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Research article

On estimates for augmented Hessian type parabolic equations on Riemannian manifolds

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Abstract: The author extends previous results to general classes of equations under weaker assumptions obtained in 2016 by Bao, Dong and Jiao concerning the study of the regularity of solutions for the first initial-boundary value problem for parabolic Hessian equations on Riemannian manifolds.

Keywords: fully nonlinear parabolic equations; *A priori* C^2 estimates; augmented Hessian equations; the first initial-boundary value problem

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M and $\bar{M} := M \cup \partial M$. Define $M_T = M \times (0, T] \subset M \times \mathbb{R}$, $\mathcal{P}M_T = BM_T \cup SM_T$ is the parabolic boundary of M_T with $BM_T = M \times \{0\}$ and $SM_T = \partial M \times [0, T]$. In [1], the authors derived C^2 estimates for solutions of the first initial-boundary value problem of parabolic Hessian equations in the form

$$f(\lambda(\nabla^2 u + \chi(x,t)), -u_t) = \psi(x,t), \tag{1.1}$$

where f is a symmetric smooth function of $n + 1$ variables.

In this paper, we apply an exponential barrier from [2] where Jiang-Trudinger treat the corresponding elliptic problems in $\mathbb{R}^n$ to study (1.1) in the general augmented Hessian form

$$f(\lambda(\nabla^2 u + A(x,t,\nabla u)), -u_t) = \psi(x,t,\nabla u) \tag{1.2}$$

in M_T with boundary condition

$$u = \varphi \text{ on } \mathcal{P}M_T, \tag{1.3}$$

where $\nabla^2 u + A(x,t,\nabla u)$ is called augmented Hessian, ∇u and $\nabla^2 u$ denote the gradient and the Hessian of $u(x,t)$ with respect to $x \in M$ respectively, $u_t = D_t u$ is the derivative of $u(x,t)$ with respect to $t \in [0, T]$,

$A[u] = A(x, t, \nabla u)$ is a $(0, 2)$ tensor on $\overline{M}$ which may depend on $t \in [0, T]$ and ∇u, and

$$\lambda(\nabla^2 u + A[u]) = (\lambda_1, \ldots, \lambda_n)$$

denotes the eigenvalues of $\nabla^2 u + A[u]$ with respect to the metric g.

As in [3], throughout the paper we assume $A[u]$ is smooth on $\overline{M_T}$ for $u \in C^\infty(\overline{M_T})$, $\psi \in C^\infty(T^*\overline{M} \times [0, T])$. We shall write $\psi = \psi(x, t, p)$ for $(x, p) \in T^*\overline{M}$ and $t \in [0, T]$. Note that for fixed $(x, t) \in \overline{M_T}$ and $p \in T_x^*M$,

$$A(x, t, p) : T_x^*M \times T_x^*M \to \mathbb{R}$$

is a symmetric bilinear map. We shall use the notation

$$A^{\xi\eta}(x, t, \cdot) := A(x, t, \cdot)(\xi, \eta), \quad \xi, \eta \in T_x^*M.$$

For a function $v \in C^2(M_T)$, we write $A[v] := A(x, t, \nabla v)$, $A^{\xi\eta}[v] := A^{\xi\eta}(x, t, \nabla v)$ and $\psi[u] := \psi(x, t, \nabla u)$.

There are many different A in conformal geometry, the optimal transportation satisfies, the isometric embedding, reflector design and other research fields, we recommend readers see subsection 3.8 in [4] and references therein for the Monge-Ampère type equations arising in applications.

We are concerned in this work with the *a priori* estimates of admissible solutions to (1.2) with boundary condition. The use of the exponential barrier allows us to relax the concavity assumption of A to Ma-Trudinger-Wang conditions(see [5]). By the perturbation method of subsolutions in [2] (see Remark 2.2 in [6] for details), we can obtain strict subsolutions from non-strict subsulutions which simplifies the proofs and relaxes some restrictions to f in the estimates of $|u_t|$.

Our treatment here will also work for parabolic equations in the form

$$f(\lambda(\nabla^2 u + A(x, t, \nabla u))) - u_t = \psi(x, t, \nabla u) \tag{1.4}$$

with slight modification. Note that we do not require *a priori* bound of $|u_t|$ in the study of (1.4).

The idea of this paper is mainly from Guan-Jiao [7] and Jiang-Trudinger [2] where those authors studied the second order estimates for the elliptic counterpart of (1.2):

$$f(\lambda(\nabla^2 u + A(x, u, \nabla u))) = \psi(x, u, \nabla u). \tag{1.5}$$

The first initial-boundary value problem for equation of form (1.4) in $\mathbb{R}^n$ with $A \equiv 0$ and $\psi = \psi(x, t)$ was studied by Ivochkina-Ladyzhenskaya in [8] (when $f = \sigma_n^{1/n}$) and [9]. In recent years, Jiao-Sui [10] treated the case that $A \equiv \chi(x, t)$ and $\psi = \psi(x, t)$ on Riemannian manifolds and Jiao [3] extend their results to the form

$$f(\lambda(\nabla^2 u + A(x, t, \nabla u))) - u_t = \psi(x, t, u, \nabla u)$$

by the method using in the corresponding elliptic problems.

Krylov in [11] treated (1.2) in the parabolic Monge–Ampère form

$$-u_t \det(\nabla^2 u + A) = \psi^{n+1}$$

in $\mathbb{R}^n$, where $A \equiv 0$ and $\psi = \psi(x, t)$. In [12], Lieberman studied the first initial–boundary value problem of (1.2) when $A = 0$ and ψ may depend on u and ∇u in a bounded domain under various conditions.

For the elliptic Hessian equations, we refer the readers to Li [13], Urbas [14, 367–377], Guan [15, 16], Guan-Jiao [17] , Jiang-Trudinger [2] and their references.

Following [18], in which the authors studied the corresponding elliptic equations in $\mathbb{R}^n$, $f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ is assumed to be defined on Γ, where Γ is an open, convex, symmetric proper subcone of $\mathbb{R}^{n+1}$ with vertex at the origin and

$$\Gamma^+ \equiv \{\lambda \in \mathbb{R}^{n+1} : \text{ each component } \lambda_i > 0\} \subseteq \Gamma,$$

and to satisfy the following structure conditions in this paper:

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \le i \le n+1, \tag{1.6}$$

$$f \text{ is concave in } \Gamma, \tag{1.7}$$

and

$$\delta_{\psi,f} \equiv \inf_{M_T} \psi - \sup_{\partial\Gamma} f > 0, \quad \text{where } \sup_{\partial\Gamma} f \equiv \sup_{\lambda_0 \in \partial\Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda). \tag{1.8}$$

Typical examples are $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \le l < k \le n$, defined in the cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k\}$$

and $f = (\mathcal{M}_k)^{1/\binom{n}{k}}$ defined in

$$M_k = \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0\},$$

where $\sigma_k(\lambda)$ are the kth elementary symmetric functions and $\mathcal{M}_k$ are the p-plurisubharmonic functions defined by

$$\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \le k \le n$$

and

$$\mathcal{M}_k(\lambda) = \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \le k \le n$$

respectively. When $k = n$, $f = \sigma_n^{\frac{1}{n}}$ is the famous Monge-Ampère equation arising in many research fields such as conformal geometry, optimal transportation, isometric embedding and reflector designs, see the survey [4] and references therein.

We define a function $u(x,t)$ to be admissible if $(\lambda(\nabla^2 u + A[u]), -u_t) \in \Gamma$ in $M \times [0, T]$. It is shown in [18] that (1.6) ensures that Eq (1.2) is parabolic for admissible solutions. (1.7) means that the function F defined by $F(A, \tau) = f(\lambda[A], \tau)$ is concave for (A, τ) with $(\lambda[A], \tau) \in \Gamma$, where A is in the set of $n \times n$ symmetric matrices $\mathcal{S}^{n\times n}$. Moreover, when $\{U_{ij}\}$ is diagonal so is $\{F^{ij}\}$, and the following identities hold

$$F^{ij} U_{ij} = \sum f_i \lambda_i, \quad F^{ij} U_{ik} U_{kj} = \sum f_i \lambda_i^2, \quad \lambda(U) = (\lambda_1, \ldots, \lambda_n).$$

We define a function $\overline{u}$ to be a admissible viscosity supersolution of (1.2) if

$$f(\lambda(\nabla^2 \phi(\hat{x}, \hat{t}) + A(\hat{x}, \hat{t}, \nabla\phi(\hat{x}, \hat{t})), -\phi_t(\hat{x}, \hat{t})) \le \psi(\hat{x}, \hat{t}, \nabla\phi(\hat{x}, \hat{t}))$$

whenever $\phi \in C^2(M_T)$ is a admissible function and $(\hat{x}, \hat{t}) \in M_T$ is a local minimum of $\overline{u} - \phi$.

In this paper we assume that there exists an admissible function $\underline{u} \in C^2(\bar{M}_T)$ satisfying

$$\begin{cases} f(\lambda(\nabla^2 \underline{u} + A[\underline{u}]), -\underline{u}_t) \geq \psi(x, t, \nabla \underline{u}) & \text{in } M \times [0, T], \\ \underline{u} = \varphi & \text{on } \partial M \times [0, T], \\ \underline{u} \leq \varphi & \text{on } M \times \{0\}. \end{cases} \tag{1.9}$$

A $(0, 2)$ tensor B is called regular (strictly regular), if

$$\sum_{i,j,k,l}^{n} B^{ij}_{p_k,p_l}(x, t, p)\xi_i\xi_j\eta_k\eta_l \geq 0 (> 0)$$

for all $(x, t, p) \in M \times [0, T] \times \mathbb{R}^n$, $\xi, \eta \in T^*_x M$ and $g(\xi, \eta) = 0$.

The regular condition, well known as MTW condition, was first introduced by Ma, Trudinger and Wang in [5] for the study of optimal transportation in its strict form, and used in [2, 19] and other relevant problems. It is natural to consider MTW conditions instead of normal concavity assumptions on A. Examples in [5] shows that there exists a tensor A, without convexity respect to p, derived from special cost functions satisfying this regular condition. There are many results about MTW conditions, see, for instance, [20–25] and references therein.

We now begin to formulate the main theorems of this paper.

Theorem 1. *Let $u \in C^4(\bar{M}_T)$ be an admissible solution of (1.2). Suppose* (1.6)–(1.8) *and* (1.9) *hold. Assume, in addition, that*

$$\psi(x, t, p) \text{ is convex in } p, \tag{1.10}$$

$$-A^{\xi\xi}(x, t, p) \text{ is regular}, \tag{1.11}$$

then

$$\max_{\bar{M}_T} |\nabla^2 u| \leq C_1(1 + \max_{\mathcal{P}M_T} |\nabla^2 u|), \tag{1.12}$$

where $C_1 > 0$ depends on $|u|_{C^1(\bar{M}_T)}$, $|u_t|_{C^0(\bar{M}_T)}$ and $|\underline{u}|_{C^2(\bar{M}_T)}$. Suppose that u also satisfies the boundary condition (1.3) *and, in addition, assume that there exists a function $\Theta \in C^2(BM_T)$ such that $\Theta = -\varphi_t$ on $\partial M \times \{0\}$ and*

$$(\lambda(\nabla^2\varphi(x, 0) + A[\varphi(x, 0)]), \Theta(x)) \in \Gamma, \quad \forall x \in \bar{M}, \tag{1.13}$$

and that

$$f(\lambda(\nabla^2\varphi(x, 0) + A[\varphi(x, 0)]), -\varphi_t(x, 0)) = \psi[\varphi(x, 0)], \quad \forall x \in \partial M, \tag{1.14}$$

for each $(x, t) \in SM_T$ and $p \in T^*_x\bar{M}$. Then there exists $C_2 > 0$ depending on $|u|_{C^1(\bar{M}_T)}$, $|u_t|_{C^0(\bar{M}_T)}$, $|\underline{u}|_{C^2(\bar{M}_T)}$ and $|\varphi|_{C^4(\mathcal{P}M_T)}$ such that

$$\max_{\mathcal{P}M_T} |\nabla^2 u| \leq C_2. \tag{1.15}$$

Combining with the gradient estimates and the estimates of $|u_t|$, we can prove the following theorem immediately.

Theorem 2. *Let $u \in C^4(\bar{M}_T)$ be an admissible solution of (1.2) in M_T with $u \geq \underline{u}$ in M_T and $u = \varphi$ on $\mathcal{P}M_T$. Suppose* (1.6)–(1.11) *and* (1.13)–(1.14) *hold. Assume, in addition, for every $C > 0$, there is a constant $R = R(C)$ such that*

$$f(R\mathbf{1}) > C, \tag{1.16}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{n+1}$. *Assume also there exist a bounded admissible viscosity supersolution* $\overline{u}$ *of* (1.2) *satisfying* $\overline{u} \geq \varphi$ *on* $\mathcal{P}M_T$. *Then we have*

$$|u|_{C^2(\bar{M}_T)} \leq C, \tag{1.17}$$

where $C > 0$ *depends on n, M and* $|\underline{u}|_{C^2(\bar{M}_T)}$ *under the additional assumptions* (3.1)–(3.4) *in Section 3.*

The assumptions of the existence of bounded viscosity supersolution and the additional conditions (3.1)–(3.4) are only used to derive C^0 and C^1 estimates. (1.16) is used in the estimates of $|u_t|$ and can be dropped if $\underline{u}$ is strict subsolution. Both (1.16) and (3.4) hold for many operators such as the famous Monge-Ampère operator or more general k-Hessian operator $\sigma_k^{1/k}$.

The outline of this paper is as follows. In Section 2, we present some preliminaries and give a proof of Lemma 4. The solution bound and the gradient bound are derived in Section 3 while an *a priori* estimates for u_t is obtained in Section 4. Finally we establish the global and boundary C^2 estimates in Sections 5 and 6 respectively.

2. Preliminaries

Throughout the paper ∇ denotes the Levi-Civita connection of (M^n, g).

Let $u \in C^4(\bar{M}_T)$ be an admissible solution of Eq (1.2). For simplicity we shall denote $U := \nabla^2 u + A(x, t, \nabla u)$ and $\underline{U} := \nabla^2 \underline{u} + A(x, t, \nabla \underline{u})$. Moreover, we denote,

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}(U, -u_t), \quad F^{\tau} = \frac{\partial F}{\partial \tau}(U, -u_t),$$

$$F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}(U, -u_t), \quad F^{ij,\tau} = \frac{\partial^2 F}{\partial h_{ij} \partial \tau}(U, -u_t), \quad F^{\tau,\tau} = \frac{\partial^2 F}{\partial^2 \tau}(U, -u_t)$$

and, under a local frame $e_1, \ldots, e_n$,

$$U_{ij} \equiv U(e_i, e_j) = \nabla_{ij} u + A^{ij}(x, t, \nabla u),$$

$$\begin{aligned} \nabla_k U_{ij} &\equiv \nabla U(e_i, e_j, e_k) = \nabla_{kij} u + \nabla_k A^{ij}(x, t, \nabla u) \\ &\equiv \nabla_{kij} u + A_k^{ij}(x, t, \nabla u) + A_{p_l}^{ij}(x, t, \nabla u) \nabla_{kl} u, \end{aligned}$$

$$\begin{aligned} (U_{ij})_t &\equiv (U(e_i, e_j))_t = (\nabla_{ij} u)_t + A_t^{ij}(x, t, \nabla u) + A_{p_l}^{ij}(x, t, \nabla u)(\nabla_l u)_t \\ &\equiv \nabla_{ij} u_t + A_t^{ij}(x, t, \nabla u) + A_{p_l}^{ij}(x, t, \nabla u) \nabla_l u_t, \end{aligned}$$

where $A^{ij} = A^{e_i e_j}$ and A_k^{ij} denotes the *partial* covariant derivative of A when viewed as depending on $x \in M$ only, while the meanings of A_t^{ij} and $A_{p_l}^{ij}$, etc are obvious. Similarly we can calculate $\nabla_{kl} U_{ij} = \nabla_k \nabla_l U_{ij} - \Gamma_{kl}^m \nabla_m U_{ij}$, etc.

It is convenient to express the regular condition of $-A$ in the equivalent form as in [26],

$$-A_{p_k p_l}^{ij} \xi_i \xi_j \eta_k \eta_l \geq -2\overline{\lambda} |\xi||\eta| g(\xi \cdot \eta), \tag{2.1}$$

for all $\xi, \eta \in \mathbb{R}^n$, where $\overline{\lambda}$ is a non-negative function in $C^0(\overline{M_T} \times \mathbb{R}^n)$, depending on $\nabla_p A$. Hence, we have, for any non-negative symmetric matrix F^{ij} and $\epsilon \in (0, 1]$,

$$-F^{ij} A^{ij}_{p_k p_l} \eta_k \eta_l \geq -\overline{\lambda}(\epsilon \sum F^{ii} |\eta|^2 + \frac{1}{\epsilon} F^{ij} \eta_i \eta_j). \tag{2.2}$$

Define the linear operator $\mathcal{L}$ locally by

$$\mathcal{L}v = F^{ij} \nabla_{ij} v + (F^{ij} A^{ij}_{p_k} - \psi_{p_k}) \nabla_k v - F^\tau v_t$$

for $v \in C^2(M_T)$.

A crucial lemma was proved by Jiang-Trudinger for elliptic type equations in Lemma 2.1(ii) in [2] for $M = \mathbb{R}^n$, we extend their results to the parabolic case. Note that their perturbation of non-strict subsolution, which make a non-strict subsolution to be strict, only holds near the boundary in the Riemannian manifolds case. Therefore we shall apply a classification technique from [7] to deal with global estimates.

Let $\mu(x, t) = \lambda(\nabla^2 \underline{u}(x, t) + A[\underline{u}])$ and note that $\{\mu(x, t) : (x, t) \in M_T\}$ is a compact subset of positive cone Γ^+ since (1.6). There exists uniform constant $\beta \in (0, \frac{1}{2\sqrt{n}})$ such that

$$\nu_\mu - 2\beta \mathbf{1} \in \Gamma^+, \ \forall x \in \bar{M}_T, \tag{2.3}$$

where $\nu_\lambda := Df(\lambda)/|Df(\lambda)|$ is the unit normal vector to the level hypersurface $\partial\Gamma^{f(\lambda)}$ for $\lambda \in \Gamma$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{n+1}$.

For fixed (x_0, t_0), we consider two cases: (i) $|\nu_\mu - \nu_\lambda| \geq \beta$ and (ii) $|\nu_\mu - \nu_\lambda| < \beta$. In case (i), we shall modify Jiang-Trduinger's Lemma 2.1 [2]. First, we need the following lemma, its proof can be found in Lemma 2.2 [27].

Lemma 3. *Let K be a compact subset of Γ and $\beta > 0$. There is a constant $\epsilon > 0$ such that, for any $\mu \in K$ and $\lambda \in \Gamma$ with $|\nu_\mu - \nu_\lambda| \geq \beta$,*

$$\sum f_i(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \epsilon\Big(1 + \sum f_i(\lambda)\Big). \tag{2.4}$$

It follows from Lemma 6.2 in [18] and Lemma (2.4) that

$$F^{ij}(\underline{U}_{ij} - U_{ij}) \geq F(\underline{U}, -\underline{u}_t) - F(U, -u_t) + \epsilon(1 + \sum F^{ii} + F^\tau). \tag{2.5}$$

We now prove the crucial lemma for case (i).

Lemma 4. *Let $u \in C^2(\bar{M}_T)$ be an admissible solution of Eq (1.2) Suppose $|\nu_\mu - \nu_\lambda| \geq \beta$. Assume F satisfies (1.6)–(1.7) and (1.9)–(1.11) hold. Then there exist positive constants K and ϵ, depending on M_T, A, $|u|_{C^1(\bar{M}_T)}$ and $|\underline{u}|_{C^1(\bar{M}_T)}$ such that*

$$\mathcal{L}\eta > \epsilon(1 + \sum F^{ii} + F^\tau), \tag{2.6}$$

where $\eta = e^{K(\underline{u} - u)}$.

Proof. By (2.5), we have

$$\begin{aligned}\mathcal{L}(\underline{u}-u) &= F^{ij}\{[\underline{U}_{ij}-U_{ij}]-F^{\tau}[\underline{u}_t-u_t]+A^{ij}_{p_k}D_k(\underline{u}-u)\\ &\quad -A^{ij}(x,t,D\underline{u})+A^{ij}(x,t,Du)\}-\psi_{p_k}\nabla_k(\underline{u}-u)\\ &\geq F(\underline{U},-\underline{u}_t)-F(U,-u_t)-\psi_{p_k}\nabla_k(\underline{u}-u)\\ &\quad -\frac{1}{2}F^{ij}A^{ij}_{p_k,p_l}(x,t,\hat{p})D_k(\underline{u}-u)D_l(\underline{u}-u)\\ &\quad +\epsilon(1+\sum F^{ii}+F^{\tau})\\ &\geq -\frac{1}{2}F^{ij}A^{ij}_{p_k,p_l}(x,t,\hat{p})D_k(\underline{u}-u)D_l(\underline{u}-u)\\ &\quad +\epsilon(1+\sum F^{ii}+F^{\tau})\end{aligned}\tag{2.7}$$

by Taylor's formula and the convexity of ψ, where $\hat{p}=\theta\nabla u+(1-\theta)\nabla\underline{u}$ for some $\theta\in(0,1)$. Thus

$$\begin{aligned}\mathcal{L}e^{K(\underline{u}-u)} &= Ke^{K(\underline{u}-u)}[\mathcal{L}(\underline{u}-u)+KF^{ij}D_i(\underline{u}-u)D_j(\underline{u}-u)]\\ &\geq Ke^{K(\underline{u}-u)}\Big\{-\frac{1}{2}F^{ij}A^{ij}_{p_k,p_l}(x,t,\hat{p})D_k(\underline{u}-u)D_l(\underline{u}-u)\\ &\quad +KF^{ij}D_i(\underline{u}-u)D_j(\underline{u}-u)+\epsilon(1+\sum F^{ii}+F^{\tau})\Big\}.\end{aligned}\tag{2.8}$$

Since A is regular, by (2.2), we obtain

$$\begin{aligned}&\epsilon\sum F^{ii}-\frac{1}{2}F^{ij}A^{ij}_{p_k,p_l}(x,t,\hat{p})D_k(\underline{u}-u)D_l(\underline{u}-u)+KF^{ij}D_i(\underline{u}-u)D_j(\underline{u}-u)\\ &\geq\Big(\epsilon-\frac{\overline{\lambda}\epsilon_1}{2}|D(\underline{u}-u)|^2\Big)\sum F^{ii}+\Big(K-\frac{\overline{\lambda}}{2\epsilon_1}\Big)F^{ij}D_i(\underline{u}-u)D_j(\underline{u}-u)\\ &\geq\frac{\epsilon}{2}\sum F^{ii}\end{aligned}$$

by successively fixing ϵ_1 and K.

Therefore, by (2.8), we have

$$\mathcal{L}e^{K(\underline{u}-u)}\geq Ke^{K(\underline{u}-u)}\Big(\frac{\epsilon}{2}(1+\sum F^{ii}+F^{\tau})\Big)\geq\epsilon_0(1+\sum F^{ii}+F^{\tau})\tag{2.9}$$

for some positive constant ϵ_0. □

Next, in case (ii), we have $\nu_\lambda-\beta\mathbf{1}\in\Gamma^+$. Thus we derive

$$F^{ii}\geq\frac{\beta}{\sqrt{n+1}}\sum F^{ii}\quad\forall 1\leq i\leq n+1.\tag{2.10}$$

Remark 1. *If $\underline{u}$ is a strict subsolution or $M=\mathbb{R}^n$, then we can derive* (2.6) *without the assumption $|\nu_\mu-\nu_\lambda|\geq\beta$. Actually, when $M=\mathbb{R}^n$, let $d(x)=dist(x,\partial M)$, by consider $\underline{u}+ae^{bx_1}$ and $\underline{u}+a(e^{bd}-1)$ for interior and near boundary respectively in $\mathbb{R}^n$, a strict subsolution can be derived from a non-strict one, see remark 2.2 in [6]. Then* (2.6) *will be obtained by Jiang-Trudinger's proof with a little modification.*

3. Gradient estimates

In this section, we derive the gradient estimates. We introduce the following growth conditions: When $|p|$ is sufficiently large,

$$p \cdot \nabla_x \psi(x,t,p),\ \ p \cdot \nabla_x A^{\xi\xi}(x,t,p)/|\xi|^2 \le \bar{\psi}_1(x,t)(1+|p|^\gamma), \tag{3.1}$$

$$|p \cdot D_p \psi(x,t,p)|,\ \ |p \cdot D_p A^{\xi\xi}(x,t,p)|/|\xi|^2 \le \bar{\psi}_2(x,t)(1+|p|^\gamma) \tag{3.2}$$

and

$$|A^{\xi\eta}(x,t,p)| \le \bar{\psi}_3(x,t)|\xi||\eta|(1+|p|^{\gamma_1}) \ \ \forall\, \xi,\eta \in T_x^*\bar{M} \tag{3.3}$$

hold for some functions $\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3 \ge 0$, and constants $\gamma \in (0,4)$ and $\gamma_1 \in (0,2)$.

By the existence of viscosity supersolution $\overline{u}$ and classical subsolution $\underline{u}$, we have

$$\max_{\bar{M}_T} |u| \le C.$$

Since u is admissible, we have

$$0 < \triangle u + \mathrm{tr} A(x,t,\nabla u) - u_t.$$

The boundary gradient estimates are derived by subsolution $\underline{u}$ for the lower bound and by (3.3) with the method of Lemma 10.1 in [12] for the upper bound.

Theorem 5. *Let $u \in C^3(\bar{M}_T)$ be an admissible solution of (1.2). Suppose* (1.6)–(1.7) *and* (3.1)–(3.3) *hold. Assume, in addition, that*

$$f_j \ge \nu_0(1+\sum_{i=1}^{n+1} f_i) \ \ \textit{for any } \lambda \in \Gamma \textit{ with } \lambda_j < 0, \tag{3.4}$$

where ν_0 is a uniform positive constant. Then

$$\max_{\bar{M}_T} |\nabla u| \le C_3(1+\max_{\mathcal{P}M_T} |\nabla u|), \tag{3.5}$$

where C_3 is a positive constant depending on $|u|_{C^0(\bar{M}_T)}$ and other known data.

Proof. Let $\phi \in C^2(\bar{M}_T)$ is a positive function to be determined. Suppose $|\nabla u|\phi^{-a}$ achieves a positive maximum at an interior point $(x_0,t_0) \in \bar{M}_T - \mathcal{P}M_T$ where $a < 1$ is a constant. Choose a smooth orthonormal local frame $e_1,\dots,e_n$ about (x_0,t_0) such that $\nabla_{e_i} e_j = 0$ at (x_0,t_0) if $i \ne j$ and $\{U_{ij}\}$ is diagonal. Define $v = \log|\nabla u| - a\log\phi$, then the function v also attains its maximum at (x_0,t_0) where, for $i = 1,\dots,n$,

$$\nabla_i v = \frac{\nabla_l u \nabla_{il} u}{|\nabla u|^2} - a\frac{\nabla_i \phi}{\phi} = 0 \tag{3.6}$$

and

$$F^\tau v_t \ge 0 \ge F^{ii}\nabla_{ii} v. \tag{3.7}$$

Thus, by (3.6) and (3.7), we have

$$\begin{aligned} 0 &\geq F^{ii}\nabla_{ii}v - F^{\tau}v_t \\ &= F^{ii}\nabla_{ii}(\log|\nabla u|) - F^{\tau}(\log|\nabla u|)_t - aF^{ii}\nabla_{ii}\log\phi + aF^{\tau}(\log\phi)_t \\ &= \frac{1}{|\nabla u|^2}F^{ii}\nabla_{il}u\nabla_{il}u + \frac{\nabla_l u}{|\nabla u|^2}\Big(F^{ii}\nabla_{iil}u - F^{\tau}\nabla_l u_t\Big) \\ &\quad + \frac{a-2a^2}{\phi^2}F^{ii}(\nabla_i\phi)^2 - \frac{a}{\phi}F^{ii}\nabla_{ii}\phi. \end{aligned} \tag{3.8}$$

Differentiating both sides of Eq (1.2) with respect to x, we obtain, at (x_0, t_0),

$$F^{ii}\nabla_k U_{ii} - F^{\tau}\nabla_k u_t = \psi_k + \psi_{p_j}\nabla_{kj}u \tag{3.9}$$

for all $k = 1, \ldots, n$.

Let $\phi = -u + \sup_{\bar{M}_T} u + 1$. Note that, at (x_0, t_0), $\nabla_{ij}u = \nabla_{ij}u$ and

$$\nabla_{ijk}u - \nabla_{jik}u = R^l_{kij}\nabla_l u. \tag{3.10}$$

By (3.1), (3.2), (3.6), (3.9) and (3.10), we have

$$\begin{aligned} \frac{\nabla_l u}{|\nabla u|^2}\Big(F^{ii}\nabla_{iil}u - F^{\tau}\nabla_l u_t\Big) &= \frac{\nabla_l u}{|\nabla u|^2}F^{ii}(\nabla_{lii}u - R^k_{iil}\nabla_k u - F^{\tau}\nabla_l u_t) \\ &\geq \frac{\nabla_l u}{|\nabla u|^2}F^{ii}(\nabla_l U_{ii} - \nabla_l(A^{ii}) - F^{\tau}\nabla_l u_t) - C \\ &\geq -C(1+|\nabla u|^{\gamma-2})(1+\sum F^{ii}). \end{aligned} \tag{3.11}$$

Therefore, by substituting (3.11) into (3.8), we have

$$\begin{aligned} 0 \geq{}& \frac{1}{|\nabla u|^2}F^{ii}\nabla_{il}u\nabla_{il}u + \frac{a-2a^2}{\phi^2}F^{ii}(\nabla_i u)^2 + \frac{a}{\phi}F^{ii}\nabla_{ii}u \\ & - C(1+|\nabla u|^{\gamma-2})(1+\sum F^{ii}). \end{aligned} \tag{3.12}$$

Notice that

$$\frac{1}{|\nabla u|^2}F^{ii}\nabla_{ii}u\nabla_{ii}u + \frac{a}{\phi}F^{ii}\nabla_{ii}u \geq -\frac{a^2|\nabla u|^2}{4\phi^2}\sum F^{ii}.$$

It follows from (3.12) that

$$\begin{aligned} 0 \geq{}& \frac{a-2a^2}{\phi^2}F^{ii}(\nabla_i u)^2 - \frac{a^2|\nabla u|^2}{4\phi^2}\sum F^{ii} \\ & - C(1+|\nabla u|^{\gamma-2})(1+\sum F^{ii}). \end{aligned} \tag{3.13}$$

Without loss of generality we may consider $\nabla_1 u(x_0, t_0) \geq \frac{1}{n}|\nabla u(x_0, t_0)| > 0$. Recall that $U_{ij}(x_0, t_0)$ is diagonal. By (3.3) and (3.6), we have

$$\begin{aligned} U_{11} &= -\frac{a}{\phi}|\nabla u|^2 + A^{11} + \frac{\sum_{l\geq 2}\nabla_l u A^{1l}}{\nabla_1 u} \\ &\leq -\frac{a}{\phi}|\nabla u|^2 + C(1+|\nabla u|^{\gamma_1}) < 0 \end{aligned} \tag{3.14}$$

provided $|\nabla u|$ is sufficiently large. The appearance of A^{1l} in the first line is due to the diagonality of $\{U_{ij}\}$. Therefore, by (3.4),

$$f_1 \geq \nu_0\Big(1 + \sum_{i=1}^{n} f_i + F^\tau\Big)$$

and a bound $|\nabla u(x_0, t_0)| \leq C_3$ follows from (3.13) by choosing a sufficiently small such that

$$\frac{a - 2a^2}{\phi^2} \cdot \frac{\nu_0}{n} - \frac{a^2}{4\phi^2} \geq c_1 > 0$$

holds for some uniform constant c_1.

□

Remark 2. *This assumptions follow from [7] and [3].* (3.3) *with* $\gamma_1 \in (0, 2)$ *is more of a technical condition here. Actually, it will be better to obtain gradient estimates with quadratic growth conditions, i.e* $\gamma_1 = 2$*, see examples in [4]. The reason why we need* (3.3) *is the regular assumption of A which make us can not use barrier* $\eta = e^{K(\underline{u}-u)}$ *in gradient estimates. From the proof of Lemma 4 you can see the proof of the barrier is based on the gradient estimates. This requirement also occurs in Theorem 1.3 (ii) in [28].*

(3.4) *is a natural assumption satisfied by many operators such as the k-Hessian operator* $\sigma_k^{\frac{1}{k}}$*. It is commonly used in deriving gradient estimate, for example in [29].*

4. The estimates for $|u_t|$

In this section, we derive the estimates for $|u_t|$.

Theorem 6. *Suppose that* (1.6)–(1.7), (1.9) *and* (1.16) *hold,* $A = A(x, t, \nabla u)$ *and* $\psi = \psi(x, t, \nabla u)$*. Let* $u \in C^3(\bar{M}_T)$ *be an admissible solution of* (1.2)-(1.3) *in* M_T*. Then there exists a positive constant* C_2 *depending on* $|u|_{C^1(\bar{M}_T)}$, $|\underline{u}|_{C^2(\bar{M}_T)}$, $|\psi|_{C^2(\bar{M}_T)}$ *and other known data such that*

$$\sup_{\bar{M}_T} |u_t| \leq C_4(1 + \sup_{\mathcal{P}M_T} |u_t|). \tag{4.1}$$

Proof. We first show that

$$\sup_{\bar{M}_T}(-u_t) \leq C_4(1 + \sup_{\mathcal{P}M_T} |u_t|) \tag{4.2}$$

for which we set

$$W = \sup_{\bar{M}_T}(-u_t)e^{\phi},$$

where ϕ is a positive function to be chosen.

We may assume that W is attained at $(x_0, t_0) \in \bar{M}_T - \mathcal{P}M_T$. As in the proof of Theorem 5, we choose an orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ and $\{U_{ij}(x_0, t_0)\}$ is diagonal. We may assume $-u_t(x_0, t_0) > 0$. Define $v = \log(-u_t) + \phi$. At (x_0, t_0), where the function v achieves its maximum, we have, for $i = 1, \ldots n$,

$$\nabla_i v = \frac{\nabla_i u_t}{u_t} + \nabla_i \phi = 0 \tag{4.3}$$

and

$$F^{\tau}v_t \geq 0 \geq F^{ii}\nabla_{ii}v = F^{ij}\nabla_{ii}v + (F^{ij}A^{ij}_{p_k} - \psi_{p_k})\nabla_k v. \tag{4.4}$$

Thus, by (4.3) and (4.4), we have

$$\begin{aligned}
0 \geq\, & F^{ii}\nabla_{ii}v - F^{\tau}v_t + (F^{ij}A^{ij}_{p_k} - \psi_{p_k})\nabla_k v \\
&= F^{ii}\nabla_{ii}\log(-u_t) - F^{\tau}(\log(-u_t))_t + F^{ii}\nabla_{ii}\phi - F^{\tau}\phi_t \\
&\quad + (F^{ij}A^{ij}_{p_k} - \psi_{p_k})\nabla_k(\log(-u_t) + \phi) \\
&= \frac{1}{u_t}\Big(F^{ii}\nabla_{ii}u_t - F^{\tau}u_{tt} + (F^{ij}A^{ij}_{p_k} - \psi_{p_k})\nabla_k u_t\Big) \\
&\quad + \mathcal{L}\phi - F^{ii}(\nabla_i\phi)^2.
\end{aligned} \tag{4.5}$$

By differentiating equation (1.2) with respect to t, we get

$$F^{ii}(U_{ii})_t - F^{\tau}u_{tt} = \psi_t + \psi_{p_k}(\nabla_k u)_t. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\begin{aligned}
0 &\geq \frac{1}{u_t}((\psi_t - F^{ii}A^{ii}_t) - F^{ii}(\nabla_i\phi)^2 + \mathcal{L}\phi \\
&\geq \frac{C}{u_t}(1 + \sum F^{ii}) - F^{ii}(\nabla_i\phi)^2 + \mathcal{L}\phi.
\end{aligned} \tag{4.7}$$

Fix a positive constant $\alpha \in (0, 1)$ and let $\phi = \frac{\delta^{1+\alpha}}{2}|\nabla u|^2 + \delta u + b\eta$, where $\eta = e^{K(\underline{u}-u)}$ as in Lemma 4 and $\delta \ll b \ll 1$ are positive constants to be determined. By straightforward calculations, we have

$$\nabla_i\phi = \delta^{1+\alpha}\sum_k \nabla_k u\nabla_{ik}u + \delta\nabla_i u + b\nabla_i\eta,$$

$$\phi_t = \delta^{1+\alpha}\sum_k \nabla_k u(\nabla_k u)_t + \delta u_t + b\eta_t,$$

$$\nabla_{ii}\phi = \delta^{1+\alpha}\sum_k(\nabla_{ik}u)^2 + \delta^{1+\alpha}\sum_k \nabla_k u\nabla_{iik}u + \delta\nabla_{ii}u + b\nabla_{ii}\eta.$$

It follows that

$$\begin{aligned}
\mathcal{L}\phi \geq\, & \delta^{1+\alpha}\nabla_k u\Big(F^{ii}\nabla_{iik}u - F^{\tau}(\nabla_k u)_t + F^{ij}A^{ij}_{p_l}\nabla_{kl}u - \psi_{p_l}\nabla_{kl}u\Big) \\
& + \frac{\delta^{1+\alpha}}{2}F^{ii}U^2_{ii} - C\delta^{1+\alpha}\sum F^{ii} + \delta\mathcal{L}u + b\mathcal{L}\eta \\
\geq & - C\delta^{1+\alpha}\Big(1 + \sum F^{ii}\Big) + \frac{\delta^{1+\alpha}}{2}F^{ii}U^2_{ii} + \delta\mathcal{L}u + b\mathcal{L}\eta
\end{aligned} \tag{4.8}$$

and

$$(\nabla_i\phi)^2 \leq C\delta^{2(1+\alpha)}U^2_{ii} + Cb^2 \tag{4.9}$$

since $b \gg \delta$. Thus, (4.7) becomes, by (4.8) and (4.9),

$$b\mathcal{L}\eta + \frac{\delta^{1+\alpha}}{4}F^{ii}U^2_{ii} + \delta\mathcal{L}u \leq -\frac{C}{u_t}(1 + \sum F^{ii}) + C\delta^{1+\alpha}\Big(1 + \sum F^{ii}\Big) + Cb^2\sum F^{ii}. \tag{4.10}$$

We first consider case (i): $|\nu_\mu - \nu_\lambda| \geq \beta$. Note that

$$\delta F^{ii}U_{ii} \geq -\frac{\delta^{1+\alpha}}{4}F^{ii}U_{ii}^2 - \delta^{1-\alpha}\sum F^{ii}.$$

It follows from that

$$\begin{aligned}\frac{\delta^{1+\alpha}}{4}F^{ii}U_{ii}^2 + \delta\mathcal{L}u \geq & - C\delta(1+\sum F^{ii}) + \frac{\delta^{1+\alpha}}{4}F^{ii}U_{ii}^2 \\ & + \delta F^{ii}U_{ii} - \delta F^\tau u_t \\ \geq & - C\delta^{1-\alpha}(1+\sum F^{ii})\end{aligned} \tag{4.11}$$

since $u_t(x_0, t_0) < 0$. Therefore, by (4.10) and (4.11), we have

$$b\mathcal{L}\eta \leq -\frac{C}{u_t}(1+\sum F^{ii}) + C\delta^{1-\alpha}\Big(1+\sum F^{ii}\Big) + Cb^2\sum F^{ii}. \tag{4.12}$$

Choosing b and δ such that $b\epsilon_0 - C\delta^{1-\alpha} - Cb^2 \geq b_1 > 0$ for a positive constant b_1, then a upper bound of $-u_t(x_0, t_0)$ derived by (2.6).

Case (ii): $|\nu_\mu - \nu_\lambda| < \beta$. We see that (2.10) holds. Note that

$$\frac{\delta^{1+\alpha}}{8}F^{ii}U_{ii}^2 + \delta F^{ii}U_{ii} \geq -2\delta^{1-\alpha}\sum F^{ii}$$

and

$$\begin{aligned}\mathcal{L}e^{K(\underline{u}-u)} &= Ke^{K(\underline{u}-u)}[\mathcal{L}(\underline{u}-u) + KF^{ij}D_i(\underline{u}-u)D_j(\underline{u}-u)] \\ &\geq Ke^{K(\underline{u}-u)}\Big\{-\frac{1}{2}F^{ij}A^{ij}_{p_k,p_l}(x,t,\hat{p})D_k(\underline{u}-u)D_l(\underline{u}-u) \\ &\quad + KF^{ij}D_i(\underline{u}-u)D_j(\underline{u}-u)\Big\} \\ &\geq -C\sum F^{ii}\end{aligned} \tag{4.13}$$

by the concavity of F and ψ, where C depends on $|u|_{C^1(\bar{M}_T)}$ and other known data. We have, by (4.10),

$$\begin{aligned}\frac{\delta^{1+\alpha}}{8}F^{ii}U_{ii}^2 - \delta F^\tau u_t \leq & -\frac{C}{u_t}(1+\sum F^{ii}) + C\delta\Big(1+\sum F^{ii}\Big) \\ & + C(\delta^{1-\alpha} + b + b^2)\sum F^{ii} \\ \leq & -\frac{C}{u_t}(1+\sum F^{ii}) + C\delta^{1-\alpha} + C\sum F^{ii}.\end{aligned} \tag{4.14}$$

Recalling that $u_t < 0$, we get

$$F^{ii}U_{ii} - F^\tau u_t \geq u_t\Big(\sum F^{ii} + F^\tau\Big) + \frac{1}{4u_t}\Big(F^{ii}U_{ii}^2 + F^\tau u_t^2\Big).$$

Therefore, by the concavity of f, we have

$$\begin{aligned}-u_t\Big(\sum F^{ii} + F^\tau\Big) &\geq f(-u_t\mathbf{1}) - f(\lambda(U), -u_t) + F^{ii}U_{ii} - F^\tau u_t \\ &\geq u_t\Big(\sum F^{ii} + F^\tau\Big) + \frac{1}{4u_t}\Big(F^{ii}U_{ii}^2 + F^\tau u_t^2\Big) \\ &\quad + f(-u_t\mathbf{1}) - \psi[u],\end{aligned} \tag{4.15}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{n+1}$.

Note that $\lim_{t\to\infty} f(t\mathbf{1}) = \sup_\Gamma f > \sup_{\bar{M}_T} \psi[u]$. It follows from (1.6) that

$$f(-u_t\mathbf{1}) - \psi[u] \geq f(-u_t\mathbf{1}) - \sup_{\bar{M}_T} \psi[u] := 2b_2 \tag{4.16}$$

provided $-u_t(x_0, t_0)$ is big enough, where b_2 is a positive constant. Therefore, by (4.15) and (4.16), we have

$$-u_t\Big(\sum F^{ii} + F^\tau\Big) \geq b_2 + \frac{1}{8u_t}\Big(F^{ii}U_{ii}^2 + F^\tau u_t^2\Big). \tag{4.17}$$

It follows from (2.10) and (4.17) that

$$\begin{aligned} -F^\tau u_t \geq & -2\gamma_0 u_t\Big(\sum F^{ii} + F^\tau\Big) \\ \geq & -\gamma_0 u_t\Big(\sum F^{ii} + F^\tau\Big) + \gamma_0 b_2 + \frac{\gamma_0}{8u_t}\Big(F^{ii}U_{ii}^2 + F^\tau u_t^2\Big) \\ \geq & -\gamma_0 u_t \sum F^{ii} + \gamma_0 b_2 + \frac{\gamma_0}{8u_t} F^{ii}U_{ii}^2, \end{aligned} \tag{4.18}$$

where $\gamma_0 := \frac{\beta}{2\sqrt{n+1}} > 0$.

Without loss of generality, we suppose $-u_t \geq \gamma_0\delta^{-\alpha}$ for fixed δ. Substituting (4.18) in (4.14) we derive

$$(-\delta\gamma_0 u_t - C)\sum F^{ii} + \delta\gamma_0 b_2 - C\delta^{1-\alpha} \leq -\frac{C}{u_t}(1 + \sum F^{ii}). \tag{4.19}$$

By (1.16), we see that b_2 can be sufficiently large, then a bound is derived from (4.19) and therefore (4.2) holds.

Similarly, we can show

$$\sup_{\bar{M}_T} u_t \leq C_4(1 + \sup_{\mathcal{P}M_T} |u_t|) \tag{4.20}$$

by letting

$$\phi = \frac{\delta^{1+\alpha}}{2}|\nabla u|^2 - \delta u + b(\underline{u} - u).$$

Combining (4.2) and (4.20), the proof is finished. □

Remark 3. *If $\underline{u}$ is a strict subsolution, then Theorem 6 follows without* (1.16). *In face, in this case we have* (2.6) *holds without classification. Let $W = \sup_{\bar{M}_T} |u_t|e^{a\phi}$ and $\phi = \eta$ in Lemma 2.6, the theorem will be proved easily.*

By (1.13) and (1.14) we can the short time existence as Theorem 15.9 in [12]. So without of loss of generality, we may assume that φ is defined on $M \times [0, t_0]$ for some small constant $t_0 > 0$ and

$$f(\lambda(\nabla^2\varphi(x, 0) + A[\varphi]), -\varphi_t(x, 0)) = \psi[\varphi] \quad \forall x \in \bar{M}. \tag{4.21}$$

Since that $u_t = \varphi_t$ on SM_T and (4.21), we can obtain the estimate

$$\sup_{\bar{M}_T} |u_t| \leq C_5. \tag{4.22}$$

5. Global estimates for second derivatives

In this section, we derive the global estimates for the second order derivatives. In particular, we prove the following maximum principle.

Theorem 7. *Let $u \in C^4(\bar{M}_T)$ be an admissible solution of* (1.2) *in M_T. Suppose that* (1.6)–(1.7) *and* (1.9)–(1.11) *hold. Then*

$$\sup_{\bar{M}_T} |\nabla^2 u| \leq C_1(1 + \sup_{\mathcal{P}M_T} |\nabla^2 u|), \tag{5.1}$$

where $C_1 > 0$ depends on $|u|_{C^1(\bar{M}_T)}$, $|\underline{u}|_{C^1(\bar{M}_T)}$, $|u_t|_{C^0(\bar{M}_T)}$, $|\psi|_{C^2(\bar{M}_T)}$ and other known data.

Proof. Set

$$W = \max_{(x,t)\in\bar{M}_T} \max_{\xi\in T_xM, |\xi|=1} (\nabla_{\xi\xi} u + A^{\xi\xi}(x, t, \nabla u))e^{\phi},$$

as in [7], where ϕ is a function to be determined. It suffices to estimate W. We may assume W is achieved at $(x_0, t_0) \in \bar{M}_T - \mathcal{P}M_T$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\nabla_i e_j = 0$, and $\{U_{ij}\}$ is diagonal at (x_0, t_0). We assume $U_{11}(x_0, t_0) \geq \ldots \geq U_{nn}(x_0, t_0)$ and, without loss of generality, we assume $U_{11} > 1$.

Define $v = \log U_{11} + \phi$. At (x_0, t_0), where the function v attains its maximum, we have, for each $i = 1, \ldots, n$,

$$\nabla_i v = \frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \phi = 0 \tag{5.2}$$

and

$$F^{\tau} v_t \geq 0 \geq F^{ii} \nabla_{ii} v. \tag{5.3}$$

Thus, by (5.3), we have

$$\begin{aligned} 0 &\geq F^{ii}\nabla_{ii} v - F^{\tau} v_t \\ &= F^{ii}\nabla_{ii}(\log U_{11}) - F^{\tau}(\log U_{11})_t + F^{ii}\nabla_{ii}\phi - F^{\tau}\phi_t \\ &= -\frac{1}{U_{11}^2} F^{ii}\nabla_i U_{11}^2 + \frac{1}{U_{11}}\Big(F^{ii}\nabla_{ii} U_{11} - F^{\tau}(U_{11})_t\Big) \\ &\quad + F^{ii}\nabla_{ii}\phi - F^{\tau}\phi_t. \end{aligned} \tag{5.4}$$

Differentiating Eq (1.2) twice, we obtain, by (1.10), (3.9). (3.10) and (5.2),

$$\begin{aligned} F^{ii}\nabla_{11} U_{ii} &+ F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} - 2F^{ij,\tau}\nabla_1 U_{ij}\nabla_1 u_t \\ &+ F^{\tau,\tau}(\nabla_1 u_t)^2 - F^{\tau}\nabla_{11} u_t \\ &\geq -CU_{11} + \psi_{p_k p_l}\nabla_{1k} u\nabla_{1l} u + \psi_{p_k}\nabla_{11l} u \\ &\geq -CU_{11} - U_{11}\psi_{p_k}\nabla_k\phi. \end{aligned} \tag{5.5}$$

Note that the regular condition of A means $A^{ii}_{p_1p_1} \leq 0$ for $i \neq 1$. Therefore by (3.9) and (5.2), we have

$$\begin{aligned} F^{ii}(\nabla_{ii} A^{11} - \nabla_{11} A^{ii}) &\geq F^{ii}(A^{11}_{p_k}\nabla_{iik} u - A^{ii}_{p_k}\nabla_{11k} u) - CU_{11}\sum F^{ii} \\ &\quad + F^{ii}(A^{11}_{p_ip_i} U_{ii}^2 - A^{ii}_{p_1p_1} U_{11}^2) \\ &\geq U_{11} F^{ii} A^{ii}_{p_k}\nabla_k\phi + F^{\tau} A^{11}_{p_k}\nabla_k u_t - CU_{11}\sum F^{ii} \\ &\quad - CU_{11} - C\sum_{i\geq 2} F^{ii} U_{ii}^2. \end{aligned} \tag{5.6}$$

Note that

$$\begin{aligned}\nabla_{ijkl}v - \nabla_{klij}v =& R^m_{ljk}\nabla_{im}v + \nabla_i R^m_{ljk}\nabla_m v + R^m_{lik}\nabla_{jm}v \\ &+ R^m_{jik}\nabla_{lm}v + R^m_{jil}\nabla_{km}v + \nabla_k R^m_{jil}\nabla_m v.\end{aligned}$$

Thus we have

$$\nabla_{ii}U_{11} \geq \nabla_{11}U_{ii} + \nabla_{ii}A^{11} - \nabla_{11}A^{ii} - CU_{11}. \tag{5.7}$$

It follows from (5.5), (5.6) and (5.7) that

$$\begin{aligned}F^{ii}\nabla_{ii}U_{11} - F^{\tau}(U_{11})_t \geq& F^{ii}\nabla_{11}U_{ii} - F^{\tau}\nabla_{11}u_t - CU_{11}\sum F^{ii} \\ &- F^{ii}(\nabla_{ii}A^{11} - \nabla_{11}A^{ii}) - F^{\tau}(A^{11})_t \\ \geq& - F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} - 2F^{ij,\tau}\nabla_1 U_{ij}\nabla_1 u_t \\ &+ F^{\tau,\tau}(\nabla_1 u_t)^2 + U_{11}(F^{ii}A^{ii}_{p_k} - \psi_{p_k})\nabla_k\phi \\ &- C\sum_{i\geq 2}F^{ii}U^2_{ii} - CU_{11}(1 + \sum F^{ii}).\end{aligned} \tag{5.8}$$

Thus, by (5.4) and (5.8), we have, at (x_0, t_0),

$$\mathcal{L}\phi \leq \frac{C}{U_{11}}\sum_{i\geq 2}F^{ii}U^2_{ii} + C(1 + \sum F^{ii}) + E, \tag{5.9}$$

where

$$E = \frac{1}{U^2_{11}}F^{ii}(\nabla_i U_{11})^2 + \frac{1}{U_{11}}(F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} - 2F^{ij,\tau}\nabla_1 U_{ij}\nabla_1 u_t + F^{\tau,\tau}(\nabla_1 u_t)^2).$$

Let $\eta = e^{K(\underline{u}-u)}$. Define

$$\phi = \frac{\delta|\nabla u|^2}{2} + b\eta,$$

where b and δ are undetermined constants such that $0 < \delta < 1 \leq b$. We find, at (x_0, t_0),

$$\nabla_i\phi = \delta\nabla_j u\nabla_{ij}u + b\nabla_i\eta = \delta\nabla_i uU_{ii} - \delta\nabla_j uA^{ij} + b\nabla_i\eta, \tag{5.10}$$

$$\phi_t = \delta\nabla_j u(\nabla_j u)_t + b\eta_t, \tag{5.11}$$

$$\nabla_{ii}\phi \geq \frac{\delta}{2}U^2_{ii} - C\delta + \delta\nabla_j u\nabla_{iij}u + b\nabla_{ii}\eta. \tag{5.12}$$

From (3.10) and (3.9), we derive

$$\begin{aligned}F^{ii}\nabla_j u\nabla_{iij}u \geq& F^{ii}\nabla_j u(\nabla_j U_{ii} - \nabla_j A^{ii}) - C|\nabla u|^2\sum F^{ii} \\ \geq& (\psi_{p_k} - F^{ii}A^{ii}_{p_k})\nabla_j u\nabla_{jk}u + F^{\tau}\nabla_j u\nabla_j(u_t) \\ &- C(1 + \sum F^{ii}).\end{aligned} \tag{5.13}$$

Therefore,

$$\mathcal{L}\phi \geq b\mathcal{L}\eta + \frac{\delta}{2}F^{ii}U^2_{ii} - C\delta(1 + \sum F^{ii}). \tag{5.14}$$

Next, by (5.10) we get

$$(\nabla_i\phi)^2 \leq C\delta^2(1 + U^2_{ii}) + 2b^2(\nabla_i(\underline{u} - u))^2 \leq C\delta^2 U^2_{ii} + Cb^2. \tag{5.15}$$

Now we estimate E as in [16] and [17] (see [1] for details). Let

$$J = \{i : U_{ii} \le -sU_{11}\}, \quad K = \{i : U_{ii} > -sU_{11}\},$$

where $0 < s \le 1/3$ is a fixed number. Using an inequality of Andrews [30] and Gerhardt [31], we have, by (5.15),

$$\begin{aligned} -F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} &\ge \sum_{i\ne j}\frac{F^{ii}-F^{jj}}{U_{jj}-U_{ii}}(\nabla_1 U_{ij})^2 \\ &\ge 2\sum_{i\ge 2}\frac{F^{ii}-F^{11}}{U_{11}-U_{ii}}(\nabla_1 U_{i1})^2 \\ &\ge \frac{2(1-s)}{(1+s)U_{11}}\sum_{i\in K}(F^{ii}-F^{11})((\nabla_i U_{11})^2 - CU_{11}^2/s). \end{aligned} \tag{5.16}$$

Thus, we obtain

$$\begin{aligned} E &\le \frac{1}{U_{11}^2}\sum_{i\in J}F^{ii}(\nabla_i U_{11})^2 + C\sum_{i\in K}F^{ii} + \frac{CF^{11}}{U_{11}^2}\sum_{i\in K}(\nabla_i U_{11})^2 \\ &\le \sum_{i\in J}F^{ii}(\nabla_i\phi)^2 + C\sum F^{ii} + CF^{11}\sum(\nabla_i\phi)^2 \\ &\le Cb^2\sum_{i\in J}F^{ii} + C\delta^2\sum F^{ii}U_{ii}^2 + C\sum F^{ii} + C(\delta^2 U_{11}^2 + b^2)F^{11}. \end{aligned} \tag{5.17}$$

Therefore, by (5.9), (5.14), (5.15) and (5.17), we have

$$\begin{aligned} b\mathcal{L}\eta \le& \Big(C\delta^2 + \frac{C}{U_{11}} - \frac{\delta}{2}\Big)F^{ii}U_{ii}^2 + Cb^2\sum_{i\in J}F^{ii} \\ &+ C(\delta^2 U_{11}^2 + b^2)F^{11} + C(1+\sum F^{ii}). \end{aligned} \tag{5.18}$$

Case (i): $|\nu_\mu - \nu_\lambda| \ge \beta$. It follows from (2.6) and (5.18) that

$$\begin{aligned} (b\varepsilon - C)(1+\sum F^{ii}) \le& \Big(C\delta^2 + \frac{C}{U_{11}} - \frac{\delta}{2}\Big)F^{ii}U_{ii}^2 + Cb^2\sum_{i\in J}F^{ii} \\ &+ C(\delta^2 U_{11}^2 + b^2)F^{11}. \end{aligned}$$

Choosing b sufficiently large such that $b\varepsilon - C \ge \frac{b\varepsilon}{2}$, we have

$$\begin{aligned} \frac{b\varepsilon}{2}(1+\sum F^{ii}) \le& \Big(C\delta^2 + \frac{C}{U_{11}} - \frac{\delta}{2}\Big)F^{ii}U_{ii}^2 + Cb^2\sum_{i\in J}F^{ii} \\ &+ C(\delta^2 U_{11}^2 + b^2)F^{11}. \end{aligned}$$

and we can get a bound $U_{11}(x_0, t_0) \le C$ by choosing δ sufficiently small since $|U_{ii}| \ge sU_{11}$ for $i \in J$. Thus we derive a bound of $U_{11}(x_0, t_0)$ and therefore (5.1) holds.

Case (ii): $|\nu_\mu - \nu_\lambda| < \beta$. For every fixed $C > 0$, choosing δ sufficiently small such that $\frac{\delta}{4} - C\delta^2 \ge \delta_0 > 0$. Without loss of generality, suppose $U_{11} \ge \frac{C}{\delta_0}$ for otherwise we are done. Then (5.18) becomes

$$b\mathcal{L}\eta + \frac{\delta}{4}F^{ii}U_{ii}^2 \le Cb^2\sum_{i\in J}F^{ii} + C(\delta^2 U_{11}^2 + b^2)F^{11} + C(1+\sum F^{ii}). \tag{5.19}$$

Next, let $\hat{\lambda} := \lambda(U(x_0, t_0))$. In the view of (4.15)–(4.17), we have

$$|\hat{\lambda}|\Big(\sum F^{ii} + F^{\tau}\Big) \geq b_3, \tag{5.20}$$

where $b_3 := \frac{1}{2}\Big(f(|\hat{\lambda}|\mathbf{1}) - \sup_{\bar{M}_T} \psi[u]\Big) > 0$ provided $|\hat{\lambda}|$ is large enough. By (2.10) and (5.20), we have

$$\frac{\delta}{4}F^{ii}U_{ii}^2 \geq 2c_2|\hat{\lambda}|^2\Big(\sum F^{ii} + F^{\tau}\Big) \geq c_2|\hat{\lambda}|^2\Big(\sum F^{ii} + F^{\tau}\Big) + c_2 b_3|\hat{\lambda}|,$$

where $c_2 = \frac{\delta\beta}{8\sqrt{n+1}}$. Therefore, it follows from (4.13) and (5.19) that

$$c_2|\hat{\lambda}|^2\Big(\sum F^{ii} + F^{\tau}\Big) + c_2 b_3|\hat{\lambda}| \leq C\delta^2 U_{11}^2 F^{11} + C(1 + \sum F^{ii}). \tag{5.21}$$

Then a bound for U_{11} is derived since $\delta \in (0, 1)$ and $U_{11} \leq |\hat{\lambda}|$.

□

6. Boundary estimates for second derivatives

In this section, we establish the estimates of second order derivatives on parabolic boundary $\mathcal{P}M_T$. We may assume $\varphi \in C^4(\bar{M}_T)$. We shall establish the estimate

$$\max_{\mathcal{P}M_T} |\nabla^2 u| \leq C_2 \tag{6.1}$$

for some positive constant C_2 depending on $|u|_{C^1\bar{M}_T}$, $|u_t|_{C^0\bar{M}_T}$, $|\underline{u}|_{C^2\bar{M}_T}$, $|\psi|_{C^4\bar{M}_T}$, and other known data.

Fix a point $(x_0, t_0) \in SM_T$. We shall choose smooth orthonormal local frames $e_1, \ldots, e_n$ around x_0 such that when restricted to ∂M, e_n is the interior normal to ∂M along the boundary when restricted to ∂M. Since $u - \underline{u} = 0$ on SM_T we have

$$\nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n(u - \underline{u})\Pi(e_\alpha, e_\beta), \quad \forall\ 1 \leq \alpha, \beta < n \ \text{ on } SM_T, \tag{6.2}$$

where Π denotes the second fundamental form of ∂M. Therefore,

$$|\nabla_{\alpha\beta}u| \leq C, \quad \forall\ 1 \leq \alpha, \beta < n \ \text{ on } \ SM_T. \tag{6.3}$$

Let $\rho(x)$ and $d(x)$ denote the distance from $x \in M$ to x_0 and ∂M respectively and set

$$M_T^\delta = \{X = (x, t) \in M \times (0, T] : \rho(x) < \delta\}.$$

Now we shall use a perturbation method to obtain a strict subsolution from a non-strict one. Let $s(x, t) = \underline{u}(x, t) + a(h(x) - 1)$ and $S = \{\nabla_{ij}s + A[s]\}$, where $h(x) = e^{bd(x)}$, a and b are constants to be determined. We wish to show $\tilde{M} = (F(S, -s_t) - \psi[s]) - (F(\underline{U}, -\underline{u}_t) - \psi[\underline{u}]) > 0$ for some a and b. Note that d is smooth near boundary and

$$S_{ij} - \underline{U}_{ij} = ab^2h\nabla_i d\nabla_j d + abh\nabla_{ij}d + abhA_{p_k}^{ij}(x, t, \hat{p}_1)\nabla_k d,$$

where $\hat{p}_1 = \nabla\underline{u} + \theta_1 abh\nabla d$ for some $\theta_1 \in (0,1)$. Therefore, if a is small enough for fixed b, s is admissible since $\underline{u}$ is admissible and Γ is open. Let $F_0^{ij} = F^{ij}(\underline{U}, -\underline{u}_t)$, there is a positive constant c_3 such that $F_0^{ij}\nabla_i d\nabla_j d \geq c_3 > 0$ since $|\nabla d(x)| \equiv 1$. Thus, we derive

$$\begin{aligned}\tilde{M} \geq\, & F_0^{ij}(ab^2h\nabla_i d\nabla_j d + abh\nabla_{ij}d + abhA_{p_k}^{ij}(x,t,\tilde{p})\nabla_k d)\\ & - abh\psi_{p_k}(x,t,\hat{p}_2)\nabla_k d\\ \geq\, & ab^2hc_3 - abC > 0,\end{aligned}$$

where $b > C/c_3 \geq C/hc_3$ and $\hat{p}_2 = \nabla\underline{u} + \theta_2 abh\nabla d$ for some $\theta_2 \in (0,1)$.

Therefore a strict admissible subsolution with same boundary condition is derived near boundary and (2.6) holds without the assumption $|\nu_\mu - \nu_\lambda| \geq \beta$, see Remark 1. For convenience, we still use $\underline{u}$ to denote the strict subsolution below.

For the mixed tangential-normal and pure normal second derivatives at (x_0, t_0), we shall use the following barrier function as in [16],

$$\Psi = A_1 v + A_2\rho^2 - A_3\sum_{l<n}|\nabla_l(u-\varphi)|^2, \tag{6.4}$$

where

$$v = 1 - \eta = 1 - e^{K(\underline{u}-u)}$$

and A_1, A_2, A_3 are positive constants to be chosen. By differentiating Eq (1.2) and

$$\nabla_{ij}(\nabla_k u) = \nabla_{ijk}u + \Gamma_{ik}^l\nabla_{jl}u + \Gamma_{jk}^l\nabla_{il}u + \nabla_{\nabla_{ij}e_k}u,$$

we obtain, by straightforward calculation,

$$\mathcal{L}(\nabla_k(u-\varphi)) \leq C\Big(1 + \sum f_i|\lambda_i| + \sum f_i + F^\tau\Big), \quad \forall\ 1 \leq k \leq n, \tag{6.5}$$

where $\lambda = \lambda(\nabla^2 u + A[u])$.

The following lemma is crucial to construct barrier functions.

Lemma 8. *Suppose that* (1.6)–(1.8) *and* (1.9)–(1.11) *hold. Then for any positive constant K_1 there exist uniform positive constants t, δ sufficiently small, and A_1, A_2, A_3 sufficiently large such that $\Psi \geq K_1\rho^2$ in $\overline{M_T^\delta}$ and*

$$\mathcal{L}\Psi \leq -K_1\Big(1 + f_i|\lambda_i| + \sum f_i + F^\tau\Big) \ \text{ in } \overline{M_T^\delta}. \tag{6.6}$$

Proof. First by Lemma 4, we have

$$\mathcal{L}v \leq -\varepsilon\Big(1 + \sum f_i + F^\tau\Big) \text{ in } M_T^\delta. \tag{6.7}$$

Similar to Proposition 2.19 of [16], we can show that

$$\sum_{l<n}F^{ij}U_{il}U_{jl} \geq \frac{1}{2}\sum_{i\neq r}f_i\lambda_i^2, \tag{6.8}$$

for some index r. It follows that

$$\begin{aligned}\sum_{l<n}\mathcal{L}|\nabla_l(u-\varphi)|^2 &\geq \sum_{l<n}F^{ij}U_{il}U_{jl} - C\Big(1+\sum f_i|\lambda_i| + \sum F^{ii} + F^\tau\Big)\\ &\geq \frac{1}{2}\sum_{i\neq r} f_i\lambda_i^2 - C\Big(1+\sum f_i|\lambda_i| + \sum F^{ii} + F^\tau\Big).\end{aligned} \tag{6.9}$$

We first consider the case that $\lambda_r \geq 0$. Notice that

$$\begin{aligned}\mathcal{L}v = -Le^{K(\underline{u}-u)} &= -Ke^{K(\underline{u}-u)}[\mathcal{L}(\underline{u}-u) + KF^{ij}D_i(\underline{u}-u)D_j(\underline{u}-u)]\\ &\geq a_0\sum f_i\lambda_i - C(1+\sum F^{ii} + F^\tau),\end{aligned}$$

where $a_0 = \inf_{\mathcal{P}M_T} Ke^{K(\underline{u}-u)}$.

By (6.7), (6.8) and (6.9), we obtain, for any $0 < B < A_1$,

$$\begin{aligned}\mathcal{L}\Psi \leq& (A_1+B)\mathcal{L}v - B\mathcal{L}v + CA_2\Big(1+\sum f_i + F^\tau\Big) - \frac{A_3}{2}\sum_{i\neq r} f_i\lambda_i^2\\ &+ CA_3\Big(1+f_i|\lambda_i| + \sum f_i + F^\tau\Big)\\ \leq& -(A_1+B)\varepsilon\Big(1+\sum f_i + F^\tau\Big) - a_0Bf_i\lambda_i + CA_3f_i|\lambda_i|\\ &- \frac{A_3}{2}\sum_{i\neq r} f_i\lambda_i^2 + C(B+A_2+A_3)\Big(1+\sum f_i + F^\tau\Big)\\ \leq& -(A_1+B)\varepsilon\Big(1+\sum f_i + F^\tau\Big) + 2a_0B\sum_{i\neq r} f_i|\lambda_i| - \frac{A_3}{2}\sum_{i\neq r} f_i\lambda_i^2\\ &- (a_0B - CA_3)f_i|\lambda_i| + C(B+A_2+A_3)\Big(1+\sum f_i + F^\tau\Big).\end{aligned} \tag{6.10}$$

Notice that

$$\frac{A_3}{2}\sum_{i\neq r} f_i\lambda_i^2 \geq 2a_0B\sum_{i\neq r} f_i|\lambda_i| - \frac{2(a_0B)^2}{A_3}\sum f_i. \tag{6.11}$$

Thus, we derive from (6.10) and (6.11) that

$$\begin{aligned}\mathcal{L}\Psi \leq& -(A_1+B)\varepsilon\Big(1+\sum f_i + F^\tau\Big) - (a_0B - CA_3)f_i|\lambda_i|\\ &+ C(B+A_2+A_3)\Big(1+\sum f_i + F^\tau\Big) + \frac{2(a_0B)^2}{A_3}\sum f_i.\end{aligned} \tag{6.12}$$

If $\lambda_r < 0$, similarly to (6.12), we have

$$\begin{aligned}\mathcal{L}\Psi \leq& -(A_1+B)\varepsilon\Big(1+\sum f_i + F^\tau\Big) - (a_1B - CA_3)f_i|\lambda_i|\\ &+ C(B+A_2+A_3)\Big(1+\sum f_i + F^\tau\Big) + \frac{2(a_1B)^2}{A_3}\sum f_i,\end{aligned} \tag{6.13}$$

where $a_1 = \sup_{\mathcal{P}M_T} Ke^{K(\underline{u}-u)}$.

Checking (6.12) and (6.13), we can choose $A_1 \gg A_2 \gg A_3 \gg 1$ and $A_1 - B \gg a_1B \geq a_0B \gg A_2 \gg A_3$ in (6.12) and (6.13) such that (6.6) holds and $\Psi \geq K_1\rho^2$ in M_T^δ. □

By (6.5) and (6.6), we can use Lemma 8 to choose suitable δ, N and $A_1 \gg A_2 \gg A_3 \gg 1$ such that in M_T^δ, $\mathcal{L}(\Psi \pm \nabla_\alpha(u-\phi)) \leq 0$, and $\Psi \pm \nabla_\alpha(u-\phi) \geq 0$ on $\mathcal{P}M_T^\delta$. Then it follows from the maximum principle that $\Psi \pm \nabla_\alpha(u-\phi) \geq 0$ in M_T^δ and therefore

$$|\nabla_{n\alpha}u(x_0,t_0)| \leq \nabla_n\Psi(x_0,t_0) \leq C, \quad \forall\ \alpha < n. \tag{6.14}$$

It remains to show that

$$\nabla_{nn}u(x_0,t_0) \leq C \tag{6.15}$$

since $\triangle u - u_t + trA > 0$. We shall use an idea of Trudinger [32] to prove that there exist uniform positive constants c_0, R_0 such that for all $R > R_0$, $(\lambda'[U], R, -u_t) \in \Gamma$ and

$$f(\lambda'[U], R, -u_t) \geq \psi[u] + c_0 \text{ on } \overline{SM_T},$$

which implies (6.15) by Lemma 1.2 in [18], where $\lambda'[U] = (\lambda'_1, \ldots, \lambda'_{n-1})$ denote the eigenvalues of the $(n-1)\times(n-1)$ matrix $\{U_{\alpha\beta}\}_{1\leq\alpha,\beta\leq(n-1)}$ and $\psi[u] = \psi(\cdot,\cdot,\nabla u)$. Define

$$\widetilde{F}(U_{\alpha\beta}, -u_t) \equiv \lim_{R\to+\infty} f(\lambda'(\{U_{\alpha\beta}\}), R, -u_t)$$

and consider

$$m \equiv \min_{(x,t)\in\overline{SM_T}} \Big(\widetilde{F}(U_{\alpha\beta}(x,t), -u_t(x,t)) - \psi[u](x,t)\Big).$$

Note that $\widetilde{F}$ is concave and m is monotonically increasing with respect to R, and that

$$c \equiv \min_{(x,t)\in\overline{SM_T}} \Big(\widetilde{F}(\underline{U}_{\alpha\beta}(x,t), -\underline{u}_t(x,t)) - \psi[\underline{u}](x,t)\Big) > 0$$

when R is sufficiently large.

We shall show $m > 0$ and we may assume $m < c/2$ (otherwise we are done) and suppose m is achieved at a point $(x_0,t_0) \in \overline{SM_T}$. Choose local orthonormal frames around x_0 as before and assume $\nabla_{nn}u(x_0,t_0) \geq \nabla_{nn}\underline{u}(x_0,t_0)$. Let $\sigma_{\alpha\beta} = \langle\nabla_\alpha e_\beta, e_n\rangle$ and

$$\widetilde{F}_0^{\alpha\beta} = \frac{\partial\widetilde{F}}{\partial r_{\alpha\beta}}(U_{\alpha\beta}(x_0,t_0), -u_t(x_0,t_0)),$$

$$\widetilde{F}_0^{\tau} = \frac{\partial\widetilde{F}}{\partial \tau}(U_{\alpha\beta}(x_0,t_0), -u_t(x_0,t_0)).$$

Note that $\sigma_{\alpha\beta} = \Pi(e_\alpha, e_\beta)$ on ∂M and by (6.2), we have, at (x_0,t_0),

$$\begin{aligned}\nabla_n(u-\underline{u})\widetilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta} &\geq \widetilde{F}(\underline{U}_{\alpha\beta}, -\underline{u}_t) - \widetilde{F}(U_{\alpha\beta}, -u_t) + \widetilde{F}_0^{\tau}(\underline{u}_t - u_t)\\ &\quad + \widetilde{F}_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}])\\ &\geq \frac{c}{2} + H[u] - H[\underline{u}]\\ &\geq \frac{c}{2} + H_{p_n}\nabla_n(u-\underline{u}),\end{aligned} \tag{6.16}$$

where $H[u] = \widetilde{F}_0^{\alpha\beta}A^{\alpha\beta}[u] - \psi[u]$. The last inequality is from the regularity of $-A$ and the convexity of ψ with respect to p.

Note that $-A$ is regular, which means $A^{\alpha\beta}$ is concave respect to p_n and $\underline{u}$ is strict subsolution near the boundary, we have $H_{p_n p_n} \leq 0$ and

$$0 < \nabla_n(u - \underline{u}) < c_4$$

for some positive constant c_4. It follows from (6.16) that, at (x_0, t_0),

$$\kappa - H_{p_n} \geq \frac{c}{2c_4} > 0, \tag{6.17}$$

where $\kappa = \widetilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}$.

Let $\vartheta(x,t) = \kappa(x,t) - H_{p_n}(x,t,\nabla'\varphi(x,t),\nabla_n u(x_0,t_0))$. Since $\nabla_\alpha u = \nabla_\alpha \underline{u} = \nabla_\alpha\varphi$ on $\overline{SM_T}$, we derive

$$\vartheta(x,t) > c_5 \;\text{ on } \partial M_T^\delta \cap \overline{SM_T} \tag{6.18}$$

for some small positive constant c_5, where $\nabla'\varphi = (\nabla_1\varphi, \ldots, \nabla_{n-1}\varphi)$.

Next, since H is concave with respect to p_n, we have

$$\begin{aligned} &H(x,t,\nabla'\varphi,\nabla_n u(x_0,t_0)) - H(x,t,\nabla'\varphi,\nabla_n u) \\ &\geq H_{p_n}(x,t,\nabla'\varphi,\nabla_n u(x_0,t_0))(\nabla_n u(x_0,t_0) - \nabla_n u) \end{aligned} \tag{6.19}$$

on $\overline{SM_T}$.

On the other hand, since $u_t = \underline{u}_t = \varphi_t$ on $\overline{SM_T}$, by the concavity of $\widetilde{F}$, we have

$$\begin{aligned} &H(x,t,\nabla'\varphi,\nabla_n u(x,t)) - H(x_0,t_0,\nabla'\varphi(x_0,t_0),\nabla_n u(x_0,t_0)) \\ &\quad + \widetilde{F}_0^{\alpha\beta}(\nabla_{\alpha\beta}u - \nabla_{\alpha\beta}u(x_0,t_0)) + \widetilde{F}_0^\tau\varphi_t - \widetilde{F}_0^\tau\varphi_t(x_0,t_0) \\ &= \widetilde{F}_0^{\alpha\beta}U_{\alpha\beta} - \psi[u] - \widetilde{F}_0^\tau u_t - \widetilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0,t_0) + \psi[u](x_0,t_0) + \widetilde{F}_0^\tau u_t(x_0,t_0) \\ &\geq \widetilde{F}(U_{\alpha\beta}, -u_t) - \psi[u] - m \geq 0 \end{aligned} \tag{6.20}$$

on $\overline{SM_T}$. It follows from (6.2), (6.19) and (6.20) that

$$\begin{aligned} &-\vartheta(\nabla_n(u-\varphi) - \nabla_n(u-\varphi)(x_0,t_0)) \\ &\geq \widetilde{F}^{\alpha\beta}[\nabla_n(u-\varphi)(x_0,t_0)(\sigma_{\alpha\beta}(x_0,t_0) - \sigma_{\alpha\beta}) + \nabla_{\alpha\beta}\varphi(x_0,t_0) - \nabla_{\alpha\beta}\varphi] \\ &\quad + H(x,t,\nabla'\varphi,\nabla_n u(x_0,t_0)) - H(x_0,t_0,\nabla'\varphi(x_0,t_0),\nabla_n u(x_0,t_0)) \\ &\quad + H_{p_n}(x,t,\nabla'\varphi,\nabla_n u(x_0,t_0))(\nabla_n\varphi(x_0,t_0) - \nabla_n\varphi) + \widetilde{F}_0^\tau\varphi_t(x_0,t_0) \\ &\quad - \widetilde{F}_0^\tau\varphi_t \\ &:= \Theta(x,t). \end{aligned} \tag{6.21}$$

From the form of the function $\Theta(x,t)$ in (6.21), since $\Theta(x_0,t_0) = 0$, we have, on $\partial M_T^\delta \cap \overline{SM_T}$,

$$\begin{aligned} \nabla_n(u-\varphi) - \nabla_n(u-\varphi)(\tilde{x}_0) &\leq \vartheta^{-1}\Theta(x,t) \\ &\leq l(\tilde{x} - \tilde{x}_0) + \tilde{C}(\rho^2 + (t-t_0)^2), \end{aligned} \tag{6.22}$$

where $\tilde{x} = (x,t)$, l is a linear function of $\tilde{x} - \tilde{x}_0$ with $l(0) = 0$, and the constant C depends on $|u|_{C^1}$ and other known data.

Define

$$\Phi = \nabla_n(u-\varphi) - \nabla_n(u-\varphi)(\tilde{x}_0) - l(\tilde{x}-\tilde{x}_0) - \tilde{C}(t-t_0)^2.$$

By extending φ smoothly to the interior near the boundary to be constant in the normal direction, By (6.5), we have

$$\mathcal{L}\Phi \le C(1+\sum f_i + \sum f_i|\lambda_i| + F^\tau).$$

We see from (6.20) and (6.2) that $\Phi \ge 0$ on $\overline{SM_T}$ and $\Phi(x_0,t_0)=0$. Therefore, by the compatibility condition (1.14), we have, when δ is sufficiently small, $\Psi \ge 0$ on $\mathcal{P}M_\delta$.

Therefore, by Lemma 8, we can choose suitable Ψ such that

$$\begin{cases} \mathcal{L}(\Psi-\Phi) \le 0 & \text{in } M_T^\delta, \\ \Psi-\Phi \ge 0 & \text{on } \mathcal{P}M_T^\delta. \end{cases} \tag{6.23}$$

By the maximum principle we find $\Psi \ge \Phi$ in M_T^δ. It follows that $\nabla_n\Phi(x_0,t_0) \le \nabla_n\Psi(x_0,t_0) \le C$.

Therefore, we have an *a priori* upper bound for all eigenvalues of $\{U_{ij}(x_0,t_0)\}$ and hence its eigenvalues are contained in a compact subset of Γ by (1.8), and we see $m>0$ by (1.6).

Consequently, there exist positive c_6 and R_0 such that

$$(\lambda'(\widetilde{U}(x,t)), R, -u_t(x,t)) \in \Gamma$$

and

$$f(\lambda'(\widetilde{U}(x,t)), R, -u_t(x,t)) \ge \psi(x,t) + c_6$$

for all $R > R_0$ and $(x,t) \in \overline{SM_T}$

For $i = 1,\dots,n-1$, Lemma 1.2 in [18] means $\lambda_i' = \lambda_i + o(1)$ if $|U_{nn}|$ tends to infinity. Therefore, we have

$$f(\lambda(U), -u_t) > \psi$$

for unbounded $|U_{nn}|$, which leads a contradiction and therefore (6.15) holds.

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References

1. G. Bao, W. Dong, H. Jiao, The first initial–boundary value problem for Hessian equations of parabolic type on Riemannian manifolds, *Nonlinear Anal. Theory Methods Appl.*, **143** (2016), 45–63. https://doi.org/10.1016/j.na.2016.05.005
2. F. Jiang, N. S. Trudinger, On the Dirichlet problem for general augmented Hessian equations, *J. Differ. Equ.*, **269** (2020), 5204–5227. https://doi.org/10.1016/j.jde.2020.04.004
3. H. Jiao, Second order estimates for Hessian equations of parabolic type on Riemannian manifolds, *J. Differ. Equ.*, **259** (2015), 7662–7680. https://doi.org/10.1016/j.jde.2015.08.031

4. N. S. Trudinger, X.-J. Wang, The Monge-Ampère equation and its geometric applications, in *Handbook of geometric analysis. No. 1*, vol. 7 of Adv. Lect. Math. (ALM), Int. Press, Somerville, MA, 2008, 467–524.
5. X. Ma, N. S. Trudinger, X. Wang, Regularity of potential functions of the optimal transportation problem, *Arch. Ration. Mech. An.*, **177** (2005), 151–183. https://doi.org/10.1007/s00205-005-0362-9
6. F. Jiang, N. S. Trudinger, X. Yang, On the Dirichlet problem for Monge-Ampère type equations, *Calc. Var. Partial Differ. Equ.*, **49** (2014), 1223–1236.
7. B. Guan, H. Jiao, The Dirichlet problem for Hessian type elliptic equations on Riemannian manifolds, *Discrete Contin. Dyn. Syst.*, **36** (2016), 701–714. https://doi.org/10.3934/dcds.2016.36.701
8. N. M. Ivochkina, O. A. Ladyzhenskaya, Parabolic equations generated by symmetric functions of the eigenvalues of the Hessian or by the principal curvatures of a surface. I. Parabolic Monge-Amère equations, *Algebra i Analiz*, **6** (1994), 141–160.
9. N. M. Ivochkina, O. A. Ladyzhenskaya, Flows generated by symmetric functions of the eigenvalues of the Hessian, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.*, **221** (1995), 127–144. https://doi.org/10.1007/BF02355587
10. H. Jiao, Z. Sui, The first initial-boundary value problem for a class of fully nonlinear parabolic equations on Riemannian manifolds, *Int. Math. Res. Notices*, **2** (2014), 151. https://doi.org/10.1093/imrn/rnu014
11. N. V. Krylov, Sequences of convex functions, and estimates of the maximum of the solution of a parabolic equation, *Sibirsk. Mat. Ž.*, **17** (1976), 290–303.
12. G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, Singapore, 1996.
13. Y. Li, Some existence results for fully nonlinear elliptic equations of Monge-Ampère type, *Comm. Pure Appl. Math.*, **43** (1990), 233–271.
14. J. Urbas, Hessian equations on compact Riemannian manifolds, In *Nonlinear problems in mathematical physics and related topics, II*, Int. Math. Ser. (N. Y.), Kluwer/Plenum, New York, New York, 2002.
15. B. Guan, The Dirichlet problem for Hessian equations on Riemannian manifolds, *Calc. Var. Partial Differ. Equ.*, **8** (1999), 45–69. https://doi.org/10.1007/s005260050116
16. B. Guan, Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, *Duke Math. J.*, **163** (2014), 1491–1524. https://doi.org/10.1215/00127094-2713591
17. B. Guan, H. Jiao, Second order estimates for Hessian type fully nonlinear elliptic equations on Riemannian manifolds, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 2693–2712. https://doi.org/10.1007/s00526-015-0880-8
18. L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. iii. Functions of the eigenvalues of the Hessian, *Acta Math.*, **155** (1985), 261–301. https://doi.org/10.1007/BF02392544

19. E. Andriyanova, On the Dirichlet problem for degenerate Monge–Ampère type equations, *Calc. Var. Partial Differ. Equ.*, **58** (2019), 1–22.

20. S. Du, Q. Li, Positivity of Ma-Trudinger-Wang curvature on Riemannian surfaces, *Calc. Var. Partial Differ. Equ.*, **51** (2014), 495–523. https://doi.org/10.1007/s00526-013-0684-7

21. A. Figalli, L. Rifford, C. Villani, On the Ma–Trudinger–Wang curvature on surfaces, *Calc. Var. Partial Differ. Equ.*, **39** (2010), 307–332. https://doi.org/10.1007/s00526-010-0311-9

22. A. Figalli, L. Rifford, C. Villani, Necessary and sufficient conditions for continuity of optimal transport maps on Riemannian manifolds, *Tohoku Math. J.*, **63** (2011), 855–876. https://doi.org/10.2748/tmj/1325886291

23. C. Goodrich, A. Scapellato, Partial regularity of minimizers of asymptotically 2 convex functionals with p(x)-growth, *Studia Math.*, **264** (2022), 71–102. https://doi.org/10.4064/sm210104-20-9

24. G. Loeper, N. S. Trudinger, Weak formulation of the MTW condition and convexity properties of potentials, *Methods Appl. Anal.*, **28** (2021), 53–60. https://doi.org/10.4310/MAA.2021.v28.n1.a4

25. M. A. Ragusa, On weak solutions of ultraparabolic equations, *Nonlinear Anal. Theory Methods Appl.*, **47** (2001), 503–511. https://doi.org/10.1016/S0362-546X(01)00195-X

26. F. Jiang, N. S. Trudinger, Oblique boundary value problems for augmented Hessian equations ii, *Nonlinear Anal. Theory Methods Appl.*, **154** (2017), 148–173. https://doi.org/10.1016/j.na.2016.08.007

27. B. Guan, S. Shi, Z. Sui, On estimates for fully nonlinear parabolic equations on Riemannian manifolds, *Anal. PDE*, **8** (2015), 1145–1164. https://doi.org/10.2140/apde.2015.8.1145

28. F. Jiang, N. S. Trudinger, Oblique boundary value problems for augmented Hessian equations I, *Bull. Math. Sci.*, **8** (2018), 353–411.

29. B. Guan, J. Spruck, Interior gradient estimates for solutions of prescribed curvature equations of parabolic type, *Indiana Univ. Math. J.*, **40** (1991), 1471–1481.

30. B. Andrews, Contraction of convex hypersurfaces in Euclidean space, *Calc. Var. Partial Differ. Equ.*, **2** (1994), 151–171. https://doi.org/10.1007/BF01191340

31. C. Gerhardt, Closed Weingarten hypersurfaces in Riemannian manifolds, *J. Differ. Geom.*, **43** (1996), 612–641. https://doi.org/10.4310/jdg/1214458325

32. N. S. Trudinger, On the Dirichlet problem for hessian equations, *Acta Math.*, **175** (1995), 151–164. https://doi.org/10.1007/BF02393303