + F_2(v, p)p)dxdt \leq C_{B,T} \sup_{s \in [0,T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2).$$

Therefore, combining the above estimates, we have

$$\begin{aligned} \int_0^T E(t)dt &\leq C_B \int_0^T \int_0^L (G_1(v_t)v_t + G_2(p_t)p_t)dxdt \\ &\quad + C_{B,T} \sup_{s \in [0,T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2) + C(E(0) + E(T)). \end{aligned} \quad (3.8)$$

for some constants  $C_B, C_{B,T} > 0$ .

Step 4. Multiplying the equations of (3.3) by  $v_t, p_t$ , respectively, and integrating over  $[0, L] \times [s, T]$ , we obtain

$$E(t) = E(s) - \int_s^T \int_0^L (G_1(v_t)v_t + G_2(p_t)p_t) dx dt - \int_s^T \int_0^L (F_1(v, p)v_t + F_2(v, p)p_t) dx dt. \quad (3.9)$$

Due to

$$- \int_s^T \int_0^L (G_1(v_t)v_t + G_2(p_t)p_t) dx dt \leq 0,$$

we obtain

$$E(t) \leq E(s) - \int_s^T \int_0^L (F_1(v, p)v_t + F_2(v, p)p_t) dx dt.$$

By the similar argument, we have

$$\begin{aligned} - \int_0^L F_1(v, p)v_t dx dt &\leq C_B(\|v\|_{2\theta} + \|p\|_{2\theta})\|v_t\|_2 \\ &\leq \zeta\|v_t\|_2^2 + \frac{C_B}{4\zeta}(\|v\|_{2\theta}^2 + \|p\|_{2\theta}^2). \end{aligned}$$

Analogously,

$$- \int_0^L F_2(v, p)p_t dx dt \leq \zeta\|p_t\|_2^2 + \frac{C_B}{4\zeta}(\|v\|_{2\theta}^2 + \|p\|_{2\theta}^2).$$

Consequently,

$$- \int_0^L (F_1(v, p)v_t + F_2(v, p)p_t) dx dt \leq \zeta E(t) + \frac{C_B}{4\zeta}(\|v\|_{2\theta}^2 + \|p\|_{2\theta}^2). \quad (3.10)$$

Let  $\zeta = \frac{1}{T}$ , we have

$$E(T) \leq \frac{1}{T} \int_0^T E(t) dt + TC_B \int_0^T (\|v\|_{2\theta}^2 + \|p\|_{2\theta}^2) dt + E(s).$$

Then, integrating in  $[0, T]$ , there exists a constant  $C_{B,T} > 0$  such that

$$TE(T) \leq 2 \int_0^T E(t) dt + C_{B,T} \sup_{s \in [0, T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2). \quad (3.11)$$

Step 5. Let (3.9) with  $s = 0$  and (3.10) with  $\zeta = 1$ , we have

$$\int_0^T \int_0^L (G_1(v_t)v_t + G_2(p_t)p_t) dx dt \leq E(0) - E(T) + \int_0^T E(t) dt + TC_{B,T} \sup_{s \in [0, T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2).$$

Combining the above estimates with (3.8), we can deduce

$$\int_0^T E(t) dt \leq C_{B,T} \sup_{s \in [0, T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2) + C_B(E(0) + E(T)).$$

Then, from (3.11)

$$TE(T) \leq C_{B,T} \sup_{s \in [0, T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2) + C_B(E(0) + E(T)).$$

Choosing  $T > 2C_B$ , letting  $m_T = \frac{C_B}{T - C_B} < 1$ , we obtain that

$$E(T) \leq m_T E(0) + C_{B,T} \sup_{s \in [0, T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2).$$

Let  $M_n = \sup_{s \in [nT, (n+1)T]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2)$ ,  $n \in \mathbb{N}$ . Then, repeating the above argument progress on any  $[nT, (n+1)T]$ , we have

$$\begin{aligned} E(nT) &= m_T^n E(0) + C_{B,T} \sum_{k=1}^n m_T^{n+1-k} M_{j-1} \\ &\leq m_T^n E(0) + \frac{C_{B,T}}{1 - m_T} \sup_{s \in [0, mT]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2). \end{aligned}$$

For any  $t \geq 0$ , there exists  $n \in \mathbb{N}, k \in [0, T)$  such that  $t = nT + k$ . Hence, we have

$$E(t) \leq E(nT) \leq m_T^{-1} m_T^{\frac{t}{T}} E(0) + \frac{C_{B,T}}{1 - m_T} \sup_{s \in [0, t]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2).$$

Consequently, letting  $\sigma = -\frac{\ln(m_T)}{T}$ ,  $\varsigma = m_T^{-1}$ ,  $C_B = \frac{C_{B,T}}{1 - m_T}$ , we have

$$E(t) \leq \varsigma E(0) e^{-\sigma t} + C_B \sup_{s \in [0, t]} (\|v(s)\|_{2\theta}^2 + \|p(s)\|_{2\theta}^2), \quad \forall t \geq 0.$$

The proof is complete.

**Lemma 3.4.** *Let  $B \subset \mathcal{H}$  be a bounded positively invariant set, then dynamical system  $(\mathcal{H}, S(t))$  is quasi-stable.*

*Proof.* Defining  $S(t)z^i = (v^i(t), p^i(t), v_t^i(t), p_t^i(t))^T$  for  $z^i \in B, i = 1, 2$ , and  $v = v^1 - v^2, p = p^1 - p^2$ . Then, it follows from (2.28) that

$$\|S(t)z^1 - S(t)z^2\|_{\mathcal{H}}^2 \leq \alpha(t) \|z^1 - z^2\|_{\mathcal{H}}^2,$$

where  $\alpha(t) = e^{cT}$ .

Now, we let  $X = H_*^1(0, L) \times H_*^1(0, L)$  and define the semi-norm

$$\chi_X(v, p) = (\|v\|_{2\theta}^2 + \|p\|_{2\theta}^2)^{\frac{1}{2}}.$$

Since  $H_*^1(\Omega) \hookrightarrow L^{2\theta}(\Omega)$ , we can obtain that  $\chi_X$  is a compact semi-norm on  $X$ .

According to Lemma 3.3, we have

$$\|S(t)z^1 - S(t)z^2\|_{\mathcal{H}}^2 \leq \mathfrak{b}(t) \|z^1 - z^2\|_{\mathcal{H}}^2 + \mathfrak{c}(t) \sup_{s \in [0, t]} [\chi_X(v(s), p(s))]^2,$$

with  $\mathfrak{b}(t) = \varsigma e^{-\sigma t}$ ,  $\mathfrak{c}(t) = C_B$ .

It is easy to verify that  $\mathfrak{b}(t) \in L^1(\mathbb{R}^+)$  and  $\lim_{t \rightarrow \infty} \mathfrak{b}(t) = 0$ .

Then, since  $B$  is a bounded subset of  $\mathcal{H}$ , we have  $\mathfrak{c}(t)$  is locally bounded on  $[0, \infty)$ . By the Definition 3.2, we have the dynamical system is quasi-stable on  $B \subset \mathcal{H}$ .



Since dynamical system  $(\mathcal{H}, S(t))$  is quasi-stable, we can give our main results as following.

**Theorem 3.1.** *Under the assumptions of Theorem 2.1, we obtain*

(1) *The dynamical system  $(\mathcal{H}, S(t))$  has a global attractor  $\mathcal{A} \subset \mathcal{H}$  which is compact and connected. Moreover, the attractor can be characterized by the unstable manifold*

$$\mathcal{A} = \mathbb{M}^+(\mathcal{N}),$$

*emanating from the set of stationary solutions established in Lemma 3.2.*

(2) *The attractor  $\mathcal{A}$  has finite fractal dimension  $\dim_{\mathcal{H}}\mathcal{A}$ .*

(3) *Every trajectory stabilizes to the set  $\mathcal{N}$ , that is,*

$$\lim_{t \rightarrow +\infty} \text{dist}_{\mathcal{H}}(S_t z, \mathcal{N}) = 0 \quad \text{for any } z \in \mathcal{H}.$$

*In particular, there exists a global minimal attractor  $\mathcal{A}_{\min}$  to the dynamical system, which is precisely characterized by the set of the stationary points  $\mathcal{N}$ , that is  $\mathcal{A}_{\min} = \mathcal{N}$ .*

(4) *The attractor is bounded in  $\mathcal{H}_1 = (H^2(0, L) \cap H_*^1(0, L))^2 \times (H_*^1(0, L))^2$ , and every trajectory  $z = (v, p, v_t, p_t)^T$  in  $\mathcal{A}$  has the property*

$$\|(v, p)\|_{(H^2(0, L) \cap H_*^1(0, L))^2}^2 + \|(v_t, p_t)\|_{(H_*^1(0, L))^2}^2 + \|(v_{tt}, p_{tt})\|_{(L^2(0, L))^2}^2 \leq R^2, \quad (3.12)$$

*for some constant  $R > 0$  independent of  $\varepsilon \in [0, 1]$ .*

*Proof.* (1) It follows from Lemma 3.4 that  $(\mathcal{H}, S(t))$  is quasi-stable. Hence, we have that  $(\mathcal{H}, S(t))$  is asymptotically smooth by Proposition 7.9.4 of [36]. Then, applying Lemmas 3.1, 3.2 and Corollary 7.5.7 of [36], we can conclude  $(\mathcal{H}, S(t))$  possesses a compact global attractor  $\mathcal{A}$ . In addition, it can be characterized by  $\mathcal{A} = \mathbb{M}^+(\mathcal{N})$ .

(2) Since the system  $(\mathcal{H}, S(t))$  is quasi-stable, applying Theorem 7.9.6 of [36], we conclude that attractor  $\mathcal{A}$  has finite fractal dimension  $\dim_{\mathcal{H}}\mathcal{A}$ .

(3) Combining Theorem 3.1(1) and Theorem 7.5.10 of [36], we can get the desired result immediately.

(4) Since  $(\mathcal{H}, S(t))$  is quasi-stable on  $\mathcal{A}$ , the arbitrary trajectory  $z = (v, p, v_t, p_t)^T$  in  $\mathcal{A}$  has the following regularity properties

$$\begin{aligned} v_t, p_t &\in L^\infty(\mathbb{R}; H_*^1(0, L)) \cap C(\mathbb{R}; L^2(0, L)), \\ v_{tt}, p_{tt} &\in L^\infty(\mathbb{R}; L^2(0, L)). \end{aligned}$$

It follows from (1.1) that

$$\begin{cases} \alpha v_{xx} = \rho v_{tt} + \gamma \beta p_{xx} + f_1(v, p) + g_1(v_t) - \varepsilon h_1, \\ \beta p_{xx} = \mu p_{tt} + \gamma \beta v_{xx} + f_2(v, p) + g_2(p_t) - \varepsilon h_2. \end{cases} \quad (3.13)$$

Then, we can deduce

$$\alpha_1 v_{xx} = \rho v_{tt} + \gamma \mu p_{tt} + \gamma f_2(v, p) + \gamma g_2(p_t) - \gamma \varepsilon h_2 + f_1(v, p) + g_1(v_t) - \varepsilon h_1.$$

Using the fact  $\alpha_1 \neq 0$ ,  $v_t, p_t \in L^\infty(\mathbb{R}; H_*^1(0, L)) \hookrightarrow L^\infty(\mathbb{R}; L^2(0, L))$  and  $f_i(v, p)$  is locally Lipschitz continuous, we have

$$\begin{aligned} v_{xx} &\in L^\infty(\mathbb{R}; L^2(0, L)), \\ v &\in L^\infty(\mathbb{R}; H^2(0, L) \cap H_*^1(0, L)). \end{aligned}$$

It follows from (3.13) that

$$\begin{aligned} \beta p_{xx} &\in L^\infty(\mathbb{R}; L^2(0, L)), \\ p &\in L^\infty(\mathbb{R}; H^2(0, L) \cap H_*^1(0, L)). \end{aligned}$$

The proof is complete.

**Theorem 3.2.** *The system  $(\mathcal{H}, S(t))$  has a generalized exponential attractor. More precisely, for any given  $\xi \in (0, 1]$ , there exists a generalized exponential attractor  $\mathcal{A}_{exp, \xi} \subset \mathcal{H}$  with finite fractal dimension in the extended space  $\mathcal{H}_{-\xi}$  which is defined as the interpolation of*

$$\mathcal{H}_0 := \mathcal{H}, \quad \mathcal{H}_{-1} := (L^2(0, L))^2 \times (H_*^{-1}(0, L))^2.$$

*Proof.* Let us take  $\mathcal{B} = \{z \in \mathcal{H} | \Phi(z) \leq R\}$  where  $\Phi$  is the strict Lyapunov functional given in Lemma 3.1. Then we can derive that for sufficiently large  $R$  that  $\mathcal{B}$  is a positively invariant bounded absorbing set, which shows that the system is quasi-stable on the set  $\mathcal{B}$ .

Then for solution  $z(t) = S(t)z_0$  with initial data  $z(0) \in \mathcal{B}$ , we can derive that, for any  $T > 0$ ,

$$\int_0^T \|z_t(s)\|_{\mathcal{H}_{-1}}^2 ds \leq C_{\mathcal{B}T}^2,$$

which shows that

$$\|S(t_1)z - S(t_2)z\|_{\mathcal{H}_{-1}} \leq \int_{t_1}^{t_2} \|z_t(s)\|_{\mathcal{H}_{-1}} ds \leq C_{\mathcal{B}T} |t_1 - t_2|^{\frac{1}{2}}$$

where  $C_{\mathcal{B}T}$  is a positive constant and  $t_1, t_2 \in [0, T]$ . Hence we obtain that for any initial data  $z_0 \in \mathcal{H}$  the map  $t \mapsto S(t)z_0$  is  $\frac{1}{2}$ -Hölder continuous in the extended phase space  $\mathcal{H}_{-1}$ . Therefore, it follows from Theorem 7.9.9 in [34] that the dynamical system  $(\mathcal{H}, S(t))$  possesses a generalized fractal exponential attractor with finite fractal dimension in the extended space  $\mathcal{H}_{-1}$ .

Furthermore, by the standard interpolation theorem, we can obtain the existence of exponential attractors in the extended space  $\mathcal{H}_{-\xi}$  with  $\xi \in (0, 1)$ . The proof is complete.

#### 4. Upper semicontinuity of global attractor

In this section, we denote the attractor obtained in Theorem 3.1 as the  $\mathcal{A}_\varepsilon$ . Then, we investigate the upper semicontinuity of the attractors  $\mathcal{A}_\varepsilon$  as  $\varepsilon \rightarrow \varepsilon_0$ .

**Definition 4.1.** [26] *Let  $\Lambda$  be a complete metric space and  $S_\sigma(t)$  a family of semigroups on  $\mathcal{H}$ , where  $\sigma \in \Lambda$ . The global attractors  $\mathcal{A}_\sigma$  is called upper semicontinuous on  $\sigma_0 \in \Lambda$  if*

$$\lim_{\sigma \rightarrow \sigma_0} \text{dist}_{\mathcal{H}}(\mathcal{A}_\sigma, \mathcal{A}_{\sigma_0}) = 0,$$

where  $\text{dist}_{\mathcal{H}}\{\mathbb{A}, \mathbb{B}\} = \sup_{a \in \mathbb{A}} \inf_{b \in \mathbb{B}} d(a, b)$  expresses the Hausdorff semi-distance in  $X$ . Similarly,  $\mathcal{A}_{\sigma}$  is lower semicontinuous on  $\sigma_0 \in \Lambda$  if

$$\lim_{\sigma \rightarrow \sigma_0} \text{dist}_{\mathcal{H}}(\mathcal{A}_{\sigma_0}, \mathcal{A}_{\sigma}) = 0.$$

Then  $\mathcal{A}_{\sigma}$  is continuous on  $\sigma_0 \in \Lambda$  if

$$\lim_{\sigma \rightarrow \sigma_0} d_{\mathcal{H}}(\mathcal{A}_{\sigma}, \mathcal{A}_{\sigma_0}) = 0,$$

where  $d_{\mathcal{H}}(\mathbb{A}, \mathbb{B}) = \max\{\text{dist}_{\mathcal{H}}(\mathbb{A}, \mathbb{B}), \text{dist}_{\mathcal{H}}(\mathbb{B}, \mathbb{A})\}$  expresses the Hausdorff metric in  $X$ .

**Proposition 4.1.** [39] Assume that

(H1)  $(\mathcal{H}, S_{\sigma}(t))$  has a global attractor  $\mathcal{A}_{\sigma}$  for any  $\sigma \in \Lambda$ ,

(H2) There exists a bounded set  $B \subset \mathcal{H}$  such that  $\mathcal{A}_{\sigma} \subset B$  for every  $\sigma \in \Lambda$ ,

(H3)  $S_{\sigma}(t)z$  is continuous in  $\sigma$  for  $t > 0$  and uniformly for  $z$  in bounded subsets of  $\mathcal{H}$ .

Then the global attractor is continuous on all  $I$ , where  $I$  is a “residual” set dense in  $\Lambda$ .

**Lemma 4.1.** There exists a set  $I$  dense in  $[0, 1]$  such that the global attractor  $\mathcal{A}_{\varepsilon}$  obtained in Theorem 3.1 is continuous at  $\varepsilon_0 \in I$ , that is

$$\lim_{\varepsilon \rightarrow \varepsilon_0} d_{\mathcal{H}}(\mathcal{A}_{\varepsilon}, \mathcal{A}_{\varepsilon_0}) = 0. \quad (4.1)$$

*Proof.* The argument is inspired by [40, 41]. We apply Proposition 4.1 with  $\Lambda = [0, 1]$ . Then Theorem 3.1 implies that the assumption (H1) holds.

It follows from (2.19) and the fact  $\sup_{z \in \mathcal{A}_{\varepsilon}} \Phi_{\varepsilon}(z) \leq \sup_{z \in \mathcal{N}_{\varepsilon}} \Phi_{\varepsilon}(z)$  that

$$\begin{aligned} \sup_{z \in \mathcal{A}_{\varepsilon}} \|z\|_{\mathcal{H}}^2 &\leq \frac{\sup_{z \in \mathcal{A}_{\varepsilon}} \Phi_{\varepsilon}(z) + C_F}{\beta_2} \\ &\leq \frac{C_F \sup_{z \in \mathcal{N}_{\varepsilon}} \|z\|_{\mathcal{H}}^{r+1} + 2C_F}{\beta_2}. \end{aligned}$$

Hence, we can derive from (3.2) that there exists a positive constant  $C$  independent of  $\varepsilon$  such that

$$\sup_{z \in \mathcal{A}_{\varepsilon}} \|z\|_{\mathcal{H}}^2 \leq C, \quad \forall \varepsilon \in [0, 1].$$

Then we have that  $\mathcal{B} = \{z \in \mathcal{H} : \|z\|_{\mathcal{H}}^2 \leq C\}$  is a bounded set which is independent of  $\varepsilon$  and  $\mathcal{A}_{\varepsilon} \subset \mathcal{B}$  for any  $\varepsilon \in [0, 1]$ . Therefore, the assumption (H2) holds.

Let  $B$  be a bounded set of  $\mathcal{H}$ . Then for any given  $\varepsilon_1, \varepsilon_2 \in [0, 1]$ ,  $z_0 \in B$ , we define

$$S_{\varepsilon_i}(t)z_0 = (v^i(t), p^i(t), v_t^i(t), p_t^i(t))^T, \quad i = 1, 2,$$

and

$$v = v^1 - v^2, p = p^1 - p^2.$$

Then  $(v, p, v_t, p_t)$  satisfies the following equations

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = (\varepsilon_1 - \varepsilon_2)h_1 - F_1(v, p) - G_1(v_t), \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = (\varepsilon_1 - \varepsilon_2)h_2 - F_2(v, p) - G_2(p_t), \end{cases} \quad (4.2)$$

where  $F_i(v, p)$  ( $i = 1, 2$ )  $G_1(v_t)$  and  $G_2(p_t)$  are constructed by the same as in Lemma 3.3. Multiplying the equations of (4.2) by  $v_t, p_t$ , respectively, and integrating over  $[0, L]$  by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|z\|_{\mathcal{H}}^2 &= - \int_0^L (F_1(v, p) + F_2(v, p)) dx - \int_0^L (G_1(v_t)v_t + G_2(p_t)) p_t dx \\ &+ (\varepsilon_1 - \varepsilon_2) \int_0^L (h_1 v_t + h_2 p_t) dx. \end{aligned} \quad (4.3)$$

Using (2.4), Hölder's inequality and  $H_*^1(0, L) \hookrightarrow L^r(0, L)$ ,  $r \in [0, \infty)$ , we can derive that

$$\begin{aligned} \int_0^L F_1(v, p) v_t dx &\leq C_f (1 + \|S_{\varepsilon_1}(t)z_0\|_{\mathcal{H}}^{r-1} + \|S_{\varepsilon_2}(t)z_0\|_{\mathcal{H}}^{r-1}) (\|v\|_{2r} + \|p\|_{2r}) \|v_t\|_2 \\ &\leq C_f (1 + \|S_{\varepsilon_1}(t)z_0\|_{\mathcal{H}}^{r-1} + \|S_{\varepsilon_2}(t)z_0\|_{\mathcal{H}}^{r-1}) (\|v_x\|_2 + \|p_x\|_2) \|v_t\|_2. \end{aligned} \quad (4.4)$$

Since  $\mathcal{E}_\varepsilon(t)$  is a non-increasing function, then for any  $z_0 \in B$ , we have

$$\|S_{\varepsilon_i}(t)z_0\|_{\mathcal{H}}^{r-1} \leq \frac{\mathcal{E}_{\varepsilon_i}(0) + C_F}{\beta_2} \leq C_B, \quad i = 1, 2.$$

Combining the above estimate with (4.4) and (2.12), we have

$$\int_0^L F_1(v, p) v_t dx \leq C_B (\|v_x\|_2 + \|p_x\|_2) \|v_t\|_2 \leq C_B (\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2) + \|v_t\|_2^2.$$

Similarly,

$$\int_0^L F_2(v, p) p_t dx \leq C_B (\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2) + \|p_t\|_2^2.$$

Therefore,

$$\int_0^L (F_1(v, p) v_t + F_2(v, p) p_t) dx \leq C_B \|z\|_{\mathcal{H}}^2. \quad (4.5)$$

It follows from the monotonicity property (2.9) that

$$- \int_0^L (G_1(v_t)v_t + G_2(p_t)p_t) dx \leq 0. \quad (4.6)$$

Moreover, it is easy to verify that

$$\begin{aligned} (\varepsilon_1 - \varepsilon_2) \int_0^L (h_1 v_t + h_2 p_t) dx &\leq \frac{1}{2} (\|v_t\|_2^2 + \|p_t\|_2^2) + \frac{1}{2} |\varepsilon_1 - \varepsilon_2|^2 (\|h_1\|_2^2 + \|h_2\|_2^2) \\ &\leq \frac{1}{2} \|z\|_{\mathcal{H}}^2 + \frac{1}{2} |\varepsilon_1 - \varepsilon_2|^2 (\|h_1\|_2^2 + \|h_2\|_2^2). \end{aligned} \quad (4.7)$$

Combining (4.5)–(4.7) with (4.3), we have

$$\frac{d}{dt} \|z\|_{\mathcal{H}}^2 \leq C_B \|z\|_{\mathcal{H}}^2 + |\varepsilon_1 - \varepsilon_2|^2 (\|h_1\|_2^2 + \|h_2\|_2^2). \quad (4.8)$$

Using the Gronwall's inequality and the fact  $\|z(0)\|_{\mathcal{H}}^2 = 0$ , we can derive from (4.8) that

$$\|z(t)\|_{\mathcal{H}}^2 \leq \frac{1}{C_B}(e^{C_B t} - 1)(\|h_1\|_2^2 + \|h_2\|_2^2)|\varepsilon_1 - \varepsilon_2|, \quad t > 0.$$

Hence we have

$$\|S_{\varepsilon_1}(t)z_0 - S_{\varepsilon_2}(t)z_0\|_{\mathcal{H}} \leq \left(\frac{1}{C_B}(e^{C_B t} - 1)(\|h_1\|_2^2 + \|h_2\|_2^2)\right)^{\frac{1}{2}}|\varepsilon_1 - \varepsilon_2|.$$

So the assumption (H3) holds. As a conclusion, we obtain that there exists a dense set  $I \subset [0, 1]$  such that (4.1) holds by Proposition 4.1.

**Theorem 4.1.** *Suppose the assumptions of Theorem 3.1 hold. Then the family of global attractors  $\mathcal{A}_\varepsilon$  is upper semicontinuous at  $\varepsilon_0$ , namely,*

$$\lim_{\varepsilon \rightarrow \varepsilon_0} \text{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon, \mathcal{A}_{\varepsilon_0}) = 0. \quad (4.9)$$

*Proof.* The argument is inspired by [42, 43]. Firstly, we suppose that (4.9) does not hold. Then there exist  $\lambda > 0$ , the sequence  $\varepsilon_n \rightarrow \varepsilon_0$  and  $z_0^n \in \mathcal{A}_{\varepsilon_n}$  such that

$$\text{dist}_{\mathcal{H}}(z_0^n, \mathcal{A}_{\varepsilon_0}) \geq \lambda > 0, \quad n \in \mathbb{N}. \quad (4.10)$$

Let  $z^n(t) = (v^n(t), p^n(t), v_t^n(t), p_t^n(t))^T$  be a bounded full trajectory from the attractor  $\mathcal{A}_{\varepsilon_n}$  with  $z^n(0) = z_0^n$ . It follows from (3.12) that  $z^n$  is bounded in  $L^\infty(\mathbb{R}; \mathcal{H}_1)$ .

Noticing  $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ , applying Simon's Compactness Theorem (see [44] for details), we can obtain that there exist  $z \in C([-T, T]; \mathcal{H})$  and a subsequence (still denote)  $\{z^n\}$  such that,

$$z^n \rightarrow z \quad \text{in } C([-T, T]; \mathcal{H}).$$

Then, we can conclude that

$$\sup_{t \in \mathbb{R}} \|z(t)\|_{\mathcal{H}} < \infty.$$

Let  $z(t) = (v(t), p(t), v_t(t), p_t(t))^T$  be a bounded full trajectory of the limiting semi-flow. Then, we can infer that  $z$  solves the limiting equation ( $\varepsilon = \varepsilon_0$ ), namely,

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + f_1(v, p) + g_1(v_t) = \varepsilon_0 h_1, \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + f_2(v, p) + g_2(p_t) = \varepsilon_0 h_2. \end{cases} \quad (4.11)$$

In fact, from (1.1), we can get  $z^n$  satisfies

$$\begin{cases} \rho v_{tt}^n - \alpha v_{xx}^n + \gamma \beta p_{xx}^n + f_1(v^n, p^n) + g_1(v_t^n) = \varepsilon^n h_1, \\ \mu p_{tt}^n - \beta p_{xx}^n + \gamma \beta v_{xx}^n + f_2(v^n, p^n) + g_2(p_t^n) = \varepsilon^n h_2. \end{cases} \quad (4.12)$$

we can use the same argument as in the proof of (H3) in Lemma 4.1, so we can infer that (4.11) is the limit case of (4.12) as  $n \rightarrow \infty$ .

Consequently, we have

$$z_0^n \rightarrow z(0) \in \mathcal{A}_{\varepsilon_0},$$

which contradicts (4.10). This completes the proof.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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