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Research article

Blowup and MLUH stability of time-space fractional reaction-diffusion equations

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Abstract: In this paper, we consider a class of nonlinear time-space fractional reaction-diffusion equations by transforming the time-space fractional reaction-diffusion equations into an abstract evolution equations in a fractional Sobolev space. Based on operator semigroup theory, the local uniqueness of mild solutions to the reaction-diffusion equations is obtained under the assumption that nonlinear function is locally Lipschitz continuous. On this basis, a blowup alternative result for unique saturated mild solutions is obtained. We further verify the Mittag-Leffler-Ulam-Hyers stability of the nonlinear time-space fractional reaction-diffusion equations.

Keywords: time-space fractional reaction-diffusion equations; Sobolev space; saturated mild solutions; local uniqueness; blowup alternative result

1. Introduction and main results

Fractional derivatives are integro-differential operators which generalize integer-order differential and integral calculus. They can describe the property of memory and heredity of various materials and processes compared with integer-order derivatives. In recent years, many scholars are committed to the research of time-fractional or space-fractional partial differential equations, see [1–7]. On the other hand, fractional diffusion models are employed for some engineering problems [8,9] with power-law memory in time and physical models considering memory effects [10–12]. There are numerous works devoted to fractional diffusion equations. We only list several of the numerous papers on the analysis for fractional diffusion equations. In [13], the author discussed well-posedness of semilinear time-fractional diffusion equations using embedding relation among spaces. Eidelman and Kochubei [14] constructed fundamental solutions of time fractional evolution equations. In [15], the author established $L^r - L^q$ estimates and weighted estimates of fundamental solutions, and obtained existence and uniqueness of mild solutions of the Keller-Segel type time-space fractional diffusion equation. In [16], Wang and Zhou introduced and discussed four types special data dependences for a class of fractional evolution equations.

In this paper, we focus on the following nonlinear time-space fractional reaction-diffusion equations with fractional Laplacian

$$\begin{cases} {}^c\mathbb{D}_t^\alpha u(x,t) + (-\Delta)^\beta u(x,t) = f(x,t,u(x,t)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a bounded open domain with smooth boundary $\partial\Omega$; $\alpha, \beta \in (0,1)$ and ${}^c\mathbb{D}_t^\alpha \cdot$ is the Caputo time-fractional derivative of order α defined as

$${}^c\mathbb{D}_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\, ds, \quad t > 0,$$

$\Gamma(\cdot)$ is the Gamma function; The spectral fractional Laplacian could be defined as

$$(-\Delta)^\beta u := \sum_{j=1}^\infty \lambda_j^\beta u_j \phi_j, \quad u_j := \int_\Omega u \phi_j\, dx, \quad j \in \mathbb{N}; \tag{1.2}$$

$f : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is the nonlinear function and the continuous initial data $u_0 : \Omega \to \mathbb{R}$. We obtain the local uniqueness of mild solutions, the blowup alternative result for saturated mild solutions and Mittag-Leffler-Ulam-Hyers stability.

The main results of this paper are as following:

Theorem 1.1. *Assume that nonlinear function $f : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies locally Lipschitz condition about the third variable, then there exists a constant $h > 0$ such that Eq (1.1) has a unique mild solution on $\Omega \times [0,h]$.*

Theorem 1.2. *Assume that all assumptions of Theorem 1.1 are satisfied, then the unique mild solution can be extended to a large time interval $[0,h^*]$ for some $h^* > h$ such that Eq (1.1) has a unique mild solution on $\Omega \times [0,h^*]$.*

Theorem 1.3. *Assume that all assumptions of Theorem 1.1 are satisfied, then there exists a maximal existence interval $[0,T_{\max})$ such that Eq (1.1) has a unique saturated mild solution $u \in C(\Omega \times [0,T_{\max}),\mathbb{R})$. Furthermore, if $T_{\max} < \infty$, then $\limsup_{t \to T_{\max}^-} \|u(t)\|_{\mathbb{H}^\beta(\Omega)} = \infty$, where $\mathbb{H}^\beta(\Omega)$ is Sobolev space introduced in the following section.*

Theorem 1.4. *Assume that all assumptions of Theorem 1.1 are satisfied, then there exists a constant $h > 0$ such that Eq (1.1) is Mittag-Leffler-Ulam-Hyers stable on $\Omega \times [0,h]$.*

2. Preliminaries

Throughout of this paper, we adopt spectral fractional Laplacian $(-\Delta)^\beta$ defined by (1.2). For each $\beta \in (0,1)$, we define the fractional Sobolev space as

$$\mathbb{H}^\beta(\Omega) := \left\{ u = \sum_{j=1}^\infty u_j \phi_j \in L^2(\Omega) :\ \|u\|^2_{\mathbb{H}^\beta(\Omega)} := \sum_{j=1}^\infty \lambda_j^\beta u_j^2 < \infty \right\}, \quad u_j = \int_\Omega u \phi_j\, dx,$$

where λ_j are the eigenvalues of $-\Delta$ with zero Dirichlet boundary conditions on Ω, ϕ_j are eigenfunctions with respect to λ_j, (λ_j, ϕ_j) is the eigen pair of $-\Delta$, for the details one can see [17]. Denote $C([0,\infty), \mathbb{H}^\beta(\Omega))$ the Banach space of all continuous $\mathbb{H}^\beta(\Omega)$-value functions on $[0,\infty)$ with norm $\|u\|_C := \sup_{t\in[0,\infty)} \|u(t)\|_{\mathbb{H}^\beta(\Omega)}$ and $A^\beta u = (-\Delta)^\beta u$. We know from [18] that $-A^\beta$ generates a Feller semigroup $T_\beta(t)(t \geqslant 0)$.

We now define two operators $\mathcal{T}_{\alpha,\beta}(t)(t \geqslant 0)$ and $\mathcal{S}_{\alpha,\beta}(t)(t \geqslant 0)$ as follows

$$\mathcal{T}_{\alpha,\beta}(t)u = \int_0^\infty h_\alpha(s) T_\beta(t^\alpha s)u\, ds, \qquad \mathcal{S}_{\alpha,\beta}(t)u = \alpha \int_0^\infty s h_\alpha(s) T_\beta(t^\alpha s)u\, ds, \quad u \in \mathbb{H}^\beta(\Omega),$$

where $h_\alpha(s) = \frac{1}{\pi\alpha}\sum_{n=1}^{\infty}(-s)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha)$ is a function of Wright type [19] defined on $(0,\infty)$ which satisfies $h_\alpha(s) \geqslant 0,\ s \in (0,\infty)$, $\int_0^\infty h_\alpha(s)ds = 1$.

Lemma 2.1. *The operators $\mathcal{T}_{\alpha,\beta}(t)(t \geqslant 0)$ and $\mathcal{S}_{\alpha,\beta}(t)(t \geqslant 0)$ have the following properties [18]:*
(i) The operators $\mathcal{T}_{\alpha,\beta}(t)(t \geqslant 0)$ and $\mathcal{S}_{\alpha,\beta}(t)(t \geqslant 0)$ are strongly continuous on $\mathbb{H}^\beta(\Omega)$;
(ii) $\|\mathcal{T}_{\alpha,\beta}(t)u\|_{\mathbb{H}^\beta(\Omega)} \leqslant \|u\|_{\mathbb{H}^\beta(\Omega)}$, $\|\mathcal{S}_{\alpha,\beta}(t)u\|_{\mathbb{H}^\beta(\Omega)} \leqslant \frac{1}{\Gamma(\alpha)}\|u\|_{\mathbb{H}^\beta(\Omega)}$;
(iii) $\mathcal{T}_{\alpha,\beta}(t)$ and $\mathcal{S}_{\alpha,\beta}(t)$ are compact operators for every $t > 0$.

Lemma 2.2. *The Gamma function $\Gamma(z) = \int_0^\infty e^{-s}s^{z-1}\, ds$, $z > 0$ and Beta function $\mathrm{B}(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1}\, ds$, $p,q > 0$ have the following equality [20]:*

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}; \quad \int_a^b (s-a)^{p-1}(b-s)^{q-1}\, ds = (b-a)^{p+q-1}\mathrm{B}(p,q), \quad b > a.$$

Lemma 2.3. (**Stirling′s Formula**) *[21] For $x \to \infty$ we have*

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}(1+o(1)).$$

Lemma 2.4. *Suppose that $a(t)$ is a nonnegative [16], nondecreasing function locally integrable on $[0,\infty)$ and $h(t)$ is a nonnegative, nondecreasing continuous function defined on $[0,\infty)$, $h(t) \leqslant \widetilde{M}$(constant), and suppose $u(t)$ is nonnegative and locally integrable on $[0,\infty)$ with*

$$u(t) \leqslant a(t) + h(t)\int_0^t (t-s)^{\alpha-1}u(s)\, ds, \quad t \in [0,\infty).$$

Then $u(t) \leqslant a(t)E_\alpha[h(t)\Gamma(\alpha)t^\alpha]$, where E_α is the Mittag-Leffer function defined by $E_\alpha[z] = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}$, $z \in \mathbb{C}$.

Let $u(t) = u(\cdot,t)$, $f(t,u(t)) = f(\cdot,t,u(\cdot,t))$, $u_0 = u_0(\cdot)$. Then the Eq (1.1) can be rewritten abstract form of fractional evolution equation in $C([0,\infty), \mathbb{H}^\beta(\Omega))$ as

$$\begin{cases} {}^c\mathbb{D}_t^\alpha u(t) + A^\beta u(t) = f(t,u(t)), & t > 0, \\ u(0) = u_0. \end{cases} \tag{2.1}$$

If the nonlinear function $f : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies locally Lipschitz condition about the third variable with Lipschitz constant L, one can derive

$$\|f(t,u(t)) - f(t,v(t))\|_{\mathbb{H}^\beta(\Omega)} \leqslant \left(\sum_{j=1}^{\infty}\lambda_j^\beta\left(\int_\Omega |f(t,u(t)) - f(t,v(t))|\phi_j\, dx\right)^2\right)^{\frac{1}{2}}$$

$$\leqslant \left(\sum_{j=1}^{\infty} \lambda_j^{\beta} \left(\int_{\Omega} L|u(t) - v(t)|\phi_j \, dx \right)^2 \right)^{\frac{1}{2}}$$
$$= L\|u(t) - v(t)\|_{\mathbb{H}^{\beta}(\Omega)}. \tag{2.2}$$

3. Local existence and uniqueness, continuation and Blowup alternative results of solutions

Definition 3.1. *A function $u \in C([0, \infty), \mathbb{H}^{\beta}(\Omega))$ is called a mild solution of* (2.1) *if it satisfies*

$$u(t) = \mathcal{T}_{\alpha,\beta}(t)u_0 + \int_0^t (t - s)^{\alpha-1} \mathcal{S}_{\alpha,\beta}(t - s) f(s, u(s)) \, ds.$$

Proof of Theorem 1.1. It follows discussions in Section 2 that Eq (1.1) can be transformed into the abstract evolution Eq (2.1) in $C([0, \infty), \mathbb{H}^{\beta}(\Omega))$. We now prove the local existence and uniqueness of the mild solution to the evolution Eq (2.1). Assume that nonlinear function f is continuous in $\Theta = \{(t, u) : 0 \leqslant t \leqslant a, \|u(t) - u_0\|_{\mathbb{H}^{\beta}(\Omega)} \leqslant b\}$ for $a > 0$ and $b > 0$, then there exists a unique mild solution to the evolution Eq (2.1) on $[0, h]$, where

$$b = 2\|u_0\|_{\mathbb{H}^{\beta}(\Omega)} + 1, \quad h = \min\left\{a, \left(\frac{\Gamma(\alpha + 1)}{M}\right)^{\frac{1}{\alpha}}\right\}, \quad M = \sup_{(t,u)\in\Theta} \|f(t, u(t))\|_{\mathbb{H}^{\beta}(\Omega)}.$$

Define $\mathbf{P} : C([0, h], \mathbb{H}^{\beta}(\Omega)) \to C([0, h], \mathbb{H}^{\beta}(\Omega))$ as

$$\mathbf{P}u(t) = \mathcal{T}_{\alpha,\beta}(t)u_0 + \int_0^t (t - s)^{\alpha-1} \mathcal{S}_{\alpha,\beta}(t - s) f(s, u(s)) \, ds. \tag{3.1}$$

From Definition 3.1, the mild solution to (2.1) on $[0, h]$ is equivalent to the fixed point of operator $\mathbf{P}$ defined by (3.1). Set $\Lambda = \{u \in C([0, h], \mathbb{H}^{\beta}(\Omega)) : \|u(t) - u_0\|_{\mathbb{H}^{\beta}(\Omega)} \leqslant b, \ t \in [0, h]\}$ is a nonempty, convex and closed subset in $C([0, h], \mathbb{H}^{\beta}(\Omega))$. Now we show the operator $\mathbf{P}$ has a fixed point in Λ by applying power compression mapping principle.

Step I. $\mathbf{P} : \Lambda \to \Lambda$. For any $u \in \Lambda$, $t \in [0, h]$, by (3.1) and Lemma 2.1 we have

$$\begin{aligned}
\|\mathbf{P}u(t) - u_0\|_{\mathbb{H}^{\beta}(\Omega)} &= \left\| \mathcal{T}_{\alpha,\beta}(t)u_0 - u_0 + \int_0^t (t - s)^{\alpha-1} \mathcal{S}_{\alpha,\beta}(t - s) f(s, u(s)) \, ds \right\|_{\mathbb{H}^{\beta}(\Omega)} \\
&\leqslant \|\mathcal{T}_{\alpha,\beta}(t)u_0\|_{\mathbb{H}^{\beta}(\Omega)} + \|u_0\|_{\mathbb{H}^{\beta}(\Omega)} + \left\| \int_0^t (t - s)^{\alpha-1} \mathcal{S}_{\alpha,\beta}(t - s) f(s, u(s)) \, ds \right\|_{\mathbb{H}^{\beta}(\Omega)} \\
&\leqslant 2\|u_0\|_{\mathbb{H}^{\beta}(\Omega)} + \frac{Mt^{\alpha}}{\Gamma(\alpha + 1)} \leqslant b.
\end{aligned}$$

Then, we get that $\mathbf{P} : \Lambda \to \Lambda$.

Step II. $\mathbf{P} : \Lambda \to \Lambda$ *is a power compression mapping*. For any $u, v \in \Lambda$, by (2.2), (3.1) and Lemma 2.1, we get

$$\begin{aligned}
\|\mathbf{P}u(t) - \mathbf{P}v(t)\|_{\mathbb{H}^{\beta}(\Omega)} &= \left\| \int_0^t (t - s)^{\alpha-1} \mathcal{S}_{\alpha,\beta}(t - s)[f(s, u(s)) - f(s, v(s))] \, ds \right\|_{\mathbb{H}^{\beta}(\Omega)} \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \|f(s, u(s)) - f(s, v(s)\|_{\mathbb{H}^{\beta}(\Omega)} \, ds
\end{aligned}$$

$$\leqslant \frac{Lt^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|_{C}. \tag{3.2}$$

By (2.2), (3.1), (3.2), Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
\|\mathbf{P}^2u(t)-\mathbf{P}^2v(t)\|_{\mathbb{H}^{\beta}(\Omega)} &= \left\|\int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)[f(s,\mathbf{P}u(s))-f(s,\mathbf{P}v(s))]\,ds\right\|_{\mathbb{H}^{\beta}(\Omega)}\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,\mathbf{P}u(s))-f(s,\mathbf{P}v(s)\|_{\mathbb{H}^{\beta}(\Omega)}\,ds\\
&\leqslant \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{Ls^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|_C\,ds\\
&= \frac{L^2}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}s^{\alpha}\,ds\|u-v\|_C\\
&= \frac{L^2t^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\mathrm{B}(\alpha+1,\alpha)\|u-v\|_C\\
&= \frac{L^2t^{2\alpha}}{\Gamma(2\alpha+1)}\|u-v\|_C.
\end{aligned}$$

Suppose $n=k-1$ we have

$$\|\mathbf{P}^{k-1}u(t)-\mathbf{P}^{k-1}v(t)\|_{\mathbb{H}^{\beta}(\Omega)} \leqslant \frac{(Lt^{\alpha})^{k-1}}{\Gamma((k-1)\alpha+1)}\|u-v\|_C. \tag{3.3}$$

Let $n=k$, by (2.2), (3.1), (3.3), Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
\|\mathbf{P}^ku(t)-\mathbf{P}^kv(t)\|_{\mathbb{H}^{\beta}(\Omega)} &= \left\|\int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)[f(s,\mathbf{P}^ku(s))-f(s,\mathbf{P}^kv(s))]\,ds\right\|_{\mathbb{H}^{\beta}(\Omega)}\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,\mathbf{P}^{k-1}u(s))-f(s,\mathbf{P}^{k-1}v(s)\|_{\mathbb{H}^{\beta}(\Omega)}\,ds\\
&\leqslant \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{(Ls^{\alpha})^{k-1}}{\Gamma((k-1)\alpha+1)}\|u-v\|_C\,ds\\
&= \frac{L^k}{\Gamma(\alpha)\Gamma((k-1)\alpha+1)}\int_0^t (t-s)^{\alpha-1}s^{(k-1)\alpha}\,ds\|u-v\|_C\\
&= \frac{L^kt^{k\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\mathrm{B}((k-1)\alpha+1,\alpha)\|u-v\|_C\\
&= \frac{L^kt^{k\alpha}}{\Gamma(k\alpha+1)}\|u-v\|_C.
\end{aligned}$$

Therefore, we have

$$\|\mathbf{P}^nu-\mathbf{P}^nv\|_C \leqslant \frac{(Lh^{\alpha})^n}{\Gamma(n\alpha+1)}\|u-v\|_C \tag{3.4}$$

for any $n\in\mathbb{N}^+$ and $t\in[0,h]$ by mathematical induction. By Lemma 2.3 we get

$$\Gamma(n\alpha+1)=\left(\frac{n\alpha}{e}\right)^{n\alpha}\sqrt{2\pi n\alpha}(1+o(1)),\quad n\to\infty,$$

which implies

$$\frac{(Lh^{\alpha})^n}{\Gamma(n\alpha+1)} \leqslant \frac{(Lh^{\alpha})^n}{\left(\frac{n\alpha}{e}\right)^{n\alpha}\sqrt{2\pi n\alpha}} \to 0 \quad \text{as} \quad n\to\infty.$$

Hence, there exists $m \in \mathbb{N}$ such that

$$\frac{(Lh^{\alpha})^m}{\Gamma(m\alpha+1)} < 1. \tag{3.5}$$

Combining (3.4) and (3.5) we have

$$\|\mathbf{P}^m u - \mathbf{P}^m v\|_C < \|u-v\|_C,$$

which means that the operator $\mathbf{P}^m$ is compressive and $\mathbf{P}$ is a power compression operator. Therefore $\mathbf{P}$ has unique fixed point $u \in \Lambda$ by power compression mapping principle, the fixed point is the unique mild solution of (2.1) on $[0,h]$. Hence, Eq (1.1) has unique mild solution $u \in C(\Omega \times [0,h], \mathbb{R})$. This completes the proof of Theorem 1.1. □

Definition 3.2. *A function u^* is a continuation mild solution of the unique mild solution $u \in C([0,h], \mathbb{H}^{\beta}(\Omega))$ to* (2.1) *on $(0,h^*]$ for some $h^* > h$ if it satisfies*

$$\begin{cases} u^*(t) = u(t), \quad t \in [0,h], \\ u^* \in C([h,h^*], \mathbb{H}^{\beta}(\Omega)) \text{ is a mild solution of (2.1) for all } t \in [h,h^*]. \end{cases}$$

Proof of Theorem 1.2. Let $u \in C([0,h], \mathbb{H}^{\beta}(\Omega))$ be the unique mild solution of (2.1), h is the constant defined in Theorem 1.1. Fix $b^* = 2\|u_0\|_{\mathbb{H}^{\beta}(\Omega)} + 2$, $M^* = \sup\{\|f(t,u^*(t))\|_{\mathbb{H}^{\beta}(\Omega)} : \|u(t)\|_{\mathbb{H}^{\beta}(\Omega)} \leqslant b^*, h \leqslant t \leqslant h+a^*\}$ for $a^* > 0$, we shall prove that $u^* : [0,h^*] \to \mathbb{H}^{\beta}(\Omega)$ is a mild solution of (2.1) for $h^* > h$. Set $\Lambda^* = \{u^* \in C([0,h^*], \mathbb{H}^{\beta}(\Omega)) : \|u(t)-u(h)\|_{C([h,h^*],\mathbb{H}^{\beta}(\Omega))} \leqslant b^*,\ t \in [h,h^*]; u^*(t) = u(t),\ t \in [0,h]\}$, where

$$h^* = \min\left\{a^*, \left(\frac{\Gamma(\alpha+1)}{M^*}\right)^{\frac{1}{\alpha}}, \left(\frac{\Gamma(\alpha+1)}{L}\right)^{\frac{1}{\alpha}}\right\}.$$

Define $\mathbf{P} : C([0,h^*], \mathbb{H}^{\beta}(\Omega)) \to C([0,h^*], \mathbb{H}^{\beta}(\Omega))$ as (3.1). Now we show the operator $\mathbf{P}$ has a fixed point in Λ^* via Banach fixed point theorem.

Step I. $\mathbf{P} : \Lambda^* \to \Lambda^*$. Let $u^* \in \Lambda^*$, if $t \in [0,h]$, from the proof of Theorem 1.1 we know equation (2.1) has unique mild solution and $u^*(t) = u(t)$. Thus $\mathbf{P}u^*(t) = \mathbf{P}u(t) = u(t)$ for all $t \in [0,h]$. Now we just consider $t \in [h,h^*]$, thus we have

$$\begin{aligned} \|\mathbf{P}u^*(t) - u^*(h)\|_{\mathbb{H}^{\beta}(\Omega)} &\leqslant \|\mathcal{T}_{\alpha,\beta}(t)u_0 - \mathcal{T}_{\alpha,\beta}(h)u_0\|_{\mathbb{H}^{\beta}(\Omega)} + \Big\| \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)f(s,u^*(s))\,ds \\ &\quad - \int_0^h (h-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(h-s)f(s,u^*(s))\,ds\Big\|_{\mathbb{H}^{\beta}(\Omega)} \\ &\leqslant 2\|u_0\|_{\mathbb{H}^{\beta}(\Omega)} + \frac{M^*t^{\alpha}}{\Gamma(\alpha+1)} + \frac{M^*h^{\alpha}}{\Gamma(\alpha+1)} \\ &\leqslant 2\|u_0\|_{\mathbb{H}^{\beta}(\Omega)} + \frac{2M^*t^{\alpha}}{\Gamma(\alpha+1)} \leqslant b^*. \end{aligned}$$

Step II. **P** *is a compression on* Λ^*. Let $u^*, v^* \in \Lambda^*$, and we have that for $t \in [0, h^*]$,

$$\begin{aligned}\|\mathbf{P}u^*(t) - \mathbf{P}v^*(t)\|_{\mathbb{H}^\beta(\Omega)} &= \left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_{\alpha,\beta}(t-s)[f(s,u^*(s)) - f(s,v^*(s))]\, ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u^*(s)) - f(s,v^*(s)\|_{\mathbb{H}^\beta(\Omega)}\, ds \\ &\leqslant \frac{Lt^\alpha}{\Gamma(\alpha+1)} \|u^* - v^*\|_{C([0,h^*],\mathbb{H}^\beta(\Omega))} \\ &< \frac{L(h^*)^\alpha}{\Gamma(\alpha+1)} \|u^* - v^*\|_{C([0,h^*],\mathbb{H}^\beta(\Omega))}.\end{aligned}$$

Then,

$$\|\mathbf{P}u^* - \mathbf{P}v^*\|_{C([0,h^*],\mathbb{H}^\beta(\Omega))} < \|u^* - v^*\|_{C([0,h^*],\mathbb{H}^\beta(\Omega))}.$$

This implies the operator **P** is compressive. By the Banach fixed point theorem it follows there exists a unique fixed point u^* of **P** in Λ^*, which is a continuation of u. The fixed point is the unique mild solution of Eq (2.1) on $[0, h^*]$. Therefore, Eq (1.1) has unique mild solution u on $\Omega \times [0, h^*]$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Repeating the methods and steps in the proof of Theorem 1.2, one can obtain that Eq (1.1) exists unique saturated mild solution on maximal interval $\Omega \times [0, T_{\max})$. Let $T_{\max} := \sup\{h > 0 : \text{the unique mild solution exits on } (0,h]\}$ and $u_0 \in \mathbb{H}^\beta(\Omega)$. Assume that $T_{\max} < \infty$ and for some $b_0 > 0$, $M_0 = \sup\{\|f(t,u(t))\|_{\mathbb{H}^\beta(\Omega)} : \|u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant b_0,\ 0 \leqslant t \leqslant T_{\max}\}$. Suppose there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0, T_{\max})$ such that $t_n \to T_{\max}$ and $\{u(t_n)\}_{n\in\mathbb{N}} \subset \mathbb{H}^\beta(\Omega)$. Let us demonstrate that $\{u(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}^\beta(\Omega)$. Indeed, for any $\epsilon > 0$, fix $N \in \mathbb{N}$ such that for all $n, m > N$, $0 < t_n < t_m < T_{\max}$, we get

$$\begin{aligned}\|u(t_m) - u(t_n)\|_{\mathbb{H}^\beta(\Omega)} &\leqslant \|\mathcal{T}_{\alpha,\beta}(t_m)u_0 - \mathcal{T}_{\alpha,\beta}(t_n)u_0\|_{\mathbb{H}^\beta(\Omega)} \\ &\quad + \left\| \int_{t_n}^{t_m} (t_m - s)^{\alpha-1} \mathcal{S}_{\alpha,\beta}(t_m - s) f(s,u(s))\, ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ &\quad + \left\| \int_0^{t_n} ((t_m - s)^{\alpha-1} - (t_n - s)^{\alpha-1}) \mathcal{S}_{\alpha,\beta}(t_m - s) f(s,u(s))\, ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ &\quad + \left\| \int_0^{t_n} (t_n - s)^{\alpha-1} (\mathcal{S}_{\alpha,\beta}(t_m - s) - \mathcal{S}_{\alpha,\beta}(t_n - s)) f(s,u(s))\, ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ &=: \|I_1\|_{\mathbb{H}^\beta(\Omega)} + \|I_2\|_{\mathbb{H}^\beta(\Omega)} + \|I_3\|_{\mathbb{H}^\beta(\Omega)} + \|I_4\|_{\mathbb{H}^\beta(\Omega)}.\end{aligned}$$

We choose $N := N(\epsilon) \in \mathbb{N}^*$ with $m \geqslant n \geqslant N$ such that $t_m - t_n$ small enough following the sequence $\{t_n\}_{n\in\mathbb{N}^*}$ is convergent. By Lemma 2.1,

$$\begin{aligned}&\|I_1\|_{\mathbb{H}^\beta(\Omega)} < \frac{\epsilon}{4}; \quad \|I_2\|_{\mathbb{H}^\beta(\Omega)} \leqslant \frac{M_0}{\Gamma(\alpha+1)} (t_m - t_n)^\alpha < \frac{\epsilon}{4}; \\ &\|I_3\|_{\mathbb{H}^\beta(\Omega)} \leqslant \frac{M_0}{\Gamma(\alpha+1)} (t_n^\alpha - t_m^\alpha + (t_m - t_n)^\alpha) \leqslant \frac{2M_0}{\Gamma(\alpha+1)} (t_m - t_n)^\alpha < \frac{\epsilon}{4}.\end{aligned}$$

Clearly see $\|I_4\|_{\mathbb{H}^\beta(\Omega)} = 0$ for $t_n = 0$, $0 < t_m < T_{\max}$. For $t_n > 0$ and $0 < \epsilon < t_n$, by Lemma 2.1 we have

$$\begin{aligned}\|I_4\|_{\mathbb{H}^\beta(\Omega)} &\leqslant \int_0^{t_n-\epsilon} (t_n - s)^{\alpha-1} \|\mathcal{S}_{\alpha,\beta}(t_m - s) - \mathcal{S}_{\alpha,\beta}(t_n - s)\|_{\mathbb{H}^\beta(\Omega)} \cdot \|f(s, u(s))\|_{\mathbb{H}^\beta(\Omega)}\, ds \\ &\quad + \int_{t_n-\epsilon}^{t_n} (t_n - s)^{\alpha-1} \|\mathcal{S}_{\alpha,\beta}(t_m - s) - \mathcal{S}_{\alpha,\beta}(t_n - s)\|_{\mathbb{H}^\beta(\Omega)} \cdot \|f(s, u(s))\|_{\mathbb{H}^\beta(\Omega)}\, ds \\ &\leqslant \sup_{s\in[0,t_n-\epsilon]} \|\mathcal{S}_{\alpha,\beta}(t_m - s) - \mathcal{S}_{\alpha,\beta}(t_n - s)\|_{\mathbb{H}^\beta(\Omega)} M_0(t_n^\alpha - \epsilon^\alpha) + \frac{2M_0\epsilon^\alpha}{\Gamma(\alpha+1)} < \frac{\epsilon}{4}.\end{aligned}$$

Therefore, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|u(t_m) - u(t_n)\|_{\mathbb{H}^\beta(\Omega)} < \epsilon$ when $m, n \geqslant N$. We arrive at that $\{u(t_n)\}_{t\in\mathbb{N}} \subset \mathbb{H}^\beta(\Omega)$ is a Cauchy sequences and for any $\{t_n\}_{n\in\mathbb{N}^*}$ the $\lim_{t\to T_{\max}^-} \|u(t)\|_{\mathbb{H}^\beta(\Omega)} < \infty$ exists. From result of Theorem 1.2 we know that the unique mild solution can be extended to larger interval. This means that u can be continued beyond $T_{\max}$, and this contradict $u \in C([0, T_{\max}), \mathbb{H}^\beta(\Omega))$ is a saturated mild solution. Therefore, we arrive at if $T_{\max} < \infty$ then $\limsup_{t\to T_{\max}^-} \|u(t)\|_{\mathbb{H}^\beta(\Omega)} = \infty$. This complete the proof of Theorem 1.3. □

4. Mittag-Leffler-Ulam-Hyers stability

In this section, we consider the Mittag-Leffler-Ulam-Hyers stability of Eq (1.1). It follows discussions in Section 2 that Eq (1.1) can be transformed into the abstract evolution Eq (2.1) in $C([0,\infty), \mathbb{H}^\beta(\Omega))$, we now verify the stability of Eq (2.1) on $[0, h]$, h is the constant defined in Theorem 1.1. Let $\varepsilon > 0$, we consider the following inequation

$$\|{}^c\mathbb{D}_t^\alpha v(t) + A^\beta v(t) - f(t, v(t))\|_{\mathbb{H}^\beta(\Omega)} \leqslant \varepsilon, \quad t \in [0, h]. \tag{4.1}$$

Definition 4.1. *Eq* (2.1) *is Mittag-Leffler-Ulam-Hyers stable with respect to* E_α*, if there exists a real number* $\delta > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $v \in C^1([0, h], \mathbb{H}^\beta(\Omega))$ *of inequation* (4.1)*, there exists a mild solution* $u \in C([0, h], \mathbb{H}^\beta(\Omega))$ *of Eq* (2.1) *with* $\|v(t) - u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant \delta\varepsilon E_\alpha[t]$*,* $t \in [0, h]$*.*

Remark 4.1. *A function* $v \in C^1([0, h], \mathbb{H}^\beta(\Omega))$ *is a solution of inequation* (4.1) *if and only if there exists a function* $w \in C([0, h], \mathbb{H}^\beta(\Omega))$ *(which depend on v) such that*
(i) $\|w(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant \varepsilon$*, for all* $t \in [0, h]$*;*
(ii) ${}^c\mathbb{D}_t^\alpha u(t) + A^\beta u(t) = f(t, u(t)) + w(t)$*,* $t \in [0, h]$*.*

Remark 4.2. *If* $v \in C^1([0, h], \mathbb{H}^\beta(\Omega))$ *is a solution of inequation* (4.1)*, then v is a solution of the following integral inequation*

$$\left\|v(t) - \mathcal{T}_{\alpha,\beta}(t)v(0) - \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s) f(s, v(s))\, ds\right\|_{\mathbb{H}^\beta(\Omega)} \leqslant \varepsilon \int_0^t (t-s)^{\alpha-1} \|\mathcal{S}_{\alpha,\beta}(t-s)\|_{\mathbb{H}^\beta(\Omega)}\, ds.$$

Proof of Theorem 1.4. Let $v \in C^1([0, h], \mathbb{H}^\beta(\Omega))$ be a solution of the inequation (4.1) and denote by $u \in C([0, h], \mathbb{H}^\beta(\Omega))$ the unique mild solution of the problem

$$\begin{cases} {}^c\mathbb{D}_t^\alpha u(t) + A^\beta u(t) = f(t, u(t)), & t \in [0, h], \\ u(0) = v(0). \end{cases}$$

We have

$$u(t) = \mathcal{T}_{\alpha,\beta}(t)v(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds, \quad t \in [0,h],$$

and by Remark 4.2 we get

$$\begin{aligned}&\left\| v(t) - \mathcal{T}_{\alpha,\beta}(t)v(0) - \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)f(s,v(s))\,ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ &\leqslant \varepsilon \int_0^t (t-s)^{\alpha-1}\|\mathcal{S}_{\alpha,\beta}(t-s)\|_{\mathbb{H}^\beta(\Omega)}\,ds \leqslant \frac{h^\alpha \varepsilon}{\Gamma(\alpha+1)}. \end{aligned} \tag{4.2}$$

It follows from (2.2) and (4.2) that

$$\begin{aligned}\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)} =& \left\| v(t) - \mathcal{T}_{\alpha,\beta}(t)v(0) - \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)f(s,u(s))\,ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ \leqslant& \left\| v(t) - \mathcal{T}_{\alpha,\beta}(t)v(0) - \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)f(s,v(s))\,ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ &+ \left\| \int_0^t (t-s)^{\alpha-1}\mathcal{S}_{\alpha,\beta}(t-s)[f(s,v(s)) - f(s,u(s))]\,ds \right\|_{\mathbb{H}^\beta(\Omega)} \\ \leqslant& \frac{h^\alpha \varepsilon}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|v(s)-u(s)\|_{\mathbb{H}^\beta(\Omega)}\,ds. \end{aligned} \tag{4.3}$$

Applying Lemma 2.4 to inequality (4.3), we get

$$\|v(t)-u(t)\|_{\mathbb{H}^\beta(\Omega)} \leqslant \frac{h^\alpha \varepsilon}{\Gamma(\alpha+1)} E_\alpha[Lt^\alpha].$$

Hence, Eq (2.1) is Mittag-Leffler-Ulam-Hyers stable. This completes the proof of Theorem 1.4.□

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Conflict of interest

The authors declare there is no conflicts of interest.

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