Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(9): 3362–3376.
DOI: 10.3934/era.2022171
Received: 14 May 2022
Revised: 01 July 2022
Accepted: 08 July 2022
Published: 15 July 2022

Research article

Summability in anisotropic mixed-norm Hardy spaces

Nan Li

Graduate School, Beijing Sport University, Beijing 100084, China

* **Correspondence:** Email: junyanane@bsu.edu.cn.

Abstract: Let $H_A^{\vec{p}}(\mathbb{R}^n)$ be the anisotropic mixed-norm Hardy space, where $\vec{p} \in (0, \infty)^n$ and A is a general expansive matrix on $\mathbb{R}^n$. In this paper, a general summability method, the so-called θ-summability is considered for multi-dimensional Fourier transforms in $H_A^{\vec{p}}(\mathbb{R}^n)$. Precisely, the author establishes the boundedness of maximal operators, induced by the so-called θ-means, from $H_A^{\vec{p}}(\mathbb{R}^n)$ to the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$. As applications, some norm and almost everywhere convergence results of the θ-means are presented. Finally, the corresponding conclusions of two well-known specific summability methods, namely, Bochner–Riesz and Weierstrass means, are also obtained.

Keywords: mixed-norm Hardy space; expansive matrix; θ-summability; maximal operator

1. Introduction

Let A be a general expansive matrix on $\mathbb{R}^n$. In 2003, Bownik [1] investigated the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ with $p \in (0, \infty)$, which includes both the classical Hardy space and the parabolic Hardy space of Calderón and Torchinsky [2] as special cases. Recently, Huang et al. [3] introduced the anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ with respect to $\vec{p} \in (0, \infty)^n$ and a general expansive matrix A, and established its various real-variable characterizations. This extends the real-variable theory of the Hardy space $H_A^p(\mathbb{R}^n)$ from [1]. For more information on mixed-norm function spaces, we refer the reader to [4–11].

On the other hand, it is well known that Stein, Taibleson and Weiss [12] proved for the Bochner–Riesz summability that the maximal operator σ_*^θ of the θ-means is bounded from the classical Hardy $H^p(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ with the index p greater than some constant p_0. This result has been extended to many other Hardy-type and other summability methods. For more progress about this topic, we refer the reader to [13–18] and references therein. In particular, Weisz [18] proved that the maximal operator, induced by the so-called θ-means, is bounded from the isotropic mixed-norm Hardy space $H^{\vec{p}}(\mathbb{R}^n)$ to the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$. However, the corresponding conclusion of summability in anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ is still unknown.

In this paper, under some conditions on θ and $\vec{p}$, we prove that the maximal operator σ_*^θ is bounded from $H_A^{\vec{p}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$. As a consequence, we prove some norm and almost everywhere convergence results for the θ-means. Moreover, sa special cases of the θ-means, we consider the well-known Bochner–Riesz and Weierstrass summations. This paper is organized as follows: As a preliminary, in Section 2, we recall some definitions of expansive matrices, mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ and anisotropic mixed-norm Hardy spaces $H_A^{\vec{p}}(\mathbb{R}^n)$. In Section 3, via borrowing some ideas from [18, Theorem 3] and [13, Theorem 7.4] as well as [14, Theorem 2.17], we prove our main result by using the known finite atomic characterization of $H_A^{\vec{p}}(\mathbb{R}^n)$ and a criterion on the boundedness of sublinear operators from $H_A^{\vec{p}}(\mathbb{R}^n)$ into $L^{\vec{p}}(\mathbb{R}^n)$. Section 4 is aimed to consider two special summability methods, namely, the Bochner–Riesz and Weierstrass summations.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\mathbf{0}$ be the *origin* of $\mathbb{R}^n$. For any $\gamma := (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\gamma| := \gamma_1 + \cdots + \gamma_n$ and $\partial^\gamma := (\frac{\partial}{\partial x_1})^{\gamma_1} \cdots (\frac{\partial}{\partial x_n})^{\gamma_n}$. We use C to denote a positive constant which is independent of the main parameters, but its value may change from line to line. In addition, we use $f \lesssim g$ to denote $f \le Cg$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. Moreover, for a given set $\Omega \subset \mathbb{R}^n$, we denote its *characteristic function* by $\mathbf{1}_\Omega$, the set $\mathbb{R}^n \setminus \Omega$ by $\Omega^\complement$ and its *n-dimensional Lebesgue measure* by $|\Omega|$. For any $t \in \mathbb{R}$, The symbol $\lfloor t \rfloor$ denotes the *largest integer not greater than t*. For each $r \in [1, \infty]$, we denote by r' its *conjugate index*, namely, $1/r + 1/r' = 1$. Moreover, if $\vec{r} := (r_1, \ldots, r_n) \in [1, \infty]^n$, we denote by $\vec{r}' := (r_1', \ldots, r_n')$ its *conjugate index*.

2. Preliminaries

In this section, we recall the notions of expansive matrices, mixed-norm Lebesgue spaces and anisotropic mixed-norm Hardy spaces.

We begin with the following notion of mixed-norm Lebesgue spaces from [19].

Definition 2.1. Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n$. The *mixed-norm Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1} \, dx_1 \right]^{\frac{p_2}{p_1}} \cdots \, dx_n \right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \ldots, n\}$.

Recall also that the notions of expansive matrices and homogeneous quasi-norms were originally introduced by Bownik in [1].

Definition 2.2. A real $n \times n$ matrix A is called an *expansive matrix* (shortly, a *dilation*) if

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

here and thereafter, $\sigma(A)$ denotes the *collection of all eigenvalues of A*.

Definition 2.3. Let A be a dilation. A measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ is called a *homogeneous quasi-norm*, associated with A, if

(i) $x \neq \mathbf{0}$ implies that $\rho(x) \in (0, \infty)$;
(ii) for each $x \in \mathbb{R}^n$, $\rho(Ax) = b\rho(x)$, here and below, $b := |\det A|$;
(iii) for any $x, y \in \mathbb{R}^n$, $\rho(x + y) \leq c[\rho(x) + \rho(y)]$, where c is a positive constant independent of x and y.

For any given dilation A, it was proved in [1, Lemma 2.2] that there exists an open set $\Delta \subset \mathbb{R}^n$ which has the following property: $|\Delta| = 1$, and we can find a constant $\tau \in (1, \infty)$ such that $\Delta \subset \tau\Delta \subset A\Delta$. For any $i \in \mathbb{Z}$, we define $B_i := A^i\Delta$. It is easy to check that $\{B_i\}_{i\in\mathbb{Z}}$ is a family of open sets around the origin, $B_i \subset \tau B_i \subset B_{i+1}$ and $|B_i| = b^i$. For any given dilation A, we use the symbol $\mathfrak{B}$ to denote the set of all *dilated balls*, namely,

$$\mathfrak{B} := \{x + B_i : \ x \in \mathbb{R}^n, \ i \in \mathbb{Z}\} \tag{2.1}$$

and

$$\omega := \inf\left\{k \in \mathbb{Z} : \ \tau^k \geq 2\right\}. \tag{2.2}$$

By [1, Lemma 2.4], we know that any two homogeneous quasi-norms associated with the same fixed dilation A are equivalent. Thus, in what follows, we always use the step homogeneous quasi-norm defined by setting, for each $x \in \mathbb{R}^n$,

$$\rho(x) := \begin{cases} b^i & \text{when} \quad x \in B_{i+1} \backslash B_i \\ 0 & \text{when} \quad x = \mathbf{0} \end{cases}$$

for convenience. Let $\lambda_-, \lambda_+ \in (1, \infty)$ be two *numbers* such that

$$\lambda_- \leq \min\{|\lambda| : \ \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \ \lambda \in \sigma(A)\} \leq \lambda_+. \tag{2.3}$$

Throughout this article, the symbol $\mathcal{S}(\mathbb{R}^n)$ denotes the space of all Schwartz functions, namely, the set of all $C^\infty(\mathbb{R}^n)$ functions ϕ satisfying that, for any $k \in \mathbb{Z}_+$ and multi-index $\beta \in \mathbb{Z}^n_+$,

$$\|\phi\|_{\beta,k} := \sup_{x\in\mathbb{R}^n}[\rho(x)]^k \left|\partial^\beta\phi(x)\right| < \infty.$$

The topology of $\mathcal{S}(\mathbb{R}^n)$ is determined by $\{\|\cdot\|_{\beta,k}\}_{\beta\in\mathbb{Z}^n_+, k\in\mathbb{Z}_+}$. Moreover, we use $\mathcal{S}'(\mathbb{R}^n)$ to denote the dual space of $\mathcal{S}(\mathbb{R}^n)$, namely, the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology. For any $N \in \mathbb{Z}_+$, let $\mathcal{S}_N(\mathbb{R}^n)$ denote the following set:

$$\left\{\phi \in \mathcal{S}(\mathbb{R}^n) : \|\phi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{\beta\in\mathbb{Z}^n_+, |\beta|\leq N} \sup_{x\in\mathbb{R}^n} \left[\left|\partial^\beta\phi(x)\right| \max\left\{1, [\rho(x)]^N\right\}\right] \leq 1\right\}.$$

For an n-dimensional vector $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n$, let

$$p_- := \min\{p_1, \ldots, p_n\}, \quad p_+ := \max\{p_1, \ldots, p_n\} \quad \text{and} \quad \underline{p} \in (0, \min\{p_-, 1\}). \tag{2.4}$$

The following definition of anisotropic mixed-norm Hardy spaces was first introduced by Huang et. al [3].

Definition 2.4. (i) Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *non-tangential maximal function $M_\phi(f)$ with respect to ϕ* is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\phi(f)(x) := \sup_{y \in x+B_k,\, k\in\mathbb{Z}} |f * \phi_k(y)|,$$

here and thereafter, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, let $\phi_k(\cdot) := b^k\phi(A^k\cdot)$. Moreover, for any given $N \in \mathbb{N}$, the *non-tangential grand maximal function $M_N(f)$* of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\phi \in \mathcal{S}_N(\mathbb{R}^n)} M_\phi(f)(x).$$

(ii) Let $\vec{p} \in (0,\infty)^n$ and $N \in \mathbb{N} \cap [\lfloor(\frac{1}{\min\{1,p_-\}} - 1)\frac{\ln b}{\ln \lambda_-}\rfloor + 2, \infty)$, where p_- is as in (2.4). The *anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$* is defined by setting

$$H_A^{\vec{p}}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \ M_N(f) \in L^{\vec{p}}(\mathbb{R}^n) \right\}$$

and, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$, let $\|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)} := \|M_N(f)\|_{L^{\vec{p}}(\mathbb{R}^n)}$.

Remark 1. (i) Observe that, in [3, Theorem 4.7], it was proved that the space $H_A^{\vec{p}}(\mathbb{R}^n)$ is independent of the choice of N as in Definition 2.4(ii).

(ii) Note that Cleanthous et al. [20] investigated another kind of anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, where $\vec{a} \in [1,\infty)^n$ and $\vec{p} \in (0,\infty)^n$; see [20, Definition 3.3]. We should point out that [3, Proposition 4] shows that, if

$$A := \begin{pmatrix} 2^{a_1} & 0 & \cdots & 0 \\ 0 & 2^{a_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 2^{a_n} \end{pmatrix} \tag{2.5}$$

with $\vec{a} := (a_1, \dots, a_n) \in [1,\infty)^n$, then the Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ in Definition 2.4(ii) and the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ from [20] coincide with equivalent quasi-norms.

3. Summability in $H_A^{\vec{p}}(\mathbb{R}^n)$

In this section, we study the so-called θ-summability for multi-dimensional Fourier transforms in the anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$.

We always use $L^1_{\rm loc}(\mathbb{R}^n)$ to denote the set of all locally integrable functions on $\mathbb{R}^n$. To present our main result, we need several technical lemmas as follows. First, the following Lemmas 3.1, 3.2 and 3.3 are just, respectively, [3, Lemmas 3.4, 3.2 and 4.4], which show some properties of the $L^{\vec{p}}(\mathbb{R}^n)$ quasi-norm, the boundedness of the anisotropic Hardy–Littlewood maximal operator $M_{\rm HL}$ on the space $L^{\vec{p}}(\mathbb{R}^n)$ and the anisotropic Fefferman–Stein vector-valued inequality on $L^{\vec{p}}(\mathbb{R}^n)$.

Lemma 3.1. Let $\vec{p} \in (0,\infty]^n$. Then, for any $s \in (0,\infty)$ and $f \in L^{\vec{p}}(\mathbb{R}^n)$,

$$\||f|^s\|_{L^{\vec{p}}(\mathbb{R}^n)} = \|f\|^s_{L^{s\vec{p}}(\mathbb{R}^n)},$$

here and below, for any $\gamma \in \mathbb{R}$, $\gamma\vec{p} := (\gamma p_1, \dots, \gamma p_n)$. In addition, for any $\lambda \in \mathbb{C}$, $r \in [0, \min\{1, p_-\}]$ with p_- as in (2.4) and $f,\ h \in L^{\vec{p}}(\mathbb{R}^n)$, $\|\lambda f\|_{L^{\vec{p}}(\mathbb{R}^n)} = |\lambda|\|f\|_{L^{\vec{p}}(\mathbb{R}^n)}$ and

$$\|f + h\|^r_{L^{\vec{p}}(\mathbb{R}^n)} \le \|f\|^r_{L^{\vec{p}}(\mathbb{R}^n)} + \|h\|^r_{L^{\vec{p}}(\mathbb{R}^n)}.$$

Lemma 3.2. Let $\vec{p} \in (1, \infty)^n$. Then there exists a positive constant C such that, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|M_{\mathrm{HL}}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \le C\|f\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where M_{HL} denotes the anisotropic Hardy–Littlewood maximal operator defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_{\mathrm{HL}}(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_k} \frac{1}{|B_k|} \int_{y+B_k} |f(z)|\, dz = \sup_{x\in B\in\mathfrak{B}} \frac{1}{|B|} \int_B |f(z)|\, dz$$

with $\mathfrak{B}$ as in (2.1).

Lemma 3.3. Let $\vec{p} \in (1, \infty)^n$ and $v \in (1, \infty]$. Then, for any sequence $\{f_k\}_{k\in\mathbb{N}}$ of measurable functions,

$$\left\|\left\{\sum_{k\in\mathbb{N}} [M_{\mathrm{HL}}(f_k)]^v\right\}^{1/v}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \le C \left\|\left(\sum_{k\in\mathbb{N}} |f_k|^v\right)^{1/v}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}$$

with the usual modification made when $v = \infty$, where C is a positive constant independent of $\{f_k\}_{k\in\mathbb{N}}$.

Let $\vec{p} \in (0, \infty)^n$ and $k \in \mathbb{Z}_+$. Then, by the fact that, for any dilated ball $B \in \mathfrak{B}$ and $q \in (0, \underline{p})$, $\mathbf{1}_{A^kB} \le b^{k/q}[M_{\mathrm{HL}}(\mathbf{1}_B)]^{1/q}$ as well as Lemmas 3.1 and 3.3, we conclude that there exists a positive constant C such that, for any sequence $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$,

$$\begin{aligned}\left\|\sum_{i\in\mathbb{N}} \mathbf{1}_{A^kB^{(i)}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} &\le C \left\|\sum_{i\in\mathbb{N}} b^{k/q} \left[M_{\mathrm{HL}}\left(\mathbf{1}_{B^{(i)}}\right)\right]^{1/q}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ &= Cb^{k/q} \left\|\left\{\sum_{i\in\mathbb{N}} \left[M_{\mathrm{HL}}\left(\mathbf{1}_{B^{(i)}}\right)\right]^{1/q}\right\}^q\right\|^{1/q}_{L^{\vec{p}/q}(\mathbb{R}^n)} \\ &\le Cb^{k/q} \left\|\sum_{i\in\mathbb{N}} \mathbf{1}_{B^{(i)}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}.\end{aligned} \tag{3.1}$$

The last inequality used Lemma 3.3 with the fact that $\vec{p}/q \in (1, \infty)^n$.

For any given $p \in (0, \infty]$ and measurable set $\Omega \subset \mathbb{R}^n$. The Lebesgue space $L^p(\Omega)$ is defined to be the set of all measurable functions f on Ω such that, when $p \in (0, \infty)$,

$$\|f\|_{L^p(\Omega)} := \left[\int_\Omega |f(x)|^p\, dx\right]^{1/p} < \infty$$

and

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty.$$

We also need the following notions of anisotropic mixed-norm $(\vec{p}, r, s)$-atoms and anisotropic mixed-norm finite atomic Hardy spaces $H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)$ from [3].

Definition 3.4. Let $\vec{p} \in (0,\infty)^n$, $r \in (1,\infty]$ and

$$s \in \left[\left\lfloor\left(\frac{1}{p_-}-1\right)\frac{\ln b}{\ln \lambda_-}\right\rfloor, \infty\right) \cap \mathbb{Z}_+ \tag{3.2}$$

with p_- as in (2.4).

(I) A measurable function a on $\mathbb{R}^n$ is called an *anisotropic mixed-norm $(\vec{p}, r, s)$-atom* if

(I)$_1$ supp $a \subset B$ with some $B \in \mathfrak{B}$, where $\mathfrak{B}$ is as in (2.1);

(I)$_2$ $\|a\|_{L^r(\mathbb{R}^n)} \le \frac{|B|^{1/r}}{\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)}}$;

(I)$_3$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le s$, $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$.

(II) The *anisotropic mixed-norm finite atomic Hardy space* $H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist $I \in \mathbb{N}$, $\{\lambda_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a finite sequence of $(\vec{p}, r, s)$-atoms, $\{a_i\}_{i\in[1,I]\cap\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i=1}^{I} \lambda_i a_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).$$

Moreover, for any $f \in H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)$, let

$$\|f\|_{H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{i=1}^{I} \left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where $\underline{p}$ is as in (2.4) and the infimum is taken over all decompositions of f as above.

Then, from [3, Theorem 8.1(i) and Remark 13], we immediately deduce the succeeding criterion on the boundedness of sublinear operators from $H^{\vec{p}}_A(\mathbb{R}^n)$ into $L^{\vec{p}}(\mathbb{R}^n)$, which plays a key role in the proof of our main result.

Lemma 3.5. Let $\vec{p} \in (0,\infty)^n$, $r \in (\max\{p_+, 1\}, \infty)$ with p_+ as in (2.4) and s be as in (3.2). Assume that $T : H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n) \to L^{\vec{p}}(\mathbb{R}^n)$ is a sublinear operator satisfying that there exists a positive constant C such that, for any $f \in H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \le C\|f\|_{H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)}.$$

Then T uniquely extends to a bounded sublinear operator from $H^{\vec{p}}_A(\mathbb{R}^n)$ into $L^{\vec{p}}(\mathbb{R}^n)$.

Recall that, for any given $p \in [1,2]$ and any $f \in L^p(\mathbb{R}^n)$, the Fourier inversion formula, namely,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(t)e^{2\pi\imath x\cdot t}\,dt, \quad \forall\, x \in \mathbb{R}^n,$$

holds true if $\widehat{f} \in L^1(\mathbb{R}^n)$, here and below, $\imath := \sqrt{-1}$, $x \cdot t := \sum_{k=1}^n x_k t_k$ for any $x := (x_1, \dots, x_n)$, $t := (t_1, \dots, t_n) \in \mathbb{R}^n$, and $\widehat{f}$ denotes the *Fourier transform* of f, which is defined by setting, for any $t \in \mathbb{R}^n$,

$$\widehat{f}(t) := \int_{\mathbb{R}^n} f(x)e^{-2\pi\imath x\cdot t}\,dx.$$

This motivates the following definition of θ-summability of Fourier transforms; see, for instance, [15–18, 21] for the classical case and [13, 14] for the anisotropic case. We always suppose that

$$\theta \in C_0(\mathbb{R}), \qquad \theta(|\cdot|) \in L^1(\mathbb{R}^n), \qquad \theta(0) = 1 \quad \text{and} \quad \theta \text{ is even}, \tag{3.3}$$

where the symbol $C_0(\mathbb{R})$ denotes the *set of all continuous functions f satisfying that* $\lim_{|x|\to\infty} |f(x)| = 0$. Let A be a given dilation, $m \in \mathbb{Z}$ and $p \in [1, 2]$. The *m-th anisotropic θ-mean*, denoted by σ_m^θ, is defined by setting, for any $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\sigma_m^\theta f(x) := \int_{\mathbb{R}^n} \theta\left(|(A^*)^{-m}u|\right) \widehat{f}(u) e^{2\pi\imath x\cdot u}\, du, \tag{3.4}$$

where A^* be the transposed matrix of A. This integral is well defined because $\theta \in L^p(\mathbb{R})$ with $p \in [1, 2]$ and $\widehat{f} \in L^{p'}(\mathbb{R}^n)$. Let $\theta_0(x) := \theta(|x|)$ for any $x \in \mathbb{R}^n$ and assume that

$$\widehat{\theta_0} \in L^1(\mathbb{R}^n). \tag{3.5}$$

Moreover, by [14, (2.17)], we can rewrite $\sigma_m^\theta f(x)$ in (3.4) as

$$\sigma_m^\theta f(x) = b^m \int_{\mathbb{R}^n} f(t)\widehat{\theta_0}\left(A^m(x - t)\right)\, dt.$$

This definition of anisotropic θ-means can be extended to any $f \in L^{\vec{p}}(\mathbb{R}^n)$ with $p_- \in [1, \infty)$ by setting, for any $x \in \mathbb{R}^n$,

$$\sigma_m^\theta f(x) := b^m \int_{\mathbb{R}^n} f(x - t)\widehat{\theta_0}(A^m t)\, dt, \tag{3.6}$$

where $m \in \mathbb{Z}$. Furthermore, (3.6) induces the definition of *maximal θ-operators* σ_*^θ as follows: for any $f \in L^{\vec{p}}(\mathbb{R}^n)$ with $p_- \in [1, \infty)$,

$$\sigma_*^\theta f := \sup_{m\in\mathbb{Z}} \left|\sigma_m^\theta f\right|.$$

Now we state the main result of this paper as follows.

Theorem 3.6. *Let θ and θ_0 be, respectively, as in* (3.3) *and* (3.5) *satisfying that there exists a positive constant $\beta \in (1, \infty)$ such that, for any $\gamma \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\left|\partial^\gamma \widehat{\theta_0}(x)\right| \le C\, |x|^{-\beta}, \tag{3.7}$$

where C is a positive constant independent of x. If $\vec{p} \in (0, \infty)^n$,

$$\beta \in \left(\frac{\ln b}{\ln \lambda_-}, \infty\right) \quad \text{and} \quad p_- \in \left(\frac{\ln b}{\beta \ln \lambda_-}, \infty\right) \tag{3.8}$$

with λ_- as in (2.3)*, then there exists a positive constant $C_{(p_-,p_+)}$, with p_- and p_+ as in* (2.4)*, such that, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$,*

$$\left\|\sigma_*^\theta f\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \le C_{(p_-,p_+)}\, \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)}.$$

Proof. By Lemma 3.5, to prove Theorem 3.6, we only need to show that, for any $f \in H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$\left\|\sigma^\theta_* f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)} \tag{3.9}$$

with s being as in (3.2) large enough and $r \in (\max\{p_+, 1\}, \infty)$ to be chosen later, where p_+ is as in (2.4).

For this purpose, suppose now $f \in H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)$. Then, by Definition 3.4(II), we find that there exist some $I \in \mathbb{N}$, $\{\lambda_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a finite sequence of $(\vec{p}, r, s)$-atoms, $\{a_i\}_{i\in[1,I]\cap\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathfrak{B}$ such that $f = \sum_{i=1}^I \lambda_i a_i$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{H^{\vec{p},r,s}_{A,\,\mathrm{fin}}(\mathbb{R}^n)} \sim \left\|\left\{\sum_{i=1}^I \left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{\vec{p}}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}, \tag{3.10}$$

where $\underline{p}$ is as in (2.4). Take two sequences $\{x_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{R}^n$ and $\{k_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{Z}$ such that, for any $i \in [1, I] \cap \mathbb{N}$, $x_i + B_{k_i} = B^{(i)}$. Then

$$\begin{aligned}\left\|\sigma^\theta_* f\right\|_{L^{\vec{p}}(\mathbb{R}^n)} &\lesssim \left\|\sum_{i=1}^I |\lambda_i|\sigma^\theta_*(a_i)\mathbf{1}_{x_i+A^\omega B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} + \left\|\sum_{i=1}^I |\lambda_i|\sigma^\theta_*(a_i)\mathbf{1}_{(x_i+A^\omega B_{k_i})^\complement}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\ &=: \mathrm{J}_1 + \mathrm{J}_2,\end{aligned} \tag{3.11}$$

where ω is as in (2.2).

For J_1, take $u \in L^{(\vec{p}/\underline{p})'}(\mathbb{R}^n)$ satisfying that $\|u\|_{L^{(\vec{p}/\underline{p})'}(\mathbb{R}^n)} \le 1$ and

$$\left\|\sum_{i=1}^I |\lambda_i|^{\underline{p}}\left[\sigma^\theta_*(a_i)\right]^{\underline{p}}\mathbf{1}_{x_i+A^\omega B_{k_i}}\right\|_{L^{\vec{p}/\underline{p}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\sum_{i=1}^I |\lambda_i|^{\underline{p}}\left[\sigma^\theta_*(a_i)(x)\right]^{\underline{p}}\mathbf{1}_{x_i+A^\omega B_{k_i}}(x)u(x)\,dx.$$

From this and the Hölder inequality, it follows that, for any $t \in (1, \infty)$ with $p_+ < t\underline{p} < r$,

$$\begin{aligned}(\mathrm{J}_1)^{\underline{p}} &\lesssim \left\|\sum_{i=1}^I |\lambda_i|^{\underline{p}}\left[\sigma^\theta_*(a_i)\right]^{\underline{p}}\mathbf{1}_{x_i+A^\omega B_{k_i}}\right\|_{L^{\vec{p}/\underline{p}}(\mathbb{R}^n)} \\ &\sim \int_{\mathbb{R}^n}\sum_{i=1}^I |\lambda_i|^{\underline{p}}\left[\sigma^\theta_*(a_i)(x)\right]^{\underline{p}}\mathbf{1}_{x_i+A^\omega B_{k_i}}(x)u(x)\,dx. \\ &\lesssim \sum_{i=1}^I |\lambda_i|^{\underline{p}}\left\|\left[\sigma^\theta_*(a_i)\right]^{\underline{p}}\mathbf{1}_{x_i+A^\omega B_{k_i}}\right\|_{L^t(\mathbb{R}^n)}\left\|\mathbf{1}_{x_i+A^\omega B_{k_i}}u\right\|_{L^{t'}(\mathbb{R}^n)} \\ &\lesssim \sum_{i=1}^I |\lambda_i|^{\underline{p}}\left\|\sigma^\theta_*(a_i)\right\|^{\underline{p}}_{L^r(\mathbb{R}^n)}\left\|\mathbf{1}_{x_i+A^\omega B_{k_i}}\right\|^{1/t}_{L^{\frac{r}{r-t\underline{p}}}(\mathbb{R}^n)}\left\|\mathbf{1}_{x_i+A^\omega B_{k_i}}u\right\|_{L^{t'}(\mathbb{R}^n)}.\end{aligned}$$

By this, the fact that σ^θ_* is bounded on $L^q(\mathbb{R}^n)$ for any $1 < q < \infty$, and Definition 3.4(I), we further conclude that

$$(\mathrm{J}_1)^{\underline{p}} \lesssim \sum_{i=1}^I |\lambda_i|^{\underline{p}}\left\|\mathbf{1}_{x_i+B_{k_i}}\right\|^{-\underline{p}}_{L^{\vec{p}}(\mathbb{R}^n)}\left|A^\omega B_{k_i}\right|^{\underline{p}/r}\left|A^\omega B_{k_i}\right|^{\frac{r-t\underline{p}}{rt}}\left\|\mathbf{1}_{x_i+A^\omega B_{k_i}}u\right\|_{L^{t'}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&\sim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-\underline{p}} \left|A^\omega B_{k_i}\right|^{1/t} \left\|\mathbf{1}_{x_i+A^\omega B_{k_i}} u\right\|_{L^{t'}(\mathbb{R}^n)} \\
&\sim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-\underline{p}} \left|A^\omega B_{k_i}\right| \left[\frac{1}{|A^\omega B_{k_i}|} \int_{x_i+A^\omega B_{k_i}} [u(x)]^{t'}\, dx\right]^{1/t'} \\
&\lesssim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-\underline{p}} \int_{\mathbb{R}^n} \mathbf{1}_{x_i+A^\omega B_{k_i}}(x) \left[M_{\mathrm{HL}}\left(u^{t'}\right)(x)\right]^{1/t'}\, dx \\
&\lesssim \left\|\sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-\underline{p}} \mathbf{1}_{x_i+A^\omega B_{k_i}}\right\|_{L^{\vec{p}/\underline{p}}(\mathbb{R}^n)} \left\|\left[M_{\mathrm{HL}}\left(u^{t'}\right)\right]^{1/t'}\right\|_{L^{(\vec{p}/\underline{p})'}(\mathbb{R}^n)}.
\end{aligned}
$$

Note that the assumption $0 < p_+/\underline{p} < t$ implies that $t' < \vec{p}/\underline{p})' \le \infty$. From this, Lemmas 3.2 and 3.1, the fact that $\|u\|_{L^{(\vec{p}/\underline{p})'}(\mathbb{R}^n)} \le 1$, (3.1) and (3.10), we deduce that

$$
\begin{aligned}
\mathrm{J}_1 &\lesssim \left\|\sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-\underline{p}} \mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}/\underline{p}}(\mathbb{R}^n)}^{1/\underline{p}} \|u\|_{L^{(\vec{p}/\underline{p})'}(\mathbb{R}^n)}^{1/\underline{p}} \\
&\sim \left\|\left\{\sum_{i=1}^{I} \left[\frac{|\lambda_i| \mathbf{1}_{x_i+B_{k_i}}}{\|\mathbf{1}_{x_i+B_{k_i}}\|_{L^{\vec{p}}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \|f\|_{H_{A,\,\mathrm{fin}}^{\vec{p},r,s}(\mathbb{R}^n)}.
\end{aligned} \tag{3.12}
$$

We next deal with J_2. To do this, we first claim that, for any $i \in [1, I] \cap \mathbb{N}$ and $x \in (x_i + A^\omega B_{k_i})^\complement$,

$$
\sigma_*^\theta(a_i)(x) \lesssim \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} \left[M_{\mathrm{HL}}\left(\mathbf{1}_{x_i+B_{k_i}}\right)(x)\right]^{\beta \ln \lambda_- / \ln b}, \tag{3.13}
$$

where β is as in (3.7) and (3.8). Assume that (3.13) holds true for the time being. Then, by (3.8), Lemmas 3.1 and 3.3 as well as (3.10), we find that

$$
\begin{aligned}
\mathrm{J}_2 &\lesssim \left\|\sum_{i=1}^{I} |\lambda_i| \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} \left[M_{\mathrm{HL}}\left(\mathbf{1}_{x_i+B_{k_i}}\right)(x)\right]^{\beta \ln \lambda_- / \ln b} \mathbf{1}_{(x_i+A^\omega B_{k_i})^\complement}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\
&\lesssim \left\|\sum_{i=1}^{I} \left[|\lambda_i|^{\ln b/(\beta \ln \lambda_-)} \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-\ln b/(\beta \ln \lambda_-)} M_{\mathrm{HL}}\left(\mathbf{1}_{x_i+B_{k_i}}\right)\right]^{\beta \ln \lambda_- / \ln b}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \\
&\lesssim \left\|\left[\sum_{i=1}^{I} |\lambda_i| \left\|\mathbf{1}_{x_i+B_{k_i}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} \mathbf{1}_{x_i+B_{k_i}}\right]^{\ln b/(\beta \ln \lambda_-)}\right\|_{L^{\vec{p}\beta \ln \lambda_- / \ln b}(\mathbb{R}^n)}^{\beta \ln \lambda_- / \ln b} \\
&\lesssim \left\|\left\{\sum_{i=1}^{I} \left[\frac{|\lambda_i| \mathbf{1}_{x_i+B_{k_i}}}{\|\mathbf{1}_{x_i+B_{k_i}}\|_{L^{\vec{p}}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \|f\|_{H_{A,\,\mathrm{fin}}^{\vec{p},r,s}(\mathbb{R}^n)}.
\end{aligned}
$$

This, together with (3.11) and (3.12), further implies that (3.9) holds true.

Thus, to complete the proof of Theorem 3.6, it suffices to verify (3.13). To this end, let a be any $(\vec{p}, r, s)$-atom supported in $x_0 + B_k \in \mathfrak{B}$, where $x_0 \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $\mathfrak{B}$ is as in (2.1). Without loss of generality, we may assume that $x_0 = \mathbf{0}$. Suppose that P is a polynomial of degree not more than

s. Then, by (3.6), Definition 3.4(I) and the Hölder inequality, we obtain that, for any $m \in \mathbb{Z}$ and $x \in (B_{k+\omega})^{\complement}$,

$$\begin{aligned}\left|\sigma_m^\theta a(x)\right| &= b^m \left|\int_{B_k} a(t)\widehat{\theta_0}(A^m(x-t))\,dt\right| \\ &= b^m \left|\int_{B_k} a(t)\left[\widehat{\theta_0}(A^m(x-t)) - P(A^m(x-t))\right]dt\right| \\ &\le b^m \|a\|_{L^r(\mathbb{R}^n)} \left[\int_{B_k} \left|\widehat{\theta_0}(A^m(x-t)) - P(A^m(x-t))\right|^{r'} dt\right]^{1/r'} \\ &\le b^m b^{k/r} \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{-m/r'} \left[\int_{A^m x + B_{k+m}} \left|\widehat{\theta_0}(y) - P(y)\right|^{r'} dy\right]^{1/r'} \\ &\le b^m b^{k/r} \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{-m/r'} b^{(k+m)/r'} \sup_{y\in A^m x + B_{k+m}} \left|\widehat{\theta_0}(y) - P(y)\right| \\ &\le b^{k+m} \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} \sup_{y\in A^m x + B_{k+m}} \left|\widehat{\theta_0}(y) - P(y)\right|.\end{aligned} \tag{3.14}$$

Assume that $x \in B_{k+\omega+\nu+1} \setminus B_{k+\omega+\nu}$ for some $\nu \in \mathbb{Z}_+$. Then, using [1, (2.11)], we have

$$A^m x + B_{k+m} \subset A^{k+m}\left(B_{\omega+\nu+1} \setminus B_{\omega+\nu} + B_0\right) \subset A^{k+m}\left(B_{2\omega+\nu+1} \setminus B_\nu\right). \tag{3.15}$$

In addition, the Taylor remainder theorem implies that

$$\sup_{y\in A^m x + B_{k+m}} \left|\widehat{\theta_0}(y) - P(y)\right| \lesssim \sup_{t,\tilde{t}\in B_{k+m}} \sup_{\gamma\in\mathbb{Z}_+^n,\,|\gamma|\le N} \left|\partial^\gamma \widehat{\theta_0}\left(A^m x + \tilde{t}\right)\right| |t|^N, \tag{3.16}$$

where $N \in [0, s+1] \cap \mathbb{Z}_+$.

For the case when $k+m \in \mathbb{Z} \setminus \mathbb{Z}_+$ and $k+m+\nu \in \mathbb{Z}_+$, by (3.16), [1, (2.2)] and (3.7), it is easy to see that

$$\begin{aligned}\sup_{y\in A^m x + B_{k+m}} \left|\widehat{\theta_0}(y) - P(y)\right| &\lesssim \lambda_-^{N(k+m)} \sup_{z\in A^m x + B_{k+m}} \sup_{\gamma\in\mathbb{Z}_+^n,\,|\gamma|\le N} \left|\partial^\gamma \widehat{\theta_0}(z)\right| \\ &\lesssim \lambda_-^{N(k+m)} \sup_{z\in A^m x + B_{k+m}} |z|^{-\beta}\end{aligned}$$

with β as in (3.7) and (3.8). From this, (3.15) and [1, (3.2)], it follows that

$$\begin{aligned}\sup_{y\in A^m x + B_{k+m}} \left|\widehat{\theta_0}(y) - P(y)\right| &\lesssim \lambda_-^{N(k+m)} \sup_{z\in A^m x + B_{k+m}} \rho(z)^{-\beta \ln\lambda_-/\ln b} \\ &\lesssim \lambda_-^{N(k+m)} b^{-\beta(k+m+\nu)\ln\lambda_-/\ln b}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\sigma_m^\theta a(x)\right| &\lesssim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{k+m} b^{N(k+m)\ln\lambda_-/\ln b} b^{-\beta(k+m+\nu)\ln\lambda_-/\ln b} \\ &\sim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{(k+m)[1+(N-\beta)\ln\lambda_-/\ln b]} b^{-\beta\nu\ln\lambda_-/\ln b}.\end{aligned} \tag{3.17}$$

Choosing N larger than β, we have

$$\sigma_*^\theta a(x) \lesssim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} \left[M_{\mathrm{HL}}\left(\mathbf{1}_{B_k}\right)(x)\right]^{\beta\ln\lambda_-/\ln b}. \tag{3.18}$$

For the case when $k+m \in \mathbb{Z}\setminus\mathbb{Z}_+$ and $k+m+\nu \in \mathbb{Z}\setminus\mathbb{Z}_+$, similarly to (3.17), it is easy to check that, for any $x \in B_{k+\omega+\nu+1}\setminus B_{k+\omega+\nu}$,

$$\begin{aligned}\left|\sigma_m^\theta a(x)\right| &\lesssim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{k+m}\lambda_-^{N(k+m)} \sup_{z\in A^m x+B_{k+m}} \rho(z)^{-\beta\ln\lambda_+/\ln b}\\ &\lesssim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{(k+m)(1+N\ln\lambda_-/\ln b-\beta\ln\lambda_+/\ln b)} b^{-\beta\nu\ln\lambda_+/\ln b}.\end{aligned}$$

This further implies that (3.18) holds true for this case by choosing N large enough such that

$$1+N\frac{\ln\lambda_-}{\ln b}-\beta\frac{\ln\lambda_+}{\ln b}>0.$$

Finally, for the case when $k+m\in\mathbb{Z}_+$, we choose $P\equiv 0$. Then, by (3.14), (3.15) and the fact that $\beta\in(\ln b/\ln\lambda_-,\infty)$, we conclude that, for any $x\in B_{k+\omega+\nu+1}\setminus B_{k+\omega+\nu}$,

$$\begin{aligned}\left|\sigma_m^\theta a(x)\right| &\lesssim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{k+m} \sup_{z\in A^m x+B_{k+m}} \rho(z)^{-\beta\ln\lambda_-/\ln b}\\ &\lesssim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{(k+m)(1-\beta\ln\lambda_-/\ln b)} b^{-\beta\nu\ln\lambda_-/\ln b}\\ &\lesssim \left\|\mathbf{1}_{B_k}\right\|_{L^{\vec{p}}(\mathbb{R}^n)}^{-1} b^{-\beta\nu\ln\lambda_-/\ln b}\end{aligned}$$

and hence (3.18) also holds true for this case. This finishes the proof of (3.13) and hence of Theorem 3.6. □

Remark 2. (i) If $A := d\,\mathrm{I}_{n\times n}$ for some $d\in\mathbb{R}$ with $|d|\in(1,\infty)$, where $\mathrm{I}_{n\times n}$ denotes the $n\times n$ *unit matrix*, then $\frac{\ln b}{\ln\lambda_-}=n$ and the Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ goes back to the isotropic mixed-norm Hardy space $H^{\vec{p}}(\mathbb{R}^n)$. In this case, Theorem 3.6 was obtained by Weisz in [18, Theorem 3]. Moreover, if $\vec{p} := (\overbrace{p,\ldots,p}^{n\text{ times}})$ with some $p\in(0,\infty)$, then Theorem 3.6 implies the well-known result, with $\beta\in(n,\infty)$ and $p\in(n/\beta,\infty)$, for the classical Hardy space $H^p(\mathbb{R}^n)$ (see Weisz [16]). This classical result was also proved in a special case, namely, for the Bochner–Riesz means, in Stein et al. [12]. For the same case, a counterexample was also given in [12] to show that the same conclusion is not true for $p\in(0,n/\beta]$.

(ii) When A is as in (2.5), the space $H_A^{\vec{p}}(\mathbb{R}^n)$ is just the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$; see Remark 1(ii). We should point out that Theorem 3.6 is new even for this case.

As applications of Theorem 3.6, we next give some consequences about the convergence of $\sigma_m^\theta f$ as follows.

Corollary 3.7. With the same assumptions as in Theorem 3.6, if $f\in H_A^{\vec{p}}(\mathbb{R}^n)$, then $\{\sigma_m^\theta f\}_{m\in\mathbb{N}}$ converges almost everywhere as well as in the $L^{\vec{p}}(\mathbb{R}^n)$-norm as $m\to\infty$.

Proof. Let h be a continuous function with compact support on $\mathbb{R}^n$. Then, by (3.6), we know that, for any $m\in\mathbb{N}$ and $x\in\mathbb{R}^n$,

$$\sigma_m^\theta h(x)=\int_{\mathbb{R}^n} h\left(x-A^{-m}t\right)\widehat{\theta_0}(t)\,dt.$$

Note that, for any $t\in\mathbb{R}^n$, $\lim_{m\to\infty}A^{-m}t=\mathbf{0}$. By (3.5), the fact that h is bounded and the Lebesgue dominated convergence theorem, we find that, for almost every $x\in\mathbb{R}^n$,

$$\lim_{m\to\infty}\sigma_m^\theta h(x)=\int_{\mathbb{R}^n} h(x)\widehat{\theta_0}(t)\,dt=h(x)\theta_0(\mathbf{0})=h(x).$$

This convergence also holds true in the $L^{\vec{p}}(\mathbb{R}^n)$ quasi–norm since $h \in H_A^{\vec{p}}(\mathbb{R}^n)$ which implies $\sigma_*^\theta h \in L^{\vec{p}}(\mathbb{R}^n)$.

On the other hand, from [22, Lemma 9(i)], we deduce that, for any given $f \in H_A^{\vec{p}}(\mathbb{R}^n)$ and $\epsilon \in (0,\infty)$, there exists a continuous function with compact support h such that

$$\|f-h\|_{H_A^{\vec{p}}(\mathbb{R}^n)} < \epsilon. \tag{3.19}$$

For any $K \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$P_K(x) := \sup_{m,k\in[K,\infty)\cap\mathbb{N}} \left|\sigma_m^\theta f(x) - \sigma_k^\theta f(x)\right| \quad \text{and} \quad P(x) := \lim_{K\to\infty} P_K(x).$$

To show Corollary 3.7, we only need to prove that $P = 0$ almost everywhere. To do this, observe that, for any $K \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$P_K(x) \le \sup_{m\in[K,\infty)\cap\mathbb{N}} \left|\sigma_m^\theta (f-h)(x)\right| + \sup_{m,k\in[K,\infty)\cap\mathbb{N}} \left|\sigma_m^\theta h(x) - \sigma_k^\theta h(x)\right| + \sup_{k\in[K,\infty)\cap\mathbb{N}} \left|\sigma_k^\theta (h-f)(x)\right|.$$

Then we have

$$P(x) \le 2\sigma_*^\theta (f-h)(x), \qquad \forall\, x \in \mathbb{R}^n.$$

Combining this, Theorem 3.6 and (3.19), we conclude that

$$\|P\|_{L^{\vec{p}}(\mathbb{R}^n)} \le 2\left\|\sigma_*^\theta (f-h)\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \lesssim \|f-h\|_{H_A^{\vec{p}}(\mathbb{R}^n)} \lesssim \epsilon.$$

Note that $\epsilon \in (0,\infty)$ is arbitrary. Then we immediately obtain $P = 0$ almost everywhere, which completes the proof of Corollary 3.7. □

The following Corollary 3.8 can be easily verified by Theorem 3.6 and an argument same as that used in the proof of [14, Corollary 2.20]; the details are omitted.

Corollary 3.8. With the same assumptions as in Theorem 3.6, if $f \in H_A^{\vec{p}}(\mathbb{R}^n)$ and there exists a subset $I \subset \mathbb{R}^n$ such that the restriction $f|_I \in L^{\vec{q}}(I)$ with $q_- \in [1,\infty)$, then

$$\lim_{m\to\infty} \sigma_m^\theta f(x) = f(x) \qquad \text{for almost every } x \in I \text{ as well as in the } L^{\vec{p}}(I) \text{ quasi-norm.}$$

Remark 3. Note that, if $p_- \in (1,\infty)$, then $H_A^{\vec{p}}(\mathbb{R}^n) = L^{\vec{p}}(\mathbb{R}^n)$ with equivalent quasi-norm (see [3, Proposition 2]). Therefore, Corollary 3.8 further implies the following result: Under the same assumptions as in Theorem 3.6, if $f \in L^{\vec{p}}(\mathbb{R}^n)$ with $p_- \in (1,\infty)$, then

$$\lim_{m\to\infty} \sigma_m^\theta f(x) = f(x) \qquad \text{for almost every } x \in \mathbb{R}^n \text{ as well as in the } L^{\vec{p}}(\mathbb{R}^n) \text{ norm.}$$

4. Two specific summability methods

As special cases, we consider two specific summability methods.

4.1. Bochner–Riesz summation

For any $\alpha \in (0, \infty)$ and $\gamma \in \mathbb{N}$, the *Bochner-Riesz summation* is defined by setting, for any $t \in \mathbb{R}^n$,

$$\theta_0(t) := \begin{cases} (1 - |t|^\gamma)^\alpha & \text{when } |t| \in [0, 1), \\ 0 & \text{when } |t| \in [1, \infty). \end{cases} \tag{4.1}$$

The next lemma comes from [16].

Lemma 4.1. Let θ_0 be as in (4.1). If $\alpha \in (\frac{n-1}{2}, \infty)$, then (3.3) and (3.5) hold true and, for any $\beta \in \mathbb{Z}_+^n$, there exists a positive constant $C_{(\alpha,\beta)}$, depending on α and β, such that, for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\left|\partial^\beta \widehat{\theta_0}(x)\right| \le C_{(\alpha,\beta)} |x|^{-n/2-\alpha-1/2}.$$

By Lemma 4.1 and Theorem 3.6, we have the following conclusion for the Bochner-Riesz summation; the details are omitted.

Theorem 4.2. *Let θ_0 be as in* (4.1) *and $\vec{p} \in (0, \infty)^n$. Assume that*

$$\alpha \in \left(\max\left\{\frac{n-1}{2}, \frac{\ln b}{\ln \lambda_-} - \frac{n+1}{2}\right\}, \infty\right) \quad \text{and} \quad p_- \in \left(\frac{\ln b}{\ln \lambda_-(n/2 + \alpha + 1/2)}, \infty\right).$$

Then, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$,

$$\left\|\sigma_*^\theta f\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \le C_{(p_-,p_+)} \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)},$$

where the positive constant $C_{(p_-,p_+)}$, with p_- and p_+ as in (2.4)*, is independent of f.*

Remark 4. Let θ_0 be as in (4.1). Then, in this special case, the corresponding conclusions in Corollaries 3.7 and 3.8 hold true as well.

4.2. Weierstrass summation

The *Weierstrass summation* is defined by setting, for any $t \in \mathbb{R}^n$,

$$\theta_0(t) := e^{-|t|^2/2}. \tag{4.2}$$

The succeeding Lemma 4.3 is just [14, Lemma 2.27].

Lemma 4.3. Let θ_0 be as in (4.2). Then (3.3) and (3.5) hold true and, for any $\beta \in (1, \infty)$ and $\alpha \in \mathbb{Z}_+^n$, there exists a positive constant $\widetilde{C}_{(\alpha,\beta)}$, depending on α and β, such that, for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\left|\partial^\alpha \widehat{\theta_0}(x)\right| \le \widetilde{C}_{(\alpha,\beta)} |x|^{-\beta}.$$

By Lemma 4.3 and Theorem 3.6, we obtain the following result for the Weierstrass summation; the details are omitted.

Theorem 4.4. *Let θ_0 be as in* (4.2) *and $\vec{p} \in (0, \infty)^n$. Then, for any $f \in H_A^{\vec{p}}(\mathbb{R}^n)$,*

$$\left\|\sigma_*^\theta f\right\|_{L^{\vec{p}}(\mathbb{R}^n)} \le \widetilde{C}_{(p_-,p_+)} \|f\|_{H_A^{\vec{p}}(\mathbb{R}^n)},$$

where the positive constant $\widetilde{C}_{(p_-,p_+)}$, with p_- and p_+ as in (2.4)*, is independent of f. Moreover, the corresponding conclusions in Corollaries 3.7 and 3.8 hold true as well.*

Acknowledgments

The author would like to thank the referees for their careful reading and helpful comments which indeed improved the presentation of this article.

Conflict of interest

The author declares that there is no conflict of interests in this manuscript.

References

1. M. Bownik, *Anisotropic Hardy Spaces and Wavelets*, American Mathematical Society, Rhode Island, 2003. https://doi.org/http://dx.doi.org/10.1090/memo/0781
2. A. P. Calderón, A. Torchinsky, Parabolic maximal functions associated with a distribution, *Adv. Math.*, **16** (1975), 1–64. https://doi.org/10.1016/0001-8708(75)90099-7
3. L. Huang, J. Liu, D. Yang, W. Yuan, Real-variable characterizations of new anisotropic mixed-norm Hardy spaces, *Commun. Pure Appl. Anal.*, **19** (2020), 3033–3082. https://doi.org/10.3934/cpaa.2020132
4. T. Chen, W. Sun, Iterated and mixed weak norms with applications to geometric inequalities, *J. Geom. Anal.*, **30** (2020), 4268–4323. https://doi.org/10.1007/s12220-019-00243-x
5. T. Chen, W. Sun, Extension of multilinear fractional integral operators to linear operators on mixed-norm Lebesgue spaces, *Math. Ann.*, **379** (2021), 1089–1172. https://doi.org/10.1007/s00208-020-02105-2
6. T. Chen, W. Sun, Hardy–Littlewood–Sobolev inequality on mixed-norm Lebesgue spaces, *J. Geom. Anal.*, **32** (2022), 43. https://doi.org/10.1007/s12220-021-00855-2
7. G. Cleanthous, A. G. Georgiadis, Mixed-norm α-modulation spaces, *Trans. Amer. Math. Soc.*, **373** (2020), 3323–3356. https://doi.org/10.1090/tran/8023
8. G. Cleanthous, A. G. Georgiadis, M. Nielsen, Molecular decomposition of anisotropic homogeneous mixed-norm spaces with applications to the boundedness of operators, *Appl. Comput. Harmon. Anal.*, **47** (2019), 447–480. https://doi.org/10.1016/j.acha.2017.10.001
9. J. Johnsen, S. Munch Hansen, W. Sickel, Characterisation by local means of anisotropic Lizorkin–Triebel spaces with mixed norms, *Z. Anal. Anwend.*, **32** (2013), 257–277. https://doi.org/10.4171/ZAA/1484
10. J. Johnsen, S. Munch Hansen, W. Sickel, Anisotropic Lizorkin–Triebel spaces with mixed norms–traces on smooth boundaries, *Math. Nachr.*, **288** (2015), 1327–1359. https://doi.org/10.1002/mana.201300313
11. T. Nogayama, T. Ono, D. Salim, Y. Sawano, Atomic decomposition for mixed Morrey spaces, *J. Geom. Anal.*, **31** (2021), 9338–9365. https://doi.org/10.1007/s12220-020-00513-z
12. E. M. Stein, M. H. Taibleson, G. Weiss, Weak type estimates for maximal operators on certain H^p classes, *Rend. Circ. Mat. Palermo*, **1** (1981), 81–97.

13. J. Liu, F. Weisz, D. Yang, W. Yuan, Variable anisotropic Hardy spaces and their applications, *Taiwanese J. Math.*, **22** (2018), 1173–1216. https://doi.org/10.11650/tjm/171101

14. J. Liu, F. Weisz, D. Yang, W. Yuan, Littlewood-Paley and finite atomic characterizations of anisotropic variable Hardy-Lorentz spaces and their applications, *J. Fourier Anal. Appl.*, **25** (2019), 874–922. https://doi.org/10.1007/s00041-018-9609-3

15. F. Weisz, *Summability of Multi-dimensional Fourier Series and Hardy Spaces*, Dordrecht: Kluwer Academic Publishers, 2002. https://doi.org/10.1007/978-94-017-3183-6

16. F. Weisz, *Convergence and Summability of Fourier Transforms and Hardy Spaces*, Basel, Birkhäuser, 2017.

17. F. Weisz, Summability of Fourier transforms in variable Hardy and Hardy–Lorentz spaces, *Jaen J. Approx.*, **10** (2018), 101–131.

18. F. Weisz, Summability in mixed-norm Hardy spaces, *Ann. Univ. Sci. Budapest. Sect. Comput.*, **48** (2018), 233–246.

19. A. Benedek, R. Panzone, The space L^p, with mixed norm, *Duke Math. J.*, **28** (1961), 301–324. https://doi.org/10.1215/S0012-7094-61-02828-9

20. G. Cleanthous, A. G. Georgiadis, M. Nielsen, Anisotropic mixed-norm Hardy spaces, *J. Geom. Anal.*, **27** (2017), 2758–2787. https://doi.org/10.1007/s12220-017-9781-8

21. F. Weisz, Summability of multi-dimensional trigonometric Fourier series, *Surv. Approx. Theory*, **7** (2012), 1–179.

22. J. Liu, L. Huang, C. Yue, Molecular characterizations of anisotropic mixed-norm Hardy spaces and their applications, *Mathematics*, **9** (2021), 2216. https://doi.org/10.3390/math9182216

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)